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PERTURBATION ANALYSIS OF THE MOORE–PENROSE METRIC GENERALIZED INVERSE WITH APPLICATIONS

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Abstract. In this article, based on some geometric properties of Banach spaces and one feature of the metric projection, we introduce a new class of bounded linear operators satisfying the so-called (α, β)-*USU* (*uniformly strong uniqueness*) *property*. This new convenient property allows us to take the study of the stability problem of the Moore–Penrose metric generalized inverse a step further. As a result, we obtain various perturbation bounds of the Moore–Penrose metric generalized inverse of the perturbed operator. They offer the advantage that we do not need the quasiadditivity assumption, and the results obtained appear to be the most general case found to date. Closely connected to the main perturbation results, one application, the error estimate for projecting a point onto a linear manifold problem, is also investigated.

1. Introduction

Let X and Y be Banach spaces. We denote by $B(X, Y)$ the Banach space of all bounded linear operators $T : X \to Y$. We use $\mathcal{N}(T)$ and $\mathcal{R}(T)$ to denote the kernel and the range of T, respectively. We always use I to denote the identity operator on X or Y. If $X = Y$, then we write $B(X) = B(X, X)$. Let $T \in B(X, Y)$, and let $b \in Y$ be a fixed vector. Considering the operator equation $Tx = b$, we know that when T is invertible, then $T^{-1}b$ is the unique solution. But, in many situations, the residual vector $r = Tx - b$ is a nonzero vector for any $x \in X$. It

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may be desired to find a vector x minimizing the norm of the residual vector r. Thus, in this case, we should consider the following least problem:

$$\inf \|x\| \quad \text{subject to} \quad \|Tx - b\| = \inf_{z \in X} \|Tz - b\|. \tag{1.1}$$

Note that if we consider the problem (1.1) in infinite-dimensional spaces, then, in general, (1.1) may not have a solution (see [10, Example 3.1]). But it is clear that if the range $\mathcal{R}(T)$ of T is finite-dimensional, then there is a solution to problem (1.1). In particular, when X and Y are finite-dimensional vector spaces, in some applications, a so-called *least squares solution* of $Tx = b$ is often used. It is a well-known fact that among the least squares solutions of $Tx = b$, there is a unique solution x^* of minimum norm which is said to be the *best approximation solution* (or *minimum norm least squares solution*) of $Tx = b$, given by $x^* = T^\dagger b$, where $T^\dagger$ denotes the Moore–Penrose generalized inverse of T.

The use of perturbation analysis for solving problem (1.1) is important in many applications. For example, problem (1.1) plays an important role in mathematical programming and numerical analysis. It is also closely related to the concept of generalized solutions of ordinary and partial differential and integral equations (see [1], [21]). The perturbation problems of (1.1) in Hilbert spaces have been widely studied with numerous results obtained using the perturbation bounds of the Moore–Penrose orthogonal projection generalized inverse $T^\dagger$ (see [9], [10], [13], [21], [29]). In order to more adequately investigate the perturbation analysis problem for the operator equation $Tx = b$ in Banach spaces, Chen and the second author [7] introduced an important notation, the so-called *stable perturbation* of operators on Banach spaces: that is, if $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^+) = \{0\}$, then $\bar{T}$ is said to be the *stable perturbation* of T. The recent literature has seen increased interest in the stable perturbation theory of generalized inverses. In a previous work, the second author and Chen [29] further investigated this concept and some of its important applications in Hilbert spaces. As a consequence, they got the following important perturbation results. That is, when X, Y are Hilbert spaces, if $T, \bar{T} = T + \delta T \in B(X, Y)$ with $\mathcal{R}(T)$ closed, $\|T^\dagger\|\|\delta T\| < 1$, and $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^\dagger) = \{0\}$, then $\bar{T}^\dagger$ exists and

$$\|\bar{T}^\dagger - T^\dagger\| \leq \frac{1 + \sqrt{5}}{2} \frac{\|T^\dagger\|^2 \|\delta T\|}{1 - \|T^\dagger\|\|\delta T\|}. \tag{1.2}$$

The perturbation bound (1.2) above has many applications, especially in solving the least problem (1.1) and the following problem for projecting a point onto a linear manifold (see [29, Proposition 8]). For the given $T \in B(X, Y)$ with $\mathcal{R}(T)$ closed, $b \in Y$ and $f \in X$, find a vector $x^* \in X$ satisfying

$$\begin{aligned} &\|f - x^*\| = \inf_{x \in S} \|f - x\| \\ &\quad \text{subject to} \quad S = \big\{x \in X : \|Tx - b\| = \inf_{z \in X} \|Tz - b\|\big\}. \end{aligned} \tag{1.3}$$

We have indicated that the problem (1.1) (see also (1.3)) may not have a solution in general infinite-dimensional spaces. But if X, Y are reflexive Banach spaces, it follows from [16, Theorem 6.2] that the problem (1.1) has solutions when $\mathcal{R}(T)$ is closed. When X is a reflexive strictly convex Banach space and Y

is a reflexive Banach space, it follows from [28, Proposition 2.3.7] that problem (1.1) also has a unique solution $x = x_m$. But, unfortunately, there is a problem associated with expressing the solution $x = x_m$ in general Banach space. In order to solve the best approximation problem for an ill-posed linear operator equation in Banach spaces, in 1976 Nashed and Votruba [21] introduced the so-called (*set-valued*) *metric generalized inverse* of a linear operator between Banach spaces.

Let $T : X \to Y$ be a linear operator, and consider an element $b \in Y$ such that $Tx = b$ has the *best approximation solution* (*b.a.s.*) in X. We define

$$T^{\partial}(b) = \{x \in X : x \text{ is b.a.s. to } Tx = b\},$$

and we call the set-valued mapping $b \to T^{\partial}(b)$ the *metric generalized inverse*. Here

$$D(T^{\partial}) = \{b \in Y : Tx = b \text{ has b.a.s. in } X\}.$$

A single-valued mapping (nonlinear in general) $T^{\sigma} : D(T^{\partial}) \to X$ such that $T^{\sigma}(b) \in T^{\partial}(b)$ is called a *selection* for the metric generalized inverse T^{∂}.

We see that the metric generalized inverse T^{∂} is a set-valued nonlinear mapping. As such, it is generally very difficult to deal with the metric generalized inverse T^{∂}. Over the years, under some assumptions, properties and applications related to the metric generalized inverse T^{∂}, such as continuity, continuous homogeneous selections, and criteria for the single-valued selections, have been studied by some authors, and many important results have been obtained (see [19], [23]). Most importantly, by using the metric projection and Chebyshev subspace in Banach spaces, Wang and Wang [25] introduced the nonlinear *Moore–Penrose metric generalized inverse* T^M (see Definition 2.5 in Section 2) for a linear operator $T \in B(X, Y)$ with closed range in Banach spaces, and gave some useful characterizations. Then Ni [22] defined and further characterized the Moore–Penrose metric generalized inverse for an arbitrary linear operator in Banach spaces. As a result, the unique solution x_m, if it exists, can be expressed as $x_m = T^M b$ (see [26], [25]).

Motivated by related perturbation results obtained in Hilbert spaces, it is natural to consider the following problems in some Banach spaces.

Problem 1.1. Let X, Y be Banach spaces. Let $T, \delta T \in B(X, Y)$ with $\mathcal{R}(T)$ closed. Let $b, \delta b \in Y$. Put $\bar{T} = T + \delta T$ and $\bar{b} = b + \delta b$. Suppose that the Moore–Penrose metric generalized inverse T^M of T exists. Then we might ask the following questions.

(i) When does $\bar{T}^M$ exist and how do we estimate $\|\bar{T}^M\|$?
(ii) If $\bar{T}^M$ exists, can we give some upper bound estimations of $\|\bar{T}^M - T^M\|$?
(iii) How do we solve problem (1.1) (resp., (1.3)) and its perturbation problem?

In recent years, under some quasiadditivity assumptions, then utilizing stable perturbation theory and the gap between closed subspaces, a number of authors have extensively studied the perturbation problems of the Moore–Penrose metric generalized inverse, making some important progress in this direction (see, e.g., [4], [11], [12], [20]). Also, from this previous work, we know that some geometric properties of Banach spaces, such as reflexivity, strict convexity, and smoothness, play a very important role in dealing with the many problems of the Moore–Penrose metric generalized inverse. It is well known that the Banach space $L^p(\Omega,\mu)$ $(1 < p < \infty)$ has many very good geometric structures, as well as very important theoretical and practical applications. Based on the stability of metric projection in $L^p(\Omega,\mu)$ $(1 < p < \infty)$, the first-named author and Zhang [5] recently obtained some error estimates of the upper bound of $\|\bar{T}^M - T^M\|$ in terms of the gap function.

But, we should emphasize that almost all the perturbation results in the previous literature were obtained using some *quasiadditivity* assumptions; thus, the obtained perturbation bounds are greatly limited in application. In this article, based on some geometric properties of Banach spaces and the uniformly strong uniqueness of the metric projection, we first introduce a new class of bounded linear operators satisfying the so-called (α,β)-*USU property* in reflexive strictly convex Banach space, then, utilizing the gap between closed subspaces and the stable perturbation theory, but without using the quasiadditivity assumption, we further study the perturbation problem of the Moore–Penrose metric generalized inverses, subsequently presenting various perturbation bounds of the Moore–Penrose metric generalized inverse of the perturbed operator. In this way, we extend many other perturbation results available in the literature.

The article is organized as follows. In Section 2 below, we recall some necessary concepts and preliminary results. In Section 3, we define and characterize the so-called (α,β)-USU operator. Our main perturbation results for the Moore–Penrose metric generalized inverse will be proved in Section 4. Applications in connection with the main perturbation results will be presented in Section 5. Finally, in Section 6, we conclude with some remarks and a discussion on future lines of research.

2. Preliminaries

Let $T : X \to Y$ be a mapping, and let D be a subset of X. Recall from [18] and [25] that D is called *homogeneous* if $\lambda x \in D$ whenever $x \in D$ and $\lambda \in \mathbb{R}$; a mapping $T\colon X \to Y$ is called a *bounded homogeneous operator* if T maps every bounded set in X into a bounded set in Y, and $T(\lambda x) = \lambda T(x)$ for every $x \in X$ and every $\lambda \in \mathbb{R}$. Let $H(X,Y)$ denote the set of all bounded homogeneous operators from X to Y. Equipped with the usual linear operations on $H(X,Y)$ and the norm on $T \in H(X,Y)$ defined by $\|T\| = \sup\{\|Tx\| \mid \|x\| = 1, x \in X\}$, we can easily prove that $(H(X,Y),\|\cdot\|)$ is a Banach space. Obviously, $B(X,Y) \subset H(X,Y)$. We need the following concept of quasiadditivity for a mapping with respect to some subset.

Definition 2.1 ([3, Definition 2.1], see also [18], [26]). Let $M \subset X$ be a subset, and let $T : X \to Y$ be a mapping. Then we say that T is *quasiadditive* on M if T satisfies

$$T(x+z) = T(x) + T(z), \quad x \in X, z \in M.$$

If T is quasiadditive on $\mathcal{R}(T)$, then we will simply say that T is a *quasilinear operator*. In general, a quasilinear operator is not a linear operator.

Now, we recall the definition of set-valued metric projection.

Definition 2.2 ([24, Definition 4.1]). Let $M \subset X$ be a subset. The set-valued mapping $P_M : X \to M$ defined by

$$P_M(x) = \{ s \in M \mid \|x - s\| = \operatorname{dist}(x, M) \}, \quad \forall x \in X$$

is called the *set-valued metric projection*, where $\operatorname{dist}(x, M) = \inf_{z \in M} \|x - z\|$.

For $M \subset X$, if $P_M(x)$ is nonempty and contains at most a singleton for each $x \in X$, then M is called a *Chebyshev set*. We denote by π_M any selection for the set-valued mapping P_M, that is, any single-valued mapping $\pi_M : \mathcal{D}(\pi_M) \to M$ with the property that $\pi_M(x) \in P_M(x)$ for any $x \in \mathcal{D}(\pi_M)$, where $\mathcal{D}(\pi_M) = \{x \in X : P_M(x) \neq \emptyset\}$. For the particular case when M is a Chebyshev set, the mapping π_M is called the *metric projector* from X onto M.

Remark 2.3 ([24, Section 3.3]). It is well known that if X is a reflexive and strictly convex Banach space, then every closed convex subset in X is a Chebyshev set, and the metric projector is just the linear orthogonal projector in Hilbert space.

We need the following important properties of the metric projection.

Lemma 2.4 ([24, Theorem 4.1])**.** *Let X be a Banach space, and let L be a Chebyshev subspace of X. Then the metric projection π_L is quasiadditive on L. Moreover, $\|x - \pi_L(x)\| \leq \|x\|$ for any $x \in X$; that is, $\|\pi_L\| \leq 2$.*

Now, we present the definition of the Moore–Penrose metric generalized inverse.

Definition 2.5 ([18, Definition 2.1], [26, Definition 4.3.1]). Let $T \in B(X, Y)$. Suppose that $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are Chebyshev subspaces of X and Y, respectively. If there exists a bounded homogeneous operator $T^M : Y \to X$ such that

$$(1) \quad TT^MT = T, \qquad (2) \quad T^MTT^M = T^M,$$

$$(3) \quad T^MT = I_X - \pi_{\mathcal{N}(T)}, \qquad (4) \quad TT^M = \pi_{\mathcal{R}(T)},$$

then T^M is called the *Moore–Penrose metric generalized inverse* of T, where $\pi_{\mathcal{N}(T)}$ and $\pi_{\mathcal{R}(T)}$ are the metric projectors onto $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively.

When X and Y are Hilbert spaces, then from Definition 2.5, we see obviously that the Moore–Penrose metric generalized inverse T^M of T is indeed the Moore–Penrose orthogonal projection generalized inverse $T^\dagger$ of T in the usual sense. It is well known that the theory of the Moore–Penrose metric generalized inverses has its genesis in the context of the so-called *ill-posed* linear problems. (See [26] and [25] for more information about Moore–Penrose metric generalized inverses and related topics.) Here we only need the following result which characterizes the existence of the Moore–Penrose metric generalized inverse in a reflexive and strictly convex Banach space.

Proposition 2.6 ([25, Corollary 2.1])**.** *Let X, Y be reflexive strictly convex Banach spaces, and let $T \in B(X, Y)$ with $\mathcal{R}(T)$ closed. Then there exists a unique Moore–Penrose metric generalized inverse T^M of T.*

We also need some basic results about the reduced minimum module and the gap between two subspaces in a Banach space. (For more information, see [15],

[17], [28].) Let X and Y be Banach spaces, and let $T \in B(X,Y)$. The *reduced minimum module* $\gamma(T)$ of T is defined by

$$\gamma(T) = \inf\big\{\|Tx\| \;\big|\; \mathrm{dist}\big(x, \mathcal{N}(T)\big) = 1, \forall x \in X\big\}.$$

Remark 2.7. From the definition of $\gamma(T)$, it is easy to see that $\|Tx\| \geq \gamma(T)\,\mathrm{dist}(x, \mathcal{N}(T))$ for any $x \in X$. Moreover, according to [15, Theorem 5.2], we know that $\mathcal{R}(T)$ is closed if and only if $\gamma(T) > 0$.

We also need the following useful inequalities between T^M and $\gamma(T)$.

Lemma 2.8 ([11, Lemma 2.14])**.** *Let X and Y be reflexive and strictly convex Banach spaces, and let $T \in B(X,Y)$ with $\mathcal{R}(T)$ closed in Y. Then T^M exists and*

$$\frac{1}{\|T^M\|} \leq \gamma(T) \leq \frac{\|TT^M\|}{\|T^M\|}.$$

Let X be a Banach space, and let M, N be two closed subspaces in X. We denote by $S(N)$ the unit sphere of N (i.e., the set of all $u \in N$ with $\|u\| = 1$). Set

$$\delta(M,N) = \begin{cases} \sup\{\mathrm{dist}(x,N) \mid x \in M, \|x\| = 1\}, & M \neq \{0\}, \\ 0 & M = \{0\}. \end{cases}$$

We call $\hat{\delta}(M,N) = \max\{\delta(M,N), \delta(N,M)\}$ the *gap* between M and N. We know from the literature that the gap function plays an important role in the research of some stability problems. One goal of this article is to establish the perturbation bounds of $\|T^M\|$ in terms of $\hat{\delta}(\mathcal{N}(T), \mathcal{N}(\bar{T}))$. Finally, in this section, we recall the following basic results about the geometric characterizations of a Banach space.

Definition 2.9 ([8, Definitions 1.10, 4.12])*.* Let X be a Banach space. The modulus of convexity δ_X is defined by

$$\delta_X(\epsilon) = \inf\Big\{1 - \frac{1}{2}\|x+y\| \;\Big|\; \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\Big\}, \quad 0 < \epsilon \leq 2.$$

The Banach space X is said to be *uniformly convex* if $\delta_X > 0$. Let $1 < p < \infty$. Then X is said to be *p-uniformly convex* (or to have a *modulus of convexity of power type* p) if there exists a constant $d_X > 0$ such that $\delta_X(\epsilon) \geq d_X \epsilon^p$, $\forall \epsilon \in (0,2]$.

From Definition 2.9, we see that if X is a p-uniformly convex Banach space, then X must be uniformly convex. It is a well-known fact that every uniformly convex Banach space is a reflexive and strictly convex Banach space; thus, the conclusion of Proposition 2.6 works well in p-uniformly convex Banach space. Also, we see that Hilbert spaces, L^p $(1 < p < \infty)$ spaces, and the Sobolev spaces W^p_m $(1 < p < \infty)$ are all p-uniformly convex.

3. The (α, β)-USU operator on reflexive strictly convex Banach spaces

Let X be a Banach space, and let $M \subset X$ be a closed subspace. Recall from [17, Definition 2.1] that the metric projection π_M is said to be *strongly unique* of order $\alpha \geq 1$ at M if for each $x \in X$, there is a constant $\gamma_M(x) \in (0,1]$ (depending only on x and M) such that for every $m \in M$,

$$\gamma_M(x)\|\pi_M x - m\|^\alpha \leq \|x-m\|^\alpha - \|x-\pi_M x\|^\alpha.$$

Now we give a stronger version of above notation as follows.

Definition 3.1. Let X be reflexive strictly convex Banach space, and let M be a closed subspace. The metric projection π_M is said to be *uniformly strongly unique* of order $\alpha \geq 1$ at M if there is a constant $\gamma_M \in (0,1]$ such that *for every* $x \in X$ *and every* $m \in M$,

$$\gamma_M\|\pi_M x - m\|^\alpha \leq \|x-m\|^\alpha - \|x-\pi_M x\|^\alpha. \tag{3.1}$$

Remark 3.2. It is straightforward to ask the following questions. What is the exact value of the constant $\gamma_M(x)$ (resp., γ_M)? And what is it when the metric projection is uniformly strongly unique of order $\alpha \geq 1$ at M? Regarding these questions, for the important Banach space $X = L^p$ $(1 < p < \infty)$ and, more generally, the p-uniformly convex Banach space, we have the following results.

(1) Let $X = L^p$ $(1 < p < \infty)$. Then by [17, Propositions 2.3 and 2.4, Corollary 2.5], $\gamma_M = c_p$, $\alpha = p$ when $p > 2$ in (3.1), where $c_p = (p-1)(1+s)^{2-p}$ and s is the unique positive zero of the function $t^{p-1} - (p-1)t - (p-2)$; and $\gamma_M = p-1$, $\alpha = 2$ when $1 < p \leq 2$.

(2) Let X be a p-uniformly convex Banach space. Then

$$1 - \frac{1}{2}\|x+y\| \geq d_X\|x-y\|^p, \quad \forall x, y \in X \text{ with } \|x\| \leq 1, \|y\| \leq 1. \tag{3.2}$$

For any $x \in X$ and $m \in M$, set $r = \|x-m\| \geq \|x - \pi_M x\|$. Then according to (3.2),

$$\begin{aligned}\|x-\pi_M x\| &\leq \Big\|x - \frac{1}{2}(\pi_M x + m)\Big\| = \frac{r}{2}\Big\|\frac{x-\pi_M x}{r} + \frac{x-m}{r}\Big\| \\ &\leq r\Big[1 - d_X\Big(\frac{\|\pi_M x - m\|}{r}\Big)^p\Big], \\ \|x-\pi_M x\|^p &\leq \|x-m\|^p\Big[1 - d_X\Big(\frac{\|\pi_M x - m\|}{\|x-m\|}\Big)^p\Big]^p \\ &\leq \|x-m\|^p - d_X\|\pi_M x - m\|^p \quad (p > 1).\end{aligned}$$

Thus, in this case, $\gamma_M = d_X$ and $\alpha = p$ in (3.1).

Lemma 3.3. *Let X be a reflexive strictly convex Banach space, and let U, V be two closed subspaces of X. Suppose that π_U is uniformly strongly unique of order α at U. Then*

(1) $\|\pi_U x - \pi_V x\| \leq 10\gamma_U^{-\frac{1}{\alpha}}\hat{\delta}(U,V)^{\frac{1}{\alpha}}\|x\|$, $\forall x \in X$;
(2) $\|\pi_U x - \pi_U y\| \leq (\alpha\gamma_U^{-1})^{\frac{1}{\alpha}} \max\{\|x-\pi_U y\|^{\frac{\alpha-1}{\alpha}}, \|y-\pi_U x\|^{\frac{\alpha-1}{\alpha}}\}\|x-y\|^{\frac{1}{\alpha}}$, $\forall x, y \in X$.

Proof. (1) This assertion comes from [17, Theorem 2.1].

(2) Note that for any $a \geq b \geq 0$ and $\alpha \geq 1$, $a^\alpha - b^\alpha \leq \alpha a^{\alpha-1}(a-b)$. Thus, from (3.1), we get that for any $x, y \in X$,

$$\|\pi_U x - \pi_U y\|^\alpha \leq \alpha\gamma_U^{-1}\|x - \pi_U y\|^{\alpha-1}\big(\|x - \pi_U y\| - \|x - \pi_U x\|\big), \tag{3.3}$$

$$\|\pi_U y - \pi_U x\|^\alpha \leq \alpha\gamma_U^{-1}\|y - \pi_U x\|^{\alpha-1}\big(\|y - \pi_U x\| - \|y - \pi_U y\|\big). \tag{3.4}$$

Since

$$\|x - \pi_U y\| \leq \|x - y\| + \|y - \pi_U y\|, \qquad \|y - \pi_U x\| \leq \|y - x\| + \|x - \pi_U x\|,$$

it follows from (3.3) and (3.4) that

$$2\|\pi_U x - \pi_U y\|^\alpha \leq \alpha\gamma_U^{-1}\max\big\{\|x - \pi_U y\|^{\alpha-1}, \|y - \pi_U x\|^{\alpha-1}\big\}\big(2\|x - y\|\big)$$

and the assertion follows. □

Inspired by the above results and the theory of strong uniqueness in the literature, now, we introduce a new class of operators as follows.

Definition 3.4. Let X and Y be reflexive strictly convex Banach spaces. Let $T \in B(X,Y)$ with $\mathcal{R}(T)$ closed in Y. We say T has the (α,β)-*USU property* if $\pi_{\mathcal{N}(T)}$ is uniformly strongly unique of order $\alpha \geq 1$ at $\mathcal{N}(T)$ and $\pi_{\mathcal{R}(T)}$ is uniformly strongly unique of order $\beta \geq 1$ at $\mathcal{R}(T)$.

Clearly, from the above definition, we see that if X is a p-uniformly convex Banach space and Y is a q-uniformly convex Banach space, then every bounded linear operator from X to Y with closed range has the (p,q)-USU property. With respect to $\pi_{\mathcal{R}(T)}$, $\pi_{\mathcal{N}(T)}$, and T^M, we have the following inequalities, which also show that $\pi_{\mathcal{R}(T)}$, $\pi_{\mathcal{N}(T)}$, and T^M are all continuous when T has the (α,β)-USU property.

Proposition 3.5. *Let X and Y be reflexive strictly convex Banach spaces. Let $T \in B(X,Y)$ with $\mathcal{R}(T)$ closed. If T has the (α,β)-USU property, then for any $x_1, x_2 \in X$ and $y_1, y_2 \in Y$, we have*

(1) $$\|TT^M y_1 - TT^M y_2\| \leq 2\Big(\frac{\beta}{2\gamma_{\mathcal{R}(T)}}\Big)^{\frac{1}{\beta}}\big[\|y_1\| + \|y_2\|\big]^{\frac{\beta-1}{\beta}}\|y_1 - y_2\|^{\frac{1}{\beta}};$$

(2) $$\|T^M T x_1 - T^M T x_2\| \leq \Big[1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big]\big[\|x_1\| + \|x_2\|\big]^{\frac{\alpha-1}{\alpha}}\|x_1 - x_2\|^{\frac{1}{\alpha}};$$

(3) $$\|T^M y_1 - T^M y_2\| \leq 2^{\frac{2\alpha-1}{\alpha}}\Big[1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big]\Big(\frac{\beta}{2\gamma_{\mathcal{R}(T)}}\Big)^{\frac{1}{\alpha\beta}}\|T^M\| \times \big[\|y_1\| + \|y_2\|\big]^{\frac{\alpha\beta-1}{\alpha\beta}}\|y_1 - y_2\|^{\frac{1}{\alpha\beta}}.$$

Proof. (1) Note that $TT^M = \pi_{\mathcal{R}(T)}$. So

$$\|y_1 - \pi_{\mathcal{R}(T)} y_2\| \leq 2\big(\|y_1\| + \|y_2\|\big), \qquad \|y_2 - \pi_{\mathcal{R}(T)} y_1\| \leq 2\big(\|y_1\| + \|y_2\|\big),$$

and hence by Lemma 3.3(2),

$$\|TT^M y_1 - TT^M y_2\| \leq 2\Big(\frac{\beta}{2\gamma_{\mathcal{R}(T)}}\Big)^{\frac{1}{\beta}}\big[\|y_1\| + \|y_2\|\big]^{\frac{\beta-1}{\beta}}\|y_1 - y_2\|^{\frac{1}{\beta}}.$$

(2) Since $T^M T = I - \pi_{\mathcal{N}(T)}$, we have by Lemma 3.3(2),

$$\begin{aligned}
&\|T^M T x_1 - T^M T x_2\| \\
&\quad \le \|x_1 - x_2\| + \|\pi_{\mathcal{N}(T)} x_1 - \pi_{\mathcal{N}(T)} x_2\| \\
&\quad \le \|x_1 - x_2\| + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}} \big[\|x_1\| + \|x_2\|\big]^{\frac{\alpha-1}{\alpha}} \|x_1 - x_2\|^{\frac{1}{\alpha}} \\
&\quad \le \Big[1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big] \big[\|x_1\| + \|x_2\|\big]^{\frac{\alpha-1}{\alpha}} \|x_1 - x_2\|^{\frac{1}{\alpha}}.
\end{aligned}$$

(3) Since $\gamma(T) > 0$ and

$$\begin{aligned}
\big\|T(T^M y_1 - T^M y_2)\big\| &\ge \gamma(T)\operatorname{dist}\big(T^M y_1 - T^M y_2, \mathcal{N}(T)\big) \\
&\ge \frac{1}{\|T^M\|}\operatorname{dist}\big(T^M y_1 - T^M y_2, \mathcal{N}(T)\big),
\end{aligned}$$

we can find $z \in \mathcal{N}(T)$ such that

$$\begin{aligned}
\|T^M y_1 - T^M y_2 - z\| &= \operatorname{dist}\big(T^M y_1 - T^M y_2, \mathcal{N}(T)\big) \\
&\le \|T^M\|\|T T^M y_1 - T T^M y_2\|.
\end{aligned}$$

Thus,

$$\|z\| \le \|T^M\|\|T T^M y_1 - T T^M y_2\| + \|T^M y_1 - T^M y_2\| \le 3\|T^M\|\big(\|y_1\| + \|y_2\|\big).$$

Note that

$$\begin{aligned}
\|T^M y_1 - T^M y_2\| &= \big\|T^M T(T^M y_1 - z) - T^M T(T^M y_2)\big\|, \\
\|T^M y_1 - z\| + \|T^M y_2\| &\le \|T^M y_1\| + \|z\| + \|T^M y_2\| \le 4\|T^M\|\big(\|y_1\| + \|y_2\|\big).
\end{aligned}$$

Therefore, by (2) and (1) in this proposition, we get

$$\begin{aligned}
\|T^M y_1 - T^M y_2\| &\le 2^{\frac{2\alpha-2}{\alpha}} \Big[1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big] \big[\|T^M\|\big(\|y_1\| + \|y_2\|\big)\big]^{\frac{\alpha-1}{\alpha}} \\
&\quad \times \|T^M y_1 - z - T^M y_2\|^{\frac{1}{\alpha}} \\
&\le 2^{\frac{2\alpha-2}{\alpha}} \Big[1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big] \|T^M\| \big[\|y_1\| + \|y_2\|\big]^{\frac{\alpha-1}{\alpha}} \\
&\quad \times \|T T^M y_1 - T T^M y_2\|^{\frac{1}{\alpha}} \\
&\le 2^{\frac{2\alpha-1}{\alpha}} \Big[1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big] \Big(\frac{\beta}{2\gamma_{\mathcal{R}(T)}}\Big)^{\frac{1}{\alpha\beta}} \|T^M\| \big[\|y_1\| + \|y_2\|\big]^{\frac{\alpha\beta-1}{\alpha\beta}} \\
&\quad \times \|y_1 - y_2\|^{\frac{1}{\alpha\beta}}
\end{aligned}$$

and we are done. $\square$

4. Perturbation analysis for the Moore–Penrose metric generalized inverse of the (α, β)-USU operator

In this section, based on some results obtained in Section 3, we mainly study the perturbation analysis problem for T^M under various conditions. Unless stated otherwise, throughout this section, we always assume that X and Y are reflexive and strictly convex Banach spaces. We need the following lemma, which presents some estimates of the perturbation of the reduced minimum module. These results have been proved in [28] for densely defined closed linear operators in general Banach space. For our purposes here, we present these results only for bounded linear operators. By using these perturbation results, we will characterize the existence of the Moore–Penrose metric generalized inverse of the perturbed operator.

Lemma 4.1 ([28, Propositions 6.1.5, 6.1.6])**.** *Let X, Y be Banach spaces, and let $T, \delta T \in B(X, Y)$. Put $\bar{T} = T + \delta T$. Then we have*

$$\gamma(\bar{T}) \geq \gamma(T) \frac{1 - \delta(\mathcal{N}(T), \mathcal{N}(\bar{T}))}{1 + \delta(\mathcal{N}(T), \mathcal{N}(\bar{T}))} - \|\delta T\|. \tag{4.1}$$

In addition, if $\dim \mathcal{N}(\bar{T}) = \dim \mathcal{N}(T) < \infty$ *or* $\dim \mathcal{R}(\bar{T}) = \dim \mathcal{R}(T) < \infty$, *then* $|\gamma(\bar{T}) - \gamma(T)| \leq \|\delta T\|$.

Let $T, \delta T \in B(X, Y)$ with $\mathcal{R}(T)$ closed. For convenience, in the following, we always let $\epsilon_T = \frac{\|\delta T\|}{\|T\|}$ and $\kappa_T = \|T^M\| \|T\|$.

Proposition 4.2. *Let $T, \delta T \in B(X, Y)$ with $\mathcal{R}(T)$ closed. Assume that $\epsilon_T \kappa_T < 1$.*

(1) *If $\delta(\mathcal{N}(T), \mathcal{N}(\bar{T})) < \frac{1 - \epsilon_T \kappa_T}{1 + \epsilon_T \kappa_T}$, then $\bar{T}^M$ exists and*

$$\|\bar{T}^M\| \leq \frac{2\|T^M\|(1 + \delta(\mathcal{N}(T), \mathcal{N}(\bar{T})))}{1 - \epsilon_T \kappa_T - (1 + \epsilon_T \kappa_T)\delta(\mathcal{N}(T), \mathcal{N}(\bar{T}))}.$$

(2) *If* $\dim \mathcal{N}(\bar{T}) = \dim \mathcal{N}(T) < \infty$ *or* $\dim \mathcal{R}(\bar{T}) = \dim \mathcal{R}(T) < \infty$, *then* $\bar{T}^M$ *exists and* $\|\bar{T}^M\| \leq \frac{2\|T^M\|}{1 - \epsilon_T \kappa_T}$.

Proof. (1) We have by (4.1),

$$\gamma(\bar{T}) \geq \frac{1 - \epsilon_T \kappa_T - (1 + \epsilon_T \kappa_T)\delta(\mathcal{N}(T), \mathcal{N}(\bar{T}))}{\|T^M\|(1 + \delta(\mathcal{N}(T), \mathcal{N}(\bar{T}))}.$$

So $\gamma(\bar{T}) > 0$ when $\delta(\mathcal{N}(T), \mathcal{N}(\bar{T})) < \frac{1 - \epsilon_T \kappa_T}{1 + \epsilon_T \kappa_T}$; that is, $\bar{T}^M$ exists. Moreover, by Lemma 2.8, we have

$$\|\bar{T}^M\| \leq \frac{\|\bar{T}\bar{T}^M\|}{\gamma(\bar{T})} \leq \frac{2\|T^M\|(1 + \delta(\mathcal{N}(T), \mathcal{N}(\bar{T})))}{1 - \epsilon_T \kappa_T - (1 + \epsilon_T \kappa_T)\delta(\mathcal{N}(T), \mathcal{N}(\bar{T}))}.$$

(2) When $\dim \mathcal{N}(\bar{T}) = \dim \mathcal{N}(T) < \infty$ or $\dim \mathcal{R}(\bar{T}) = \dim \mathcal{R}(T) < \infty$, by Lemma 4.1, we have $|\gamma(\bar{T}) - \gamma(T)| \leq \|\delta T\|$. Thus,

$$\gamma(\bar{T}) \geq \gamma(T) - \|\delta T\| \geq \frac{1}{\|T^M\|}(1 - \epsilon_T \kappa_T) > 0$$

and consequently,

$$\|\bar{T}^M\| \le \frac{\|\bar{T}\bar{T}^M\|}{\gamma(\bar{T})} \le \frac{2\|T^M\|}{1-\epsilon_T\kappa_T}.$$

This completes the proof. □

Remark 4.3. Let $T, \delta T \in B(X,Y)$ with $\mathcal{R}(T)$ closed, and let $\epsilon_T\kappa_T < 1$. Put $\bar{T} = T + \delta T$. Suppose that $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$. Then under the further assumption that $T^M\delta T$ is linear—that is, T^M is quasiadditive on $\mathcal{R}(\delta T)$—we have proved that (see [3, Proposition 3.6]) $\bar{T}^M$ exists and that $\|\bar{T}^M\| \le \frac{2\|T^M\|}{1-\epsilon_T\kappa_T}$.

But our following theorem shows that when $T^M\delta T$ is not linear, we can also give some estimate of the upper bound of $\|\bar{T}^M\|$ under some suitable conditions. Thus, in a certain sense, the following estimate extends many previous perturbation bounds in this field (see, e.g., [2], [4], [5], [11], [12], [20]).

Theorem 4.4. *Let $T, \delta T \in B(X,Y)$ with $\mathcal{R}(T)$ closed and δT compact. Suppose that T has the (α,β)-USU property. Put $\bar{T} = T + \delta T$. If $\epsilon_T\kappa_T < \frac{1}{3}$ and $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$, then $\bar{T}^M$ exists and $\|\bar{T}^M\| \le \frac{2\|T^M\|}{1-3\epsilon_T\kappa_T}$.*

Proof. Let $x_0 \in X\backslash\{0\}$. For any $x \in X$, put $Sx = x_0 - T^M\delta Tx$. Then, from Proposition 3.5(3), we see that S is continuous. Set $R = \frac{\|x_0\|\epsilon_T\kappa_T}{1-\epsilon_T\kappa_T}$, $K = \{x \in X | \|x - x_0\| \le R\}$. Then for any $x \in K$,

$$\|Sx - x_0\| \le \epsilon_T\kappa_T\|x\| \le \epsilon_T\kappa_T\big(\|x_0\| + R\big) = R.$$

This means that S is a continuous map from the closed convex set K to itself. Since δT is compact and T^M is continuous, also noting that K is bounded, it follows that the closure $\overline{S(K)}$ of $S(K)$ is a compact subset of K. Therefore, applying Schauder's fixed point theorem to S, we can find $z \in K$ such that $Sz = z$. This indicates that $\mathcal{R}(I + T^M\delta T) = X$.

Set $G = \{x \in X | (I + T^M\delta T)x \in \mathcal{N}(T)\}$. We will prove that $G = \mathcal{N}(\bar{T})$.

First, let $x \in \mathcal{N}(\bar{T})$. Then $\delta Tx = -Tx$. So $(I+T^M\delta T)x = (I-T^MT)x \in \mathcal{N}(T)$ and hence $\mathcal{N}(\bar{T}) \subset G$. On the other hand, let $x \in G$. Note that the metric projection $TT^M = \pi_{\mathcal{R}(T)}$ is quasiadditive on $\mathcal{R}(T)$, thus,

$$Tx = -TT^M\delta Tx = -TT^M(\bar{T}x - Tx) = -TT^M\bar{T}x + TT^MTx$$

and so that $TT^M\bar{T}x = 0$, which implies that $\bar{T}x \in \mathcal{N}(T^M)$. Since $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$, we get that $\bar{T}x = 0$; that is, $G \subset \mathcal{N}(\bar{T})$.

Now let $x \in \mathcal{N}(T)$ with $\|x\| = 1$. Then there is $z \in \mathcal{N}(\bar{T})$ such that $(I + T^M\delta T)z = x$. Thus,

$$1 = \|x\| \ge \|z\| - \epsilon_T\kappa_T\|z\|$$
$$\operatorname{dist}\big(x, \mathcal{N}(\bar{T})\big) \le \|x - z\| \le \epsilon_T\kappa_T\|z\|$$

and consequently,

$$\delta\big(\mathcal{N}(T), \mathcal{N}(\bar{T})\big) \le \frac{\epsilon_T\kappa_T}{1-\epsilon_T\kappa_T} \le \frac{1-\epsilon_T\kappa_T}{1+\|T^M\|\|\delta T\|}$$

when $\epsilon_T\kappa_T \le \frac{1}{3}$. By (4.1),

$$\gamma(\bar{T}) \ge \frac{1}{\|T^M\|}\frac{1-\frac{\epsilon_T\kappa_T}{1-\epsilon_T\kappa_T}}{1+\frac{\epsilon_T\kappa_T}{1-\epsilon_T\kappa_T}} - \|\delta T\| \ge \frac{1}{\|T^M\|}(1-3\epsilon_T\kappa_T) > 0.$$

Therefore, $\bar{T}^M$ exists by Proposition 2.6 and $\|\bar{T}^M\| \le \frac{2\|T^M\|}{1-3\epsilon_T\kappa_T}$ by Lemma 2.8. □

Lemma 4.5. *Let $T, \delta T \in B(X,Y)$ with $\mathcal{R}(T)$ closed. Put $\bar{T} = T + \delta T$. Suppose that $\bar{T}^M$ exists. Then*

(1) *for any $y \in Y$,*

$$\operatorname{dist}\big(\bar{T}^M y - T^M y, \mathcal{N}(T)\big) \le \|T^M\|\|\bar{T}^M\|\|\delta T\|\|y\| + \|T^M\|\|\bar{T}\bar{T}^M y - TT^M y\|;$$

(2) *for any $y \in Y$, there is $z_y \in \mathcal{N}(T)$ such that*

$$\|\bar{T}^M y - T^M y - z_y\| = \operatorname{dist}\big(\bar{T}^M y - T^M y, \mathcal{N}(T)\big)$$
$$\text{with } \|\bar{T}^M y\| + \|T^M y + z_y\| \le \big(3\|\bar{T}^M\| + \|T^M\|\big)\|y\|.$$

Proof. (1) We have

$$\begin{aligned}\operatorname{dist}\big(\bar{T}^M y - T^M y, \mathcal{N}(T)\big) &\le \frac{1}{\gamma(T)}\big\|T(\bar{T}^M y - T^M y)\big\| \\ &\le \|T^M\|\|\bar{T}\bar{T}^M y - TT^M y - \delta T\bar{T}^M y\| \\ &\le \|T^M\|\|\bar{T}^M\|\|\delta T\|\|y\| + \|T^M\|\|\bar{T}\bar{T}^M y - TT^M y\|.\end{aligned}$$

(2) Choose $z_y \in \mathcal{N}(T)$ such that $\|\bar{T}^M y - T^M y - z_y\| = \operatorname{dist}(\bar{T}^M y - T^M y, \mathcal{N}(T))$. Using

$$\|\bar{T}^M y - T^M y - z_y\| = \operatorname{dist}\big(\bar{T}^M y - T^M y, \mathcal{N}(T)\big) \le \|\bar{T}^M y - T^M y\|,$$

we get that

$$\begin{aligned}\|\bar{T}^M y\| + \|T^M y + z_y\| &\le \|\bar{T}^M y\| + \|\bar{T}^M y - T^M y - z_y\| + \|\bar{T}^M y\| \\ &\le \big(3\|\bar{T}^M\| + \|T^M\|\big)\|y\|.\end{aligned}$$

This completes the proof. □

Let X be Banach space, and let $T, \delta T \in B(X)$ such that T^M exists. Put $\bar{T} = T + \delta T$. From Lemma 2.8, we know that $\|T^M\| \ge \gamma(T)^{-1}$, and then similarly to [28, Lemma 1.3.5], we can check that

$$\delta\big(\mathcal{R}(T), \mathcal{R}(\bar{T})\big) \le \|T^M\|\|\delta T\|, \qquad \delta\big(\mathcal{N}(\bar{T}), \mathcal{N}(T)\big) \le \|T^M\|\|\delta T\|.$$

Thus, if $\bar{T}^M$ exists, by symmetry, we can get

$$\hat{\delta}\big(\mathcal{N}(T), \mathcal{N}(\bar{T})\big) \le \max\big\{\|\bar{T}^M\|\|\delta T\|, \|T^M\|\|\delta T\|\big\} = t\|\delta T\|, \tag{4.2}$$

$$\hat{\delta}\big(\mathcal{R}(T), \mathcal{R}(\bar{T})\big) \le \max\big\{\|\bar{T}^M\|\|\delta T\|, \|T^M\|\|\delta T\|\big\} = t\|\delta T\|, \tag{4.3}$$

where $t = \max\{\|T^M\|, \|\bar{T}^M\|\}$.

Lemma 4.6. *Let $T, \delta T \in B(X,Y)$ with $\mathcal{R}(T)$ closed. Put $\bar{T} = T + \delta T$, and assume that $\bar{T}^M$ exists and that T has the (α,β)-USU property. Set $t = \max\{\|T^M\|, \|\bar{T}^M\|\}$. Then*

$$\begin{aligned}&\|\bar{T}^M - T^M\| \\ &\quad \le t^{1+\frac{1}{\alpha\beta}} \|\delta T\|^{\frac{1}{\alpha\beta}} \Big[\frac{10}{(\gamma_{\mathcal{N}(T)})^{\frac{1}{\alpha}}} \big(t\|\delta T\|\big)^{\frac{\beta-1}{\alpha\beta}} \\ &\qquad + 2^{\frac{2\alpha-2}{\alpha}} \Big(1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big) \Big(\big(t\|\delta T\|\big)^{\frac{\beta-1}{\alpha\beta}} + \frac{10^{\frac{1}{\alpha}}}{(\gamma_{\mathcal{R}(T)})^{\frac{1}{\alpha\beta}}}\Big)\Big].\end{aligned}$$

Proof. Since $\mathcal{R}(T)$ is closed and $\bar{T}^M$ exists, it follows from Lemmas 4.5 and 3.3(1) that for any $y \in Y$ there is $z_y \in \mathcal{N}(T)$ such that

$$\begin{aligned}&\|\bar{T}^M y - T^M y - z_y\| \\ &\quad \le \|T^M\| \|\bar{T}^M\| \|\delta T\| \|y\| + \|T^M\| \|\bar{T}\bar{T}^M y - TT^M y\| \\ &\quad \le \|T^M\| \Big(\|\bar{T}^M\| \|\delta T\| + \frac{10}{(\gamma_{\mathcal{R}(T)})^{\frac{1}{\beta}}} \big[\hat{\delta}\big(\mathcal{R}(T), \mathcal{R}(\bar{T})\big)\big]^{\frac{1}{\beta}} \Big) \|y\|. \qquad (4.4)\end{aligned}$$

Thus, by Lemma 3.3(1) and Proposition 3.5(1), (2),

$$\begin{aligned}&\|\bar{T}^M y - T^M y\| \\ &\quad = \big\|\bar{T}^M \bar{T}(\bar{T}^M y) - T^M T(T^M y + z_y)\big\| \\ &\quad \le \big\|(I - \bar{T}^M \bar{T})\bar{T}^M y - (I - T^M T)\bar{T}^M y\big\| + \big\|T^M T \bar{T}^M y - T^M T(T^M y + z_y)\big\| \\ &\quad \le \frac{10}{(\gamma_{\mathcal{N}(T)})^{\frac{1}{\alpha}}} \big[\hat{\delta}\big(\mathcal{N}(T), \mathcal{N}(\bar{T})\big)\big]^{\frac{1}{\alpha}} \|\bar{T}^M\| \|y\| \\ &\qquad + \Big(1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big) \big(\|\bar{T}^M y\| + \|T^M y + z_y\|\big)^{\frac{\alpha-1}{\alpha}} \|\bar{T}^M y - T^M y - z_y\|^{\frac{1}{\alpha}}.\end{aligned}$$

By Lemma 4.5 and (4.4),

$$\begin{aligned}&\|\bar{T}^M y - T^M y\| \\ &\quad \le \frac{10}{(\gamma_{\mathcal{N}(T)})^{\frac{1}{\alpha}}} \big[\hat{\delta}\big(\mathcal{N}(T), \mathcal{N}(\bar{T})\big)\big]^{\frac{1}{\alpha}} \|\bar{T}^M\| \|y\| \\ &\qquad + \Big(1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big) \big(3\|\bar{T}^M\| + \|T^M\|\big)^{\frac{\alpha-1}{\alpha}} \|T^M\|^{\frac{1}{\alpha}} \\ &\qquad \times \Big(\big(\|\bar{T}^M\| \|\delta T\|\big)^{\frac{1}{\alpha}} + \Big(\frac{10^\beta}{\gamma_{\mathcal{R}(T)}}\Big)^{\frac{1}{\alpha\beta}} \big[\hat{\delta}\big(\mathcal{R}(T), \mathcal{R}(\bar{T})\big)\big]^{\frac{1}{\alpha\beta}} \Big) \|y\|. \qquad (4.5)\end{aligned}$$

Now, applying (4.2) and (4.3) to (4.5), we can obtain the assertion. □

We can now look at the stability problem of the Moore–Penrose metric generalized inverse for (α,β)-(USU) property operators in greater detail thanks mainly to the estimate formulas obtained in Lemma 4.6.

Theorem 4.7. *Let $T, \delta T \in B(X,Y)$ with $\mathcal{R}(T)$ closed. Suppose that T has the (α,β)-USU property. Put $\bar{T} = T + \delta T$, and set*

$$A(\alpha,\beta) = \frac{2^{\frac{2\alpha-1}{\alpha}} 5^{\frac{1}{\alpha}}}{(\gamma_{\mathcal{R}(T)})^{\frac{1}{\alpha\beta}}}\Big[1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big],$$
$$B(\alpha) = \frac{10}{(\gamma_{\mathcal{N}(T)})^{\frac{1}{\alpha}}} + 2^{\frac{2\alpha-2}{\alpha}}\Big[1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big].$$

(1) *If $\epsilon_T\kappa_T < 1$ and $\delta(\mathcal{N}(T),\mathcal{N}(\bar{T})) < \frac{1-\epsilon_T\kappa_T}{1+\epsilon_T\kappa_T}$, then $\bar{T}^M$ exists and*

$$\frac{\|\bar{T}^M - T^M\|}{\|T^M\|} \le \Big(A(\alpha,\beta) + \frac{2^{\frac{2\beta-2}{\alpha\beta}} B(\alpha)}{[1-\epsilon_T\kappa_T - (1+\epsilon_T\kappa_T)\delta(\mathcal{N}(T),\mathcal{N}(\bar{T}))]^{\frac{\beta-1}{\alpha\beta}}}\Big) \times \Big[\frac{2(1+\delta(\mathcal{N}(T),\mathcal{N}(\bar{T})))}{1-\epsilon_T\kappa_T - (1+\epsilon_T\kappa_T)\delta(\mathcal{N}(T),\mathcal{N}(\bar{T}))}\Big]^{\frac{1}{\alpha\beta}} (\epsilon_T\kappa_T)^{\frac{1}{\alpha\beta}}.$$

(2) *Assume that $\epsilon_T\kappa_T < 1$. If $\dim \mathcal{N}(\bar{T}) = \dim \mathcal{N}(T) < \infty$ or $\dim \mathcal{R}(\bar{T}) = \dim \mathcal{R}(T) < \infty$, then $\bar{T}^M$ exists and*

$$\frac{\|\bar{T}^M - T^M\|}{\|T^M\|} \le \Big(A(\alpha,\beta) + \frac{2^{\frac{\beta-1}{\alpha\beta}} B(\alpha)}{(1-\epsilon_T\kappa_T)^{\frac{\beta-1}{\alpha\beta}}}\Big)\Big[\frac{2\epsilon_T\kappa_T}{1-\epsilon_T\kappa_T}\Big]^{\frac{1}{\alpha\beta}}.$$

(3) *If δT is compact and $\epsilon_T\kappa_T < \frac{1}{3}$, $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$, then $\bar{T}^M$ exists and*

$$\frac{\|\bar{T}^M - T^M\|}{\|T^M\|} \le \Big(A(\alpha,\beta) + \frac{2^{\frac{\beta-1}{\alpha\beta}} B(\alpha)}{(1-3\epsilon_T\kappa_T)^{\frac{\beta-1}{\alpha\beta}}}\Big)\Big[\frac{2\epsilon_T\kappa_T}{1-3\epsilon_T\kappa_T}\Big]^{\frac{1}{\alpha\beta}}.$$

Proof. (1) By Proposition 4.2(1), $\bar{T}^M$ exists and

$$\|\bar{T}^M\| \le \frac{2\|T^M\|(1+\delta(\mathcal{N}(T),\mathcal{N}(\bar{T})))}{1-\epsilon_T\kappa_T - (1+\epsilon_T\kappa_T)\delta(\mathcal{N}(T),\mathcal{N}(\bar{T}))}$$

when $\delta(\mathcal{N}(T),\mathcal{N}(\bar{T})) < \frac{1-\epsilon_T\kappa_T}{1+\epsilon_T\kappa_T}$. In this case,

$$t \le \frac{2\|T^M\|(1+\delta(\mathcal{N}(T),\mathcal{N}(\bar{T})))}{1-\epsilon_T\kappa_T - (1+\epsilon_T\kappa_T)\delta(\mathcal{N}(T),\mathcal{N}(\bar{T}))} \le \frac{4\|T^M\|}{1-\epsilon_T\kappa_T - (1+\epsilon_T\kappa_T)\delta(\mathcal{N}(T),\mathcal{N}(\bar{T}))}. \tag{4.6}$$

Then applying (4.6) to Lemma 4.6, we get the assertion.

(2) By Proposition 4.2(2), $t \le \frac{2\|T^M\|}{1-\epsilon_T\kappa_T}$ under the assumptions. Combining this with Lemma 4.6, we can obtain the assertion.

(3) By Theorem 4.4, we see that $t \le \frac{2\|T^M\|}{1-3\epsilon_T\kappa_T}$ under our assumptions. Then, the assertion follows from Lemma 4.6. □

Note that from Remark 3.2 and Theorem 4.7, it is easy to obtain the following corollary in L^p $(1 < p < \infty)$ space.

Corollary 4.8. *Let $X = L^p(\Omega,\mu)$ $(1 < p < \infty)$, and let $T, \delta T \in B(X,X)$ with $\mathcal{R}(T)$ closed. Put $\bar{T} = T + \delta T$, and set*

$$A_p = \begin{cases} 2^{\frac{3}{2}} 5^{\frac{1}{2}} (p-1)^{-\frac{1}{4}} [1 + 2(p-1)^{-\frac{1}{2}}] & (1 < p \le 2), \\ 2^{\frac{2p-1}{p}} 5^{\frac{1}{p}} (c_p)^{-\frac{1}{p^2}} [1 + 2(\frac{p}{2c_p})^{\frac{1}{p}}] & (2 < p < \infty), \end{cases}$$

$$B_p = \begin{cases} 10(p-1)^{-\frac{1}{2}} + 2[1 + 2(p-1)^{-\frac{1}{2}}] & (1 < p \le 2), \\ 10(c_p)^{-\frac{1}{p}} + 2^{\frac{2p-2}{p}} [1 + 2(\frac{p}{2c_p})^{\frac{1}{p}}] & (2 < p < \infty). \end{cases}$$

(1) *If $\epsilon_T\kappa_T < 1$ and $\delta(\mathcal{N}(T), \mathcal{N}(\bar{T})) < \frac{1-\epsilon_T\kappa_T}{1+\epsilon_T\kappa_T}$, then $\bar{T}^M$ exists and*

$$\frac{\|\bar{T}^M - T^M\|}{\|T^M\|} \le \begin{cases} \left(A_p + \frac{2^{\frac{1}{2}} B_p}{[1-\epsilon_T\kappa_T-(1+\epsilon_T\kappa_T)\delta(\mathcal{N}(T),\mathcal{N}(\bar{T}))]^{\frac{1}{4}}}\right) \\ \quad \times \left[\frac{2(1+\delta(\mathcal{N}(T),\mathcal{N}(\bar{T})))}{1-\epsilon_T\kappa_T-(1+\epsilon_T\kappa_T)\delta(\mathcal{N}(T),\mathcal{N}(\bar{T}))}\right]^{\frac{1}{4}} (\epsilon_T\kappa_T)^{\frac{1}{4}} & (1 < p \le 2), \\ \left(A_p + \frac{2^{\frac{2p-2}{p^2}} B_p}{[1-\epsilon_T\kappa_T-(1+\epsilon_T\kappa_T)\delta(\mathcal{N}(T),\mathcal{N}(\bar{T}))]^{\frac{p-1}{p^2}}}\right) \\ \quad \times \left[\frac{2(1+\delta(\mathcal{N}(T),\mathcal{N}(\bar{T})))}{1-\epsilon_T\kappa_T-(1+\epsilon_T\kappa_T)\delta(\mathcal{N}(T),\mathcal{N}(\bar{T}))}\right]^{\frac{1}{p^2}} (\epsilon_T\kappa_T)^{\frac{1}{p^2}} & (2 < p < \infty). \end{cases}$$

(2) *Suppose that $\epsilon_T\kappa_T < 1$. If $\dim \mathcal{N}(\bar{T}) = \dim \mathcal{N}(T) < \infty$ or $\dim \mathcal{R}(\bar{T}) = \dim \mathcal{R}(T) < \infty$, then $\bar{T}^M$ exists and*

$$\frac{\|\bar{T}^M - T^M\|}{\|T^M\|} \le \begin{cases} \left(A_p + \frac{2^{\frac{1}{4}} B_p}{(1-\epsilon_T\kappa_T)^{\frac{1}{4}}}\right)\left[\frac{2\epsilon_T\kappa_T}{1-\epsilon_T\kappa_T}\right]^{\frac{1}{4}} & (1 < p \le 2), \\ \left(A_p + \frac{2^{\frac{p-1}{p^2}} B_p}{(1-\epsilon_T\kappa_T)^{\frac{p-1}{p^2}}}\right)\left[\frac{2\epsilon_T\kappa_T}{1-\epsilon_T\kappa_T}\right]^{\frac{1}{p^2}} & (2 < p < \infty). \end{cases}$$

(3) *If δT is compact and $\epsilon_T\kappa_T < \frac{1}{3}$, $\mathcal{R}(\bar{T}) \cap \mathcal{N}(T^M) = \{0\}$, then $\bar{T}^M$ exists and*

$$\frac{\|\bar{T}^M - T^M\|}{\|T^M\|} \le \begin{cases} \left(A_p + \frac{2^{\frac{1}{4}} B_p}{(1-3\epsilon_T\kappa_T)^{\frac{1}{4}}}\right)\left[\frac{2\epsilon_T\kappa_T}{1-3\epsilon_T\kappa_T}\right]^{\frac{1}{4}} & (1 < p \le 2), \\ \left(A_p + \frac{2^{\frac{p-1}{p^2}} B_p}{(1-3\epsilon_T\kappa_T)^{\frac{p-1}{p^2}}}\right)\left[\frac{2\epsilon_T\kappa_T}{1-3\epsilon_T\kappa_T}\right]^{\frac{1}{p^2}} & (2 < p < \infty). \end{cases}$$

5. Applications

In this section, we give applications of the main perturbation results in Section 4. We always assume that X and Y are reflexive strictly convex Banach spaces in this section. Let $T, \delta T \in B(X,Y)$ with $\mathcal{R}(T)$ closed, and let $b \in Y$. We consider the following problem for projecting a point onto a linear manifold.

For the given $T \in B(X,Y)$ with $\mathcal{R}(T)$ closed, $b \in Y$, and $f \in X$, find a vector $x^* \in X$ satisfying

$$\|f - x^*\| = \inf_{x \in S} \|f - x\|$$
$$\text{subject to} \quad S = \big\{x \in X : \|Tx - b\| = \inf_{z \in X} \|Tz - b\|\big\}. \tag{5.1}$$

Solving this problem (5.1) is important in many applications. For example, when $f = 0$, then the problem (5.1) is just the usual minimum norm least squares problem; if $b = 0$, then the problem (5.1) is that of projecting the vector f to the null space $\mathcal{N}(A)$, which is a key step in the development of interior-point projective algorithms for linear programming initiated with Karmarkar's pioneering work (see [14]). When X and Y are finite-dimensional vector spaces or infinite-dimensional Hilbert spaces, it is well known that a unique optimal solution to the problem (5.1) exists; indeed, $x^* = T^\dagger b + (I - T^\dagger T)f$, where $T^\dagger$ is the Moore–Penrose orthogonal projection generalized inverse of A.

Now, we give T (resp., b and f) a small perturbation δT (resp., δb and δf). Put $\bar{T} = T + \delta T$, $\bar{b} = b + \delta b$ and $\bar{f} = f + \delta f$. Then the problem (5.1) is perturbed to the following:

$$\begin{aligned} &\|\bar{f} - y^*\| = \inf \|\bar{f} - y\| \\ &\quad \text{subject to} \quad \tilde{S} = \big\{x \in X : \|\bar{T}y - \bar{b}\| = \inf_{z \in X} \|\bar{T}z - \bar{b}\|\big\}. \end{aligned} \tag{5.2}$$

Clearly, if $\bar{T}^\dagger$ exists, then the problem (5.2) has a unique optimal solution $y^* = \bar{T}^\dagger \bar{b} + (I - T^\dagger T)\bar{f}$. When X and Y are finite-dimensional vector spaces or infinite-dimensional Hilbert spaces, the problem (5.1) and its perturbation problem (5.2) have been considered by many authors in the literature (see [6], [9], [10], [27]). In particular, the authors of [29], by using the so-called *stable perturbation theory of operators*, first obtained (see [29, Proposition 8]) the following perturbation estimate for problems (5.1) and (5.2) in Hilbert spaces:

$$\begin{aligned} \frac{\|y^* - x^*\|}{\|x^*\|} \le \frac{\kappa}{1 - \kappa\epsilon_T}\Big(\epsilon_T + \frac{\|b - Tx^*\|}{\|T\|\|x^*\|}\kappa\epsilon_T + \frac{\|b\|}{\|T\|\|x^*\|}\epsilon_b\Big) \\ + \frac{\|f - x^*\|}{\|x^*\|}\kappa\epsilon_T + \frac{\|f\|}{\|x^*\|}\epsilon_f, \end{aligned} \tag{5.3}$$

where $\epsilon_b = \frac{\|\delta b\|}{\|b\|}$, $\epsilon_T = \frac{\|\delta T\|}{\|T\|}$, $\epsilon_f = \frac{\|\delta f\|}{\|f\|}$, and $\kappa = \|T\|\|T^\dagger\|$.

In the following, by using some perturbation bounds for the Moore–Penrose metric generalized inverse obtained in Section 4, we further study problems (5.1) and (5.2) in p-uniform convex Banach spaces. Similarly as in Hilbert spaces, we can establish the following existence and uniqueness result for the problem (5.1).

Lemma 5.1. *Suppose that $\mathcal{R}(T) \subset Y$ is closed. Then the unique optimal solution to the problem (5.1) exists, and can be expressed as*

$$x^* = T^M b + \pi_{\mathcal{N}(T)} f.$$

Proof. Since $\mathcal{R}(T)$ is closed, it follows from [28, Proposition 2.3.7] that the problem (5.1) has solutions. Moreover, from Definition 2.5, we see that the feasible solution x to problem (5.1) can be expressed as $T^M b + \pi_{\mathcal{N}(T)} u$ for any vector $u \in X$, and then from Remark 2.3, we know that the optimal solution to problem (5.1) exists and is unique. Using Definition 2.5 again, we see that $x^* = T^M b + \pi_{\mathcal{N}(T)} f$. □

The optimal solutions to problems (5.1) and (5.2) will be denoted by x^* and y^*, respectively.

Theorem 5.2. *Let $T, \delta T \in B(X,Y)$ with $\mathcal{R}(T)$ closed, and let $\epsilon_T\kappa_T < 1$. Put $\bar{T} = T + \delta T$. If T has the (α,β)-USU property and $\delta(\mathcal{N}(T),\mathcal{N}(\bar{T})) < \frac{1-\epsilon_T\kappa_T}{1+\epsilon_T\kappa_T}$, then*

$$\begin{aligned}
&\|y^* - x^*\| \\
&\quad \le 2^{\frac{2\alpha-1}{\alpha}}\Big[1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big]\Big(\frac{\beta}{2\gamma_{\mathcal{R}(T)}}\Big)^{\frac{1}{\alpha\beta}}\|T^M\|\big[2\|b\| + \|\delta b\|\big]^{\frac{\alpha\beta-1}{\alpha\beta}}\|\delta b\|^{\frac{1}{\alpha\beta}} \\
&\qquad + \bigg(A(\alpha,\beta) + \frac{2^{\frac{2\beta-2}{\alpha\beta}}B(\alpha)}{[1-\epsilon_T\kappa_T - (1+\epsilon_T\kappa_T)\hat{\delta}(\mathcal{N}(T),\mathcal{N}(\bar{T}))]^{\frac{\beta-1}{\alpha\beta}}}\bigg) \\
&\qquad \times \Big[\frac{2(1+\hat{\delta}(\mathcal{N}(T),\mathcal{N}(\bar{T})))}{1-\epsilon_T\kappa_T - (1+\epsilon_T\kappa_T)\hat{\delta}(\mathcal{N}(T),\mathcal{N}(\bar{T}))}\Big]^{\frac{1}{\alpha\beta}}(\epsilon_T\kappa_T)^{\frac{1}{\alpha\beta}}\|T^M\|\big(\|b\| + \|\delta b\|\big) \\
&\qquad + \Big[1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big]\big[2\|f\| + \|\delta f\|\big]^{\frac{\alpha-1}{\alpha}}\|\delta f\|^{\frac{1}{\alpha}} \\
&\qquad + 10\gamma_{\mathcal{N}(T)}^{-\frac{1}{\alpha}}\hat{\delta}\big(\mathcal{N}(T),\mathcal{N}(\bar{T})\big)^{\frac{1}{\alpha}}\big(\|f\| + \|\delta f\|\big) + \|\delta f\|.
\end{aligned}$$

Proof. From Proposition 4.2(1), we know that $\bar{T}^M$ exists. Then from Lemma 5.1, we have

$$x^* = T^M b + (I_X - T^M T)f, \qquad y^* = \bar{T}^M\bar{b} + (I_X - \bar{T}^M\bar{T})\bar{f}.$$

Subtracting the second equality from the first equality above, we have

$$\begin{aligned}
y^* - x^* &= \bar{T}^M\bar{b} - T^M b - \bar{T}^M\bar{T}\bar{f} + T^M T f + \delta f \\
&= (\bar{T}^M\bar{b} - T^M\bar{b}) + (T^M\bar{b} - T^M b) \\
&\quad - (\bar{T}^M\bar{T}\bar{f} - T^M T\bar{f}) - (T^M T\bar{f} - T^M T f) + \delta f. \qquad (5.4)
\end{aligned}$$

Note that from Proposition 4.2(1), we also have

$$\|\bar{T}^M\| \le \frac{2\|T^M\|(1+\hat{\delta}(\mathcal{N}(T),\mathcal{N}(\bar{T})))}{1-\epsilon_T\kappa_T - (1+\epsilon_T\kappa_T)\hat{\delta}(\mathcal{N}(T),\mathcal{N}(\bar{T}))}.$$

Consequently, using Proposition 3.5(3), Theorem 4.7(1), Proposition 3.5(2), and Lemma 3.3(1), respectively, we get

$$\begin{aligned}
&\|T^M\bar{b} - T^M b\| \\
&\quad \le 2^{\frac{2\alpha-1}{\alpha}}\Big[1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big]\Big(\frac{\beta}{2\gamma_{\mathcal{R}(T)}}\Big)^{\frac{1}{\alpha\beta}}\|T^M\|\big[2\|b\| + \|\delta b\|\big]^{\frac{\alpha\beta-1}{\alpha\beta}}\|\delta b\|^{\frac{1}{\alpha\beta}}, \qquad (5.5) \\
&\|\bar{T}^M\bar{b} - T^M\bar{b}\| \\
&\quad \le \bigg(A(\alpha,\beta) + \frac{2^{\frac{2\beta-2}{\alpha\beta}}B(\alpha)}{[1-\epsilon_T\kappa_T - (1+\epsilon_T\kappa_T)\hat{\delta}(\mathcal{N}(T),\mathcal{N}(\bar{T}))]^{\frac{\beta-1}{\alpha\beta}}}\bigg) \\
&\qquad \times \Big[\frac{2(1+\hat{\delta}(\mathcal{N}(T),\mathcal{N}(\bar{T})))}{1-\epsilon_T\kappa_T - (1+\epsilon_T\kappa_T)\hat{\delta}(\mathcal{N}(T),\mathcal{N}(\bar{T}))}\Big]^{\frac{1}{\alpha\beta}} \\
&\qquad \times (\epsilon_T\kappa_T)^{\frac{1}{\alpha\beta}}\|T^M\|\big(\|b\| + \|\delta b\|\big), \qquad (5.6)
\end{aligned}$$

$$\|T^M T\bar{f} - T^M Tf\| \leq \Big[1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big]\big[2\|f\| + \|\delta f\|\big]^{\frac{\alpha-1}{\alpha}}\|\delta f\|^{\frac{1}{\alpha}}, \tag{5.7}$$

$$\|\bar{T}^M\bar{T}\bar{f} - T^M T\bar{f}\| \leq 10\gamma_{\mathcal{N}(T)}^{-\frac{1}{\alpha}}\hat{\delta}\big(\mathcal{N}(T),\mathcal{N}(\bar{T})\big)^{\frac{1}{\alpha}}\big(\|f\| + \|\delta f\|\big), \tag{5.8}$$

where, $A(\alpha,\beta)$ and $B(\alpha)$ are the same as in Theorem 4.7. Now, applying (5.5), (5.6), (5.7), and (5.8) to (5.4), we can get the desired estimate. □

As immediate consequences of Theorem 5.2 above, we have the following corollaries.

Corollary 5.3. *Under the same conditions of Theorem 5.2, if, in addition, $b = 0$ and $\delta b = 0$, that is, the problem (5.1) is that of projecting f to the null space of T, then*

$$\|y^* - x^*\| \leq \Big[1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big]\big[2\|f\| + \|\delta f\|\big]^{\frac{\alpha-1}{\alpha}}\|\delta f\|^{\frac{1}{\alpha}} + 10\gamma_{\mathcal{N}(T)}^{-\frac{1}{\alpha}}\hat{\delta}\big(\mathcal{N}(T),\mathcal{N}(\bar{T})\big)^{\frac{1}{\alpha}}\big(\|f\| + \|\delta f\|\big) + \|\delta f\|.$$

Corollary 5.4. *Under the same conditions of Theorem 5.2, if, in addition, $f = 0$ and $\delta f = 0$, that is, the problem (5.1) is the best approximate solution problem, then*

$$\begin{aligned}
&\|y^* - x^*\| \\
&\leq 2^{\frac{2\alpha-1}{\alpha}}\Big[1 + 2\Big(\frac{\alpha}{2\gamma_{\mathcal{N}(T)}}\Big)^{\frac{1}{\alpha}}\Big]\Big(\frac{\beta}{2\gamma_{\mathcal{R}(T)}}\Big)^{\frac{1}{\alpha\beta}}\|T^M\|\big[2\|b\| + \|\delta b\|\big]^{\frac{\alpha\beta-1}{\alpha\beta}}\|\delta b\|^{\frac{1}{\alpha\beta}} \\
&\quad + \Big(A(\alpha,\beta) + \frac{2^{\frac{2\beta-2}{\alpha\beta}}B(\alpha)}{[1 - \epsilon_T\kappa_T - (1+\epsilon_T\kappa_T)\hat{\delta}(\mathcal{N}(T),\mathcal{N}(\bar{T}))]^{\frac{\beta-1}{\alpha\beta}}}\Big) \\
&\quad \times \Big[\frac{2(1+\hat{\delta}(\mathcal{N}(T),\mathcal{N}(\bar{T}))}{1 - \epsilon_T\kappa_T - (1+\epsilon_T\kappa_T)\hat{\delta}(\mathcal{N}(T),\mathcal{N}(\bar{T}))}\Big]^{\frac{1}{\alpha\beta}}(\epsilon_T\kappa_T)^{\frac{1}{\alpha\beta}}\|T^M\|\big(\|b\| + \|\delta b\|\big).
\end{aligned}$$

Remark 5.5. It should be pointed out that, by using Lemma 4.6 and the bounds (2) and (3) in Theorem 4.7, we can get the corresponding perturbation bounds similar to Theorem 5.2. Also, by using Corollary 4.8, we can get some applications in L^p $(1 < p < \infty)$. But, we see that the method is straightforward and almost the same as the one above, so we leave these applications to the interested reader.

6. Concluding remarks

In this article, by using the feature of metric projection and some geometric properties of Banach spaces, for the class of so-called (α,β)-*USU operators*, we have obtained various perturbation bounds of the Moore–Penrose metric generalized inverse in reflexive and strictly convex Banach spaces. One advantage associated with our results here is that they do not need the *quasiadditivity* assumption. We have also presented applications in connection with the main perturbation results. Consequently, the results obtained in this paper make some progress on the problem posed by the second author (see [28, p. 243] for more information). We believe that such results have direct applications to error estimates of some ill-posed operator equations with some weaker assumptions.

Finally, note that a large number of operators which arise naturally in applications such as mathematical physics, quantum mechanics, and partial differential equations are unbounded. Thus, it would also be interesting to extend our results from bounded linear operators to more general closed operators (see [12], [15], [21]) since the differential operators or partial differential operators are always unbounded closed linear operators. We would like to propose this extension as a project that might be of interest for further research.

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