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A GENERALIZED SCHUR COMPLEMENT FOR NONNEGATIVE OPERATORS ON LINEAR SPACES

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ABSTRACT. Extending the corresponding notion for matrices or bounded linear operators on a Hilbert space, we define a generalized Schur complement for a nonnegative linear operator mapping a linear space into its dual, and we derive some of its properties.

1. Introduction

In the present article we construct a Schur complement of a nonnegative linear operator mapping the direct sum of two linear spaces over $\mathbb{C}$ into the direct sum of their dual spaces. We show that this object has features that extend the corresponding properties of the generalized Schur complement of 2×2 block matrices with matrix entries. In contrast to most other studies, we mainly discuss nonnegative operators on spaces without topology so that topological restrictions, and particularly continuity questions, play a secondary role.

The Schur complement and its generalizations occur in various mathematical fields. For a comprehensive exposition of its history, theory, and diverse applications, we refer to [24]. Moreover, the generalized Schur complement is closely related to the so-called *shorted operator*, which was first introduced by Kreĭn in [11] and has found interesting applications in electrical network theory (see [2]). Shmulyan [17] investigated nonnegative bounded linear operators acting in

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the orthogonal sum of two Hilbert spaces, and in [6] some results were given concerning the case of Banach spaces.

To define our notion of a generalized Schur complement, we need a suitable definition of a square root of a nonnegative operator. Section 3 deals with this important and useful concept, which was studied by several authors. Section 4 contains definitions and basic properties of the Schur complement and the shorted operator in a slightly wider context than nonnegative operators. In Section 5 further results on generalized Schur complements are derived. Among other things we extend the Crabtree–Haynsworth quotient formula from [4]. One of the most useful results concerning 2×2 block matrices is Albert's nonnegativity criterion (see [1]). A generalization to nonnegative operators on linear spaces and some of its consequences are given in Section 6. The special class of extremal operators, which was also introduced by Kreĭn [11], is the subject of Section 7.

As mentioned above, many results concerning the generalized Schur complement were for bounded linear operators on Hilbert spaces obtained by Shmulyan. Many of them were proved independently or rediscovered later by other mathematicians. The present paper is strongly influenced by Shmulyan's work and was written to illustrate his contribution to the theory of generalized Schur complements. Thus most of our assertions of Sections 4–6 are generalizations of results in [17] about nonnegative operators on linear spaces.

2. Basic definitions and notation

In the present paper, all linear spaces are spaces over $\mathbb{C}$, the field of complex numbers, and the zero element is denoted by 0. For a linear space X, let X' denote its dual space of all antilinear functionals on X, and let $\langle x', x\rangle_X := \langle x', x\rangle$ denote the value of $x' \in X'$ at $x \in X$. If $X^\sim$ is a subspace of X', then an arbitrary $x \in X$ defines an element jx of $(X^\sim)'$ according to

$$\langle jx, x^\sim\rangle_{X'} := \overline{\langle x^\sim, x\rangle}_X, \quad x^\sim \in X^\sim,$$

where $\bar{\alpha}$ stands for the complex conjugate of $\alpha \in \mathbb{C}$.

Convention (CN). If for all $x \in X \setminus \{0\}$ there exists $x^\sim \in X^\sim$ such that $\langle x^\sim, x\rangle \neq 0$, then we identify X with its isomorphic image under the map j, and we write

$$\langle jx, x^\sim\rangle_{X'} =: \langle x, x^\sim\rangle_{X'}, \quad x \in X, x^\sim \in X^\sim.$$

The linear space of all linear operators from X into a linear space Y is denoted by $\mathscr{L}(X, Y)$, and I is the identity operator in case $X = Y$. If $A \in \mathscr{L}(X, Y)$ and X_1 is a subspace of X, then the symbols $\ker A$, $\operatorname{ran} A$, and $A\restriction_{X_1}$ stand for the null space, range, and restriction of A to X_1, respectively. Set $AX_1 := \operatorname{ran} A\restriction_{X_1}$. The dual operator $A' \in \mathscr{L}(Y', X')$ is defined by the relation $\langle y', Ax\rangle_Y = \langle A'y', x\rangle_X$, $x \in X$, $y' \in Y'$.

Examples.

1. If Z is a linear space and $A \in \mathscr{L}(X, Y)$, $B \in \mathscr{L}(Y, Z)$, then $(BA)' = A'B'$.
2. If $A \in \mathscr{L}(X, Y)$, then $A'' \in \mathscr{L}(X'', Y'')$ and $A = A''\restriction_X$ according to (CN).

3. If $A \in \mathscr{L}(X, X')$, then $A' \in \mathscr{L}(X'', X')$. Taking into account (CN), we get $\langle x_2, Ax_1\rangle_{X'} = \overline{\langle Ax_1, x_2\rangle}_X$ and $\langle x_2, Ax_1\rangle_{X'} = \langle A'x_2, x_1\rangle_X$, and hence,

$$\langle Ax_1, x_2\rangle_X = \overline{\langle A'x_2, x_1\rangle}_X, \quad x_1, x_2 \in X. \tag{2.1}$$

An operator $A \in \mathscr{L}(X, X')$ is called *Hermitian* if $\langle Ax_1, x_2\rangle = \overline{\langle Ax_2, x_1\rangle}$ and nonnegative if $\langle Ax_1, x_1\rangle \geq 0$, $x_1, x_2 \in X$. The sets of all Hermitian and all nonnegative operators are denoted by $\mathscr{L}^h(X, X')$ and $\mathscr{L}^{\geq}(X, X')$, respectively. The polarization identity implies that A is Hermitian if and only if $\langle Ax, x\rangle$ is real for all $x \in X$. Thus $\mathscr{L}^{\geq}(X, X') \subseteq \mathscr{L}^h(X, X')$ and the space $\mathscr{L}^h(X, X')$ can be provided with Loewner's semiordering (i.e., for $A, D \in \mathscr{L}^h(X, X')$ we write $A \leq D$ if and only if $\langle Ax, x\rangle \leq \langle Dx, x\rangle$, $x \in X$). Recall the Cauchy inequality

$$\big|\langle Ax_1, x_2\rangle\big|^2 \leq \langle Ax_1, x_1\rangle\langle Ax_2, x_2\rangle, \quad x_1, x_2 \in X, \tag{2.2}$$

if $A \in \mathscr{L}^{\geq}(X, X')$.

3. Square roots

Let H be a complex Hilbert space with norm $\|\cdot\| := \|\cdot\|_H$ and inner product $(\cdot\,|\,\cdot) := (\cdot\,|\,\cdot)_H$, which is assumed to be antilinear with respect to the second component. Let $R \in \mathscr{L}(X, H)$. Identifying H with the space of continuous antilinear functionals on H in the usual way, we have $H \subseteq H'$ and

$$(h\,|\,Rx) = \langle R'h, x\rangle, \quad x \in X, h \in H. \tag{3.1}$$

Set $R^* := R'\restriction_H$. From (3.1) it follows that $\ker R^*$ is equal to the orthogonal complement of $(\operatorname{ran} R)^c$, where M^c denotes the closure of a subset M of a topological space. It follows that R^* is one-to-one if and only if $\operatorname{ran} R$ is dense in H and that

$$\operatorname{ran} R^* = R^*(\operatorname{ran} R)^c. \tag{3.2}$$

Therefore, we can define a generalized inverse $R^{*[-1]}$ of R^* by

$$R^{*[-1]}x' := (R^*\restriction_{(\operatorname{ran} R)^c})^{-1}x', \quad x' \in \operatorname{ran} R^*.$$

Lemma 3.1. *Let $R \in \mathscr{L}(X, H)$. An element $x' \in X'$ belongs to* $\operatorname{ran} R^*$ *if and only if the following conditions are satisfied:*

(i) *if $x \in \ker R$, then $\langle x', x\rangle = 0$;*
(ii) $\sup_{x\in X} \frac{|\langle x', x\rangle|^2}{\|Rx\|_H^2} < \infty$ *(with convention $\frac{0}{0} := 0$ in the left-hand side).*

Proof. If $x' \in R^*h$ for some $h \in H$, then

$$\big|\langle x', x\rangle\big| = \big|\langle R^*h, x\rangle\big| = \big|(h\,|\,Rx)\big| \leq \|h\|\|Rx\|,$$

which yields (i) and (ii). Conversely, assume that (i) and (ii) are satisfied for some $x' \in X'$. Set $\varphi(Rx) := \langle x', x\rangle$, $x \in X$. Because of (i) φ is well defined, and (ii) implies that φ is continuous, so that φ is a continuous antilinear functional on $\operatorname{ran} R$. Thus, there exists $h \in H$ such that $\langle x', x\rangle = (h\,|\,Rx) = \langle R^*h, x\rangle$ for all $x \in X$, which yields $x' = R^*h \in \operatorname{ran} R^*$. □

Definition 3.2. Let $A \in \mathscr{L}(X, X')$. A pair (R, H) of a Hilbert space H and an operator $R \in \mathscr{L}(X, H)$ is called a *square root* of A if $A = R^*R$, and a *minimal square root* if, addionally, $\operatorname{ran} R$ is dense in H.

Note that there exists a square root of A if and only if there exists a minimal one. The following result is basic to our considerations and generalizes the fact concerning the existence of a square root of a nonnegative selfadjoint operator in a Hilbert space. Its well-known short proof is recapitulated for convenience of the reader.

Theorem 3.3. *An operator $A \in \mathscr{L}(X, X')$ possesses a square root if and only if it is nonnegative.*

Proof. Let $A \in \mathscr{L}^{\geq}(X, X')$. The Cauchy inequality (2.2) implies that

$$N := \{x \in X : \langle Ax, x\rangle = 0\}$$

is a subspace of X. Define an inner product on the quotient space X/N by

$$(x_1 + N \,|\, x_2 + N) := \langle Ax_1, x_2\rangle, \quad x_1, x_2 \in X,$$

and denote by H the completion of the corresponding inner product space. Set $Rx := x + N$, $x \in X$. It follows that $R \in \mathscr{L}(X, H)$, $(\operatorname{ran} R)^c = H$, and that

$$\langle Ax_1, x_2\rangle = (Rx_1 \,|\, Rx_2) = \langle R^*Rx_1, x_2\rangle, \quad x_1, x_2 \in X.$$

Therefore, (R, H) is a minimal square root of A. The “only if” part of the assertion is obvious. □

The notion of a square root of a nonnegative operator acting between spaces more general than Hilbert spaces has been discussed and applied by many authors. Most of them deal with a topological space X, and in that case continuity problems also arise. Some properties of square roots for operators of special type were obtained by Vaĭnberg and Engel’son [21]. For a Banach space X, the construction of the proof of Theorem 3.3 was published as an appendix to [22] and attributed to Chobanyan (see also [15] and [23]). Another but related construction was proposed by Sebestyén [16] (see also [19]). Górniak [7] and Górniak and Weron [9] dealt with the existence of a continuous square root when X is a topological linear space. Górniak, Makagon, and Weron [8] investigated square roots of nonnegative operator-valued measures. Pusz and Woronowicz [14] extended the construction of the proof of Theorem 3.3 to pairs of nonnegative sequilinear forms (see [20] for further generalizations).

Lemma 3.4. *If $A \in \mathscr{L}(X, X')$ and (R, H) is a square root of A, then*

$$\ker R = \ker A = \{x \in X : \langle Ax, x\rangle = 0\}.$$

Proof. The result follows from a chain of conclusions:

$$\langle Ax, x\rangle = 0 \Rightarrow \langle R^*Rx, x\rangle = 0 \Rightarrow (Rx \,|\, Rx) = 0 \Rightarrow Rx = 0,$$

and conversely

$$Rx = 0 \Rightarrow R^*Rx = 0 \Rightarrow Ax = 0 \Rightarrow \langle Ax, x\rangle = 0. \qquad □$$

The preceding results can be used to derive a version of a part of Douglas's theorem in [5] (see also [18]).

Proposition 3.5. *Let $A, D \in \mathscr{L}^{\geq}(X, X')$, and let (R_A, H_A) and (R_D, H_D) be square roots of A and D, respectively. The following assertions are equivalent:*

(i) $A \leq \alpha^2 D$ *for some* $\alpha \in [0, \infty)$,

(ii) *there exists a bounded operator* $W \in \mathscr{L}(H_A, H_D)$ *with operator norm* $\|W\| \leq \alpha$ *and such that* $R_A^* = R_D^* W$.

If (i) or (ii) are satisfied, then there exists a unique W so that $W \subseteq (\operatorname{ran} R_D)^c$. Moreover, $\ker W = \ker R_A^*$ *for this operator W.*

Proof. Since $R_A^* = R_D^* W$ yields $R_A = R_A^{*}{}' \restriction_X = W' R_D^{*}{}' \restriction_X = W^* R_D$ by (CN), from (ii) it follows that

$$\langle Ax, x \rangle = \|R_A x\|^2 = \|W^* R_D x\|^2 \leq \alpha^2 \|R_D x\|^2 = \alpha^2 \langle Dx, x \rangle, \quad x \in X,$$

and hence, (i). Let $W_j \in \mathscr{L}(H_A, H_D)$ be such that $R_A^* = R_D^* W_j$ and $\operatorname{ran} W_j \subseteq (\operatorname{ran} R_D)^c$, $j = 1, 2$. Then $\operatorname{ran}(W_1 - W_2) \subseteq \ker R_D^*$ and $\operatorname{ran}(W_1 - W_2) \subseteq (\operatorname{ran} R_D)^c$, which shows that $W_1 = W_2$. Now assume that (i) is true. One has $\ker R_D \subseteq \ker R_A$ by Lemma 3.4, and hence, $\operatorname{ran} R_A^* \subseteq \operatorname{ran} R_D^*$ by Lemma 3.1. The operator $W := R_D^{*[-1]} R_A^* \in \mathscr{L}(H_A, H_D)$ satisfies $R_D^* W = R_A^*$, $\ker W = \ker R_A^*$ and $\operatorname{ran} W \subseteq (\operatorname{ran} R_D)^c$. The inclusion $\operatorname{ran} R_D^{*[-1]} \subseteq (\operatorname{ran} R_D)^c$ implies that $W^* h = R_A^{*\prime} (R_D^{*[-1]})^* h = 0$ if h is orthogonal to $\operatorname{ran} R_D$. Therefore, from

$$\|W^* R_D x\|^2 = \|R_A x\|^2 = \langle Ax, x \rangle \leq \alpha^2 \langle Dx, x \rangle = \alpha^2 \|R_D x\|^2, \quad x \in X,$$

one can conclude that $\|W\| = \|W^*\| \leq \alpha$. □

As a by-product of Proposition 3.5, we obtain the following corollary.

Corollary 3.6. *Let A, D, (R_A, H_A), and (R_D, H_D) be as in Proposition 3.5.*

(i) *If* $A \leq \alpha^2 D$ *for some* $\alpha \in [0, \infty)$, *then* $\operatorname{ran} R_A^* \subseteq \operatorname{ran} R_D^*$.

(ii) *If* $\beta^2 D \leq A \leq \alpha^2 D$ *for some* $\alpha, \beta \in (0, \infty)$, *then* $\operatorname{ran} R_A^* = \operatorname{ran} R_D^*$.

(iii) *If* (S_A, G_A) *is a square root of A, then* $\operatorname{ran} R_A^* = \operatorname{ran} S_A^*$.

Corollary 3.7. *Let H_j be Hilbert spaces and let $R_j \in \mathscr{L}(X, H_j)$, $j = 1, 2$. If (R, H) is a square root of the nonnegative operator $A := R_1^* R_1 + R_2^* R_2$, then* $\operatorname{ran} R^* = \operatorname{ran} R_1^* + \operatorname{ran} R_2^*$.

Proof. Let G be the orthogonal sum of H_1 and H_2 and let $S \in \mathscr{L}(X, G)$ be defined by $S = \binom{R_1}{R_2}$. Since $S^* = (R_1^*, R_2^*)$ and $S^* S = A$, we get that $\operatorname{ran} S^* = \operatorname{ran} R_1^* + \operatorname{ran} R_2^*$ and that (S, G) is a square root of A. Now apply Corollary 3.6(iii). □

Lemma 3.8. *Let (R, H) be a square root and let (S, G) be a minimal square root of $A \in \mathscr{L}^{\geq}(X, X')$. There exists an isometry $U \in \mathscr{L}(G, H)$ such that $US = R$.*

Proof. By Lemma 3.4 there exists an operator $\tilde{U}$ satisfying $\tilde{U} S x = Rx$, $x \in X$. From $\|\tilde{U} S x\|^2 = \|Rx\|^2 = \langle Ax, x \rangle = \|Sx\|^2$ it follows that $\tilde{U}$ is isometric and can be extended to an isometry $U \in \mathscr{L}(G, H)$. □

4. Generalized Schur complements and shorted operators of operators of positive type

Let X and Y be linear spaces.

Definition 4.1. A pair (A, B) of an operator $A \in \mathscr{L}^{\geq}(X, X')$ and $B \in \mathscr{L}(Y, X')$ is called a *positive pair* if $\operatorname{ran} B \subseteq \operatorname{ran} R^*$ for some square root (and, hence, for all square roots) (R, H) of A.

The following criterion is an immediate consequence of Lemma 3.1.

Lemma 4.2. *Let $A \in \mathscr{L}^{\geq}(X, X')$ and $B \in \mathscr{L}(Y, X')$. The pair (A, B) is a positive pair if and only if for all $y \in Y$ the following conditions are satisfied:*

(i) *if $x \in \ker A$, then $\langle By, x\rangle = 0$;*
(ii) $\sup_{x\in X} \frac{|\langle By,x\rangle|^2}{\langle Ax,x\rangle} < \infty$ *(with convention $\frac{0}{0} := 0$).*

According to (CN), the space X can be considered as a subspace of the domain of B'. To abbreviate the notation, we set

$$B^{\sim} := B'\restriction_X .$$

Note that $\langle By, x\rangle = \overline{\langle B^{\sim}x, y\rangle}$, $x \in X$, $y \in Y$. Thus condition (i) of Lemma 4.2 is equivalent to the inclusion $\ker A \subseteq \ker B^{\sim}$.

If (A, B) is a positive pair and (R, H) is a square root of A, then the operators

$$T := R^{*[-1]}B \tag{4.1}$$

and

$$\omega(A, B) := T^*T = (R^{*[-1])}B)^*R^{*[-1]}B \tag{4.2}$$

can be defined. Note that $B = R^*T$. The following lemma is obvious.

Lemma 4.3. *If (A, B) is a positive pair and (R, H) is a square root of A, then for all $y \in Y$*

$$\inf\{\|Ty - Rx\| : x \in X\} = 0.$$

Equivalently, $\operatorname{ran} T \subseteq (\operatorname{ran} R)^c$.

Recall that the dual space of $X \times Y$ can be written as a direct product

$$(X \times Y)' = X' \times Y',$$

where $\langle\cdot,\cdot\rangle_{X\times Y} = \langle\cdot,\cdot\rangle_X + \langle\cdot,\cdot\rangle_Y$. Also, it should not cause confusion if we identify the subspace $X \times \{0\}$ of $X \times Y$ with X. An operator $\mathbf{A}$ of $\mathscr{L}(X \times Y, (X \times Y)')$ can be represented as a 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in \mathscr{L}(X, X')$, $B, C \in \mathscr{L}(Y, X')$, $D \in \mathscr{L}(Y, Y')$. It is not hard to see that $\mathbf{A}$ is Hermitian if and only if A and D are Hermitian and $C = B^{\sim}$. To abbreviate the notation, we set $\mathscr{L}^h(X\times Y, (X\times Y)') =: \mathscr{L}^h$ and $\mathscr{L}^{\geq}(X\times Y, (X\times Y)') =: \mathscr{L}^{\geq}$.

Definition 4.4. An operator $\left(\begin{smallmatrix} A & B \\ B^{\sim} & D \end{smallmatrix}\right) \in \mathscr{L}^h$ is called an operator of positive type if (A, B) is a positive pair. The set of operators of positive type is denoted by $\mathscr{L}^+(X \times Y, (X \times Y)') =: \mathscr{L}^+$.

Definition 4.5. Let $\mathbf{A} = \left(\begin{smallmatrix} A & B \\ B^{\sim} & D \end{smallmatrix}\right) \in \mathscr{L}^+$. The operator $\sigma(\mathbf{A}) := D - \omega(A, B)$ is called a *generalized Schur complement* of $\mathbf{A}$ and the operator

$$\mathscr{S}(\mathbf{A}) := \begin{pmatrix} 0 & 0 \\ 0 & \sigma(\mathbf{A}) \end{pmatrix}$$

is called a *shorted operator.*

The following result is a generalization of [2, Corollary 1 to Theorem 3].

Proposition 4.6. *If* $\mathbf{A} = \left(\begin{smallmatrix} A & B \\ B^{\sim} & D \end{smallmatrix}\right) \in \mathscr{L}^+$*, then* $\operatorname{ran} \mathbf{A} \cap Y' \subseteq \operatorname{ran} \mathscr{S}(\mathbf{A})$*.*

Proof. Let $y' \in Y'$ be such that $\mathbf{A}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ y' \end{smallmatrix}\right)$ for some $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in X \times Y$. Let (R, H) be a minimal square root of A. Since

$$\mathbf{A} = \begin{pmatrix} R^*R & R^*T \\ T^*R & T^*T \end{pmatrix} + \mathscr{S}(\mathbf{A}),$$

one has that $R^*Rx + R^*Ty = 0$, and hence, that $Rx + Ty = 0$ and $T^*Rx + T^*Ty = 0$, which yields

$$\begin{pmatrix} 0 \\ y' \end{pmatrix} = \mathscr{S}(\mathbf{A}) \begin{pmatrix} x \\ y \end{pmatrix} \in \operatorname{ran} \mathscr{S}(\mathbf{A}).$$ □

The next result is a simple but useful consequence of Lemma 4.3.

Proposition 4.7. *Let* $\mathbf{A} = \left(\begin{smallmatrix} A & B \\ B^{\sim} & D \end{smallmatrix}\right) \in \mathscr{L}^+$*. For* $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in X \times Y$*,*

$$\left\langle \mathscr{S}(\mathbf{A}) \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \inf_{z \in X} \left\langle \mathbf{A} \begin{pmatrix} x - z \\ y \end{pmatrix} \begin{pmatrix} x - z \\ y \end{pmatrix} \right\rangle . \tag{4.3}$$

In particular $\sigma(\mathbf{A})$ *and* $\mathscr{S}(\mathbf{A})$ *do not depend on the choice of the square root of* A*.*

Proof. Since (4.3) is independent of $x \in X$, it is enough to prove it for $x = 0$. From Lemma 4.3 it follows that

$$\begin{aligned}
&\inf_{z \in X} \left\langle \mathbf{A} \begin{pmatrix} -z \\ y \end{pmatrix}, \begin{pmatrix} -z \\ y \end{pmatrix} \right\rangle \\
&\quad = \inf_{z \in X} \left\langle \begin{pmatrix} R^*R & R^*T \\ T^*R & T^*T \end{pmatrix} \begin{pmatrix} -z \\ y \end{pmatrix}, \begin{pmatrix} -z \\ y \end{pmatrix} \right\rangle + \left\langle \mathscr{S}(\mathbf{A}) \begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle \\
&\quad = \inf_{z \in X} \|Ty - Rz\|^2 + \left\langle \mathscr{S}(\mathbf{A}) \begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle \\
&\quad = \left\langle \mathscr{S}(\mathbf{A}) \begin{pmatrix} 0 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ y \end{pmatrix} \right\rangle .
\end{aligned}$$ □

Corollary 4.8.

(i) *If* $\mathbf{A} \in \mathscr{L}^+$*, then* $\mathscr{S}(\mathbf{A}) \leq \mathbf{A}$ *and* $\ker \mathbf{A} \subseteq \ker \mathscr{S}(\mathbf{A})$*.*
(ii) *If* $\mathbf{A}, \mathbf{A}_1 \in \mathscr{L}^+$ *and* $\mathbf{A} \leq \mathbf{A}_1$*, then* $\mathscr{S}(\mathbf{A}) \leq \mathscr{S}(\mathbf{A}_1)$*.*

Proof. The first assertion of (i) as well as (ii) are immediately clear from Proposition 4.7. To prove the second assertion of (i), let $\binom{x}{y} \in \ker \mathbf{A}$. If $z \in X$, then we have

$$\left\langle \mathbf{A} \begin{pmatrix} x-z \\ y \end{pmatrix}, \begin{pmatrix} x-z \\ y \end{pmatrix} \right\rangle = \langle Az, z \rangle \geq 0,$$

which implies that the infimum at the right-hand side of (4.3) is equal to 0. Since $\mathscr{S}(\mathbf{A}) \leq \mathbf{A}$ and

$$\left\langle (\mathbf{A} - \mathscr{S}(\mathbf{A})) \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = 0,$$

it follows that $\binom{x}{y} \in \ker(\mathbf{A} - \mathscr{S}(\mathbf{A}))$ by Lemma 3.4, and hence, $\binom{x}{y} \in \ker \mathscr{S}(\mathbf{A})$. ☐

Corollary 4.9. *If $\mathbf{A} \in \mathscr{L}^{\geq}$, then $\mathscr{S}(\mathbf{A}) \in \mathscr{L}^{\geq}$.*

5. Further applications of square roots

First we express the generalized Schur complement of an operator of $\mathscr{L}^{\geq}$ with the aid of its square root, and we derive a range description (see [2, Corollary 4 to Theorem 1]). Let $\mathbf{A} \in \mathscr{L}^{\geq}$ and (R, H) be a square root of $\mathbf{A}$. Let L be the orthogonal complement of $(RX)^c$, and let P be the orthoprojection onto L. Note that L can be characterized by $L = \{h \in H : R^*h \in Y'\}$, which yields $R^*L = \operatorname{ran} R^* \cap Y'$.

Proposition 5.1. *If $\mathbf{A} \in \mathscr{L}^{\geq}$, then $\mathscr{S}(\mathbf{A}) = R^*PR$.*

Proof. Let $\binom{x}{y} \in X \times Y$. An application of (4.3) gives

$$\left\langle \mathscr{S}(\mathbf{A}) \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \inf_{z \in X} \left\| R \begin{pmatrix} x \\ y \end{pmatrix} - R \begin{pmatrix} z \\ 0 \end{pmatrix} \right\|^2,$$

which shows that $\langle \mathscr{S}(\mathbf{A})\binom{x}{y}, \binom{x}{y} \rangle$ is the squared distance of $R\binom{x}{y}$ to RX. Therefore,

$$\left\langle \mathscr{S}(\mathbf{A}) \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \left\| PR \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 = \left\langle R^*PR \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$

and the assertion follows from the polarization identity. ☐

Proposition 5.2. *If $\mathbf{A} \in \mathscr{L}^{\geq}$ and (R, H) and (S, G) are square roots of $\mathbf{A}$ and $\mathscr{S}(\mathbf{A})$, respectively, then* $\operatorname{ran} S^* = \operatorname{ran} R^* \cap Y'$.

Proof. Setting $R_X := R{\restriction}_X$ and $R_Y := R{\restriction}_Y$, we get

$$\mathbf{A} = \begin{pmatrix} R_X^* \\ R_Y^* \end{pmatrix} (R_X \ R_Y) = \begin{pmatrix} R_X^* R_X & R_X^* R_Y \\ R_Y^* R_X & R_Y^* R_Y \end{pmatrix},$$

and hence,

$$\sigma(\mathbf{A}) = R_Y^* R_Y - (R_X^{*[-1]} R_X^* R_Y)^* R_X^{*[-1]} R_X^* R_Y = R_Y^* P R_Y$$

since $R_X^{*[-1]} R_X^* = I - P$. Thus $\mathscr{S}(\mathbf{A}) = R^*PR$ and (PR, H) is a square root of $\mathscr{S}(\mathbf{A})$. If $\binom{0}{y'} \in X' \times Y'$ is such that $R^*h = \binom{0}{y'}$ for some $h \in H$, then $R_X^* h = 0$, and hence, $Ph = h$ and $(PR)^*h = R^*h = \binom{0}{y'}$, which implies that $\operatorname{ran} R^* \cap$

$Y' \subseteq \operatorname{ran}(PR)^* = \operatorname{ran} S^*$ by Corollary 3.6(iii). Since, obviously, $\operatorname{ran} S^* \subseteq Y'$ and $\operatorname{ran} S^* \subseteq \operatorname{ran} R^*$ by Corollaries 4.8(i) and 3.6(i), the assertion is proved. □

Our next result is a generalization of the Crabtree–Haynsworth quotient formula [4, pp. 365–366]. To give it a nice form, let us denote $\sigma(\mathbf{A}) =: \mathbf{A}/A$.

Proposition 5.3. *Let X, Y, and Z be linear spaces, with*

$$\mathbf{D} := \begin{pmatrix} A & B & B_X \\ B^\sim & D & B_Y \\ B_X^\sim & B_Y^\sim & D_1 \end{pmatrix} \in \mathscr{L}^{\geq}(X \times Y \times Z, X' \times Y' \times Z'),$$

and $\mathbf{A} := \left(\begin{smallmatrix} A & B \\ B^\sim & D \end{smallmatrix}\right)$. The operator $\mathbf{A}/A$ is the left upper corner of $\mathbf{D}/A$ and

$$\mathbf{D}/A \,\big/\, \mathbf{A}/A = \mathbf{D}/\mathbf{A}.$$

Proof. Let (R, H) be a minimal square root of $\mathbf{A}$, $R_X := R\restriction_X$, $R_Y := R\restriction_Y$,

$$E := (R^*)^{-1} \begin{pmatrix} B_X \\ B_Y \end{pmatrix},$$

and hence, $R_X^* E = B_X$, $R_Y^* E = B_Y$. From $R_X^{*[-1]} R_X^* = I - P$ we obtain

$$\begin{aligned} \mathbf{D}/A &= \begin{pmatrix} R_Y^* R_Y & R_Y^* E \\ E^* R_Y & D_1 \end{pmatrix} - \left(R_X^{*[-1]}(R_X^* R_Y, R_X^* E)\right)^* R_X^{*[-1]}(R_X^* R_Y, R_X^* E) \\ &= \begin{pmatrix} R_Y^* P R_Y & R_Y^* P E \\ E^* P R_Y & D_1 - E^*(I-P)E \end{pmatrix} \end{aligned}$$

and

$$\mathbf{A}/A = R_Y^* R_Y - (R_X^{*[-1]} R_X^* R_Y)^* R_X^{*[-1]} R_X^* R_Y = R_Y^* P R_Y,$$

which shows that $\mathbf{A}/A$ is the left upper corner of $\mathbf{D}/A$. Since (PR_Y, H) is a square root of $\mathbf{A}/A$, we can compute

$$\begin{aligned} \mathbf{D}/A \,\big/\, \mathbf{A}/A &= D_1 - E^*(I-P)E - \left((PR_Y)^{*[-1]} R_Y^* P E\right)^* (PR_Y)^{*[-1]} R_Y^* P E \\ &= D_1 - E^*(I-P)E - E^* Q E, \end{aligned}$$

where Q denotes the orthoprojection onto $(\operatorname{ran} PR_Y)^c$. Comparing this with

$$\mathbf{D}/\mathbf{A} = D_1 - \left((R^*)^{-1} R^* E\right)^* (R^*)^{-1} R^* E = D_1 - E^* E,$$

we can conclude that the assertion will be proved if we can show that the restriction of $I - P + Q$ to $\operatorname{ran} R$ is the identity. If $h \in \operatorname{ran} R$, then

$$h = R_X x + R_Y y = R_X x + (I-P) R_Y y + P R_Y y$$

for some $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in X \times Y$. Since $R_X x + (I-P) R_Y y \in (\operatorname{ran} R_X)^c$, there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of X such that $\lim_{n\to\infty} R_X x_n = R_X x + (I-P) R_Y y$. For $h_n := R_X x_n + P R_Y y$, we have

$$(I-P+Q) h_n = (I-P+Q)(R_X x_n + P R_Y y) = R_X x_n + P R_Y y = h_n$$

and therefore

$$\begin{aligned}(I-P+Q)h &= \lim_{n\to\infty}(I-P+Q)h_n \\ &= \lim_{n\to\infty}(R_X x_n + PR_Y y) = R_X x + R_Y y = h.\end{aligned}$$ □

We conclude this section with a criterion for nonnegativity of operators of $\mathscr{L}^h$.

Proposition 5.4. *Let* $\mathbf{A} = \begin{pmatrix} A & B \\ B^\sim & D \end{pmatrix} \in \mathscr{L}^h$. *The operator* $\mathbf{A}$ *is nonnegative if and only if the following two conditions are satisfied:*

(i) *the operators A and D are nonnegative;*

(ii) *for any square roots (R_A, H_A) and (R_D, H_D) of A and D, respectively, there exists a contraction $K \in \mathscr{L}(H_D, H_A)$ such that $B = R_A^* K R$ and* $\operatorname{ran} K \subseteq (\operatorname{ran} R_A)^c$.

Proof. If $\mathbf{A}$ is nonnegative, assertion (i) is trivial. To prove (ii), let (R, H) be a square root of $\mathbf{A}$ and $R_X := R\restriction_X$, $R_Y = R\restriction_Y$, and hence,

$$\mathbf{A} = \begin{pmatrix} R_X^* R_X & R_X^* R_Y \\ R_Y^* R_X & R_Y^* R_Y \end{pmatrix}.$$

Let (S_A, G_A) and (S_D, G_D) be minimal square roots of A and D, respectively. According to Lemma 3.8 there exist isometries $U_A \in \mathscr{L}(G_A, H_A)$, $V_A \in \mathscr{L}(G_A, H)$, $U_D \in \mathscr{L}(G_D, H_D)$, and $V_D \in \mathscr{L}(G_D, H)$ satisfying $U_A S_A = R_A$, $V_A S_A = R_X$, $U_D S_D = R_D$, $V_D S_D = R_Y$. It follows that

$$B = R_X^* R_Y = R_A^* U_A V_A^* V_D U_D^* R_D = R_A^* K R_D,$$

where $K := U_A V_A^* V_D U_D^* \in \mathscr{L}(H_D, H_A)$ is a contraction with $\operatorname{ran} K \subseteq (\operatorname{ran} R_A)^c$. Conversely, if (i) and (ii) are satisfied, then

$$\mathbf{A} = \begin{pmatrix} R_A^* R_A & R_A^* K R_D \\ (R_A^* K R_D)^\sim & R_D^* R_D \end{pmatrix}.$$

Since

$$(R_A^* K R_D)^\sim = (R_A' K R_D)'\restriction_X = R_D' K' R_A''\restriction_X = R_D^* K^* R_A,$$

one obtains

$$\mathbf{A} = \begin{pmatrix} R_A^* & 0 \\ 0 & R_D^* \end{pmatrix} \begin{pmatrix} I & K \\ K^* & I \end{pmatrix} \begin{pmatrix} R_A & 0 \\ 0 & R_D \end{pmatrix},$$

which implies that $\mathbf{A}$ is nonnegative. □

6. Albert's theorem

An application of Proposition 5.4 leads to a generalization of an important criterion for nonnegativity (see [1]), which is often called *Albert's theorem* in matrix theory. It should be mentioned that Shmulyan [17, Theorem 1.7] proved a similar assertion even for bounded operators in Hilbert spaces ten years earlier (see also [3] and [10]).

Theorem 6.1. *An operator* $\mathbf{A} = \begin{pmatrix} A & B \\ B^\sim & D \end{pmatrix} \in \mathscr{L}^h$ *is nonnegative if and only if it is of positive type and* $\sigma(\mathbf{A})$ *is nonnegative.*

Proof. If $\mathbf{A} \in \mathscr{L}^{\geq}$, then Proposition 5.4 implies that $\mathbf{A}$ is of positive type and $R_A^{*[-1]}B = KR_D$ for some contraction $K \in \mathscr{L}(H_D, H_A)$. It follows that

$$\sigma(\mathbf{A}) = R_D^* R_D - (KR_D)^* KR_D = R_D^*(I - K^*K)R_D \geq 0.$$

Conversely, let $\mathbf{A} \in \mathscr{L}^+$ and $\sigma(\mathbf{A}) \in \mathscr{L}^{\geq}(Y, Y')$. If (R_A, H_A) and (R_D, H_D) are square roots of A and D, respectively, one has

$$\|R_A^{*[-1]}By\|^2 = \big\langle \omega(A,B)y, y\big\rangle \leq \langle Dy, y\rangle = \|R_D y\|^2, \quad y \in Y,$$

which yields $KR_D = R^{*[-1]}B$, and hence, $R_A^* KR_D = B$ for some contraction $K \in \mathscr{L}(H_D, H_A)$. An application of Proposition 5.4 completes the proof. □

Theorem 6.1 can be used to study the set $\mathscr{L}^{\geq}$ as well as the set $\mathscr{L}^+$, and to establish interrelations between these two sets. A first result is the inclusion $\mathscr{L}^{\geq} \subseteq \mathscr{L}^+$. For a positive pair (A, B), set

$$\mathbf{A}_{\text{ex}} := \begin{pmatrix} A & B \\ B^{\sim} & \omega(A,B) \end{pmatrix} \in \mathscr{L}^+.$$

Corollary 6.2. *Two operators $A \in \mathscr{L}(X, X')$ and $B \in \mathscr{L}(Y, X')$ form a positive pair if and only if the set*

$$\mathscr{A} := \left\{ \mathbf{A} \in \mathscr{L}^{\geq} : \mathbf{A} = \begin{pmatrix} A & B \\ B^{\sim} & D \end{pmatrix} \text{ for some } D \in \mathscr{L}^{\geq}(Y, Y') \right\}$$

is nonempty. If (A, B) is a positive pair, then the operator $\mathbf{A}_{\text{ex}}$ is the minimal element of $\mathcal{A}$.

Corollary 6.3. *If $\mathbf{A} \in \mathscr{L}^{\geq}$, then the set*

$$\mathscr{A}_1 := \{\mathbf{A}_1 \in \mathscr{L}^{\geq} : \mathbf{A}_1 \leq \mathbf{A} \text{ and } X \subseteq \ker \mathbf{A}_1\}$$

is nonempty and $\mathscr{S}(\mathbf{A})$ is its maximal element.

Proof. Corollaries 4.8(i) and 4.9 imply that $\mathscr{S}(\mathbf{A}) \in \mathscr{A}_1$. If $\mathbf{A} = \left(\begin{smallmatrix} A & B \\ B^{\sim} & D \end{smallmatrix}\right)$ and $\mathbf{A}_1 \in \mathscr{A}_1$, then $\mathbf{A}_1$ has representation $\mathbf{A}_1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & D_1 \end{smallmatrix}\right)$ and $D - \omega(A,B) - D_1 \geq 0$, and hence, $\mathbf{A}_1 \leq \mathscr{S}(\mathbf{A})$ by Theorem 6.1. □

Corollary 6.4. *Let $\mathbf{A} = \left(\begin{smallmatrix} A & B \\ B^{\sim} & D \end{smallmatrix}\right) \in \mathscr{L}^h$. The operator $\mathbf{A}$ belongs to $\mathscr{L}^+$ if and only if there exists an operator $\mathbf{A}_1 \in \mathscr{L}^h$ satisfying $X \subseteq \ker \mathbf{A}_1$ and $\mathbf{A}_1 \leq \mathbf{A}$.*

Proof. If $\mathbf{A} \in \mathscr{L}^+$, then the operator $\mathbf{A}_1 := \mathscr{S}(\mathbf{A})$ has all the properties claimed. Conversely, if there exists an operator $\mathbf{A}_1$ satisfying all conditions, then it has the form $\mathbf{A}_1 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & D_1 \end{smallmatrix}\right)$, where $D_1 \in \mathscr{L}(Y, Y')$ and $\mathbf{A} - \mathbf{A}_1 = \left(\begin{smallmatrix} A & B \\ B^{\sim} & D - D_1 \end{smallmatrix}\right) \in \mathscr{L}^{\geq}$. It follows from Theorem 6.1 that (A, B) is a positive pair, and hence, $\mathbf{A} \in \mathscr{L}^+$. □

Another application of Theorem 6.1 gives an expression of the supremum occuring in Lemma 4.2.

Corollary 6.5. *If (A, B) is a positive pair, then*

$$\sup_{x \in X} \frac{|\langle By, x\rangle|^2}{\langle Ax, x\rangle} = \big\langle \omega(A,B)y, y\big\rangle, \quad y \in Y. \tag{6.1}$$

Proof. Let $y \in Y$. Since $\mathbf{A}_{\mathrm{ex}} \in \mathscr{L}^{\geq}$ by Corollary 6.2, we have

$$\left|\langle By, x\rangle\right|^2 \leq \langle Ax, x\rangle\big\langle \omega(A,B)y, y\big\rangle,$$

which yields

$$\frac{|\langle By, x\rangle|^2}{\langle Ax, x\rangle} \leq \big\langle \omega(A,B)y, y\big\rangle, \quad x \in X,$$

if we take into account the convention $\frac{0}{0} := 0$. Thus, (6.1) has been proved if $Ty = R^{*[-1]}By = 0$, where (R,H) is a minimal square root of A. Now assume that $Ty \neq 0$. There exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X such that $Rx_n \neq 0$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} Rx_n = Ty$. It follows that

$$\begin{aligned}\lim_{n\to\infty} \frac{|\langle By, x_n\rangle|^2}{\langle Ax_n, x_n\rangle} &= \lim_{n\to\infty} \frac{|\langle R^*Ty, x_n\rangle|^2}{\langle R^*Rx_n, x_n\rangle} \\ &= \lim_{n\to\infty} \frac{|(Ty\,|\,Rx_n)|^2}{\|Rx_n\|^2} \\ &= \|Ty\|^2 \\ &= \big\langle \omega(A,B)y, y\big\rangle. \end{aligned}$$

□

Corollary 6.6. *Let (A_j, B_j) with $A_j \in (X, X')$, $B_j \in (Y, X')$, $j = 1, 2$, be positive pairs. Then $(A_1 + A_2, B_1 + B_2)$ is a positive pair and*

$$\omega(A_1 + A_2, B_1 + B_2) \leq \omega(A_1, B_1) + \omega(A_2, B_2). \tag{6.2}$$

Proof. Since the operators $(\mathbf{A}_j)_{\mathrm{ex}}$, $j = 1, 2$ are nonnegative, it follows that

$$\begin{pmatrix} A_1 + A_2 & B_1 + B_2 \\ B_1^{\sim} + B_2^{\sim} & \omega(A_1, B_1) + \omega(A_2, B_2) \end{pmatrix} \in \mathscr{L}^{\geq},$$

and hence, (6.2) by Corollary 6.2. □

Corollary 6.7. *If $\mathbf{A}_j \in \mathscr{L}^+$, $j = 1, 2$, then $\mathbf{A}_1 + \mathbf{A}_2 \in \mathscr{L}^+$ and*

$$\mathscr{S}(\mathbf{A}_1) + \mathscr{S}(\mathbf{A}_2) \leq \mathscr{S}(\mathbf{A}_1 + \mathbf{A}_2).$$

A subset $\mathscr{A}$ of $\mathscr{L}^h$ is said to be *bounded below* if there exists $\mathbf{A}_1 \in \mathscr{L}^h$ such that $\mathbf{A}_1 \leq \mathbf{A}$ for all $\mathbf{A} \in \mathscr{A}$. An operator $\mathbf{A}_0 \in \mathscr{L}^h$ is called an *infimum* of $\mathscr{A}$ if the following conditions are satisfied:

(a) $\mathbf{A}_0 \leq \mathbf{A}$ for all $\mathbf{A} \in \mathscr{A}$,
(b) $\mathbf{A}_1 \leq \mathbf{A}_0$ for all $\mathbf{A}_1 \in \mathscr{L}^h$ such that $\mathbf{A}_1 \leq \mathbf{A}$, $\mathbf{A} \in \mathscr{A}$.

If an infimum of $\mathscr{A}$ exists, it is unique. Recall that any set $\mathscr{A}$ which is bounded from below and directed downwards (i.e., for all $\mathbf{A}_1, \mathbf{A}_2 \in \mathscr{A}$ there exists $\mathbf{A} \in \mathscr{A}$ such that $\mathbf{A} \leq \mathbf{A}_1$ and $\mathbf{A} \leq \mathbf{A}_2$) possesses an infimum. In particular, if $\{\mathbf{A}_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of operators of $\mathscr{L}^h$ which is bounded from below, then there exists an infimum $\mathbf{A}_0$ and $\langle \mathbf{A}_0 z_1, z_2\rangle = \lim_{n\to\infty}\langle \mathbf{A}_n z_1, z_2\rangle$ for all $z_1, z_2 \in X \times Y$.

Corollary 6.8. *Let $\mathscr{A}$ be a subset of $\mathscr{L}^h$, which has an infimum $\mathbf{A}_0$. The operator $\mathbf{A}_0$ belongs to $\mathscr{L}^+$ if and only if the set $\mathcal{S}(\mathscr{A}) := \{\mathscr{S}(\mathbf{A}) : \mathbf{A} \in \mathscr{A}\}$ is bounded from below. In this case ,$\mathscr{S}(\mathbf{A}_0)$ is the infimum of $\mathcal{S}(\mathscr{A})$.*

Proof. If $\mathbf{A}_0 \in \mathscr{L}^+$, then the set $\mathcal{S}(\mathscr{A})$ is bounded from below since $\mathscr{S}(\mathbf{A}_0) \leq \mathscr{S}(\mathbf{A})$, $\mathbf{A} \in \mathscr{A}$, by Corollary 4.8(ii). Conversely, assume that there exists $\mathbf{A}_1 \in \mathscr{L}^h$ such that $\mathbf{A}_1 \leq \mathscr{S}(\mathbf{A})$ for all $\mathbf{A} \in \mathscr{A}$. It follows that $-\mathbf{A}_1 \geq -\mathscr{S}(\mathbf{A})$, which yields $-\mathbf{A}_1 \in \mathscr{L}^+$ by Corollary 6.4 and $\mathscr{S}(-\mathbf{A}_1) \geq \mathscr{S}(-\mathscr{S}(\mathbf{A})) = -\mathscr{S}(\mathbf{A})$, and hence, $-\mathscr{S}(-\mathbf{A}_1) \leq \mathscr{S}(\mathbf{A}) \leq \mathbf{A}$, $\mathbf{A} \in \mathscr{A}$, by Corollary 4.8. We obtain $-\mathscr{S}(-\mathbf{A}_1) \leq \mathbf{A}_0$ and therefore $\mathbf{A}_0 \in \mathscr{L}^+$ by Corollary 6.4. Moreover, $\mathscr{S}(\mathbf{A}_0) \leq \mathscr{S}(\mathbf{A})$, $\mathbf{A} \in \mathscr{A}$, and

$$\mathbf{A}_1 = -(-\mathbf{A}_1) \leq -\mathscr{S}(-\mathbf{A}_1) = \mathscr{S}\big(-\mathscr{S}(-\mathbf{A}_1)\big) \leq \mathscr{S}(\mathbf{A}_0)$$

by Corollary 4.8, which implies that $\mathscr{S}(\mathbf{A}_0)$ is the infimum of $\mathcal{S}(\mathscr{A})$. □

7. Extremal operators

An operator $\mathbf{A} \in \mathscr{L}^+$ was called an *extremal operator* by Kreĭn [11] if $\mathscr{S}(\mathbf{A}) = 0$. Since $\mathbf{A} = \mathscr{S}(\mathbf{A}) + \mathbf{A}_{\text{ex}}$, an operator is extremal if and only if it has the form

$$\mathbf{A} = \mathbf{A}_{\text{ex}} = \begin{pmatrix} A & B \\ B^{\sim} & \omega(A,B) \end{pmatrix}$$

for some positive pair (A, B). In particular, any extremal operator is nonnegative. Applying Proposition 4.7, we can give several criteria for an operator to be extremal.

Lemma 7.1. *Let $\mathbf{A} \in \mathscr{L}^{\geq}$. The following assertions are equivalent:*

(i) *the operator is extremal,*

(ii) *for all $\binom{x}{y} \in X \times Y$ and arbitrary $\varepsilon > 0$ there exists $z \in X$ such that*

$$\left\langle \mathbf{A}\begin{pmatrix} x-z \\ y \end{pmatrix}, \begin{pmatrix} x-z \\ y \end{pmatrix} \right\rangle < \varepsilon,$$

(iii) *for any square root (R, H) of $\mathbf{A}$ the spaces $(RX)^c$ and $(\operatorname{ran} R)^c$ coincide,*

(iv) *for any square root (R, H) of $\mathbf{A}$ we have $\operatorname{ran} R^* \cap Y' = \{0\}$.*

Proof. The equivalence of (i) and (ii) is an immediate consequence of (4.3). To prove (i) $\Leftrightarrow$ (iii), choose a minimal square root (R, H) of $\mathbf{A}$ and let L and P be defined as in Proposition 5.1. Then $\mathscr{S}(\mathbf{A}) = R^*PR = 0$ if and only if $P = 0$ or, equivalently, $L = \{0\}$, which in turn is equivalent to $(RX)^c = (\operatorname{ran} R)^c$. The equivalence of (iii) and (iv) follows from the equality $R^*L = \operatorname{ran} R^* \cap Y'$. □

Let (A, B) be a positive pair and let (R, H) be a square root of A. Recall the notation (4.1) of the operator $T := R^{*[-1]}B$. Moreover, let P_B be the orthoprojection onto $(\operatorname{ran} T)^c$. Since the operator $\mathbf{A}_{\text{ex}}$ is nonnegative, from Theorem 6.1 one can conclude that $(\omega(A,B), B^{\sim})$ is a positive pair as well changing the roles of X and Y. Thus, the operators $T^{*[-1]}B^{\sim}$ and

$$\omega\big(\omega(A,B), B^{\sim}\big) = [T^{*[-1]}B^{\sim}]^* T^{*[-1]}B^{\sim}$$

can be defined.

Lemma 7.2. *The equalities $T^{*[-1]}B^{\sim} = P_B R$ and $\omega(\omega(A,B), B^{\sim}) = R^* P_B R$ hold true.*

Proof. The second equality is an immediate consequence of the first one. To prove the first equality, we will show that

$$(T^{*[-1]}B^{\sim}x\,|\,h) = (P_B Rx\,|\,h) \quad \text{for all } x \in X \text{ and } h \in H. \tag{7.1}$$

Since $\operatorname{ran} T^{*[-1]}B^{\sim} \subseteq (\operatorname{ran} T)^c$, it is enough to prove (7.1) for $x \in X$ and $h \in \operatorname{ran} T$. If $h = Ty$ for some $y \in Y$, then we get

$$\begin{aligned}(T^{*[-1]}B^{\sim}x\,|\,h) &= (T^{*[-1]}B^{\sim}x\,|\,Ty)\\ &= \langle T^*T^{*[-1]}B^{\sim}x, y\rangle = \langle B^{\sim}x, y\rangle\end{aligned}$$

and

$$\begin{aligned}(P_B Rx\,|\,h) &= (P_B Rx\,|\,Ty)\\ &= (Rx\,|\,Ty)\\ &= \overline{\langle R^*Ty, x\rangle}\\ &= \overline{\langle By, x\rangle}\\ &= \langle B^{\sim}x, y\rangle,\end{aligned}$$

and hence, (7.1). □

From Corollary 6.2 it follows that $\omega(\omega(A,B),B^{\sim})$ is a minimal element of the set

$$\left\{A_1 \in \mathscr{L}(X,X') : \begin{pmatrix} A_1 & B \\ B^{\sim} & \omega(A,B)\end{pmatrix} \in \mathscr{L}^{\geq}\right\}.$$

Note also that

$$\omega\big(\omega\big(\omega(A,B),B^{\sim}\big),B\big) = \omega(A,B)$$

(see [12, Proposition 1.4(A)]). We call an extremal operator $\mathbf{A}_{\text{ex}} = \left(\begin{smallmatrix} A & B \\ B^{\sim} & \omega(A,B)\end{smallmatrix}\right)$ *doubly extremal* if $\omega(\omega(A,B),B^{\sim}) = A$.

In the case of bounded operators on Hilbert spaces, the remaining results of the present section were proved by Pekarev and Shmulyan [13, pp. 369–371] and partly rediscovered by Niemiec [12, Proposition 1.4]. We mention that Niemiec's proofs are based on Douglas's theorem and do not make explicit use of 2×2 block operators.

Proposition 7.3. *An operator* $\mathbf{A}_{\text{ex}}$ *is doubly extremal if and only if*

$$\ker T^* = \ker R^* \tag{7.2}$$

for any square root (R,H) *of* A.

Proof. According to Lemma 7.2, $\mathbf{A}$ is doubly extremal if and only if $R^*P_BR = R^*R$. If $\ker T^* = \ker R^*$ or, equivalently, $(\operatorname{ran} T)^c = (\operatorname{ran} R)^c$, it follows that $P_BR = R$, and hence, $R^*P_BR = R^*R$. Conversely, assume that $R^*P_BR = R^*R$, which yields $\|P_BRx\| = \|Rx\|$, $x \in X$, and hence, $(\operatorname{ran} R)^c \subseteq \operatorname{ran} P_B$ and $\ker P_B \subseteq \ker R^*$. Since $\operatorname{ran} P_B = (\operatorname{ran} T)^c$ or $\ker P_B = \ker T^*$, we get

$$\ker T^* \subseteq \ker R^*. \tag{7.3}$$

On the other hand, if $h \in \ker R^*$, then h is orthogonal to $(\operatorname{ran} R)^c$ and

$$0 = (h \,|\, Ty) = \langle T^* h, y \rangle, \quad y \in Y,$$

and thus, $\ker R^* \subseteq \ker T^*$. Taking into account (7.3) we obtain the desired equality. □

The preceding assertion shows that the equality (7.2) does not depend on the choice of the square root (R, H) of A and that, in the case of a minimal square root, the operator $\mathbf{A}_{\mathrm{ex}}$ is doubly extremal if and only if $\ker T^* = \{0\}$. Moreover, writing (7.2) in the equivalent form $(\operatorname{ran} T)^c = (\operatorname{ran} R)^c$, we obtain a generalization of [13, Theorem 1.6]. This means that $\mathbf{A}_{\mathrm{ex}}$ is doubly extremal if and only if the inverse image of $\operatorname{ran} B$ under the map R^* is dense in H.

Corollary 7.4. *If* $\operatorname{ran} R^* = \operatorname{ran} B$, *then the operator* $\mathbf{A}_{\mathrm{ex}}$ *is doubly extremal.*

To give another criterion for $\mathbf{A}_{\mathrm{ex}}$ to be doubly extremal, we equip the space Y' with the $\sigma(Y', Y)$-topology (i.e., the smallest topology such that for arbitrary $y \in Y$, the functional $y' \mapsto \langle y', y \rangle$ is continuous on Y'). Let (A, B) be a positive pair and let (R, H) be a square root of A. Denote by H_1 the subspace of all $h \in H$ such that there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of X with the following properties:

(a) $\lim_{n \to \infty} R x_n = h$ with respect to the norm topology of H,
(b) $\lim_{n \to \infty} B^{\sim} x_n = 0$ with respect to the $\sigma(Y', Y)$-topology.

Lemma 7.5. *The space* H_1 *is equal to* $(\operatorname{ran} R)^c \cap \ker T^*$.

Proof. An element $h \in H$ belongs to H_1 if and only if there exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of X such that $\lim_{n \to \infty} R x_n = h$ and for all $y \in Y$,

$$\begin{aligned} \langle T^* h, y \rangle &= \lim_{n \to \infty} (R x_n \,|\, Ty) \\ &= \lim_{n \to \infty} (x_n \,|\, By) \\ &= \lim_{n \to \infty} (B^{\sim} x_n \,|\, y) \\ &= 0. \end{aligned}$$

□

Proposition 7.6. *An operator* $\mathbf{A}_{\mathrm{ex}}$ *is doubly extremal if and only if* $H_1 = \{0\}$.

Proof. Since $H_1 = \{0\}$ if and only if $\ker R^* = \ker T^*$ by Lemma 7.5, the assertion follows from Proposition 7.3. □

Corollary 7.7. *If an operator* $\mathbf{A}_{\mathrm{ex}}$ *is doubly extremal, then* $\ker A = \ker B^{\sim}$. *If* $\ker A = \ker B^{\sim}$ *and* $\operatorname{ran} R$ *is closed, then* $\mathbf{A}_{\mathrm{ex}}$ *is doubly extremal.*

Proof. If $\mathbf{A}_{\mathrm{ex}}$ is doubly extremal, then $(T^{*[-1]} B^{\sim}, H)$ is a square root of A, and hence, $\ker B^{\sim} \subseteq \ker T^{*[-1]} B^{\sim} = \ker A$ by Lemma 3.4. The first assertion of the corollary follows since $\ker A \subseteq \ker B^{\sim}$ by Lemma 4.2. Now assume that $\ker A = \ker B^{\sim}$ and that $\operatorname{ran} R$ is closed. If $h \in H_1$, then there exist $x \in X$ and a sequence $\{x_n\}_{n \in \mathbb{N}}$ of elements of X such that $h = Rx = \lim_{n \to \infty} R x_n$ and

$\lim_{n\to\infty}\langle B^{\sim}x_n, y\rangle = 0$ for all $y \in Y$. It follows that

$$\begin{aligned}\langle B^{\sim}x, y\rangle &= \overline{\langle R^*Ty, x\rangle} \\ &= (Rx\,|\,Ty) \\ &= \lim_{n\to\infty}(Rx_n\,|\,Ty) \\ &= \lim_{n\to\infty}\overline{\langle By, x_n\rangle} \\ &= \lim_{n\to\infty}\langle B^{\sim}x_n, y\rangle \\ &= 0, \quad y \in Y,\end{aligned}$$

which implies that $x \in \ker B^{\sim} = \ker A = \ker R$ and $h = 0$. An application of Proposition 7.6 completes the proof. □

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