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TOEPLITZ OPERATORS ON WEIGHTED PLURIHARMONIC BERGMAN SPACE

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Abstract. In this article, we consider some algebraic properties of Toeplitz operators on weighted pluriharmonic Bergman space on the unit ball. We characterize the commutants of Toeplitz operators whose symbols are certain separately radial functions or holomorphic monomials, and then give a partial answer to the finite-rank product problem of Toeplitz operators.

1. Introduction

Let $\mathbb{B}_n$ denote the open unit ball of $\mathbb{C}^n$, and let v be the normalized Lebesgue volume measure on this unit ball. Fix a real number $\alpha > -1$. The weighted Lebesgue measure v_α on $\mathbb{B}_n$ is defined by $dv_\alpha(z) = c_\alpha(1-|z|^2)^\alpha\, dv(z)$, where c_α is a normalizing constant so that $v_\alpha(\mathbb{B}_n) = 1$. A direct computation shows that

$$c_\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

Let L^2_α denote $L^2(\mathbb{B}_n, dv_\alpha)$, and let $\langle\cdot,\cdot\rangle_\alpha$ denote its inner product.

The weighted Bergman space A^2_α consists of all functions in L^2_α which are holomorphic on $\mathbb{B}_n$. It is well known that A^2_α is a closed subspace of L^2_α. We denote the orthogonal projection from L^2_α onto A^2_α by P_α.

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The weighted pluriharmonic Bergman space b^2_α is the Hilbert space consisting of all pluriharmonic functions on $\mathbb{B}_n$ which are also in L^2_α. It is easy to verify that

$$b^2_\alpha = A^2_\alpha + \overline{A^2_\alpha},$$

where $\overline{A^2_\alpha} = \{\bar{f} : f \in A^2_\alpha, f(0) = 0\}$. For $z, w \in \mathbb{B}_n$, let

$$K_z(w) = \frac{1}{(1 - \langle w, z\rangle)^{(n+\alpha+1)}}$$

be the reproducing kernel of A^2_α. Then the reproducing kernel of b^2_α is

$$R_z(w) = K_z(w) + \overline{K_z(w)} - 1, \quad z, w \in \mathbb{B}_n.$$

Let Q_α denote the orthogonal projection from L^2_α onto b^2_α. For a function $\varphi \in L^\infty(\mathbb{B}_n, dv_\alpha)$, the Toeplitz operator T_φ with symbol φ is defined by

$$T_\varphi(f) = Q_\alpha(\varphi f) = \int_{\mathbb{B}_n} f(w)\varphi(w)\overline{R_z(w)}\, dv_\alpha(w), \quad f \in b^2_\alpha.$$

On the Hardy space of the unit disk, Brown and Halmos [4, Theorem 9] first showed that two Toeplitz operators are commuting if and only if either both symbols of these operators are analytic, or both symbols are conjugate analytic, or a nontrivial linear combination of the symbols is constant. Recently, Ding, Sun, and Zheng [9, Theorem 1.5] made progress on the commuting problem for the Hardy space of the bidisk and obtained an analogous result to the Brown and Halmos theorem as above, although their result is a little more complicated. On the polydisk, Lee [17, Main Theorem] obtained a concise result when one of the symbols of the operators is pluriharmonic.

On the Bergman space of the unit disk, Axler and Čučković [2, Theorem 1] showed that a result similar to that of the Brown and Halmos theorem holds for Toeplitz operators with bounded harmonic symbols. Although the commuting problem of Toeplitz operators with general bounded symbols is still far from its solution, some results for special symbols were obtained (see [20], [7]). Another problem that deserves consideration is the commutant problem. Čučković [6] first showed that the commutant of a Toeplitz operator with the monomial symbol $z^n (n \geq 1)$ consists of analytic Toeplitz operators. Several years later, Axler, Čučković, and Rao [3] obtained the same result when replacing the monomial symbol with a nonconstant analytic symbol. Čučković and Rao [7] gave a necessary and sufficient condition for a Toeplitz operator to commute with another Toeplitz operator whose symbol is a monomial $z^s\bar{z}^t(|s|+|t| > 0)$, and they proved that the commutant of a Toeplitz operator with a radial symbol just consists of Toeplitz operators with radial symbols.

On the Bergman space of several complex variables, the situation is much more complicated. Zheng [22] studied commuting Toeplitz operators with pluriharmonic symbols on the unit ball. Recently, Zhou and Dong [23] studied the commuting problem of Toeplitz operators whose symbols are quasihomogeneous functions. In that paper, they showed that the commutant of a radial Toeplitz operator includes nonradial Toeplitz operators, which is different from the one-variable case. Later on, they completely characterized the commutant of a radial

Toeplitz operator in [10], which was also obtained by Trieu Le [13] using a different method.

On the harmonic Bergman space, the commuting problem is harder, but some progress has been made in the literature (see [5], [11] and the references therein). Dong and Zhou [11] also investigated the commutant problem of Toeplitz operators whose symbols are radial functions or (conjugate-)analytic monomials.

On the pluriharmonic Bergman space of the unit ball, Lee and Zhu [18] and Lee [16] separately studied the commuting problem of Toeplitz operators, and obtained some results analogous to the harmonic Bergman space of the unit disk. To make some new progress, we will investigate the commutants of Toeplitz operators whose symbols are certain separately radial functions or holomorphic monomials.

For the finite-rank product problem, Aleman and Vukotić [1] showed that the product of finitely many Toeplitz operators on the Hardy space of the unit disk is of finite rank if and only if at least one of the operators is zero. On the Hardy space of the polydisk, Ding [8] proved a similar conclusion for Toeplitz operators with pluriharmonic symbols. In the settings of the Bergman space of the unit disk (see [15]) and the unit ball (see [14]), Trieu Le solved the problem for Toeplitz operators (except possibly one) diagonal with respect to the standard orthonormal basis. In the following, we will investigate this problem for Toeplitz operators on the pluriharmonic Bergman space of the unit ball.

Our article is organized as follows. In Section 2, we introduce some notation which will be used later. In Section 3, we characterize the commutant of the Toeplitz operator T_g, where g is a certain separately radial function (see Theorem 3.3). As a corollary, we will give an example to show that the Toeplitz operator commuting with a radial Toeplitz operator is not necessarily a radial one. This is a different phenomenon from the case of one variable. We also characterize the commutants of the Toeplitz operator T_{z^k}, where k is a nonzero multi-index (see Theorem 3.13). In Section 4, we investigate the finite-rank product problem of Toeplitz operators (except possibly one) whose symbols are of the form $z^s\bar{z}^t\varphi$, where $s, t \in \mathbb{N}^n$ and φ is a nonzero separately radial function (see Theorem 4.5).

2. Preliminaries

First we introduce some notation. For any multi-index $m = (m_1, \dots, m_n) \in \mathbb{N}^n$, we write $|m| = m_1 + \cdots + m_n$, $m! = m_1! \cdots m_n!$, $z^m = z_1^{m_1} \cdots z_n^{m_n}$, and $\bar{z}^m = \bar{z}_1^{m_1} \cdots \bar{z}_n^{m_n}$ for $z = (z_1, \dots, z_n) \in \mathbb{B}_n$. For two multi-indexes $m = (m_1, \dots, m_n)$ and $k = (k_1, \dots, k_n)$, we write $m \succeq k$ if $m_i \geq k_i, i = 1, \dots, n$, and we write $m \not\succeq k$ otherwise. We also write $m \succ k$ if $m \succeq k$ and there exists at least one subscript i such that $m_i > k_i$. If $m \succeq k$, then define $m - k = (m_1 - k_1, \dots, m_n - k_n)$. The standard orthonormal basis for the weighted Bergman space A^2_α is $\{e_m\}_{m\succeq 0}$, where

$$e_m(z) = \Big[\frac{\Gamma(n + |m| + \alpha + 1)}{m!\Gamma(n + \alpha + 1)}\Big]^{1/2} z^m, \quad m \in \mathbb{N}^n, z \in \mathbb{B}_n.$$

As a result, the standard orthonormal basis for the weighted pluriharmonic Bergman space b^2_α is $\{e_m\}_{m\succeq 0} \cup \{\bar{e}_m\}_{m\succ 0}$.

For any bounded measurable function g on $\mathbb{B}_n$, any $m \in \mathbb{N}^n$, and $\alpha > -1$, define

$$\tilde{g}(m) = \langle T_g e_m, e_m \rangle_\alpha = \int_{\mathbb{B}_n} g(z) e_m(z) \bar{e}_m(z)\, dv_\alpha(z).$$

It is clear that $\tilde{g}(m) = \langle T_g \bar{e}_m, \bar{e}_m \rangle_\alpha$ and $\tilde{\bar{g}}(m) = \overline{\tilde{g}(m)}$ for $m \in \mathbb{N}^n$.

For any $1 \le j \le n$, let $\sigma_j : \mathbb{N} \times \mathbb{N}^{n-1} \to \mathbb{N}^n$ be the map defined by the formula $\sigma_j(s, (r_1, \ldots, r_{n-1})) = (r_1, \ldots, r_{j-1}, s, r_{j+1}, \ldots, r_{n-1})$ for all $s \in \mathbb{N}$ and $(r_1, \ldots, r_{n-1}) \in \mathbb{N}^{n-1}$. If $\mathcal{S}$ is a subset of $\mathbb{N}^n$ and $1 \le j \le n$, then we define

$$\widetilde{\mathcal{S}}_j = \Big\{ \tilde{r} = (r_1, \ldots, r_{n-1}) \in \mathbb{N}^{n-1} : \sum_{\substack{s \in \mathbb{N} \\ \sigma_j(s, \tilde{r}) \in \mathcal{S}}} \frac{1}{s+1} = \infty \Big\}.$$

The following definition comes from [13].

Definition 2.1 ([13, Definition 3.1]). We say that $\mathcal{S}$ has *property (P)* if one of the following statements holds:

(1) $\mathcal{S} = \emptyset$, or
(2) $\mathcal{S} \neq \emptyset, n = 1$ and $\sum_{s \in \mathcal{S}} \frac{1}{s+1} < \infty$, or
(3) $\mathcal{S} \neq \emptyset, n \ge 2$ and for any $1 \le j \le n$, the set $\widetilde{\mathcal{S}}_j$ has property (P) as a subset of $\mathbb{N}^{n-1}$.

Remark 2.2. By the preceding definition, we can immediately get the following statements.

(1) If $\mathcal{S} \subset \mathbb{N}$ and $\mathcal{S}$ does not have property (P), then $\sum_{s \in \mathcal{S}} \frac{1}{s+1} = \infty$. If $\mathcal{S} \subset \mathbb{N}^n$ with $n \ge 2$ does not have property (P), then $\widetilde{\mathcal{S}}_j$ does not have property (P) as a subset of $\mathbb{N}^{n-1}$ for some $1 \le j \le n$.
(2) If $\mathcal{S}_1$ and $\mathcal{S}_2$ are subsets of $\mathbb{N}^n$ that both have property (P), then $\mathcal{S}_1 \cup \mathcal{S}_2$ also has property (P).
(3) If $\mathcal{S} \subset \mathbb{N}^n$ has property (P) and $l \in \mathbb{Z}^n$, then $(\mathcal{S} + l) \cap \mathbb{N}^n$ also has property (P). Here, $\mathcal{S} + l = \{m + l : m \in \mathcal{S}\}$.
(4) If $\mathcal{S} \subset \mathbb{N}^n$ has property (P), then $\mathbb{N} \times \mathcal{S}$ also has property (P) as a subset of $\mathbb{N}^{n+1}$. This follows by induction on n.
(5) The set $\mathbb{N}^n$ does not have property (P) for all $n \ge 1$. This together with (2) shows that if $\mathcal{S} \subset \mathbb{N}^n$ has property (P), then $\mathbb{N}^n \setminus \mathcal{S}$ does not have property (P).
(6) For any $k = (k_1, \ldots, k_n)$ in $\mathbb{N}^n$, the set $\mathcal{S} = \{m \in \mathbb{N}^n : m \not\succeq k\}$ has property (P). This follows from (2), (4), and the fact that

$$\mathcal{S} \subset \bigcup_{j=1}^{n} \mathbb{N} \times \cdots \times \mathbb{N} \times \{0, \ldots, k_j - 1\} \times \mathbb{N} \times \cdots \times \mathbb{N}.$$

3. Commutants of Toeplitz operators

3.1. **Toeplitz operators with certain separately radial symbols.** In this section, we investigate commutants of Toeplitz operators with certain separately radial symbols on the weighted pluriharmonic Bergman space of the unit ball $\mathbb{B}_n$.

Recall that a function ψ on $\mathbb{B}_n$ is *radial* if $\psi(z)$ depends only on $|z|$. In contrast to radial functions, a function φ on $\mathbb{B}_n$ is called a *separately radial function* if $\varphi(z_1, \ldots, z_n) = \varphi(|z_1|, \ldots, |z_n|)$. In order to state the main result in this section, we need the following two lemmas from [13].

Lemma 3.1 ([13, Corollary 3.5])**.** *Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be an n-tuple of integers, and let f be in $L^1(\mathbb{B}_n, dv)$. If for almost all $z \in \mathbb{B}_n$, $f(e^{i\gamma_1\theta}z_1, \ldots, e^{i\gamma_n\theta}z_n) = f(z)$ for almost all $\theta \in \mathbb{R}$, then whenever $l = (l_1, \ldots, l_n) \in \mathbb{Z}^n$ with $\gamma_1 l_1 + \cdots + \gamma_n l_n \neq 0$, we have $\int_{\mathbb{B}_n} f(z) z^{m+l} \bar{z}^m \, dv(z) = 0$ for all $m \in \mathbb{N}^n$ with $m + l \succeq 0$.*

Lemma 3.2 ([13, Proposition 3.6])**.** *Suppose that $g(z) = |z_1|^{2s_1} \cdots |z_n|^{2s_n} h(|z|)$ for $z \in \mathbb{B}_n$, where $s_1, \ldots, s_n \geq 0$ and $h : [0, 1) \to \mathbb{C}$ is a bounded measurable function. Assume that g is not a constant function on $\mathbb{B}_n$. Then for $l = (l_1, \ldots, l_n) \in \mathbb{Z}^n$ with $\sum l = 0$ and $s_1 l_1 = \cdots = s_n l_n = 0$, we have $\tilde{g}(m + l) = \tilde{g}(m)$ for all $m \in \mathbb{N}^n$ with $m + l \succeq 0$.*

Let g be of the form defined in Lemma 3.2. We now can characterize the commutant of T_g on weighted pluriharmonic Bergman space.

Theorem 3.3. *For $f \in L^\infty$, $T_f T_g = T_g T_f$ on b^2_α if and only if for $1 \leq j \leq n$ with $s_j \neq 0$, $f(e^{i\theta} z) = f(z)$ and $f(z_1, \ldots, z_{j-1}, |z_j|, \ldots, z_n) = f(z)$ for almost all $\theta \in \mathbb{R}$ and almost all $z \in \mathbb{B}_n$.*

Proof. Since $g(z_1, \ldots, z_n) = g(|z_1|, \ldots, |z_n|)$ for almost all $z \in \mathbb{B}_n$, it is easy to verify that T_g is diagonal with respect to the standard orthonormal basis and that $T_g e_m = \tilde{g}(m) e_m, T_g \bar{e}_m = \tilde{g}(m) \bar{e}_m$ for all $m \in \mathbb{N}^n$. It is clear that $T_f T_g = T_g T_f$ on b^2_α if and only if for all $l \in \mathbb{Z}^n$ and $m \in \mathbb{N}^n$ with $m + l \succeq 0$, the following four identities hold true:

(a) $0 = \langle (T_f T_g - T_g T_f) e_{m+l}, e_m \rangle_\alpha = (\tilde{g}(m + l) - \tilde{g}(m)) \langle T_f e_{m+l}, e_m \rangle_\alpha$,
(b) $0 = \langle (T_f T_g - T_g T_f) e_{m+l}, \bar{e}_m \rangle_\alpha = (\tilde{g}(m + l) - \tilde{g}(m)) \langle T_f e_{m+l}, \bar{e}_m \rangle_\alpha$,
(c) $0 = \langle (T_f T_g - T_g T_f) \bar{e}_{m+l}, e_m \rangle_\alpha = (\tilde{g}(m + l) - \tilde{g}(m)) \langle T_f \bar{e}_{m+l}, e_m \rangle_\alpha$,
(d) $0 = \langle (T_f T_g - T_g T_f) \bar{e}_{m+l}, \bar{e}_m \rangle_\alpha = (\tilde{g}(m + l) - \tilde{g}(m)) \langle T_f \bar{e}_{m+l}, \bar{e}_m \rangle_\alpha$.

Suppose that $T_f T_g = T_g T_f$ on b^2_α. Since T_g is diagonal, A^2_α is a reducing subspace of T_g and hence $PT_g = T_g P = PT_g P$, where P denotes the orthogonal projection from b^2_α onto A^2_α. If T_f and T_g commute as operators on b^2_α, then it follows that $PT_f P$ commutes with $PT_g P$. Since $PT_f P$ (resp., $PT_g P$) is in fact the Toeplitz operator with symbol f (resp., g) acting on A^2_α, it follows from [13, Theorem 1.2] that for $1 \leq j \leq n$ with $s_j \neq 0$, $f(e^{i\theta} z) = f(z)$ and $f(z_1, \ldots, z_{j-1}, |z_j|, \ldots, z_n) = f(z)$ for almost all $\theta \in \mathbb{R}$ and almost all $z \in \mathbb{B}_n$.

Now suppose that for $1 \leq j \leq n$ with $s_j \neq 0$, $f(e^{i\theta} z) = f(z)$ and $f(z_1, \ldots, z_{j-1}, |z_j|, \ldots, z_n) = f(z)$ for almost all $\theta \in \mathbb{R}$ and almost all $z \in \mathbb{B}_n$. Let $l = (l_1, \ldots, l_n) \in \mathbb{Z}^n$. If $\sum l \neq 0$ or $s_j l_j \neq 0$(hence $s_j \neq 0$ and $l_j \neq 0$) for some $1 \leq j \leq n$, then Lemma 3.1 shows that $\langle T_f e_{m+l}, e_m \rangle_\alpha = 0$ for all $m \in \mathbb{N}^n$ with $m + l \succeq 0$. If $\sum l = 0$ and $s_1 l_1 = \cdots = s_n l_n = 0$, then Lemma 3.2 tells us that $\tilde{g}(m+l) = \tilde{g}(m)$ for all $m \in \mathbb{N}^n$ with $m+l \succeq 0$. Therefore, (a) holds for all $l \in \mathbb{Z}^n$ and $m \in \mathbb{N}^n$ with $m + l \succeq 0$.

Since $f(e^{i\theta}z) = f(z)$, we have

$$\int_{\mathbb{B}_n} f(z)z^{2m+l}\, dv_\alpha(z) = \int_{\mathbb{B}_n} f(e^{i\theta}z)z^{2m+l}\, dv_\alpha(z)$$
$$= e^{-i(2|m|+\sum l)\theta} \int_{\mathbb{B}_n} f(z)z^{2m+l}\, dv_\alpha(z).$$

It is clear from the above equations that for all $m \succ 0$ with $m + l \succeq 0$, $\int_{\mathbb{B}_n} f(z)z^{2m+l}\, dv_\alpha(z) = 0$, which implies that (b) holds. Similarly, (c) holds for all $m \in \mathbb{N}^n$ with $m + l \succ 0$.

Finally, $\langle T_f \bar{e}_{m+l}, \bar{e}_m\rangle_\alpha = \overline{\langle T_{\bar{f}} e_{m+l}, e_m\rangle_\alpha}$. Under the assumption of f, for $1 \le j \le n$ with $s_j \ne 0$, $\bar{f}(e^{i\theta}z) = \bar{f}(z)$ and $\bar{f}(z_1, \dots, z_{j-1}, |z_j|, \dots, z_n) = \bar{f}(z)$ for almost all $\theta \in \mathbb{R}$ and almost all $z \in \mathbb{B}_n$. By an argument similar to the proof of (a), (d) also holds for all $m \in \mathbb{N}^n$ with $m + l \succeq 0$. Therefore, $T_f T_g = T_g T_f$ on b^2_α. □

In the case $n = 1$, Dong and Zhou [11, Theorem 4.3] proved that if a Toeplitz operator commutes with another Toeplitz operator with a radial symbol, then its symbol is also radial. The following corollary shows that the situation is different when $n > 1$.

Corollary 3.4. *Let g be a nonconstant radial function on $\mathbb{B}_n$. Then for $f \in L^\infty$, $T_f T_g = T_g T_f$ on b^2_α if and only if $f(e^{i\theta}z) = f(z)$ for almost all $\theta \in \mathbb{R}$ and almost all $z \in \mathbb{B}_n$.*

Example 3.5. Let $f(z) = z_1\bar{z}_2$ be a function on $\mathbb{B}_2$. Then $f(e^{i\theta}z) = f(z)$ for almost all $\theta \in \mathbb{R}$ and almost all $z \in \mathbb{B}_2$, but f is obviously not a radial function.

3.2. **Toeplitz operators with holomorphic monomial symbols.** Next we investigate commutants of Toeplitz operators with holomorphic monomial symbols on the weighted pluriharmonic Bergman space of the unit ball. Recall that the Mellin transform $\hat{\varphi}$ of a function $\varphi \in L^1([0,1], r\,dr)$ is defined by

$$\hat{\varphi}(z) = \int_0^1 \varphi(s)s^{z-1}\, ds.$$

It is clear that $\hat{\varphi}$ is well defined on the right half-plane $\{z : \operatorname{Re} z > 2\}$. It is important and helpful to know that the Mellin transform $\hat{\varphi}$ is uniquely determined by its value on an arithmetic sequence of integers. In fact, we have the following classical conclusion (see [21, p. 102]).

Lemma 3.6. *Suppose that f is a bounded analytic function on $\{z : \operatorname{Re} z > 0\}$ which vanishes at the pairwise distinct points $z_1, z_2, \dots,$ where*

(1) $\inf\{|z_k|\} > 0$, *and*
(2) $\sum_{k\ge1} \operatorname{Re}(1/z_k) = \infty$.

Then f vanishes identically on $\{z : \operatorname{Re} z > 0\}$.

Remark 3.7. By the above lemma, if $\varphi \in L^1([0,1], r\,dr)$ and there exists a sequence $(n_k)_{k\ge0} \subset \mathbb{N}$ such that

$$\hat{\varphi}(n_k) = 0, \qquad \sum_{k\geq 0} \frac{1}{n_k} = \infty,$$

then $\hat{\varphi}(z) = 0$ for all $z \in \{z : \operatorname{Re} z > 2\}$, and so $\varphi = 0$.

For two multi-indexes $p = (p_1, \ldots, p_n)$ and $s = (s_1, \ldots, s_n)$, the notation $p \perp s$ means that $p_1 s_1 + \cdots + p_n s_n = 0$. It is clear that if $p \perp s$, then $m + p \succeq s$ is equivalent to $m \succeq s$ for any multi-index m.

Definition 3.8. Let $l \in \mathbb{Z}^n$, and let f be a function in $L^1(\mathbb{B}_n, dv_\alpha)$. Then we say that f is a *quasihomogeneous function of quasihomogeneous degree* l if f is of the form $\xi^l \varphi$, where φ is a radial function; that is,

$$f(r\xi) = \xi^l \varphi(r)$$

for any ξ in the unit sphere $\mathbb{S}_n$ and $r \in [0, 1)$.

Remark 3.9. Clearly, any $l \in \mathbb{Z}^n$ can be uniquely written as $p - s$, where p and s are two multi-indexes such that $p \perp s$. Thus in this article, we always define the function

$$\xi^l = \xi^p \bar{\xi}^s, \quad \xi \in \mathbb{S}_n,$$

for any $l \in \mathbb{Z}^n$.

The following lemma will be used later.

Lemma 3.10. *Suppose that p, s are two multi-indexes and that φ is an integrable radial function such that $T_{\xi^p \bar{\xi}^s \varphi}$ is a bounded operator. Then for any multi-index m,*

$$T_{\xi^p\bar{\xi}^s\varphi}(z^m) = \begin{cases} \frac{(p+m)!(n-1+|p|+|m|-|s|)!}{(p+m-s)!(n-1+|m|+|p|)!} \frac{[\widehat{(1-r^2)^\alpha\varphi}](2n+2|m|+|p|-|s|)}{\widehat{(1-r^2)^\alpha}(2n+2|m|+2|p|-2|s|)} z^{p+m-s} & p+m \succeq s, \\ \frac{s!(n-1+|s|-|m|-|p|)!}{(s-m-p)!(n-1+|s|)!} \frac{[\widehat{(1-r^2)^\alpha\varphi}](2n+|s|-|p|)}{\widehat{(1-r^2)^\alpha}(2n+2|s|-2|m|-2|p|)} \bar{z}^{s-m-p} & p+m \preceq s, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. For multi-indexes m and k,

$$\begin{aligned} &\left\langle P_\alpha[\xi^p\bar{\xi}^s\varphi z^m], z^k\right\rangle_\alpha \\ &= \int_{\mathbb{B}_n} \xi^p\bar{\xi}^s\varphi(z) z^m \bar{z}^k \, dv_\alpha(z) \\ &= \int_{[0,1)} 2nc_\alpha(1-r^2)^\alpha\varphi(r) r^{2n+|m|+|k|-1}\, dr \int_{\mathbb{S}_n} \xi^p\bar{\xi}^s\xi^m\bar{\xi}^k \, d\sigma(\xi) \\ &= \begin{cases} \frac{2n! c_\alpha (p+m)!}{(n-1+|m|+|p|)!} [\widehat{(1-r^2)^\alpha\varphi}](2n+2|m|+|p|-|s|) & p+m-s = k, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

If $p+m \succeq s$, we have

$$\begin{aligned}
&\langle z^{p+m-s}, z^k\rangle_\alpha \\
&\quad = \int_{\mathbb{B}_n} z^{p+m-s}\bar{z}^k \, dv_\alpha(z) \\
&\quad = \int_{[0,1)} 2nc_\alpha(1-r^2)^\alpha r^{2n+|p|+|m|-|s|+|k|-1}\, dr \int_{\mathbb{S}_n} \xi^{p+m-s}\bar{\xi}^k \, d\sigma(\xi) \\
&\quad = \begin{cases} \frac{2n! c_\alpha (p+m-s)!}{(n-1+|p|+|m|-|s|)!} \widehat{(1-r^2)^\alpha}(2n+2|p|+2|m|-2|s|) & p+m-s=k, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Furthermore, $\langle P_\alpha[\xi^p\bar{\xi}^s\varphi z^m], \bar{z}^k\rangle_\alpha = 0 = \langle z^{p+m-s}, \bar{z}^k\rangle_\alpha$ holds for all $k \succ 0$. So we obtain

$$\begin{aligned}
&P_\alpha[\xi^p\bar{\xi}^s\varphi z^m] \\
&\quad = \begin{cases} \frac{(p+m)!(n-1+|p|+|m|-|s|)!}{(p+m-s)!(n-1+|p|+|m|)!} \frac{[\widehat{(1-r^2)^\alpha\varphi}](2n+2|m|+|p|-|s|)}{\widehat{(1-r^2)^\alpha}(2n+2|p|+2|m|-2|s|)} z^{p+m-s} & p+m \succeq s, \\ 0 & p+m \not\succeq s. \end{cases}
\end{aligned}$$

Note that $\bar{\varphi}$ is still radial, so by a similar calculation, we have

$$\begin{aligned}
&P_\alpha[\bar{\xi}^p\xi^s\bar{\varphi}\bar{z}^m] \\
&\quad = \begin{cases} \frac{s!(n-1+|s|-|m|-|p|)!}{(s-m-p)!(n-1+|s|)!} \frac{\overline{[\widehat{(1-r^2)^\alpha\varphi}]}(2n+|s|-|p|)}{\widehat{(1-r^2)^\alpha}(2n+2|s|-2|m|-2|p|)} z^{s-m-p} & p+m \preceq s, \\ 0 & p+m \not\preceq s. \end{cases}
\end{aligned}$$

Thus

$$\begin{aligned}
T_{\xi^p\bar{\xi}^s\varphi}(z^m) &= P_\alpha[\xi^p\bar{\xi}^s\varphi z^m] + \overline{P_\alpha[\bar{\xi}^p\xi^s\bar{\varphi}\bar{z}^m]} - P_\alpha[\xi^p\bar{\xi}^s\varphi z^m](0) \\
&= \begin{cases} \frac{(p+m)!(n-1+|p|+|m|-|s|)!}{(p+m-s)!(n-1+|m|+|p|)!} \frac{[\widehat{(1-r^2)^\alpha\varphi}](2n+2|m|+|p|-|s|)}{\widehat{(1-r^2)^\alpha}(2n+2|m|+2|p|-2|s|)} z^{p+m-s} & p+m \succeq s, \\ \frac{s!(n-1+|s|-|m|-|p|)!}{(s-m-p)!(n-1+|s|)!} \frac{[\widehat{(1-r^2)^\alpha\varphi}](2n+|s|-|p|)}{\widehat{(1-r^2)^\alpha}(2n+2|s|-2|m|-2|p|)} \bar{z}^{s-m-p} & p+m \preceq s, \\ 0 & \text{otherwise.} \end{cases} \quad \square
\end{aligned}$$

Dong and Zhou [11, Theorem 4.2] showed that, on the harmonic Bergman space of the unit disk, if f is a bounded function, then T_f and T_{z^k} commute if and only if a nontrivial linear combination of f and z^k is constant. What is the situation on weighted pluriharmonic Bergman space of the unit ball? We will give a partial answer to this question in Theorem 3.13.

Theorem 3.11. *Let f be a nonconstant bounded holomorphic function, and let g be a nonzero bounded quasihomogeneous function. If T_f and T_g commute on b^2_α, then g is a monomial. Moreover, if $\alpha = 0$, then T_f and T_g commute if and only if $f = \lambda g + \mu$ for some constants λ, μ.*

Proof. Let $f = \sum_{\beta \succeq 0} f_\beta z^\beta$ be the power series representation of f, and let $g = \xi^l\varphi = \xi^p\bar{\xi}^s\varphi$, where p, s are two multi-indexes such that $p \perp s$ and $l = p - s$. If

T_f and $T_{\xi^p\bar{\xi}^s\varphi}$ commute, then $T_fT_{\xi^p\bar{\xi}^s\varphi}z^m = T_{\xi^p\bar{\xi}^s\varphi}T_fz^m$ for every multi-index m. By Lemma 3.10, we have

$$T_fT_{\xi^p\bar{\xi}^s\varphi}z^m = \begin{cases} \sum_{\beta\succeq 0} T_{f_\beta z^\beta}T_{\xi^p\bar{\xi}^s\varphi}z^m & p+m\succeq s, \\ \sum_{\beta+m+p\succeq s} + \sum_{\beta+m+p\preceq s} T_{f_\beta z^\beta}T_{\xi^p\bar{\xi}^s\varphi}z^m & p+m\preceq s, \\ 0 & \text{otherwise} \end{cases}$$

and

$$T_{\xi^p\bar{\xi}^s\varphi}T_fz^m = \sum_{\beta+m+p\succeq s} T_{\xi^p\bar{\xi}^s\varphi}T_{f_\beta z^\beta}z^m + \sum_{\beta+m+p\preceq s} T_{\xi^p\bar{\xi}^s\varphi}T_{f_\beta z^\beta}z^m.$$

Claim. $T_{f_\beta z^\beta}T_{\xi^p\bar{\xi}^s\varphi}z^m = T_{\xi^p\bar{\xi}^s\varphi}T_{f_\beta z^\beta}z^m$ for any multi-index β.

We will discuss three cases.

Case 1. If $p+m\succeq s$, then $p+m+\beta\succeq s, \forall\beta\succeq 0$. Since $T_fT_{\xi^p\bar{\xi}^s\varphi}z^m = T_{\xi^p\bar{\xi}^s\varphi}T_fz^m$, we have

$$T_{f_\beta z^\beta}T_{\xi^p\bar{\xi}^s\varphi}z^m = T_{\xi^p\bar{\xi}^s\varphi}T_{f_\beta z^\beta}z^m, \quad \forall\beta\succeq 0.$$

Case 2. If $p+m\preceq s$, then $T_{f_\beta z^\beta}T_{\xi^p\bar{\xi}^s\varphi}z^m = T_{\xi^p\bar{\xi}^s\varphi}T_{f_\beta z^\beta}z^m$ when $p+m+\beta\succeq s$ or $p+m+\beta\preceq s$. Since $T_{f_\beta z^\beta}T_{\xi^p\bar{\xi}^s\varphi}z^m = T_{\xi^p\bar{\xi}^s\varphi}T_{f_\beta z^\beta}z^m = 0$ when $p+m+\beta\not\succeq s$ and $p+m+\beta\not\preceq s$, we have

$$T_{f_\beta z^\beta}T_{\xi^p\bar{\xi}^s\varphi}z^m = T_{\xi^p\bar{\xi}^s\varphi}T_{f_\beta z^\beta}z^m, \quad \forall\beta\succeq 0.$$

Case 3. If $p+m\not\succeq s$ and $p+m\not\preceq s$, then $T_{\xi^p\bar{\xi}^s\varphi}z^m = 0$, so $T_{f_\beta z^\beta}T_{\xi^p\bar{\xi}^s\varphi}z^m = 0, \forall\beta\succeq 0$. Since $T_{\xi^p\bar{\xi}^s\varphi}T_fz^m = T_fT_{\xi^p\bar{\xi}^s\varphi}z^m = 0$ and $T_{\xi^p\bar{\xi}^s\varphi}T_{f_\beta z^\beta}z^m = 0$ whenever $p+m+\beta\not\succeq s$ and $p+m+\beta\not\preceq s$, we have $T_{\xi^p\bar{\xi}^s\varphi}T_{f_\beta z^\beta}z^m = 0$ for $p+m+\beta\succeq s$ or $p+m+\beta\preceq s$, and thus

$$T_{f_\beta z^\beta}T_{\xi^p\bar{\xi}^s\varphi}z^m = T_{\xi^p\bar{\xi}^s\varphi}T_{f_\beta z^\beta}z^m = 0, \quad \forall\beta\succeq 0.$$

Because f is nonconstant, there exists some γ with $|\gamma|\geq 1$ such that $f_\gamma\neq 0$. By the above claim, we get $T_{z^\gamma}T_{\xi^p\bar{\xi}^s\varphi}z^m = T_{\xi^p\bar{\xi}^s\varphi}T_{z^\gamma}z^m$. It follows from Lemma 3.10 that

(1) $[\widehat{(1-r^2)^\alpha\varphi}](2n+2|m|+2|\gamma|+|p|-|s|) = 0$, if $p+m+\gamma\succeq s$ and $p+m\not\succeq s, p+m\not\preceq s$;
(2) $[\widehat{(1-r^2)^\alpha\varphi}](2n+2|m|+2|\gamma|+|p|-|s|) = d_m[\widehat{(1-r^2)^\alpha\varphi}](2n+2|m|+|p|-|s|)$, if $p+m\succeq s$,

where

$$d_m = \frac{(p+m)!(p+m+\gamma-s)!(n-1+|p|+|m|-|s|)!(n-1+|m|+|\gamma|+|p|)!}{(p+m+\gamma)!(p+m-s)!(n-1+|m|+|\gamma|+|p|-|s|)!(n-1+|m|+|p|)!} \times \frac{\widehat{(1-r^2)^\alpha}(2n+2|m|+2|\gamma|+2|p|-2|s|)}{\widehat{(1-r^2)^\alpha}(2n+2|m|+2|p|-2|s|)}.$$

We first prove $s=0$. Otherwise, $|s|\geq 1$ and we can consider two cases.

Case 1. Suppose that $\gamma_{i_0} \neq 0$ and $s_{i_0} \neq 0$ for some $i_0 \in \{1, \dots, n\}$. Let $m' = (s_1+1, \dots, s_{i_0-1}+1, s_{i_0}-1, s_{i_0+1}+1, \dots, s_n+1)$. Then $p+m'+\gamma \succeq s$ and $p+m' \not\succeq s, p+m' \not\preceq s$. Then it follows from (1) that

$$\big[\widehat{(1-r^2)^\alpha\varphi}\big]\big(2n+2|m'|+2|\gamma|+|p|-|s|\big)=0.$$

Using (2) repeatedly gives

$$\big[\widehat{(1-r^2)^\alpha\varphi}\big]\big(2n+2|m'|+2j|\gamma|+|p|-|s|\big)=0$$

for $j \geq 1$. It is clear that $\sum_{j\geq 1}\frac{1}{2|m'|+2j|\gamma|-|s|}=\infty$. Then it follows from Remark 3.7 that $(1-r^2)^\alpha\varphi=0$ and so $\varphi=0$.

Case 2. Suppose that $\gamma \perp s$. Obviously, for any multi-index $p+m \succeq s$, $(p+m)!(p+m+\gamma-s)! = (p+m+\gamma)!(p+m-s)!$. It follows from (2) that

$$\frac{[\widehat{(1-r^2)^\alpha\varphi}](2n+2|m|+2|\gamma|+|p|-|s|)\widehat{(1-r^2)^\alpha}(2n+2|m|+2|p|-2|s|)}{(n-1+|m|+|\gamma|+|p|)\cdots(n-1+|m|+|\gamma|+|p|-|s|+1)}$$
$$=\frac{[\widehat{(1-r^2)^\alpha\varphi}](2n+2|m|+|p|-|s|)\widehat{(1-r^2)^\alpha}(2n+2|m|+2|\gamma|+2|p|-2|s|)}{(n-1+|m|+|p|)\cdots(n-1+|m|+|p|-|s|+1)}.$$

Denote

$$F(z)=\frac{[\widehat{(1-r^2)^\alpha\varphi}](2z+2|\gamma|+|p|+|s|)\widehat{(1-r^2)^\alpha}(2z+2|p|)}{(z+|\gamma|+|p|+|s|-1)\cdots(z+|\gamma|+|p|)}$$
$$-\frac{[\widehat{(1-r^2)^\alpha\varphi}](2z+|p|+|s|)\widehat{(1-r^2)^\alpha}(2z+2|\gamma|+2|p|)}{(z+|p|+|s|-1)\cdots(z+|p|)}.$$

Then F is analytic and bounded on $\{z : \operatorname{Re} z > n\}$ since $(1-r^2)^\alpha\varphi, (1-r^2)^\alpha \in L^1([0,1), r^{2n-1}\,dr)$. By the above equation, $F(n+|m|-|s|)=0$ for $p+m \succeq s$. Note that $p+m \succeq s \Leftrightarrow m \succeq s$ since $p \perp s$ and $\sum_{m\succeq s}\frac{1}{n+|m|-|s|}=\infty$. Then Lemma 3.6 implies that $F=0$. Thus

$$\frac{[\widehat{(1-r^2)^\alpha\varphi}](2z+2|\gamma|+|p|+|s|)}{(z+|\gamma|+|p|+|s|-1)\cdots(z+|\gamma|+|p|)\widehat{(1-r^2)^\alpha}(2z+2|\gamma|+2|p|)}$$
$$=\frac{[\widehat{(1-r^2)^\alpha\varphi}](2z+|p|+|s|)}{(z+|p|+|s|-1)\cdots(z+|p|)\widehat{(1-r^2)^\alpha}(2z+2|p|)}.$$

Denote

$$G(z)=\frac{[\widehat{(1-r^2)^\alpha\varphi}](2z+|p|+|s|)}{(z+|p|+|s|-1)\cdots(z+|p|)\widehat{(1-r^2)^\alpha}(2z+2|p|)}.$$

Then $G(z)$ is a periodic function with period $|\gamma|$ on $\{z : \operatorname{Re} z > n\}$, and thus can be extended to the whole plane $\mathbb{C}$ as an entire function. By the definition of the Mellin transform and the infinite products representation

$$\frac{1}{\Gamma(z)}=ze^{\delta z}\prod_{k=1}^{\infty}\Big(1+\frac{z}{k}\Big)e^{-z/k},$$

where δ is the Euler's constant, we have

$$\begin{aligned}\big|G(z)\big| &\le \|\varphi\|_\infty \frac{\int_0^1 (1-r^2)^\alpha r^{2\operatorname{Re} z+|p|+|s|-1}\,dr}{|\int_0^1 (1-r^2)^\alpha r^{2z+2|p|-1}\,dr|} \times \Big|\frac{1}{(z+|p|+|s|-1)\cdots(z+|p|)}\Big| \\ &= O\Big(\frac{1}{(\operatorname{Re} z)^{\alpha+1}|z|^{|s|}}\Big)\end{aligned}$$

By Liouville's theorem, we obtain $G(z)=0$, which implies that $\varphi=0$. Now we have proved that if $s\neq 0$, then $\varphi=0$, which is a contradiction, so $s=0$. Then it follows from (2) that

$$\begin{aligned}&\big[\widehat{(1-r^2)^\alpha\varphi}\big]\big(2n+2|m|+2|\gamma|+|p|\big)\widehat{(1-r^2)^\alpha}\big(2n+2|m|+2|p|\big) \\ &\quad= \big[\widehat{(1-r^2)^\alpha\varphi}\big]\big(2n+2|m|+|p|\big)\widehat{(1-r^2)^\alpha}\big(2n+2|m|+2|\gamma|+2|p|\big),\end{aligned}$$

which implies that

$$\begin{aligned}&\big[\widehat{(1-r^2)^\alpha\varphi}\big]\big(z+2|\gamma|\big)\big[\widehat{(1-r^2)^\alpha r^{|p|}}\big](z) \\ &\quad= \big[\widehat{(1-r^2)^\alpha\varphi}\big]\big(z+2|\gamma|\big)\widehat{(1-r^2)^\alpha}\big(z+|p|\big) \\ &\quad= \big[\widehat{(1-r^2)^\alpha\varphi}\big](z)\widehat{(1-r^2)^\alpha}\big(z+2|\gamma|+|p|\big) \\ &\quad= \big[\widehat{(1-r^2)^\alpha\varphi}\big](z)\big[\widehat{(1-r^2)^\alpha r^{|p|}}\big]\big(z+2|\gamma|\big).\end{aligned}$$

By [19, Lemma 6], there exists some constant c such that $\varphi(r)=cr^{|p|}$, and this implies that $g=cz^p$. Moreover, if $\alpha=0$, then our assumption that T_f and T_g commute together with [18, Theorem 11] gives $f=\lambda g+\mu$ for some constants λ,μ, which is obviously sufficient for the commutativity. □

Remark 3.12. Since Theorem 11 in [18] only dealt with the case $\alpha=0$, and we are presently unable to give a proof for the weighted case, the second part of Theorem 3.11 is still open for $\alpha\neq 0$.

Theorem 3.13. *Let $f(r\xi)=\sum_{l\in\mathbb{Z}^n}\xi^l f_l(r)\in L^\infty$. If T_f commutes with T_{z^k} on b^2_α, where k is a nonzero multi-index, then f is holomorphic on $\mathbb{B}_n$. Moreover, if $\alpha=0$, then T_f and T_g commute if and only if $f=\lambda g+\mu$ for some constants λ,μ.*

Proof. If T_f and T_{z^k} commute, then $T_fT_{z^k}z^m=T_{z^k}T_fz^m$ for any multi-index m. It follows from Lemma 3.10 that

$$T_fT_{z^k}z^m=\sum_{m+k+l\succeq 0}+\sum_{m+k+l\preceq 0} T_{\xi^l f_l(r)}T_{z^k}z^m$$

and

$$T_{z^k}T_fz^m=\sum_{m+l\succeq 0}+\sum_{m+l\preceq 0,m+l+k\succeq 0}+\sum_{m+l+k\preceq 0} T_{z^k}T_{\xi^l f_l(r)}z^m.$$

Since T_f and T_{z^k} commute, the above two equations imply that

$$T_{\xi^l f_l(r)} T_{z^k} z^m = \begin{cases} 0 & m+l \not\succeq 0, m+l \not\preceq 0, m+l+k \succeq 0, \\ T_{z^k} T_{\xi^l f_l(r)} z^m & \begin{cases} m+l \succeq 0, \\ m+l \preceq 0, \quad\quad m+l+k \succeq 0, \\ m+l+k \preceq 0. \end{cases} \end{cases}$$

But $T_{z^k} T_{\xi^l f_l(r)} z^m = 0$ whenever $m+l \not\succeq 0, m+l \not\preceq 0$, so for each multi-index m,

$$T_{\xi^l f_l(r)} T_{z^k} z^m = T_{z^k} T_{\xi^l f_l(r)} z^m.$$

Let $l = p_l - s_l$, where $p_l \perp s_l$. Then by Theorem 3.11, $s_l = 0$ and there exist some constants c_l such that $\xi^l f_l(r) = c_l z^{p_l}$, thus $f = \sum_{l=p_l-s_l} c_l z^{p_l}$ is holomorphic on $\mathbb{B}_n$. Now suppose that $\alpha = 0$. Then it follows from [18, Theorem 11] again that T_f and T_g commute if and only if $f = \lambda g + \mu$ for some constants λ, μ. □

4. Finite-rank product of Toeplitz operators

Next we are going to investigate the finite-rank product problem of Toeplitz operators (except possibly one) whose symbols are of the form $z^s \bar{z}^t \varphi$, where $s, t \in \mathbb{N}^n$ and $\varphi \in L^\infty$ is a nonzero separately radial function on $\mathbb{B}_n$.

The following lemma is proved in [14], which will be used later.

Lemma 4.1 ([14, Theorem 2.3])**.** *Suppose that $\mathcal{S} \subset \mathbb{N}^n$ is a set that has property (P). Let $\mathcal{N}$ be the linear space spanned by the monomials $\{z^m : m \in \mathbb{N}^n \setminus \mathcal{S}\}$. Let $L^*(\mathcal{N}, \mathbb{C})$ denote the space of all conjugate-linear functionals on $\mathcal{N}$. Suppose that μ is a complex regular Borel measure on $\mathbb{C}^n$ with compact support. Let $L_\mu : \mathcal{N} \to L^*(\mathcal{N}, \mathbb{C})$ be the operator defined by $(L_\mu f)(g) = \int_{\mathbb{C}^n} f \bar{g}\, d\mu$ for $f, g \in \mathcal{N}$. If L_μ has finite rank and μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}^n$, then μ is the zero measure.*

Theorem 4.2. *Let S_1, S_2 be two bounded operators on b^2_α. Suppose there is a set $\mathcal{S} \subset \mathbb{N}^n$ having property (P) such that $\mathrm{Ker}(S_2) \subset \overline{\mathcal{M}}$ and $\mathcal{N} \subset \mathrm{Ran}(S_1)$, where $\overline{\mathcal{M}}$ is the closed subspace $\mathrm{cl}\{z^m : m \in \mathcal{S}\} \oplus \overline{A^2_\alpha}$, and $\mathcal{N}$ is the linear subspace spanned by $\{z^m : m \in \mathbb{N}^n \setminus \mathcal{S}\}$. Suppose that $f \in L^2_\alpha$ makes $S_2 T_f S_1$ a finite-rank operator. Then $f = 0$ almost everywhere on $\mathbb{B}_n$.*

Proof. Since $S_2 T_f S_1$ has finite rank and $\mathcal{N} \subset \mathrm{Ran}(S_1)$, $S_2 T_f(\mathcal{N})$ is a finite-dimensional linear subspace of b^2_α. Let $\{h_1, \ldots, h_k\}$ be a basis for this subspace, and let $g_i \in b^2_\alpha$ satisfy $S_2 g_i = h_i$ for $1 \le i \le k$. Then $T_f(\mathcal{N})$ is contained in $\mathrm{span}\{\mathrm{Ker}(S_2) \cup \{g_1, \ldots, g_k\}\}$, which is a subspace of $\mathrm{span}\{\overline{\mathcal{M}} \cup \{g_1, \ldots, g_k\}\}$ by our assumption. Let $Q_{\overline{\mathcal{M}}}$ denote the orthogonal projection from b^2_α onto $\overline{\mathcal{M}}$. Replacing g_i by $g_i - Q_{\overline{\mathcal{M}}} g_i$ if necessary, we may assume that $g_i \perp \overline{\mathcal{M}}$ for $1 \le i \le k$. Furthermore, we may also assume that $\{g_1, \ldots, g_k\}$ is an orthonormal subset of b^2_α by using the Gram–Schmidt process.

For any p in $\mathcal{N}$, we have

$$T_f p = Q_{\overline{\mathcal{M}}} T_f p + \sum_{i=1}^{k} \langle T_f p, g_i \rangle_\alpha g_i$$
$$= Q_{\overline{\mathcal{M}}} T_f p + \sum_{i=1}^{k} \langle f p, g_i \rangle_\alpha g_i.$$

By our assumption, for $q \in \mathcal{N}$, $q \perp \overline{\mathcal{M}}$, so we obtain

$$\int_{\mathbb{B}_n} f p \bar{q}\, dv_\alpha = \langle T_f p, q \rangle_\alpha = \langle Q_{\overline{\mathcal{M}}} T_f p, q \rangle_\alpha + \sum_{i=1}^{k} \langle f p, g_i \rangle_\alpha \langle g_i, q \rangle_\alpha$$
$$= \sum_{i=1}^{k} \langle f p, g_i \rangle_\alpha \langle g_i, q \rangle_\alpha.$$

Let $d\mu = f\, dv_\alpha$. Then the above equations tell us that the map L_μ from $\mathcal{N}$ into $L^*(\mathcal{N}, \mathbb{C})$ defined by $(L_\mu p)(q) = \int_{\mathbb{B}_n} p\bar{q}\, d\mu = \int_{\mathbb{B}_n} p\bar{q} f\, dv_\alpha$ has rank at most k. It then follows from Lemma 4.1 that μ is the zero measure, which in turn implies that $f = 0$ almost everywhere on $\mathbb{B}_n$. □

Lemma 4.3. *Suppose that $f \in L^2_\alpha$ is such that the set*

$$M(f) = \left\{ m \in \mathbb{N}^n : \int_{\mathbb{B}_n} f(z) z^m \bar{z}^m\, dv_\alpha(z) = 0 \right\}$$

does not have property (P). Then $M(f) = \mathbb{N}^n$. Moreover, if f is separately radial, then $\tilde{f}(m) = \langle T_f e_m, e_m \rangle_\alpha = \langle T_f \bar{e}_m, \bar{e}_m \rangle_\alpha = 0$ for all $m \in \mathbb{N}^n$, which implies that $T_f = 0$ and hence $f = 0$.

Proof. The first assertion is an easy corollary of Lemma 3.3 in [13]. The second assertion follows from Theorems 12 and 13 in [12]. □

Lemma 4.4. *For $1 \le j \le N$, suppose that $f_j(z) = z^{s_j} \bar{z}^{t_j} \varphi_j(z)$, where $\varphi_1, \ldots, \varphi_N$ are nonzero separately radial functions in L^∞, and $s_1, \ldots, s_N$ and $t_1, \ldots, t_N$ are multi-indexes. Let $S = T_{f_N} \cdots T_{f_1}$. Then there exist two subsets $\mathcal{J}$ and $\mathcal{I}$ of $\mathbb{N}^n$ having property (P) such that $P_\alpha(\operatorname{Ker} S)$ is contained in the closure in A^2_α of span$\{e_m : m \in \mathcal{J}\}$ and span$\{e_k : k \in \mathbb{N}^n \setminus \mathcal{I}\}$ is a subspace of $S(b^2_\alpha)$.*

Proof. Suppose that $\varphi \in L^\infty$ is a nonzero separately radial function on $\mathbb{B}_n$. Let $\tilde{\varphi}(m) = \langle T_\varphi e_m, e_m \rangle_\alpha$ for $m \in \mathbb{N}^n$. By Lemma 4.3, the set $M(\varphi) = \{m \in \mathbb{N}^n : \tilde{\varphi}(m) = 0\}$ has property (P). Now let s, t be in $\mathbb{N}^n$, and let $f(z) = z^s \bar{z}^t \varphi(z)$ for $z \in \mathbb{B}_n$. For multi-indexes m, k, l, we have

$$\langle T_f e_m, e_k \rangle_\alpha = a_k \langle \varphi e_{m+s}, e_{k+t} \rangle_\alpha = \begin{cases} 0 & \text{if } m + s \neq k + t, \\ a_k \tilde{\varphi}(m+s) & \text{if } m + s = k + t, \end{cases}$$

$$\langle T_f e_m, \bar{e}_l \rangle_\alpha = b_l \langle \varphi e_{m+s+l}, e_t \rangle_\alpha = \begin{cases} 0 & \text{if } m + s + l \neq t, \\ b_l \tilde{\varphi}(t) & \text{if } m + s + l = t, \end{cases}$$

$$\langle T_f\bar{e}_m, e_k\rangle_\alpha = c_k\langle \varphi\bar{e}_{m+t+k}, \bar{e}_s\rangle_\alpha = \begin{cases} 0 & \text{if } m+t+k \neq s, \\ c_k\tilde{\varphi}(s) & \text{if } m+t+k = s, \end{cases}$$

and

$$\langle T_f\bar{e}_m, \bar{e}_l\rangle_\alpha = d_l\langle \varphi\bar{e}_{m+t}, \bar{e}_{s+l}\rangle_\alpha = \begin{cases} 0 & \text{if } m+t \neq s+l, \\ d_l\tilde{\varphi}(m+t) & \text{if } m+t = s+l, \end{cases}$$

where a_k is a constant depending on m, s, t, k, n, α. For convenience, we only keep the "crucial" subscript k. Similarly, b_l, c_k, d_l are all defined in this way. This shows that

$$\begin{aligned} T_f e_m &= \sum_{k\in\mathbb{N}^n} \langle T_f e_m, e_k\rangle_\alpha e_k + \sum_{l\in\mathbb{N}^n\setminus\{0\}} \langle T_f e_m, \bar{e}_l\rangle_\alpha \bar{e}_l \\ &= \begin{cases} b_{t-m-s}\tilde{\varphi}(t)\bar{e}_{t-m-s} & \text{if } m+s-t \preceq 0, \\ a_{m+s-t}\tilde{\varphi}(m+s)e_{m+s-t} & \text{if } m+s-t \succeq 0, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} T_f\bar{e}_m &= \sum_{k\in\mathbb{N}^n} \langle T_f\bar{e}_m, e_k\rangle_\alpha e_k + \sum_{l\in\mathbb{N}^n\setminus\{0\}} \langle T_f\bar{e}_m, \bar{e}_l\rangle_\alpha \bar{e}_l \\ &= \begin{cases} c_{s-m-t}\tilde{\varphi}(s)e_{s-m-t} & \text{if } m+t-s \preceq 0, \\ d_{m+t-s}\tilde{\varphi}(m+t)\bar{e}_{m+t-s} & \text{if } m+t-s \succeq 0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.2}$$

As a result, for multi-index $m \succeq \sum_{j=1}^N (s_j+t_j)$, we obtain two positive constants C_1, C_2 (depending on $m, s_1, \dots, s_N, t_1, \dots, t_N, n$ and α) such that

$$Se_m = C_1\prod_{j=1}^N \tilde{\varphi}_j\Big(m + \sum_{i=1}^{j-1}(s_i - t_i) + s_j\Big)e_{m+\sum_{j=1}^N (s_j-t_j)} \tag{4.3}$$

and

$$S\bar{e}_m = C_2\prod_{j=1}^N \tilde{\varphi}_j\Big(m + \sum_{i=1}^{j-1}(t_i - s_i) + t_j\Big)\bar{e}_{m+\sum_{j=1}^N (t_j-s_j)}. \tag{4.4}$$

Define

$$\begin{aligned} \mathcal{J} &= \Big\{m : m \not\succeq \sum_{j=1}^N (s_j+t_j)\Big\} \cup \Big\{m : \prod_{j=1}^N \tilde{\varphi}_j\Big(m + \sum_{i=1}^{j-1}(s_i-t_i)+s_j\Big) = 0\Big\} \\ &= \Big\{m : m \not\succeq \sum_{j=1}^N (s_j+t_j)\Big\} \cup \bigcup_{j=1}^N \Big(M(\varphi_j) - \Big(\sum_{i=1}^{j-1}(s_i-t_i)+s_j\Big)\Big). \end{aligned}$$

It follows from (4.1) and (4.2) that for multi-index $m \not\succeq \sum_{j=1}^N (s_j+t_j)$, Se_m is a multiple of $e_{m+\sum_{j=1}^N (s_j-t_j)}$ or $\bar{e}_{\sum_{j=1}^N (t_j-s_j)-m}$, and $S\bar{e}_m$ is a multiple of $e_{\sum_{j=1}^N (s_j-t_j)-m}$

or $\bar{e}_{m+\sum_{j=1}^N(t_j-s_j)}$. Combining this with (4.3) and (4.4) gives

$$\Big\{Se_m : m \succeq \sum_{j=1}^N (s_j+t_j)\Big\} \perp \Big\{Se_m : m \not\succeq \sum_{j=1}^N (s_j+t_j)\Big\}$$

and

$$\Big\{Se_m : m \succeq \sum_{j=1}^N (s_j+t_j)\Big\} \perp \{S\bar{e}_m : m \in \mathbb{N}^n\}.$$

By statements (3) and (6) of Remark 2.2, $\mathcal{J}$ has property (P). For $m \in \mathbb{N}^n \setminus \mathcal{J}, Se_m \neq 0$ and $e_{m+\sum_{j=1}^N(s_j-t_j)}$ is a multiple of Se_m. Suppose that $h = u+\bar{v} \in b_\alpha^2$ such that $Sh = 0$, where $u, v \in A_\alpha^2$. Then we have

$$\begin{aligned} 0 = Sh &= S\Big(\sum_{m\in\mathbb{N}^n} \langle h, e_m\rangle_\alpha e_m + \sum_{l\in\mathbb{N}^n\setminus\{0\}} \langle h, \bar{e}_l\rangle_\alpha \bar{e}_l\Big) \\ &= \sum_{m\in\mathbb{N}^n} \langle h, e_m\rangle_\alpha Se_m + \sum_{l\in\mathbb{N}^n\setminus\{0\}} \langle h, \bar{e}_l\rangle_\alpha S\bar{e}_l. \end{aligned}$$

So for any $m \in \mathbb{N}^n\setminus\mathcal{J}$, $\langle u, e_m\rangle_\alpha = \langle h, e_m\rangle_\alpha = 0$. Therefore, $P_\alpha(\operatorname{Ker} S)$ is contained in the closure in A_α^2 of $\operatorname{span}\{e_m : m \in \mathcal{J}\}$. Now define

$$\mathcal{I} = \Big\{k \in \mathbb{N}^n : k \not\succeq \sum_{j=1}^N (s_j-t_j)\Big\} \cup \Big(\mathbb{N}^n \cap \Big(\mathcal{J} + \sum_{j=1}^N (s_j-t_j)\Big)\Big).$$

Then $\mathcal{I}$ has property (P) and for any $k \in \mathbb{N}^n \setminus \mathcal{I}, m = k - \sum_{j=1}^N (s_j-t_j)$ belongs to $\mathbb{N}^n \setminus \mathcal{J}$. Hence, $e_k = e_{m+\sum_{j=1}^N(s_j-t_j)}$ is a multiple of Se_m, and it follows that $\operatorname{span}\{e_k : k \in \mathbb{N}^n \setminus \mathcal{I}\} \subset \operatorname{Ran}(S)$. □

Theorem 4.5. *Let N_1, N_2 be two positive integers, let $\varphi_1, \ldots, \varphi_{N_1+N_2}$ be bounded separately radial functions, and let $s_1, \ldots, s_{N_1+N_2}, t_1, \ldots, t_{N_1+N_2}$ be multi-indexes. For each $1 \le j \le N_1+N_2$, define $f_j(z) = z^{s_j}\bar{z}^{t_j}\varphi_j(z)$ for $z \in \mathbb{B}_n$. If $f \in L_\alpha^2$ makes $T_{f_{N_1+N_2}}\cdots T_{f_{N_1+1}} T_f T_{f_{N_1}} \cdots T_{f_1}$ (which is densely defined on b_α^2) a finite-rank operator, then f is the zero function.*

Proof. Let $S_1 = T_{f_{N_1}} \cdots T_{f_1}$ and $S_2 = T_{f_{N_1+N_2}} \cdots T_{f_{N_1+1}}$. By Lemma 4.4, there exist two subsets $\mathcal{J}$ and $\mathcal{I}$ of $\mathbb{N}^n$ having property (P) such that $P_\alpha(\operatorname{Ker} S_2)$ is contained in the closure in A_α^2 of $\operatorname{span}\{e_m : m \in \mathcal{J}\}$, and $\operatorname{span}\{e_k : k \in \mathbb{N}^n\setminus\mathcal{I}\}$ is a subspace of $S_1(b_\alpha^2)$. Let $\mathcal{S} = \mathcal{J}\cup\mathcal{I}$. Then $\mathcal{S}$ has property (P), $\operatorname{Ker} S_2 \subset \overline{\mathcal{M}}$, and $\mathcal{N} \subset S_1(b_\alpha^2)$, where $\overline{\mathcal{M}} = \operatorname{cl}\{e_m : m \in \mathcal{S}\} \oplus \overline{A_\alpha^2}$, $\mathcal{N}$ is the linear subspace spanned by $\{e_m : m \in \mathbb{N}^n \setminus \mathcal{S}\}$. If $S_2 T_f S_1$ has finite rank, then Theorem 4.2 implies that f is the zero function. □

Remark 4.6. Note that the functions $f_1, \ldots, f_{N_1+N_2}$ in the last theorem are no longer separately radial, so the Toeplitz operators induced by them are not diagonal, which means that the Bergman space A_α^2 is not a reducing subspace of these operators. Consequently, the approach by considering the compression and restriction of the Toeplitz operators on the Bergman space is not available as in the proof of Theorem 3.3. Hence the result is not so obvious.

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