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LOCAL MATRIX HOMOTOPIES AND SOFT TORI

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Abstract. We present solutions to local connectivity problems in matrix representations of the form $C([-1,1]^N) \to C^*(u_\varepsilon, v_\varepsilon)$, with $C_\varepsilon(\mathbb{T}^2) \twoheadrightarrow C^*(u_\varepsilon, v_\varepsilon)$ for any $\varepsilon \in [0,2]$ and any integer $n \geq 1$, where $C^*(u_\varepsilon, v_\varepsilon) \subseteq M_n$ is an arbitrary matrix representation of the universal C^*-algebra $C_\varepsilon(\mathbb{T}^2)$ that denotes the *soft torus*. We solve the connectivity problems by introducing the so-called *toroidal matrix links*, which can be interpreted as normal contractive matrix analogies of free homotopies in differential algebraic topology.

To deal with the locality constraints, we have combined some techniques introduced in this article with some techniques from matrix geometry, combinatorial optimization, and classification and representation theory of C^*-algebras.

1. Introduction

In this article, we study the solvability of some local connectivity problems via constrained normal matrix homotopies in C^*-representations of the form

$$C(\mathbb{T}^N) \longrightarrow M_n \tag{1.1}$$

for a fixed but arbitrary integer $N \geq 1$ and any integer $n \geq 1$. In particular, we study local normal matrix homotopies which preserve commutativity and also satisfy some additional constraints, like being rectifiable or piecewise analytic.

We build on some homotopic techniques introduced initially by Bratteli, Elliott, Evans, and Kishimoto in [3] and generalized by Lin in [19] and [20]. We combine

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homotopic techniques with some first introduced here and various other techniques from matrix geometry and noncommutative topology developed by Loring [21], Loring and Shulman [23], Bhatia [2], Chu [7], Brockett [4], Choi [5], Choi and Effros [6], Eilers and Exel [10], Elsner [12], Pryde [27], and McIntosh, Pryde, and Ricker [26] to construct the so-called *toroidal matrix links*, which we use to obtain the main theorems presented in Section 4, and which consist of local connectivity results in matrix representations of the form (1.1) and also of the form

$$C\big([-1,1]^N\big) \longrightarrow M_n. \tag{1.2}$$

Toroidal matrix links can be interpreted as noncommutative analogies of free homotopies in algebraic topology and topological deformation theory. We introduce them in Section 3 together with a number of other matrix and geometric objects.

In Section 4.3, we present a connectivity technique which provides us with information on the local uniform connectivity in matrix representations of the form $C(\mathbb{T}^2) \to M_n$.

Given $\delta > 0$, a function $\varepsilon : \mathbb{R} \to \mathbb{R}_0^+$, and two matrices x, y in a set $S \subseteq M_n$ such that $\|x - y\| \leq \delta$, by an $\varepsilon(\delta)$-*local matrix homotopy* between x and y, we mean a matrix path $X \in C([0,1], M_n)$ such that $X_0 = x$, $X_1 = y$, $X_t \in S$, and $\|X_t - y\| \leq \varepsilon(\delta)$ for each $t \in [0,1]$. We write $x \rightsquigarrow_{\varepsilon,S} y$ to denote that there is an ε-local matrix homotopy between x and y relative to S.

The motivation and inspiration to study local normal matrix homotopies which preserve commutativity in C^*-representations of the form (1.1) and (1.2) came from mathematical physics (see [14, Section 3]) and matrix approximation theory (see [1], [8], [13]). In particular, by the results presented in [8] we can think of local matrix homotopies as continuous analogies of spectral refinements (in the sense of [1]) of Jacobi-type matrix flows that can be interpreted as continuous analogies of Jacobi-type simultaneous block diagonalization algorithms in the sense of [25].

From this point forward, we will write $C_\varepsilon(\mathbb{T}^2)$ to denote the *soft torus* (whose definition is restated in Section 2) and $\mathcal{N}(n)(\mathbb{D}^2)$ to denote the set of $(n \times n)$–normal matrix contractions with complex entries. In this article, we begin our study of the relation between the *soft tori* and numerical algorithms for approximate joint diagonalization of normal matrices by considering the soft torus as an environment algebra for some particular types of local matrix homotopies. More specifically, we prove that, given $\varepsilon > 0$, any $n \in \mathbb{Z}^+$, any $N \in \mathbb{Z}^+$, and any two N-tuples of pairwise commuting normal contractions $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ such that $\|X_j - Y_j\| \leq \delta$, one can find/construct a unitary/normal contraction u_ε, a unitary v_ε in M_n, and a family of piecewise analytic matrix paths $Z^j \in C([0,1], \mathcal{N}(n)(\mathbb{D}^2))$ that connect X_j to Y_j for each $1 \leq j \leq N$, such that $C_\varepsilon(\mathbb{T}^2) \twoheadrightarrow C^*(u_\varepsilon, v_\varepsilon)$, $Z_t^j \in C^*(u_\varepsilon, v_\varepsilon)$, and $[Z_t^j, (Z_t^j)^*] = [Z_t^i, Z_t^j]$ for each $1 \leq i, j \leq N$ and each $0 \leq t \leq 1$.

This study was motivated by inverse spectral problems from mathematical physics, which consist of finding—for a certain set of matrices $X_1, \ldots, X_N$ that *approximately satisfy* a set of polynomial constraints $\mathcal{R}(x_1, \ldots, x_N)$ on N noncommutative variables—a set of *nearby matrices* $\tilde{X}_1, \ldots, \tilde{X}_N$ that approximate

$X_1, \ldots, X_N$ and exactly satisfy the constraints $\mathcal{R}(x_1, \ldots, x_N)$. To be more precise, let us consider for instance a pair of pairwise commuting diagonal (or easily diagonalizable) Hermitian matrices X and Y with prescribed eigenvalues. Let us consider the problem of finding a pair of commuting Hermitian matrices $\tilde{X}$ and $\tilde{Y}$ that are close (in the metric induced by the operator norm) to X and Y, respectively, have the same (or approximately the same) eigenvalues, need not commute with X and Y in general, and that also satisfy some additional algebraic constraints. It can be seen that the solution of problems of this type has a natural connection with inverse eigenvalue problems.

The problems from matrix approximation theory that we considered here consist of all those that can be reduced to the study of the solvability conditions for approximate and exact joint diagonalization problems for N-tuples of normal matrix contractions. In particular, these types of problems have applications in *multivariate statistical signal processing*, more specifically in *blind source separation* in the sense of [17] and [29].

The problems motivating the research in this article are topological in nature and involve the study of the local piecewise analytic connectivity of matrix representations of the form $C([-1,1]^N) \to C^*(u_\varepsilon, v_\varepsilon) \subseteq M_n$, with $C_\varepsilon(\mathbb{T}^N) \twoheadrightarrow C^*(u_\varepsilon, v_\varepsilon) \subseteq M_n$. Here and in what follows, the expression $C^*(u_\varepsilon, v_\varepsilon) \subseteq M_n$, with u_ε unitary/normal contraction and v_ε unitary in M_n, is used to denote an arbitrary matrix representation of the universal C^*-algebra $C_\varepsilon(\mathbb{T}^2)$ that is defined in Section 3.3. Sometimes we obtain the C^*-representation $C_\varepsilon(\mathbb{T}^2) \twoheadrightarrow C^*(u_\varepsilon, v_\varepsilon) \subseteq M_n$ by factoring with $C_\varepsilon(\mathbb{J} \times \mathbb{T}^1) \twoheadrightarrow C^*(u_\varepsilon, v_\varepsilon) \subseteq M_n$, where $C_\varepsilon(\mathbb{J} \times \mathbb{T}^1)$ is the universal C^*-algebra that we use to denote the *soft cylinder* in the sense of [23] and is defined in Section 3.3 as well. We investigated several variations of problems of the following form.

Problem 1 (Lifted connectivity problem). Given $\varepsilon > 0$, is there $\delta > 0$ such that the following conditions hold? For any integer $n \geq 1$, some prescribed sequence of linear compressions $\kappa_n : M_{mn} \to M_n$ for some $m \geq 1$, and any two families of N pairwise commuting normal contractions $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ in M_n which satisfy the constraints $\|X_j - Y_j\| \leq \delta$, $1 \leq j \leq N$, there are two families of N pairwise commuting normal contractions $\tilde{X}_1, \ldots, \tilde{X}_N$ and $\tilde{Y}_1, \ldots, \tilde{Y}_N$ in M_{mn} which satisfy the relations $\kappa_n(\tilde{X}_j) = X_j$, $\kappa_n(\tilde{Y}_j) = Y_j$ and $\|\tilde{X}_j - \tilde{Y}_j\| \leq \varepsilon$, $1 \leq j \leq N$. Moreover, there are N piecewise analytic ε-local homotopies of normal contractions $\mathbf{X}^1, \ldots, \mathbf{X}^N \in C([0,1], M_{mn})$ between the corresponding pairs $\tilde{X}_j$, $\tilde{Y}_j$ in M_{mn} which satisfy the relations $\mathbf{X}^j_t \mathbf{X}^k_t = \mathbf{X}^k_t \mathbf{X}^j_t$, for each $1 \leq j, k \leq N$ and each $0 \leq t \leq 1$.

By solving Problem 1, we learned about the local connectivity of arbitrary δ-close N-tuples of pairwise commuting normal contractions $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ in M_n, which was the main motivation for this research. We also obtained some results concerning the geometric structure of the joint spectra (in the sense of [26]) of the N-tuples.

For a given $\delta > 0$, the study of the solvability conditions of problems such as those described in Problem 1 provided us with geometric information about local deformations of particular representations of the form $C(\mathbb{T}^N) \to A_0 :=$

$C^*(U_1,\dots,U_N)\subseteq M_n$ and $C(\mathbb{T}^N)\to A_1:=C^*(V_1,\dots,V_N)\subseteq M_n$, where $U_1,\dots,U_N,V_1,\dots,V_N\in\mathbb{U}(n)$ are pairwise commuting unitary matrices such that $\|U_j-V_j\|\leq\delta$. By *local deformations*, we mean a family $\{A_t\}_{t\in[0,1]}\subseteq M_n$ of Abelian C^*-algebras, with $A_t:=C^*(\mathbf{X}_t^1,\dots,\mathbf{X}_t^N)$ and where $\mathbf{X}_t^1,\dots,\mathbf{X}_t^N\in C([0,1],\mathbb{U}(n))$ are $\varepsilon(\delta)$-local matrix homotopies between $U_1,\dots,U_N$ and $V_1,\dots,V_N$ for some function $\varepsilon:\mathbb{R}\to\mathbb{R}_0^+$.

The main results are presented in Section 4. In Section 4.2, we use toroidal matrix links to obtain some local piecewise analytic connectivity results which are nonuniform in dimension. In Section 4.2.1, we derive a uniform approximate connectivity technique via matrix homotopy lifting, and in Section 4.4 we present a connectivity lemma that can be used to derive various uniform connectivity results between matrix representations of finite sets of universal algebraic contractions. We will provide further details of these constructions in forthcoming work.

2. Preliminaries and notation

2.1. **Matrix sets and operations.**

Given two elements x, y in a C^*-algebra A, we will write $[x,y]$ and $\mathrm{Ad}[x](y)$ to denote the operations $[x,y]:=xy-yx$ and $\mathrm{Ad}[x](y):=xyx^*$, respectively.

Given any C^*-algebra A and any element x in $M_n(A)$, we will denote by $\mathrm{diag}_n[x]$ the operation defined by the expression

$$\begin{aligned}M_n(A)&\to M_n(A),\\ x&\mapsto \mathrm{diag}_n[x],\end{aligned}$$
$$\begin{pmatrix}x_{11}&x_{12}&\cdots&x_{1n}\\ x_{21}&x_{22}&\cdots&x_{2n}\\ \vdots&\vdots&\ddots&\vdots\\ x_{n1}&x_{n2}&\cdots&x_{nn}\end{pmatrix}\mapsto\begin{pmatrix}x_{11}&0&\cdots&0\\ 0&x_{22}&\cdots&0\\ \vdots&\vdots&\ddots&\vdots\\ 0&0&\cdots&x_{nn}\end{pmatrix}.$$

Given a C^*-algebra A, we will write $\mathcal{N}(A)$, $\mathbb{H}(A)$, and $\mathbb{U}(A)$ to denote the sets of normal, Hermitian, and unitary elements in A, respectively. We will write $\mathcal{N}(n)$, $\mathbb{H}(n)$, and $\mathbb{U}(n)$ instead of $\mathcal{N}(M_n)$, $\mathbb{H}(M_n)$, and $\mathbb{U}(M_n)$. A normal element u in a C^*-algebra A is called a *partial unitary* if the element $uu^*=p$ is an orthogonal projection in A, that is, p satisfies the relations $p=p^*=p^2$. We denote by $\mathbb{PU}(A)$ the set of partial unitaries in A, and we write $\mathbb{PU}(n)$ instead of $\mathbb{PU}(M_n)$.

We will write $\mathbb{I}$, $\mathbb{J}$, $\mathbb{T}^1$, and $\mathbb{D}^2$ to denote the sets $\mathbb{I}:=[0,1]$, $\mathbb{J}=[-1,1]$, $\mathbb{T}^1:=\{z\in\mathbb{C}\mid |z|=1\}$, and $\mathbb{D}^2:=\{z\in\mathbb{C}\mid |z|\leq 1\}$. For some arbitrary matrix set $S\subseteq M_n$ and some arbitrary compact set $\mathbb{X}\subset\mathbb{C}$, we will write $S(\mathbb{X})$ to denote the subset of elements in S described by the expression

$$S(\mathbb{X}):=\{x\in S\mid \sigma(x)\subseteq\mathbb{X}\}.$$

For instance, as mentioned in Section 1, we can write $\mathcal{N}(n)(\mathbb{D}^2)$ to denote the set of normal contractions. We will denote by $\mathcal{M}_\infty$ the C^*-algebra described by

$$\mathcal{M}_\infty:=\overline{\bigcup_{n\in\mathbb{Z}^+}M_n}^{\|\cdot\|}.$$

In this article, we will write $\mathbf{1}_n$ and $\mathbf{0}_n$ to denote the identity and zero matrix in M_n, respectively. The symbol $\mathbf{N}_n$ will be used to denote the diagonal matrices

$$\mathbf{N}_n := \mathrm{diag}[n, n-1, \ldots, 2, 1]$$

We will write Ω_n and Σ_n to denote the unitary matrices defined by

$$\Omega_n := e^{\frac{2\pi i}{n}\mathbf{N}_n} = \mathrm{diag}[1, e^{\frac{2\pi i(n-1)}{n}}, \ldots, e^{\frac{4\pi i}{n}}, e^{\frac{2\pi i}{n}}]$$

and

$$\Sigma_n := \begin{pmatrix} 0 & \mathbf{1}_{n-1} \\ 1 & 0 \end{pmatrix}.$$

Remark 2.1. The unitary matrices Ω_n and Σ_n are related by the equation

$$\Omega_n = \mathscr{F}_n^* \Sigma_n \mathscr{F}_n,$$

where $\mathscr{F}_N := (\frac{1}{\sqrt{N}} e^{\frac{2\pi i(j-1)(k-1)}{N}})_{1 \leq j,k \leq N}$ is the discrete Fourier transform unitary matrix.

Given an abstract object (group or C^*-algebra) A, we will write A^{*N} to denote the operation consisting of taking the free product of N copies of A.

Definition 2.1 (Local preservers). Given a linear mapping $K : M_N \to M_n$ with $n \leq N$ and a set $S \subseteq M_n$, we say that K *locally preserves* S with respect to some set $T \subseteq M_N$ if we have that $K(T) \subseteq S$ (omitting the explicit reference to T when it is clear from the context). If in particular $K(T) \subseteq \mathcal{N}(n)$, we say that K *locally preserves normality*.

Example 2.1. The linear compression $\kappa : M_{2n} \to M_n$ defined by

$$\kappa : \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto x_{11}$$

locally preserves normality with respect to the set $T := \{X \in M_{2n} \mid x_{11} \in \mathcal{N}(n)\}$.

Example 2.2. The linear map $\phi : M_n \to M_n, x \mapsto \mathbf{D}x$ with $n \geq 1$ and $\mathbf{D} = \frac{1}{n}\,\mathrm{diag}[1, \ldots, n]$ locally preserves commutativity with respect to the set $C^*(\mathbf{D})$.

2.2. **Joint spectral variation.**

2.2.1. Clifford operators. Using the same notation used by Pryde [27], let $\mathbb{R}_{(N)}$ denote the Clifford algebra over $\mathbb{R}$ with generators $e_1, \ldots, e_N$ and relations $e_i e_j = -e_j e_i$ for $i \neq j$ and $e_i^2 = -1$. Then $\mathbb{R}_{(N)}$ is an associative algebra of dimension 2^N. Let $S(N)$ denote the set $\mathscr{P}(\{1, \ldots, N\})$. Then the elements $e_S = e_{s_1} \cdots e_{s_k}$ form a basis when $S = \{s_1, \ldots, s_k\}$ and $1 \leq s_1 < \cdots < s_k \leq N$. Elements of $\mathbb{R}_{(N)}$ are denoted by $\lambda = \sum_S \lambda_S e_S$, where $\lambda_S \in \mathbb{R}$. Under the inner product $\langle \lambda, \mu \rangle = \sum_S \lambda_S \mu_S$, $\mathbb{R}_{(N)}$ becomes a Hilbert space with orthonormal basis $\{e_S\}$.

The *Clifford operator* of N elements $X_1, \ldots, X_N \in M_n$ is the operator defined in $M_n \otimes \mathbb{R}_{(N)}$ by

$$\mathrm{Cliff}(X_1, \ldots, X_N) := \sqrt{-1} \sum_{j=1}^{N} X_j \otimes e_j.$$

Each element $T = \sum_S T_S \otimes e_S \in M_n \otimes \mathbb{R}_{(N)}$ acts on elements $x = \sum_S x_S \otimes e_S \in \mathbb{C}^n \otimes \mathbb{R}_{(N)}$ by $T(x) := \sum_{S,S'} T_s(x_{S'}) \otimes e_S e_{S'}$. So $\operatorname{Cliff}(X_1, \ldots, X_N) \in M_n \otimes \mathbb{R}_{(N)} \subseteq \mathcal{L}(\mathbb{C}^n \otimes \mathbb{R}_{(N)})$. By $\|\operatorname{Cliff}(X_1, \ldots, X_N)\|$ we will mean the operator norm of $\operatorname{Cliff}(X_1, \ldots, X_N)$ as an element of $\mathcal{L}(\mathbb{C}^n \otimes \mathbb{R}_{(N)})$. As observed by Elsner in [12, (5.2)], we have that

$$\big\|\operatorname{Cliff}(X_1, \ldots, X_N)\big\| \leq \sum_{j=1}^{N} \|X_j\|. \tag{2.1}$$

2.2.2. Joint spectral matchings. It is often convenient to have N-tuples (or $2N$-tuples) of matrices with real spectra. For this purpose, we use the following construction initiated in [26]. If $X = (X_1, \ldots, X_N)$ is an N-tuple of $(n \times n)$-matrices, then we can always decompose X_j in the form $X_j = X_{1j} + iX_{2j}$ where the X_{kj} all have real spectra. We write $\pi(X) := (X_{11}, \ldots, X_{1N}, X_{21}, \ldots, X_{2N})$ and call $\pi(X)$ a *partition* of X. If the X_{kj}'s all commute, then we say that $\pi(X)$ is a *commuting partition*, and if the X_{kj}'s are simultaneously triangularizable, then we say that $\pi(X)$ is a *triangularizable partition*. If the X_{kj}'s are all semisimple (diagonalizable), then $\pi(X)$ is called a *semisimple partition*.

We say that N normal matrices $X_1, \ldots, X_N \in M_n$ are *simultaneously diagonalizable* if there is a unitary matrix $Q \in M_n$ such that $Q^* X_j Q$ is diagonal for each $j = 1, \ldots, N$. In this case, for $1 \leq k \leq n$, let $\Lambda^{(k)}(X_j) := (Q^* X_j Q)_{kk}$ be the (k,k) element of $Q^* X_j Q$, and set $\Lambda^{(k)}(X_1, \ldots, X_N) := (\Lambda^{(k)}(X_1), \ldots, \Lambda^{(k)}(X_N)) \in \mathbb{C}^N$. The set

$$\Lambda(X_1, \ldots, X_N) := \big\{\Lambda^{(k)}(X_1, \ldots, X_N)\big\}_{1 \leq k \leq N}$$

is called the *joint spectrum* of $X_1, \ldots, X_N$. We will write $\Lambda(X_j)$ to denote the j-component of $\Lambda(X_1, \ldots, X_N)$. In other words, we will have that

$$\Lambda(X_j) = \operatorname{diag}\big[\Lambda^{(1)}(X_j), \ldots, \Lambda^{(n)}(X_j)\big].$$

The following theorem was proved by McIntosh, Pryde, and Ricker [26].

Theorem 2.1 ([26, pp. 56–57])**.** *Let $X = (X_1, \ldots, X_N)$ and $Y = (Y_1, \ldots, Y_N)$ be N-tuples of commuting $(n \times n)$-normal matrices. Then there exists a permutation τ of the index set $\{1, \ldots, n\}$ such that*

$$\begin{aligned} &\big\|\Lambda^{(k)}(X_1, \ldots, X_N) - \Lambda^{(\tau(k))}(Y_1, \ldots, Y_N)\big\| \\ &\qquad \leq e_{N,0}\big\|\operatorname{Cliff}(X_1 - Y_1, \ldots, X_N - Y_N)\big\| \end{aligned} \tag{2.2}$$

for all $k \in \{1, \ldots, n\}$.

In this theorem, $e_{N,0}$ is an explicit constant depending only on N as defined in [26, (2.4)].

2.3. **Amenable C^*-algebras and Bott elements.** The following lemma was proved by Lin in [18].

Lemma 2.1 ([18, Lemma 2.6.11])**.** *For any $\varepsilon > 0$ and $d > 0$, there exists $\delta > 0$ satisfying the following. Suppose that A is a unital C^*-algebra and that $u \in A$ is* *a unitary such that* $\mathbb{T}^1\backslash\sigma(u)$ *contains an arc of length* d. *Suppose that* $a \in A$ *with* $\|a\| \le 1$ *such that*

$$\|ua - au\| < \delta.$$

Then there is a self-adjoint element $h \in A$ *such that* $u = e^{ih}$,

$$\|ha - ah\| < \varepsilon \qquad \text{and} \qquad \|e^{ith}a - ae^{ith}\| < \varepsilon$$

for all $t \in \mathbb{I}$. *If, furthermore,* $a = p$ *is a projection, then we have*

$$\Big\|pup - p + \sum_{n=1}^{\infty} \frac{(iphp)^n}{n!}\Big\| < \varepsilon.$$

The following lemma was proved by Lin in [20] by using Lemma 2.1, since for any integer $n \ge 1$ and any $u \in \mathbb{U}(n)$, we will have that $\mathbb{T}^1\backslash\sigma(u)$ contains an arc of length at least $2\pi/n$.

Lemma 2.2 ([20, Lemma 3.3])**.** *Let* $\varepsilon > 0$, *let* $n \ge 1$ *be an integer, and let* $M > 0$. *There exists* $\delta > 0$ *satisfying the following. For any finite set* $\mathscr{F} \subset M_n$ *with* $\|a\| \le M$ *for all* $a \in \mathscr{F}$, *and a unitary* $u \in M_n$ *such that*

$$\|ua - au\| < \delta \quad \text{for all } a \in \mathscr{F},$$

there exists a continuous path of unitaries $\{u(t)\}_{t\in\mathbb{I}} \subset M_n$ *with* $u(0) = u$ *and* $u(1) = \mathbf{1}_n$ *such that*

$$\big\|u(t)a - au(t)\big\| < \varepsilon \quad \text{for all } a \in \mathscr{F}.$$

Furthermore,

$$\operatorname{Length}\big(\{u(t)\}\big) \le 2\pi.$$

Definition 2.2 (The obstruction $\operatorname{Bott}(u,v)$). Given two unitaries in a K_1-simple real rank zero C^*-algebra A that almost commute, the obstruction $\operatorname{Bott}(u,v)$ is the Bott element associated to the two unitaries as defined by Loring in [21]. It is defined whenever $\|uv - vu\| \le \nu_0$, where ν_0 is a universal constant. It is defined as the K_0-class

$$\operatorname{Bott}(u,v) = \big[\chi_{[1/2,\infty)}\big(e(u,v)\big)\big] - \left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right],$$

where $e(u,v)$ is a self-adjoint element of $M_2(A)$ of the form

$$e(u,v) = \begin{pmatrix} f(v) & h(v)u + g(v) \\ h(v)u^* + g(v) & 1 - f(v) \end{pmatrix},$$

where f, g, h are universal real-valued continuous functions on $\mathbb{T}^1$ defined as follows:

$$f(e^{2\pi\sqrt{-1}\theta}) = \begin{cases} 1 - 2\theta & \text{if } 0 \le \theta \le 1/2, \\ -1 + 2\theta & \text{if } 1/2 \le \theta \le 1, \end{cases}$$

$$g = \chi_{[0,1/2]}\sqrt{f - f^2},$$

$$h = \chi_{[1/2,1]}\sqrt{f - f^2},$$

where χ_X denotes the characteristic function of the set X.

For details on the subject of K-theory for C^*-algebras, we refer the reader to [28]. As observed by Bratteli, Elliott, Evans, and Kishimoto in [3], given a pair $u, v \in \mathbb{U}(A)$ we have that the obstruction $\mathrm{Bott}(u,v)$ needs to vanish to be able to solve the problem $uvu^* \rightsquigarrow_{\varepsilon(\delta),\mathbb{U}(n)} v$ by deforming $u \in \mathbb{U}_0(A)$ to 1 continuously in $\mathbb{U}(A)$, when $\|uv - vu\| \leq \delta$.

3. Matrix varieties and toroidal matrix links

Let us denote by $\mathcal{H}$ a universal separable Hilbert space, by $\mathbb{B}(\mathcal{H})$ the C^*-algebra of bounded operators on $\mathcal{H}$, and, for any given $S \subseteq \mathbb{B}(\mathcal{H})$, let us denote by $\mathbf{B}_r(S)$ the closed r-ball in S defined by $\mathbf{B}_r(S) := \{x \in S \mid \|x\| \leq r\}$.

Given $N \in \mathbb{Z}^+$ and a set $\mathcal{R}(S) = \mathcal{R}(y_1, \ldots, y_N)$ of normed polynomial relations on the N-set $S := \{y_1, \ldots, y_N\}$ of noncommutative variables, we will call the set $\mathcal{Z}[\mathcal{R}]$ described by

$$\mathcal{Z}[\mathcal{R}] := \big\{x_1, \ldots, x_N \bigm| \mathcal{R}(x_1, \ldots, x_N)\big\} \tag{3.1}$$

with $x_1, \ldots, x_N \in \mathbf{B}_1(\mathbb{B}(\mathcal{H}))$, a *noncommutative semialgebraic set.*

Example 3.1. As an example of normed noncommutative polynomial relations, we can consider the set $\mathcal{R}(x,y) := \{\|x^4 - 1\| \leq 10^{-10}, \|y^7 - 1\| \leq 10^{-10}, \|xy - yx\| \leq \frac{1}{8}, xx^* = x^*x = 1, yy^* = y^*y = 1\}$.

Given a noncommutative semialgebraic set $\mathcal{Z}[\mathcal{R}]$, we will use the symbol $\mathcal{E}\mathcal{Z}[\mathcal{R}]$ to denote the universal C^*-algebra

$$\mathcal{E}\mathcal{Z}[\mathcal{R}] := C^*\big\langle x_1, \ldots, x_N \bigm| \mathcal{R}(x_1, \ldots, x_N)\big\rangle, \tag{3.2}$$

which we call the *environment C^*-algebra* of $\mathcal{Z}[\mathcal{R}]$. (For details on universal C^*-algebras described in terms of generators and relations, we refer the reader to [22].)

Definition 3.1 (Semialgebraic matrix varieties). Given $J \in \mathbb{Z}^+$, a system of J polynomials $p_1, \ldots, p_J \in \Pi_{\langle N\rangle} = \mathbb{C}\langle x_1, \ldots, x_N\rangle$ in N noncommutative variables, and J real numbers $\varepsilon_j \geq 0$, $1 \leq j \leq J$, a matrix representation of the noncommutative semialgebraic set $\mathcal{Z}_n(p_1, \ldots, p_J)$ described by

$$\mathcal{Z}_n(p_1, \ldots, p_J) := \big\{X_1, \ldots, X_N \in M_n \bigm| \big\|p_j(X_1, \ldots, X_N)\big\| \leq \varepsilon_j, 1 \leq j \leq J\big\}$$

will be called an *n-semialgebraic matrix variety.* If each $\varepsilon_j = 0$, we will refer to the set as a *matrix variety*, and we may replace the normed polynomial relations by polynomial relations.

Example 3.2. As a first example, we have that the matrix set

$$\mathbf{Z}_n := \left\{(X_1, \ldots, X_N) \in M_n^N \;\middle|\; \begin{array}{l} X_jX_k - X_kX_j = \mathbf{0}_n, \\ X_j^*X_j = X_jX_j^* = \mathbf{1}_n, \end{array} 1 \leq j,k \leq N\right\}$$

is a matrix variety. If for some $\delta > 0$, we now set

$$\mathbf{Z}_{n,\delta} := \left\{(X_1, \ldots, X_N) \in M_n^N \;\middle|\; \begin{array}{l} \|X_jX_k - X_kX_j\| \leq \delta, \\ \|X_j^*X_j - X_jX_j^*\| = 0, 1 \leq j,k \leq N \\ \|X_j\| \leq 1 \end{array}\right\},$$

where the set $\mathbf{Z}_{n,\delta}$ is a matrix semialgebraic variety.

Example 3.3. Another example of a matrix semialgebraic variety which has proved useful in understanding the geometric nature of the problems solved in this article is described by the matrix set $\mathbf{Iso}_\delta(x,y)$, defined for some given $\delta \geq 0$ and any two normal contractions x and y in M_n by the expression

$$\mathbf{Iso}_\delta(x,y) := \left\{ (z,w) \in \mathcal{N}(n)(\mathbb{D}^2) \times \mathbb{U}(n) \;\middle|\; \begin{array}{l} \|xw - wz\| = 0, \\ \|[z,y]\| = 0, \\ \|z - y\| \leq \delta \end{array} \right\}.$$

3.1. **Toroidal matrix links.**

3.1.1. Finsler manifolds, matrix paths, and toroidal matrix links.

Definition 3.2 (Finsler manifold). A *Finsler manifold* is a pair (M, F), where M is a manifold and $F : TM \to [0, \infty)$ is a function (called a *Finsler norm*) such that

- F is smooth on $TM\backslash\{0\} = \bigcup_{x\in M}\{T_xM\backslash\{0\}\}$,
- $F(v) \geq 0$ with equality if and only if $v = 0$,
- $F(\lambda v) = \lambda F(v)$ for all $\lambda \geq 0$,
- $F(v + w) \leq F(v) + F(w)$ for all w on the same tangent space with v.

Given a Finsler manifold (M, F), the length of any rectifiable curve $\gamma : [a, b] \to M$ is given by the length functional

$$L[\gamma] = \int_a^b F\big(\gamma(t), \partial_t\gamma(t)\big)\, dt,$$

where $F(x, \cdot)$ is the Finsler norm on each tangent space T_xM.

The pair $(\mathcal{N}, \|\cdot\|)$ is a Finsler manifold, where $\mathcal{N}$ denotes the set of normal matrices $\mathcal{N}$ (of any size) and $\|\cdot\|$ denotes the operator norm.

Definition 3.3 (Matrix path curvature). Given a piecewise C^2-matrix path $\gamma : [0,1] \to \mathcal{N}$ such that $\|\partial_t\gamma(t)\| > 0$, we define its *curvature* $\kappa[\gamma]$ to be

$$\kappa[\gamma] := \frac{1}{\|\partial_t\gamma(t)\|}\Big\|\partial_t\Big(\frac{\partial_t\gamma(t)}{\|\partial_t\gamma(t)\|}\Big)\Big\|.$$

Definition 3.4 (Matrix flows). Given $n \geq 1$, a mapping $\phi : \mathbb{R}^+_{\not{0}} \times M_n \to M_n$, $(t,x) \mapsto x_t$ will be called a *matrix flow*. If we have in addition that $\sigma(x_t) = \sigma(x_s)$ for every $t, s \geq 0$, we say that the matrix flow is *isospectral*.

Definition 3.5 (Interpolating path). Given two matrices x and y in M_n and a matrix flow $\phi : \mathbb{I} \times M_n \to M_n$ such that $\phi_0(x) = x$ and $\phi_1(x) = y$, we say that the corresponding path $\{x_t\}_{t\in\mathbb{I}} := \{\phi_t(x)\}_{t\in\mathbb{I}} \subseteq M_n$ is a *solvent path* for the interpolation problem $x \rightsquigarrow y$.

Definition 3.6 ($\circledast$ operation). Given two matrix paths $X, Y \in C([0,1], M_n)$, we write $X \circledast Y$ to denote the concatenation of X and Y, which is the matrix path defined in terms of X and Y by the expression

$$X \circledast Y_s := \begin{cases} X_{2s}, & 0 \leq s \leq \frac{1}{2}, \\ Y_{2s-1}, & \frac{1}{2} \leq s \leq 1. \end{cases}$$

Definition 3.7. ($\ell_{\|\cdot\|}$). Given a matrix path $\{x_t\}_{t\in\mathbb{I}}$ in M_n, we will write $\ell_{\|\cdot\|}(x_t)$ to denote the length of $\{x_t\}_{t\in\mathbb{I}}$ with respect to the operator norm which is defined by the expression

$$\ell_{\|\cdot\|}(x_t) := \sup \sum_{k=0}^{m-1} \|x_{t_{k+1}} - x_{t_k}\|,$$

where the supremum is taken over all partitions of $\mathbb{I}$ as $0 = t_0 < \cdots < t_m = b$. If the function $x \in C(\mathbb{I}, M_n)$ is a piecewise C^1-function, then

$$\ell_{\|\cdot\|}(x_t) = \int_{\mathbb{I}} \|\partial_t x_t\| \, dt.$$

Definition 3.8. ($\|\cdot\|$-flatness). A set $\mathcal{S}$ of M_n is said to be $\|\cdot\|$-*flat* if any two points $x, y \in \mathcal{S}$ can be connected by a path $\{x_t\}_{t\in\mathbb{I}} \subseteq \mathcal{S}$ such that $\ell_{\|\cdot\|}(x_t) = \|x - y\|$.

Definition 3.9 (Toroidal matrix link). Given any two normal contractions x, y in M_n, a *toroidal matrix link* is any piecewise analytic normal path $x_t := \mathbb{K}[T_t(\lessdot(x))]$ induced by a locally normal piecewise analytic matrix flow $T : \mathbb{I} \times M_N \to M_N$ with $N \geq n$, together with a locally normal compression $\mathbb{K} : M_N \to M_n$ with relative lifting map $\lessdot : M_n \to M_N$, which satisfy the interpolating conditions $\mathbb{K}[T_0(\lessdot(x))] = x$ and $\mathbb{K}[T_1(\lessdot(x))] = y$ together with the constraints $\|\mathbb{K}[T_t(\lessdot(x))]\| \leq 1$ for each $t \in \mathbb{I}$.

Remark 3.1. In the particular case where $[\mathbb{K}(T_t(\lessdot(x))), \mathbb{K}(T_t(\lessdot(y)))] = 0$ for each $t \in \mathbb{I}$, whenever $[x, y] = 0$, we call T a *toral matrix link*.

Remark 3.2. The curved nature of the matrix varieties (as *Finsler submanifolds of* $\mathcal{N}$), whose local connectivity we study in this article, induces an obstruction to local connectivity via entirely *flat* toroidal matrix links in general. The toroidal matrix links $\mathbf{T} \subset C([0,1], \mathcal{N})$ we have used to solve the connectivity problems which motivated this study satisfy the constraint

$$0 \leq \kappa[T] \leq \frac{2}{\ell_{\|\cdot\|}(T)}, \quad \forall T \in \mathbf{T}.$$

3.2. **Embedded matrix flows in solid tori.** Given some fixed but arbitrary $W \in \mathbb{U}(n)$, using the operation $\mathrm{diag}_n : M_n \to M_n$ one can define the mapping $\mathscr{D} : \mathbb{U}(n) \times M_n \to \mathbb{D}^2$ determined by the following expression:

$$\mathbb{U}(n) \times M_n \to \mathbb{D}^2, \tag{3.3}$$
$$(W, x) \mapsto \mathscr{D}_{\mathbb{T}}[W](x), \tag{3.4}$$
$$(W, x) \mapsto \left\{ \left(\mathrm{diag}_n[WxW^*]\right)_{k,k} \right\}_{1 \leq k \leq n}. \tag{3.5}$$

It can be seen that, for any $(W, x) \in \mathbb{U}(n) \times M_n$, the map $\mathscr{D}$ induced by the operation

$$\mathscr{D}_{\mathbb{T}}[W](x) := \{x_1, \ldots, x_n\} \subseteq \mathbb{D}^2$$

takes values in the set of finite sequences with n elements in $\mathbb{D}^2$, where each $\{x_1, \ldots, x_n\}$ consists of the diagonal entries of WxW^* counted with multiplicity. It is clear that $\mathrm{diag}[\mathscr{D}_{\mathbb{T}}[W](x)] = \mathrm{diag}_n[WxW^*]$ and that $\mathrm{diag}[\mathscr{D}_{\mathbb{T}}[\mathbf{1}_n](x)] = \mathrm{diag}_n[x]$. Because of this, when $W = \mathbf{1}_n$ we will write $\mathscr{D}(x)$ instead of $\mathscr{D}_{\mathbb{T}}[\mathbf{1}_n](x)$.

Given a matrix flow $\mathbb{I}\times\mathcal{N}(n)(\mathbb{D}^2)\to\mathcal{N}(n)(\mathbb{D}^2),(t,x)\mapsto X_t(x)$, one can identify X with the set of flow lines in $\mathbb{D}^2\times\mathbb{T}^1$ determined by $\{(\mathscr{D}(X_t(x)),e^{2\pi it})\}_{t\in\mathbb{I}}$. The geometric picture determined by the mapping cylinder $\mathcal{N}(n)(\mathbb{D}^2)\times\mathbb{I}\to\mathbb{D}^2\times\mathbb{T}^1,(x,t)\mapsto(\mathscr{D}(X_t(x)),e^{2\pi it})$ will be called the *embedded matrix mapping cylinder* relative to the flow X. We can think of the embedded matrix mapping cylinder in topological terms as a deformation described by the expression $\mathcal{D}_{X,Z_2}$, which is defined as

$$\mathcal{D}_{X,Z_2}[Z_1\times\mathbb{I}]:=\frac{(Z_1\times\mathbb{I})\sqcup Z_2}{Z_1\times\{1\}\rightsquigarrow_{X_1}Z_2},$$

where Z_1 and Z_2 are some prescribed (matrix) *point sets* in a matrix variety Z such that $X_1(x)\in Z_2$ for each $x\in Z_1$.

Example 3.4 (Graphical example in M_3). Let us set $\hat{u}_3:=e^{\frac{2\pi i}{3}f(\mathbf{N}_3)}$, where $f\in C(\mathbb{I},\mathbb{I})$. Given $W_3\in\mathbb{U}(3)$, we can obtain a graphical example of a particular geometric picture of the computation of the embedded matrix mapping cylinder relative to the interpolating flow $\mathbf{U}$, which solves the problem $\hat{u}_3\rightsquigarrow W_3\hat{u}_3W_3^*$ relative to the matrix variety $Z_3:=\{z\in M_n\mid zz^*=z^*z=\mathbf{1}_3\}=\mathbb{U}(3)$.

Let us set

$$\begin{aligned}Z_1&:=\big\{z\in\mathbb{U}(3)\;\big|\;[\hat{u}_3,z]=0\big\},\\ Z_2&:=\big\{z\in\mathbb{U}(3)\;\big|\;[W_3\hat{u}_3W_3^*,z]=0\big\}.\end{aligned}$$

Using projective methods, we can trace specific flow lines along the matrix flows corresponding to the dynamical deformation $\mathcal{D}_{\mathbf{U},Z_2}[Z_1\times\mathbb{I}]$, which solves the interpolation problem $\hat{u}_3\rightsquigarrow W_3\hat{u}_3W_3^*$.

A particular (approximate) geometric picture of the matrix deformation induced by the toral matrix link $\{\mathbf{U}_t\}_{t\in\mathbb{I}}$ in M_3, projected in $\mathbb{D}^2\times\mathbb{T}^1$ for each $t\in\mathbb{I}$ via $\mathscr{D}_{\mathbb{T}}(\mathbf{U}_t)$, is presented in Figures 1–3.

Alternative methods for tracing particular flow lines on mapping cylinders can be obtained using matrix homotopies, which can be done using similar methods to the ones implemented in [7].

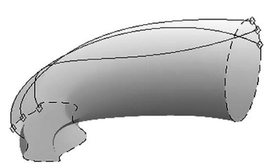

FIGURE 1. Projected matrix mapping cylinder corresponding to the path $\mathbf{U}_{[0,\frac{1}{2}]}(\hat{u}_3)$ in M_3.

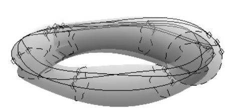

FIGURE 2. Projected matrix mapping cylinder corresponding to the path $\mathbf{U}_{\mathbb{I}}(\hat{u}_3)$ in M_3.

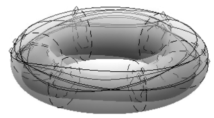

FIGURE 3. Embedded matrix mapping cylinder corresponding to the path $\mathbf{U}_{\mathbb{I}}(\hat{u}_3)$ in M_3.

3.3. **Environment algebras.**

Definition 3.10 (Environment algebra (of a matrix algebra)). Given a matrix algebra $A \subseteq M_n$, a universal C^*-algebra $\mathcal{E}_A := C_1^*\langle x_1, \dots, x_m \mid \mathcal{R}(x_1, \dots, x_m)\rangle$ for which there is a matrix representation $\mathcal{E}_A \twoheadrightarrow \mathbf{E}_A \subseteq M_n$ such that $A \subseteq \mathbf{E}_A$, will be called an *environment algebra* for A.

Let us consider the universal C^*-algebras $C(\mathbb{J})$, $C(\mathbb{T}^1)$, $C(\mathbb{T}^1) *_{\mathbb{C}} C(\mathbb{T}^1)$, $C_\delta(\mathbb{T}^2)$, $C_\delta(\mathbb{J} \times \mathbb{T}^1)$, and $C_\varepsilon^*\langle \mathbb{Z}/2 \times \mathbb{Z}\rangle$ defined in terms of generators and relations by the expressions

$$\begin{aligned}
C(\mathbb{J}) &:= C_1^*\big\langle h \;\big|\; h^* = h, \|h\| \leq 1 \big\rangle, \\
C(\mathbb{T}^1) &:= C_1^*\langle u \;\big|\; uu^* = u^*u = 1\rangle, \\
C(\mathbb{T}^1) *_{\mathbb{C}} C(\mathbb{T}^1) &:= C_1^* \left\langle u, v \;\middle|\; \begin{array}{l} uu^* = u^*u = 1, \\ vv^* = v^*v = 1 \end{array} \right\rangle, \\
C_\delta(\mathbb{T}^2) &:= C_1^* \left\langle u, v \;\middle|\; \begin{array}{l} uu^* = u^*u = 1, \\ vv^* = v^*v = 1, \\ \|uv - vu\| \leq \delta \end{array} \right\rangle, \\
C_\delta(\mathbb{J} \times \mathbb{T}^1) &:= C_1^* \left\langle h, u \;\middle|\; \begin{array}{l} h^* = h, \|h\| \leq 1 \\ uu^* = u^*u = 1, \\ \|hu - uh\| \leq \delta \end{array} \right\rangle, \\
C_\varepsilon^*\langle \mathbb{Z}/2 \times \mathbb{Z}\rangle &:= C_1^* \left\langle u, v \;\middle|\; \begin{array}{l} uu^* = u^*u = u^2 = 1, \\ vv^* = v^*v = 1, \\ \|uv - vu\| \leq \varepsilon \end{array} \right\rangle.
\end{aligned}$$

Let us now consider a local matrix representation result that we will use later in the construction of particular representation schemes.

Lemma 3.1. *For every integer $n \geq 1$, there are $s_2, u_n, v_n \in \mathbb{U}(\mathcal{M}_\infty)$ such that the diagram*

$$\begin{tikzcd}
C(\mathbb{T}^1)^{*2} \arrow[r, two heads] \arrow[d] & C^*\left\langle (\mathbb{Z}/n)^{*2}\right\rangle \arrow[r, two heads] & C_n^*(u_n, v_n) \arrow[d, equal] \\
C^*\left\langle \mathbb{Z}/n * \mathbb{Z}/2\right\rangle \arrow[r, two heads] & C_n^*(s_2, v_n) \arrow[r, equal] & M_n
\end{tikzcd}$$

commutes, where $s_2 \in \mathbb{H}(n)$, u_n and v_n are unitary elements in M_n.

Proof. Since we have that $C(\mathbb{T}^1)^{*2} \simeq C^* \langle \mathbb{F}_2 \rangle \simeq C^*(\mathbb{Z}^{*2})$, by universality of the C^*-representations

$$C^*(\mathbb{Z}^{*2}) \simeq C^* \left\langle u, v \;\middle|\; \begin{array}{l} uu^* = u^*u = \mathbf{1}, \\ vv^* = v^*v = \mathbf{1} \end{array} \right\rangle,$$

$$C^*\big((\mathbb{Z}/n)^{*2}\big) \simeq C^* \left\langle u, v \;\middle|\; \begin{array}{l} uu^* = u^*u = \mathbf{1}, \\ vv^* = v^*v = \mathbf{1}, \\ u^n = v^n = \mathbf{1} \end{array} \right\rangle,$$

$$C^*(\mathbb{Z}/n * \mathbb{Z}/2) \simeq C^* \left\langle u, v \;\middle|\; \begin{array}{l} uu^* = u^*u = \mathbf{1}, \\ vv^* = v^*v = \mathbf{1}, \\ u^n = v^2 = \mathbf{1} \end{array} \right\rangle,$$

and by the structural properties of M_n, it is enough to find for any $n \in \mathbb{Z}^+$, up to unitary congruence in M_n, three unitaries $s_2, u_n, v_n \in \mathbb{U}(n)$ such that $C^*(s_2, v_n) = M_n = C^*(u_n, v_n)$ and $u_n^n = v_n^n = s_2^2 = \mathbf{1}_n$. This can be done by taking for any $n \in \mathbb{Z}^+$ the orthogonal projection $p := \mathrm{diag}[1, 0, \ldots, 0] \in \mathbb{H}(n)$ and the matrix $s_2 = \mathbf{1} - 2p \in \mathbb{H}(n)$, setting $u_n := \Omega_n$ and $v_n := \Sigma_n$ for $n \geq 2$ and $u_1 = v_1 = 1$ for $n = 1$. By functional calculus and direct computations, it is easy to verify that $s_2, u_n, v_n \in \mathbb{U}(n)$ for every $n \in \mathbb{Z}^+$ and that $s_2 = s_2^*$. It is also easy to verify that the system of matrix units $\{ e_{i,j,n} \}_{1 \leq i,j \leq n}$ and u_n can be expressed as words in $C^*(s_2, v_n)$ for every $n \in \mathbb{Z}^+$. It is also clear that $p = e_{1,1,n}$, and hence, s_2 can be written as linear combinations of words in $C^*(u_n, v_n)$. We will then have that $C^* \langle \mathbb{Z}/n * \mathbb{Z}/2 \rangle \twoheadrightarrow C^*(v_n, s_2)$ and $C^* \langle \mathbb{Z}/n^{*2} \rangle \twoheadrightarrow C^*(u_n, v_n)$ by the universal properties of $C^* \langle \mathbb{Z}/2 * \mathbb{Z}/n \rangle$ and $C^* \langle \mathbb{Z}/n^{*2} \rangle$, respectively, since it can be easily verified that

$$u_n^n = v_n^n = s_2^2 = \mathbf{1}_n.$$

The result follows from these facts and the universal property of $C(\mathbb{T}^1)^{*2} \simeq C^* \langle \mathbb{F}_2 \rangle \simeq C^* \langle \mathbb{Z}^{*2} \rangle$. ☐

Remark 3.3. It can be seen that for any matrix C^*-subalgebra $A \subseteq M_n$, there is $\delta > 0$ such that both $C(\mathbb{T}^1) *_{\mathbb{C}} C(\mathbb{T}^1)$ and $C_\delta(\mathbb{T}^2)$ are environment algebras of A. It can also be seen that for any Abelian C^*-subalgebra $D \subseteq M_n$, $C(\mathbb{T}^1)$ is an environment algebra of D.

4. Local matrix connectivity

4.1. **Topologically controlled linear algebra and soft tori.**

Definition 4.1 (Controlled sets of matrix functions). Given $\delta > 0$, a function $\varepsilon : \mathbb{R} \to \mathbb{R}_0^+$, a finite set of functions $F \subseteq C(\mathbb{T}^1, \mathbb{D}^2)$, and two unitary matrices $u, v \in M_n$ such that $\|uv - vu\| \leq \delta$, we say that the set F is *δ-controlled* by $\mathrm{Ad}[v]$ if $\|f(vuv^*) - f(u)\| \leq \varepsilon(\delta)$, for each $f \in F$.

Remark 4.1. The C^*-homomorphism $C_\delta(\mathbb{T}^2) \to C^*(u, v)$ allows us to see that the soft torus $C_\delta(\mathbb{T}^2)$ provides an environment algebra for any δ-controlled set of matrix functions.

Lemma 4.1 (Existence of isospectral approximants)**.** *Given $\varepsilon > 0$, there is $\delta > 0$ such that, for any two families of N pairwise commuting $(n\times n)$–normal matrices $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ which satisfy the constraints $\|x_j - y_j\| \leq \delta$ for each $1 \leq j \leq N$, there is a C^*-homomorphism $\Psi : M_n \to M_n$ such that $\sigma(\Psi(x_j)) = \sigma(x_j)$, $[\Psi(x_j), y_k] = \mathbf{0}_n$, and $\max\{\|\Psi(x_j) - y_j\|, \|\Psi(x_j) - x_j\|\} \leq \varepsilon$, for each $1 \leq j, k \leq N$.*

Proof. By changing the basis if necessary, we can assume that $y_1, \ldots, y_N$ are diagonal matrices. From Theorem 2.1, we will have that there is a permutation τ of the index set $\{1, \ldots, n\}$ such that for each $1 \leq k \leq n$, we have that

$$\begin{aligned} |\Lambda^{(k)}(x_j) - \Lambda^{(\tau(k))}(y_j)| &\leq \left\|\Lambda^{(k)}(x_1, \ldots, x_N) - \Lambda^{(\tau(k))}(y_1, \ldots, y_N)\right\| \\ &\leq e_{N,0}\left\|\mathrm{Cliff}(x_1 - y_1, \ldots, x_N - y_N)\right\|. \end{aligned} \tag{4.1}$$

Using (2.1), and as a consequence of (4.1), we can find a permutation matrix $\mathcal{T} \in \mathbb{U}(n)$ such that

$$\begin{aligned} \left\|\mathcal{T}^* \operatorname{diag}\left[\Lambda(x_j)\right]\mathcal{T} - \operatorname{diag}\left[\Lambda(y_j)\right]\right\| &\leq e_{N,0}\left\|\mathrm{Cliff}(x_1 - y_1, \ldots, x_N - y_N)\right\| \\ &\leq e_{N,0}N\delta, \quad 1 \leq j \leq N. \end{aligned} \tag{4.2}$$

Let us set $c_N := e_{N,0}N$. For the matrices $x_1, \ldots, x_N$, there is a unitary joint diagonalizer $W \in M_n$ such that $W \operatorname{diag}[\Lambda(x_j)]W^* = x_j$, $1 \leq j \leq N$, and

$$\begin{aligned} \left\|W \operatorname{diag}\left[\Lambda(x_j)\right]W^* - \mathcal{T}^* \operatorname{diag}\left[\Lambda(x_j)\right]\mathcal{T}\right\| &\leq \left\|W \operatorname{diag}\left[\Lambda(x_j)\right]W^* - y_j\right\| \\ &\quad + \left\|y_j - \mathcal{T}^* \operatorname{diag}\left[\Lambda(x_j)\right]\mathcal{T}\right\| \\ &\leq (1 + c_N)\|x_j - y_j\| \\ &\leq (1 + c_N)\delta. \end{aligned} \tag{4.3}$$

If we set $V := W\mathcal{T}$ and $\varepsilon = (1 + c_N)\delta$, we will have that, by (4.2) and (4.3), the inner C^*-automorphism $\Psi := \mathrm{Ad}[V^*]$ satisfies the constraints in the statement of this lemma, and we are done. ☐

Remark 4.2. The C^*-automorphism Ψ from Lemma 4.1 is called an *isospectral approximant* for the two N-tuples $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$. If $\Psi := \mathrm{Ad}[W^*]$ for some $W \in \mathbb{U}(n)$, then we will have that its inverse $\Psi^\dagger$ will be given by the expression $\Psi^\dagger = \mathrm{Ad}[W]$.

Remark 4.3. The constant c_N in the proof of Lemma 4.1 depends only on the number N of matrices in each family. It does not depend on the matrix size.

4.2. **Local piecewise analytic connectivity.** In this section, we will present some piecewise analytic local connectivity results in matrix representations of the form $C(\mathbb{T}^N) \to C^*(u_\varepsilon, v_\varepsilon)$ and $C([-1,1]^N) \to C^*(u_\varepsilon, v_\varepsilon)$, with $C_\varepsilon(\mathbb{T}^2) \twoheadrightarrow C^*(u_\varepsilon, v_\varepsilon)$.

Theorem 4.1 (Local normal toral connectivity)**.** *Given $\varepsilon > 0$, any $n \in \mathbb{Z}^+$, and $N \in \mathbb{Z}^+$, there is $\delta > 0$ such that, for any $2N$ normal contractions $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ in M_n which satisfy the relations*

$$\begin{cases} [x_j, x_k] = [y_j, y_k] = 0, & 1 \leq j, k \leq N, \\ \|x_j - y_j\| \leq \delta, & 1 \leq j \leq N, \end{cases}$$

there exist N toral matrix links $X^1,\ldots,X^N$ in M_n, which solve the problems

$$x_j \rightsquigarrow y_j, \quad 1\leq j\leq N,$$

and satisfy the constraints

$$\left\|X_t^j(x_j)-y_j\right\|\leq\varepsilon,$$

for each $1\leq j\leq N$ and each $t\in\mathbb{I}$. Moreover, $\ell_{\|\cdot\|}(X_t^j(x_j))\leq\varepsilon$, $1\leq j\leq N$.

Proof. By Lemmas 2.1, 2.2, and 4.1, we will have that, given $\varepsilon>0$, there are $0<\delta\leq\nu\leq\varepsilon/2$ and an isospectral approximant $\Psi:=\mathrm{Ad}[W^*]$ (with $W\in\mathbb{U}(n)$) for $x_1,\ldots,x_N$ and $y_1,\ldots,y_N$ such that $\max\{\|x_j-\Psi(x_j)\|,\|y_j-\Psi(x_j)\|\}\leq\nu$ and $[\Psi(x_j),y_j]=0$ for each $1\leq j\leq N$. We will also have that there is a unitary path $\mathcal{W}\in C(\mathbb{I},M_n)$ which is defined by the expression $\mathcal{W}_t:=e^{-itH_W}$ for each $t\in\mathbb{I}$, where $H_W\in M_n$ is a Hermitian matrix such that $e^{iH_W}=W$ and $\|[H_W,x_j]\|\leq\varepsilon/2$ for each $1\leq j\leq N$, and which is defined by $H_W:=h(W)$, for some function $h:\Omega^\alpha_{d,s}\to[-1,1]$, and where $\sigma(W)\subset\Omega^\alpha_{d,s}:=\{e^{i(\pi t+\alpha)}\mid -1+s<t<1-s\}\subset\mathbb{T}^1$, with $s,\alpha\in\mathbb{R}$ chosen in such a way that $\mathbb{T}^1\backslash\Omega^\alpha_{d,s}$ contains an arc of length d (with $d\geq 2\pi/n$). Moreover, we can choose δ and ν in such a way that the path $\mathcal{W}$ satisfies the inequalities $\|[\mathcal{W}_t,\Psi(x_j)]\|\leq\varepsilon/2$ for each $t\in[0,1]$ and each $1\leq j\leq N$.

It can be seen that the paths $\breve{X}_t^j:=\mathrm{Ad}[\mathcal{W}_t](x_j)$ will solve the local interpolation problem $x_j\rightsquigarrow_{\varepsilon/2,\mathcal{N}(n)(\mathbb{D}^2)}\Psi(x_j)$ for each $1\leq j\leq N$. Let us set $\bar{X}_t^j:=(1-t)\Psi(x_j)+ty_j$. We can now construct N toroidal matrix links of the form $X^j:=\breve{X}^j\circledast\bar{X}^j$ which solve the problems $x_j\rightsquigarrow y_j$, locally preserve normality and commutativity, and satisfy the $\|\cdot\|$-distance constraints

$$\begin{aligned}\|X_t^j-y_j\|&\leq\left\|X_t^j-\Psi(x_j)\right\|+\left\|y_j-\Psi(x_j)\right\|\\&\leq\frac{\varepsilon}{2}+\nu\\&\leq\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\end{aligned}$$

together with the $\|\cdot\|$-length constraints

$$\begin{aligned}\ell_{\|\cdot\|}(X_t^j)&\leq\ell_{\|\cdot\|}(\breve{X}_t^j)+\left\|\Psi(x_j)-y_j\right\|\\&=\int_{\mathbb{I}}\left\|\partial_t\,\mathrm{Ad}[\mathcal{W}_t](x_j)\right\|dt+\left\|\Psi(x_j)-y_j\right\|\\&=\left\|\left[H_W,\Psi(x_j)\right]\right\|+\left\|\Psi(x_j)-y_j\right\|\\&\leq\frac{\varepsilon}{2}+\nu\leq\varepsilon,\end{aligned}$$

which hold whenever $\|x_j-y_j\|\leq\delta$, $1\leq j\leq N$, and we are done. □

Remark 4.4. We note that the solvent matrix links $X^1,\ldots,X^N$, whose existence is stated in Theorem 4.1, are factored in the form $X^j=\breve{X}^j\circledast\bar{X}^j$. We call $\breve{X}^j$ and $\bar{X}^j$ the *curved* and *flat* factors of X^j, respectively.

We now derive two corollaries from the proof of Theorem 4.1.

Corollary 4.1 (Local Hermitian toral connectivity). *Given $\varepsilon > 0$, any integer $n \geq 1$, and $N \in \mathbb{Z}^+$, there is $\delta > 0$ such that, for any $2N$ Hermitian contractions $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ in M_n which satisfy the relations*

$$\begin{cases} [x_j, x_k] = [y_j, y_k] = 0, & 1 \leq j, k \leq N, \\ \|x_j - y_j\| \leq \delta, & 1 \leq j \leq N, \end{cases}$$

there exist N toral matrix links $X^1, \ldots, X^N$ in M_n, which solve the problems

$$x_j \rightsquigarrow y_j, \quad 1 \leq j \leq N,$$

and satisfy the constraints

$$\begin{cases} X_t^j(x_j) = (X_t^j(x_j))^*, \\ \|X_t^j(x_j) - y_j\| \leq \varepsilon, \end{cases}$$

for each $1 \leq j \leq N$ and each $t \in \mathbb{I}$. Moreover, $\ell_{\|\cdot\|}(X_t^j(x_j)) \leq \varepsilon$, $1 \leq j \leq N$.

Proof. Since for any $\alpha \in \mathbb{R}$, any pair of Hermitian matrices $x, y \in \mathbb{H}(n)$, and any partial unitary $z \in \mathbb{PU}(n)$, we have that $x + \alpha(y - x)$ and zxz^* are also in $\mathbb{H}(n)$. The result follows as a consequence of Lemma 4.1 and Theorem 4.1. $\square$

Corollary 4.2 (Local unitary toral connectivity). *Given any $\varepsilon \geq 0$, any integer $n \geq 1$, and any $N \in \mathbb{Z}^+$, there is $\delta \geq 0$ such that, given any $2N$ unitary matrices $U_1, \ldots, U_N$, $V_1, \ldots, V_N$ in M_n which satisfy the relations*

$$\begin{cases} [U_j, U_k] = [V_j, V_k] = 0, \\ \|U_k - V_k\| \leq \delta, \end{cases}$$

for each $1 \leq j, k \leq N$, there are toral matrix links $u^1, \ldots, u^N$ in M_n which solve the interpolation problems

$$U_k \rightsquigarrow V_k, \quad 1 \leq k \leq N,$$

and also satisfy the relations

$$\begin{cases} (u_t^j)^* u_t^j = u_t^j (u_t^j)^* = \mathbf{1}_n, \\ \|u_t^j - V_j\| \leq \varepsilon, \end{cases}$$

for each $t \in \mathbb{I}$ and each $1 \leq j \leq N$. Moreover, $\ell_{\|\cdot\|}(u_t^j) \leq \varepsilon$, $1 \leq j \leq N$.

Proof. Since for any C^*-automorphisms Ψ we have that $\Psi(\mathbb{U}(n)) \subseteq \mathbb{U}(n)$, and since any two commuting unitaries U and V can be connected by a flat unitary path, $\bar{U}_t := Ue^{t \ln(U^*V)}$, for $0 \leq t \leq 1$. We will have that the result can be derived using a similar argument to the one implemented in the proof of Theorem 4.1. $\square$

4.2.1. Lifted local piecewise analytic connectivity. Let us denote by κ the matrix compression $M_{2n} \to M_n$ defined by the mapping

$$\kappa : M_{2n} \to M_n, \qquad \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto x_{11}.$$

Let us write $\imath_2 : M_n \to M_{2n}$ to denote the C^*-homomorphism defined by the expression $\imath_2(x) := x \oplus x = \mathbf{1}_2 \otimes x$.

Definition 4.2 (Standard dilations). Given a C^*-automorphism $\Psi := \mathrm{Ad}[W]$ (with $W \in \mathbb{U}(n)$) in M_n, we will denote by $\Psi^{[s]}$ the C^*-automorphism in M_{2n} defined by the expression $\Psi^{[s]} := \mathrm{Ad}[\mathbf{1}_2 \otimes W] = \mathrm{Ad}[W \oplus W]$. We call $\Psi^{[s]}$ a *standard dilation* of Ψ.

Definition 4.3 ($\mathbb{Z}/2$-dilations). Given a C^*-automorphism $\Psi := \mathrm{Ad}[W]$ (with $W \in \mathbb{U}(n)$) in M_n, we will denote by $\Psi^{[2]}$ the C^*-automorphism in M_{2n} defined by the expression $\Psi^{[2]} := \mathrm{Ad}[(\Sigma_2 \otimes \mathbf{1}_n)(W^* \oplus W)]$. We call $\Psi^{[2]}$ a $\mathbb{Z}/2$-*dilation* of Ψ.

Remark 4.5. It can be seen that $\kappa(\imath_2(x)) = x$ for any $x \in M_{2n}$. It can also be seen that $\kappa(\Psi^{[2]}(\imath_2(x))) = \kappa(\Psi^{[s]}(\imath_2(x)))$.

Theorem 4.2 (Lifted local toral connectivity)**.** *Given $\varepsilon > 0$, there is $\delta > 0$ such that, for any $2N$ normal contractions $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ in M_n which satisfy the relations*

$$\begin{cases} [x_j, x_k] = [y_j, y_k] = 0, & 1 \le j,k \le N, \\ \|x_j - y_j\| \le \delta, & 1 \le j \le N, \end{cases}$$

there is a C^*-homomorphism $\Phi : M_n \to M_{2n}$ and N toral matrix links $X^1, \ldots, X^N$ in $C(\mathbb{I}, M_{2n})$ which solve the problems

$$\Phi(x_j) \rightsquigarrow y_j \oplus y_j, \quad 1 \le j \le N,$$

and satisfy the constraints

$$\begin{cases} \kappa(\Phi(x_j)) = x_j, \\ \|\Phi(x_j) - x_j \oplus x_j\| \le \varepsilon, \\ \|X^j_t - y_j \oplus y_j\| \le \varepsilon, \end{cases}$$

for each $1 \le j \le N$ and each $t \in \mathbb{I}$. Moreover, $\ell_{\|\cdot\|}(X^j_t) \le \varepsilon$, $1 \le j \le N$.

Proof. By Lemma 4.1, we will have that, given $\varepsilon > 0$, there are $0 < \delta \le \nu = \frac{\varepsilon}{2\pi}$ and an isospectral approximant $\Psi := \mathrm{Ad}[W^*]$ (with $W \in \mathbb{U}(n)$) for $x_1, \ldots, x_N$ and $y_1, \ldots, y_N$ such that $\max\{\|x_j - \Psi(x_j)\|, \|y_j - \Psi(x_j)\|\} \le \nu$. By setting $\Phi := (\Psi^\dagger)^{[2]} \circ \imath_2 \circ \Psi$, Definition 4.3, Definition 4.2, and Remark 4.5 show that $\Phi : M_n \to M_{2n}$ is a C^*-homomorphism such that $\|\Phi(x_j) - \imath_2(x_j)\| = \|\Phi(x_j) - x_j \oplus x_j\| \le \varepsilon$, for each $1 \le j \le N$.

Since $(\Psi^\dagger)^{[2]} := \mathrm{Ad}[\hat{W}_s]$ with $\hat{W}_s := (\Sigma_2 \otimes \mathbf{1}_n)(W^* \oplus W)$ and since $\hat{W}_s \in \mathbb{U}(2n) \cap \mathbb{H}(2n)$, we will have that $\hat{W}_s$ can be represented as $\hat{W}_s = e^{i\frac{\pi}{2}(\hat{W}_s - \mathbf{1}_{2n})}$ for any $n \ge 1$. If we set $\tilde{X}_j := \Psi^{[s]}(\imath_2(x_j))$, $1 \le j \le N$, we also have that there is a unitary path $\{\mathcal{W}_t\}_{t\in\mathbb{I}} \subset M_{2n}$ with $\mathcal{W}_t := e^{i\frac{\pi(1-t)}{2}(\hat{W}_s - \mathbf{1}_{2n})}$, which satisfies the conditions $\mathcal{W}_0 = \hat{W}_s$, $\mathcal{W}_1 = \mathbf{1}_{2n}$, together with the normed estimates

$$\begin{aligned}\|\mathcal{W}_t\tilde{X}_j - \tilde{X}_j\mathcal{W}_t\| &= |\cos(\pi t/2)|\|\hat{W}_s\tilde{X}_j - \tilde{X}_j\hat{W}_s\| \\ &\leq \|\hat{W}_s\tilde{X}_j - \tilde{X}_j\hat{W}_s\| \leq \nu,\end{aligned}$$

for each $1 \leq j \leq N$ and each $0 \leq t \leq 1$. Moreover, for each $1 \leq j \leq N$ we have that the paths $\breve{X}_t^j := \mathrm{Ad}[\mathcal{W}_t](\tilde{X}_j)$ satisfy the normed estimates

$$\begin{aligned}\ell_{\|\cdot\|}(\breve{X}_t^j) &= \int_{\mathbb{I}} \left\|\partial_t \mathrm{Ad}[\mathcal{W}_t](\tilde{X}_j)\right\| dt \\ &= \frac{\pi}{2}\|\hat{W}_s\tilde{X}_j - \tilde{X}_j\hat{W}_s\| \leq \nu.\end{aligned}$$

For each $1 \leq j \leq N$, we can now use the flat paths $\bar{X}_t^j := (1-t)\tilde{X}_j + t\imath_2(y_j)$ together with the previously described curved paths $\breve{X}^j$ to construct the solvent toral matrix links $X^1, \ldots, X^N \in C([0,1], M_{2n})$ we are looking for, and which can be defined by $X^j := \breve{X}^j \circledast \bar{X}^j$ for each $1 \leq j \leq N$, and we are done. $\square$

Remark 4.6. It can be seen that, by using the technique implemented in the proof of Theorem 4.2, one can obtain lifted versions of Corollaries 4.2 and 4.1.

Remark 4.7. As a consequence of Theorem 4.2, we can derive simple detection methods to identify families of pairwise commuting matrices in M_n that can be connected uniformly via piecewise analytic toral matrix links. The existence of these detection methods raises some interesting questions for further studies.

Remark 4.8. We can interpret Theorem 4.2 as an existence theorem of solutions to lifted connectivity problems defined on matrix representations of the form

$$\begin{tikzcd} & C^*_\varepsilon\langle \mathbb{Z}/2\times\mathbb{Z}\rangle \arrow[r] & C^*(\hat{U}_s,\hat{V}) \arrow[r] & M_{2n} \arrow[d] \\ C^*\langle\mathbb{F}_2\rangle \arrow[ur] \arrow[r] & C_\delta(\mathbb{T}^2) \arrow[r] & C^*(U,V) \arrow[u] \arrow[r] & M_n \end{tikzcd}$$

with $\hat{U}_s = (\Sigma_2 \otimes \mathbf{1}_n)(U^* \oplus U)$ and $\hat{V} = V \oplus V$.

4.2.2. Matrix Klein bottles: Local matrix deformations and special symmetries. Using Theorem 4.2, we can solve all connectivity problems (together with their softened versions) in M_n that can be reduced to connectivity problems of the form $x \rightsquigarrow_\varepsilon x^*$ in $\mathcal{N}(n)(\mathbb{D}^2)$, with $x^* = TxT$ and $T^2 = \mathbf{1}_n$.

Remark 4.9. For each $\varepsilon \in [0,2]$, we can use the previously described symmetries and $\mathscr{D}_{\mathbb{T}}$ to interpret $\bigcup_{x\in M_n}\{x \rightsquigarrow_{\varepsilon, C^*(x)} x^*\}$ for $x \in \mathcal{N}(n)(\mathbb{D}^2)$ as matrix analogies of the *Klein bottle*.

By a *softened matrix Klein bottle* we mean that the symmetries are softened. In particular, we can consider the connectivity problems $x \rightsquigarrow_\varepsilon x^*$ and $y \rightsquigarrow_\varepsilon y^*$ in $\mathcal{N}(n)(\mathbb{D}^2)$ subject to the normed constraints $\|xy - yx\| \leq \delta$, $\|x^* - TxT\|, \|xT - Ty\| \leq \delta$, and $T^2 = \mathbf{1}_n$. The details regarding the solvability of these local connectivity problems will be addressed in future work.

4.3. C^0**-uniform local connectivity of pairs of unitaries and piecewise analytic approximants.** The technique presented in this section can be used to solve local connectivity problems in matrix representations of the form $C(\mathbb{T}^2) \to C^*(u_\varepsilon, v_\varepsilon) \subseteq M_n$ uniformly via C^0-unitary paths.

Suppose that U_t and V_t are unitary matrices in $\mathbf{M}_n(\mathbb{C})$ for $t = 0$ and $t = 1$, and suppose that we define

$$U_t = U_0 e^{t \ln(U_0^* U_1)} \tag{4.4}$$

and

$$V_t = V_0 e^{t \ln(V_0^* V_1)}. \tag{4.5}$$

For $t = 0$ or $t = 1$ the C^*-algebra generated by U_t and V_t is Abelian, so select a maximal Abelian subalgebra (MASA) $C_t \cong \mathbb{C}^n$ in each case. Let

$$A(C_0, C_1) = \left\{ X \in C\big([0,1], \mathbf{M}_n(\mathbb{C})\big) \;\middle|\; X(0) \in C_0 \text{ and } X(1) \in C_1 \right\}.$$

Lemma 4.2. *The C^*-algebra $A(C_0, C_1)$ has stable rank one.*

Proof. Starting with X continuous with $X(t)$ in C_t at the endpoints, we can adjust this by a small amount, leaving the endpoints in C_t, to get X piecewise linear, with the endpoints of every linear segment having no spectral multiplicity and being invertible. Using Kato's theory of analytic paths, we can get a piecewise continuous unitary U_t and piecewise analytic scalar paths $\lambda_n(t)$ so that the new path $Y \approx X$ satisfies

$$Y(t) = U_t \begin{bmatrix} \lambda_1(t) & & \\ & \ddots & \\ & & \lambda_n(t) \end{bmatrix} U_t^*.$$

There may be finitely many places where $Y(t)$ is not invertible. These places will be in the interior of the segment and so in an open interval where U_t is continuous. A small deformation of some of the λ_j's will take the path through invertibles. We have not moved the endpoints in the second adjustment, so the constructed element is in $A(C_0, C_1)$ and close to X. $\square$

Lemma 4.3. *The endpoint-restriction map $\rho : A(C_0, C_1) \to C_0 \oplus C_1$ induces an injection on K_0.*

Proof. The kernel of ρ is $C_0((0,1), \mathbf{M}_n(\mathbb{C}))$ which has trivial K_0-group. So this result follows from the exactness of the usual six-term sequence in K-theory. $\square$

Lemma 4.4. *Given unitaries U and V in $A(C_0, C_1)$, with $\|[U, V]\| \leq \nu_0$ as in Definition 2.2 (so the Bott index makes sense),* $\mathrm{Bott}(U, V)$ *is the trivial element of $K_0(A(C_0, C_1))$.*

Proof. By the previous lemma, we need only calculate $\mathrm{Bott}(\rho(U), \rho(V))$. These unitaries are in a commutative C^*-algebra, so they have trivial Bott index. $\square$

Theorem 4.3. *Given $\epsilon > 0$, there exists $\delta > 0$ so that for all n, given unitary matrices U_0, U_1, V_0, V_1 in $\mathbf{M}_n(\mathbb{C})$ with $U_0V_0 = V_0U_0$, $U_1V_1 = V_1U_1$, $\|U_0 - U_1\| \leq \delta$, and $\|V_0 - V_1\| \leq \delta$, then there exist continuous paths U_t and V_t between the given pairs of unitaries with each U_t and V_t unitary, and with $U_tV_t = V_tU_t$, $\|U_t - U_0\| \leq \epsilon$ and $\|V_t - V_0\| \leq \epsilon$ for all t.*

Proof. The paths U_t and V_t defined in (4.4) and (4.5) will be almost commuting unitary elements of $A(C_0, C_1)$. By Lemma 4.2, we may apply [11, Theorem 8.1.1] regarding approximating in $A(C_0, C_1)$ by commuting unitaries. Lemma 4.4 tells us there is no invariant to worry about, so we can find A_t and B_t close to U_t and V_t that are commuting continuous paths of unitaries with A_t and B_t in C_t for $t = 0, 1$. The unitary elements in the commutative C_t are locally connected, so we can find a short path from U_0 and V_0 to A_0 and B_0, and likewise at the other end. Concatenating, we get paths of commuting unitary matrices from U_0 and V_0 to U_1 and V_1 so that at every point we are close to some pair (U_t, V_t). These then are all close to U_0 and V_0. □

By combining Theorem 4.2, Corollary 4.2, and Theorem 4.3, we can add the following remark.

Remark 4.10 (Piecewise analytic approximants of C^0-interpolants). Given $\epsilon > 0$, there exists $\delta > 0$ so that for all n, given unitary matrices U_0, U_1, V_0, V_1 in $\mathbf{M}_n(\mathbb{C})$ with $U_0V_0 = V_0U_0$, $U_1V_1 = V_1U_1$, $\|U_0 - U_1\| \leq \delta$, and $\|V_0 - V_1\| \leq \delta$, there exist continuous (interpolant) paths U_t and V_t in M_{2n} which solve the problems $U_0 \oplus U_0 \rightsquigarrow U_1 \oplus U_1$ and $V_0 \oplus V_0 \rightsquigarrow V_1 \oplus V_1$ with each U_t and V_t unitary, and with $U_tV_t = V_tU_t$, $\|U_t - U_0 \oplus U_0\| \leq \epsilon$ and $\|V_t - V_0 \oplus V_0\| \leq \epsilon$ for all t. There is also a C^*-homomorphism $\Psi : M_n \to M_{2n}$ such that

$$\begin{aligned}\max\{&\|\Psi(U_0) - U_1 \oplus U_1\|, \|\Psi(U_0) - U_0 \oplus U_0\|,\\ &\|\Psi(V_0) - V_1 \oplus V_1\|, \|\Psi(V_0) - V_0 \oplus V_0\|\} \leq \epsilon,\end{aligned}$$

and there exist two piecewise analytic unitary pairwise commuting paths $\hat{U}, \hat{V} \in C([0,1], M_{2n})$ which solve the problems $\Psi(U_0) \rightsquigarrow U_1 \oplus U_1$, $\Psi(V_0) \rightsquigarrow V_1 \oplus V_1$ with $\max\{\|\hat{U}_t - U_t\|, \|\hat{V}_t - V_t\|\} \leq \epsilon$ for each $0 \leq t \leq 1$. Moreover, $\ell_{\|\cdot\|}(\hat{U}_t) \leq \epsilon$ and $\ell_{\|\cdot\|}(\hat{V}_t) \leq \epsilon$.

4.4. **Jointly compressible matrix sets.** Given $0 < \delta \leq \varepsilon$, we can now consider an alternative approach to the local connectivity problem involving two N-sets of pairwise commuting normal matrix contractions $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ such that $\|X_j - Y_j\| \leq \delta$ for each $1 \leq j \leq N$. The approach that we take in this section consists of considering the existence of a normal contraction $\hat{X}$ such that $X_1, \ldots, X_N \in C^*(\hat{X})$, and which also satisfies the constraint $\|\hat{X} - X_j\| \leq \varepsilon$ for some $1 \leq j \leq N$. A matrix $\hat{X}$ which satisfies the previous conditions will be called a *nearby generator* for $X_1, \ldots, X_N$. It can be seen that for any $\delta \leq \nu \leq \varepsilon$, one can find a flat analytic path $\bar{X} \in C([0,1], \mathcal{M}_\infty)$ that performs the deformation $X_j \rightsquigarrow_{\nu, \mathcal{N}(n)(\mathbb{D}^2)} \hat{X}$, where $\hat{X}$ is a nearby generator for $X_1, \ldots, X_N$.

Given any joint isospectral approximant (JIA) Ψ with respect to the families of normal contractions described in the previous paragraph, along the lines of the program that we have used to derive the connectivity results Theorem 4.1 and Theorem 4.2, we can use Lemma 4.1 to find a C^*-automorphism which solves the extension problem described by the diagram

$$\begin{tikzcd} & C^*(\hat{X}) \arrow[d, dashed, "\hat{\Psi}"] \\ C^*(X_1,\ldots,X_N) \arrow[ur, hook] \arrow[r, "\Psi"'] & C^*(Y_1,\ldots,Y_N)' \end{tikzcd} \tag{4.6}$$

and satisfies the relations $\Psi(X_j) = \hat{\Psi}(X_j)$ for each $1 \leq j \leq N$ together with the normed constraints

$$\max\{\|\hat{\Psi}(\hat{X}) - \hat{X}\|, \max_j\{\|\hat{\Psi}(X_j) - X_j\|, \|\hat{\Psi}(X_j) - Y_j\|\}\} \leq \varepsilon.$$

We refer to the C^*-automorphism $\hat{\Psi}$ in (4.6) as a *compression* of Ψ or a *compressive joint isospectral approximant* (CJIA) for the N-sets of normal contractions. Let us now consider a special type of inner C^*-automorphism that can be described as follows.

Definition 4.4 (Uniformly compressible JIA). Given $0 < \delta \leq \varepsilon$ and two N-sets of pairwise commuting normal contractions $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ in $\mathcal{M}_\infty$ such that $\|X_j - Y_j\| \leq \delta$, $1 \leq j \leq N$, a joint isospectral approximant Ψ of the N-sets is said to be *uniformly compressible* if there are a nearby generator $\hat{X}$ for $X_1, \ldots, X_N$, a compression $\hat{\Psi} := \mathrm{Ad}[W]$ (with $W \in \mathbb{U}(\mathcal{M}_\infty)$) of Ψ, and a unitary $\hat{W} \in \hat{\Psi}(C^*(\hat{X}))'$ such that $\|W - \hat{W}\| \leq \varepsilon$. We refer to the $2N$ normal contractions $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ for which there exists a uniformly compressible JIA (UCJIA) as *uniformly jointly compressible* (UJC).

Lemma 4.5 (Local connectivity of UJC matrix sets)**.** *Given $\varepsilon > 0$, there is $\delta > 0$ such that for any two N-sets of UJC pairwise commuting normal contractions $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ in $\mathcal{M}_\infty$ such that $\|X_j - Y_j\| \leq \delta$ for each $1 \leq j \leq N$, we will have that there are N toral matrix links $\mathbf{X}^1, \ldots, \mathbf{X}^N \in C([0,1], \mathcal{M}_\infty)$ that solve the interpolation problem $X_j \rightsquigarrow_{\varepsilon,\mathcal{N}(n)(\mathbb{D}^2)} Y_j$, for each $1 \leq j \leq N$.*

Proof. Since the N-sets of pairwise commuting normal contractions $X_1, \ldots, X_N$ and $Y_1, \ldots, Y_N$ are UJC, we have that, given $0 < \delta \leq \nu \leq \varepsilon/2 < 1$, there are a normal contraction $\hat{X} \in \mathcal{M}_\infty$ which commutes with each X_j together with a UCJIA $\hat{\Psi} = \mathrm{Ad}[W]$ for some $W \in \mathbb{U}(\mathcal{M}_\infty)$ and a unitary $\hat{W} \in \hat{\Psi}(C^*(\hat{X}))'$ such that

$$\|\mathbf{1} - \hat{W}^*W\| = \|W - \hat{W}\| \leq \nu < 1. \tag{4.7}$$

Let us set $Z := \hat{W}^*W$. As a consequence of (4.7), we will have that there is a Hermitian matrix $-\mathbf{1} \leq H_Z \leq \mathbf{1}$ in $\mathcal{M}_\infty$ such that $e^{\pi i H_Z} = Z$. By using (4.7) again, we see that we can now use the curved paths $\breve{\mathbf{X}}^j := \mathrm{Ad}[e^{\pi i t H_Z}](X_j)$ to solve the problems $X_j \rightsquigarrow_{\varepsilon/2,\mathcal{N}(\mathcal{M}_\infty)(\mathbb{D}^2)} \hat{\Psi}(X_j)$, and then we can solve the problems $\hat{\Psi}(X_j) \rightsquigarrow_{\nu,\mathcal{N}(\mathcal{M}_\infty)(\mathbb{D}^2)} Y_j$ by using the flat paths $\bar{\mathbf{X}}^j := (1-t)\hat{\Psi}(X_j) + tY_j$. We can construct the solvent toral matrix links by setting $\mathbf{X}^j := \breve{\mathbf{X}}^j \circledast \bar{\mathbf{X}}^j$ for each $1 \leq j \leq N$. This completes the proof. $\square$

5. Hints and future directions

The detection matrix representations of universal C^*-algebras that can be connected uniformly via piecewise analytic paths induce interesting problems which are topological/K-theoretical and computational in nature. Motivated by the C^0-connectivity technique, we are interested in the application of Theorem 4.2, Corollary 4.2, and Lemma 4.5 to the study of the question: Is $C^*\langle \mathbb{F}_2 \times \mathbb{F}_2 \rangle$ residually finite-dimensional? (This is equivalent to *Connes's embedding problem*.)

A better understanding of the geometric and approximate combinatorial nature of toroidal matrix links would provide a mutually beneficial interaction between matrix flows in the sense of Brockett [4] and Chu [8], topologically controlled linear algebra in the sense of Freedman and Press [13], and spectral refinement in the sense of [1]. These connections seem promising for the development and analysis of novel generic numerical methods to study and compute approximate solutions to systems of polynomial equations (in the sense of [9]) that involve large scale matrices. Using a similar approach, we plan to use Theorem 4.2 and Lemma 4.5 to answer some questions in topologically controlled linear algebra in the sense of [13], raised by Freedman. The connections between toroidal matrix links and refinement subroutines (in the sense of [1]) for approximate joint diagonalization algorithms (in the sense of [17], [25], and [29]) will be further studied in future work.

Some generalizations of Theorem 4.2 and particular applications of Lemma 4.5 to the study of matrix equations on words (in the sense of [15] and [16]) will also be the subject of future study. In particular, the combination of toroidal matrix links with some matrix lifting techniques along the same lines of the proof of Theorem 4.2 combined with Lemma 4.5 also seems promising with regard to the solvability of some conjectures studied numerically in [24].

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