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REDUCING SUBSPACES FOR A CLASS OF NONANALYTIC TOEPLITZ OPERATORS

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Abstract. In this paper, we give a uniform characterization for the reducing subspaces for T_φ with the symbol $\varphi(z) = z^k + \bar{z}^l$ $(k, l \in \mathbb{Z}_+^2)$ on the Bergman spaces over the bidisk, including the known cases that $\varphi(z_1, z_2) = z_1^N z_2^M$ and $\varphi(z_1, z_2) = z_1^N + \bar{z}_2^M$ with $N, M \in \mathbb{Z}_+$. Meanwhile, the reducing subspaces for $T_{z^N+\bar{z}^M}$ $(N, M \in \mathbb{Z}_+)$ on the Bergman space over the unit disk are also described. Finally, we state these results in terms of the commutant algebra $\mathcal{V}^*(\varphi)$.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$, and let $\mathbb{D}^d$ be the Cartesian product of d copies of $\mathbb{D}$. Let $\mathbb{Z}$ denote the set of all integers, let $\mathbb{Z}_+$ denote the set of all nonnegative integers, let $\mathbb{Z}^d$ denote the set of all $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\alpha_i \in \mathbb{Z}$, and let $\mathbb{Z}_+^d$ denote the set of all $\alpha \in \mathbb{Z}^d$ with $\alpha_i \in \mathbb{Z}_+$ for $1 \le i \le d$. If $z = (z_1, z_2, \ldots, z_n) \in \mathbb{D}^d$ and $\alpha \in \mathbb{Z}_+^d$, then we write

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d}.$$

The Bergman space $L_a^2(\mathbb{D}^d)$ is a Hilbert space consisting of all holomorphic functions over $\mathbb{D}^d$, which are square-integrable with respect to the normalized volume measure $dA(z) = dA(z_1)\, dA(z_2) \cdots dA(z_d)$. The inner product in $L_a^2(\mathbb{D}^d)$

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is denoted by $\langle f,g\rangle = \int_{\mathbb{D}^d} f\overline{g}\,dA(z)$. Given an essentially bounded measurable function ϕ on the polydisk, the Toeplitz operator with symbol ϕ is defined by

$$T_\phi f = P(\phi f), \quad \forall f \in L_a^2(\mathbb{D}^d),$$

where P is the orthogonal projection from $L^2(\mathbb{D}^2)$ onto $L_a^2(\mathbb{D}^2)$.

Let S be a bounded linear operator on a Hilbert space $\mathcal{H}$. A closed subspace $\mathcal{M}$ is said to be a reducing subspace for S if $S\mathcal{M} \subseteq \mathcal{M}$ and $S\mathcal{M}^\perp \subseteq \mathcal{M}^\perp$. Or equivalently, $\mathcal{M}$ is a reducing subspace for S if and only if $SP_\mathcal{M} = P_\mathcal{M}S$, where $P_\mathcal{M}$ is the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$. In addition, $\mathcal{M}$ is called *minimal* if there is no nonzero reducing subspace $\mathcal{N}$ which is contained in $\mathcal{M}$ properly. The operator S is said to be completely reducible if its lattice of reducing subspaces has no nonzero minimal elements (see [9]).

For every $\phi \in L^\infty(\mathbb{D}^d)$, denote by $\mathcal{W}^*(\phi)$ the von Neumann algebra generated by T_ϕ, and let $\mathcal{V}^*(\phi) = \mathcal{W}^*(\phi)'$ be the commutant algebra. As is given in [3], $\mathcal{V}^*(\phi)$ is a von Neumann algebra and is the norm-closed linear span of its projections. On the other side, the range of projections in $\mathcal{V}^*(\phi)$ and the reducing subspaces for T_ϕ are in one-to-one correspondence. Therefore, in some sense, determining the structure of the reducing subspaces for T_ϕ is equivalent to studying the structure of the commutant algebra $\mathcal{V}^*(\phi)$.

The study of the commutant began in earlier research on analytic Toeplitz operators on the Hardy space of the unit disk, especially work in the 1970s by Deddens and Wong [4], Thomson [22]–[24], and Cowen [2]. In particular, one of their main results is that if the inner factor of $f - f(c)$ is a finite Blaschke product for some c in the disk, then there is a Blaschke product B such that $\{T_f\}' = \{T_B\}'$. This result is usually referred to as the *Cowen–Thomson theorem*. Furthermore, it also holds on the Bergman space, and a detailed proof is given by Guo and Huang [12]. These indicate that the Toeplitz operators induced by finite Blaschke products play an important role in studying the structure of reducing subspaces for the analytic Toeplitz operators. In recent years, a lot of nice and deep work on the structures of reducing subspaces for Toeplitz operators with finite Blaschke products symbols has been done on the Bergman space over the unit disk in [6], [7], [11], [13], [19], [21], [20], [26].

For higher-dimensional domains, studies on reducing-subspace problems began with some special monomial symbols. The second author and Zhou [16] completely characterized the structure of the reducing subspaces for $T_{z_1^k z_2^k}$ on the weighted Bergman space over $\mathbb{D}^2$. The second author and the third author [18] found all the minimal reducing subspaces for $T_{z_i^k z_j^l}$ $(k \neq l, i \neq j)$ on $L_a^2(\mathbb{D}^d, dA_\alpha(z_1)\cdots dA_\alpha(z_d))$, where $dA_\alpha(z_i) = (1+\alpha)(1-|z_i|^2)^\alpha\, dA(z_i)$ for $z = (z_1,\ldots,z_d) \in \mathbb{D}^d$ and $\alpha > -1$, and showed that the unweighted case $\alpha = 0$ has more minimal reducing subspaces than the weighted case $\alpha \neq 0$. Guo and Huang [12] generalized these results to T_{z^a} with $a \in \mathbb{Z}_+^d$ on multi-dimensional separable Hilbert spaces by a different approach and gave the structure of $\mathcal{V}^*(z^a)$. Furthermore, Gu [10] characterized the reducing subspaces of weighted shifts with operator weights as wandering invariant subspaces of the shifts with additional structures, and pointed out that the operators $T_{z_1^N}$ and $T_{z_1^N z_2^M}$ on $L_a^2(\mathbb{D}^2, dA_\alpha(z_1)\,dA_\alpha(z_2))$ are unitarily equivalent to weighted shifts with invertible diagonal operator weights. For analytic polynomial $\varphi(z) = \alpha z_1^k + \beta z_2^l$, the reducing subspaces for T_φ and the structure of $\mathcal{V}^*(\varphi)$ are investigated on $L_a^2(\mathbb{D}^2)$ in [8] and [25]. Additionally, on the weighted Dirichlet space over the bidisk, Lin, Hu, and the second author [15] obtained partial results about the reducing subspaces for analytic Toeplitz operators with monomial symbols.

Recall that the study of the reducing-subspace problems of the nonanalytic Toeplitz operators over the bidisk began with Albaseer, the second author, and the third author in [1], in which the structure of the reducing subspaces for $T_{z_1^k \bar{z}_2^l}$ on $L_a^2(\mathbb{D}^2)$ was solved. Stimulated by [8] and [25], we have considered the structure of $\mathcal{V}^*(\alpha z_1^k + \beta \overline{z_2}^1)$ with $\alpha\beta \neq 0$ in [5]. It is proved that $L_{a,b}$ ($a, b \in \mathbb{Z}_+$, $a \leq k-1$, and $b \leq l-1$) are exactly all the minimal reducing subspaces for $T_{\alpha z_1^k + \beta \overline{z_2}^1}$, where $L_{a,b} = \overline{\operatorname{span}}\{z_1^{a+nk} z_2^{b+ml} | n, m \in \mathbb{Z}_+\}$. Furthermore, $\mathcal{V}^*(\varphi)$ is $*$-isomorphic to $\bigoplus_{i=1}^{kl} \mathbb{C}$, and then $\mathcal{V}^*(\varphi)$ is Abelian.

In this paper, we keep on considering the reducing subspaces for T_φ with $\varphi(z) = z^k + \bar{z}^l$ ($k, l \in \mathbb{Z}_+^2$) over the bidisk. Since $T_\varphi = T_{z^k} + T_{z^l}^*$, then a common reducing subspace for T_{z^k} and T_{z^l} is clearly a reducing subspace for T_φ. We will show that there are no other reducing subspaces other than the common reducing subspaces for the case $k \neq l$. Meanwhile, we also describe the reducing subspaces on the unit disk. The main conclusions imply the related results in [5], [16], [18], and [19].

This article is organized as follows. In Section 2, some preliminaries are presented. In Section 3, under the assumptions $k \neq l$ and $k_i^2 + l_i^2 \neq 0$ ($i = 1, 2$), we determine all the minimal reducing subspaces for $T_{z^k + \bar{z}^l}$ on the Bergman space over the bidisk. Moreover, we show that $T_{z^k + \bar{z}^k}$ is completely reducible on $L_a^2(\mathbb{D}^d)$ ($d \in \mathbb{Z}_+$). In Section 4, we describe the reducing subspaces for $T_{z^k + \bar{z}^l}$ with nonnegative integers k, l on the unit disk. Along with this result, the structure of reducing subspaces for $T_{z_i^{k_i} + \bar{z}_i^{l_i}}$ ($i = 1, 2$) over the bidisk is also characterized. In Section 5, using the conclusion in Guo and Huang [12], we obtain the structure of $\mathcal{V}^*(z^k + \bar{z}^l)$ on $L_a^2(\mathbb{D}^d)$ for $d = 1, 2$.

2. Preliminaries

Denote the partial order $\succeq$ in $\mathbb{Z}_+^2$ as: $a \succeq b$ if $a_1 \geq b_1$ and $a_2 \geq b_2$. Otherwise, we write $a \not\succeq b$. Let $\varphi(z) = z^k + \bar{z}^l$ with $k, l \in \mathbb{Z}_+^2$. Put

$$T = T_\varphi^* T_\varphi - T_\varphi T_\varphi^*.$$

Let

$$\Omega_1 = \{n \in \mathbb{Z}_+^2 : n \not\succeq k, n \not\succeq l\}, \qquad \Omega_2 = \{n \in \mathbb{Z}_+^2 : n \succeq k, n \not\succeq l\},$$
$$\Omega_3 = \{n \in \mathbb{Z}_+^2 : n \not\succeq k, n \succeq l\}, \qquad \Omega_4 = \{n \in \mathbb{Z}_+^2 : n \succeq k, n \succeq l\}.$$

Note that if $l \succeq k$ (or $k \succeq l$), then Ω_3 (or Ω_2) makes no sense. Write

$$\gamma_i = \|z^i\|^2 = \frac{1}{(1+i_1)(1+i_2)}, \quad \forall i \in \mathbb{Z}_+^2.$$

By an easy computation, we get

$$T_\varphi z^n = \begin{cases} z^{n+k}, & n \not\succeq l, \\ z^{n+k} + \frac{\gamma_n}{\gamma_{n-l}} z^{n-l}, & n \succeq l. \end{cases}$$

Similarly, we have

$$T_\varphi^* z^n = \begin{cases} z^{n+l}, & n \not\succeq k, \\ z^{n+l} + \frac{\gamma_n}{\gamma_{n-k}} z^{n-k}, & n \succeq k. \end{cases}$$

Furthermore,

$$Tz^n = \omega_n z^n,$$

where

$$\omega_n = \begin{cases} \prod_{i=1}^2 \frac{n_i+1}{n_i+k_i+1} - \prod_{i=1}^2 \frac{n_i+1}{n_i+l_i+1}, & n \in \Omega_1, \\ \prod_{i=1}^2 \frac{n_i+1}{n_i+k_i+1} - \prod_{i=1}^2 \frac{n_i+1}{n_i+l_i+1} - \prod_{i=1}^2 \frac{n_i-k_i+1}{n_i+1}, & n \in \Omega_2, \\ \prod_{i=1}^2 \frac{n_i+1}{n_i+k_i+1} - \prod_{i=1}^2 \frac{n_i+1}{n_i+l_i+1} + \prod_{i=1}^2 \frac{n_i-l_i+1}{n_i+1}, & n \in \Omega_3, \\ \prod_{i=1}^2 \frac{n_i+1}{n_i+k_i+1} - \prod_{i=1}^2 \frac{n_i+1}{n_i+l_i+1} & \\ \quad + \prod_{i=1}^2 \frac{n_i-l_i+1}{n_i+1} - \prod_{i=1}^2 \frac{n_i-k_i+1}{n_i+1}, & n \in \Omega_4. \end{cases} \tag{2.1}$$

For every $n \in \mathbb{Z}_+^2$, let

$$\begin{aligned} Q_n(\lambda) = & \prod_{i=1}^2 \frac{n_i + 1 + \lambda(k_i + l_i)}{n_i + k_i + 1 + \lambda(k_i + l_i)} - \prod_{i=1}^2 \frac{n_i + 1 + \lambda(k_i + l_i)}{n_i + l_i + 1 + \lambda(k_i + l_i)} \\ & + \prod_{i=1}^2 \frac{n_i - l_i + 1 + \lambda(k_i + l_i)}{n_i + 1 + \lambda(k_i + l_i)} - \prod_{i=1}^2 \frac{n_i - k_i + 1 + \lambda(k_i + l_i)}{n_i + 1 + \lambda(k_i + l_i)} \end{aligned} \tag{2.2}$$

defined on $\mathbb{Z}_+$. Clearly, $\lim_{\lambda\to+\infty} Q_n(\lambda) = 0$.

Let $\mathcal{V}^*(\varphi)$ denote the commutant algebra of the von Neumann algebra generated by T_φ and $A \in \mathcal{V}^*(\varphi)$. Denote by $H_\beta^0 = \overline{\text{span}}\{z^m : m \neq \beta + p(k+l), p \in \mathbb{Z}, \beta + p(k+l) \succeq (0,0)\}$. Because $AT = TA$,

$$Az^\alpha = \sum_{\beta \in \mathbb{Z}_+^2} c_\beta z^\beta \quad \text{with } \omega_\beta = \omega_\alpha, \forall \alpha \in \mathbb{Z}_+^2.$$

To determine the expression of Az^α, we provide some useful lemmas. In the following, denote by $\mathbb{N}$ the set of all the positive integers.

Lemma 2.1. *Let $\alpha, \beta \in \mathbb{Z}_+^2$, let $\alpha \not\succeq k + l$, and let $A \in \mathcal{V}^*(\varphi)$. If $\Delta_1 = \{p \in \mathbb{Z}_+ : \langle Az^\alpha, z^{\beta+p(k+l)}\rangle \neq 0\}$ is not empty and finite, then $\omega_{\alpha+h(k+l)} = \omega_{\beta+(p_0+h)(k+l)}$ for every $h \in \mathbb{Z}_+$, where $p_0 = \max \Delta_1$.*

Proof. Since Δ_1 is finite and $p_0 = \max\{p : p \in \Delta_1\}$, we may set

$$Az^{\alpha} = c_1 z^{\beta+p_0(k+l)} + p_1(z) + h_1(z),$$

where $c_1 \in \mathbb{C}$, $c_1 \neq 0$, $p_1(z) \in \overline{\text{span}}\{z^{\beta+p(k+l)} : p < p_0, \beta + p(k+l) \succeq (0,0)\}$ and $h_1(z) \in H^0_{\beta}$. Together with $AT^*_{\varphi}T_{\varphi}z^{\alpha} = T^*_{\varphi}T_{\varphi}Az^{\alpha}$, we obtain

$$A\Big(z^{\alpha+k+l} + \frac{\gamma_{\alpha+k}}{\gamma_{\alpha}}z^{\alpha} + c\frac{\gamma_{\alpha}}{\gamma_{\alpha-l}}z^{\alpha}\Big) = c_1 z^{\beta+(p_0+1)(k+l)} + P_1(z) + H_1(z),$$

where the constant $c = 1$ for $\alpha \succeq l$; $c = 0$ for $\alpha \not\succeq l$; $P_1(z) \in \overline{\text{span}}\{z^{\beta+p(k+l)} : p < p_0 + 1, \beta + p(k+l) \succeq (0,0)\}$; $H_1(z) \in H^0_{\beta}$. By the maximality of p_0, we have $\langle Az^{\alpha}, z^{\beta+(p_0+1)(k+l)}\rangle = 0$. So $\max\{p \in \mathbb{Z}_+ : \langle Az^{\alpha+k+l}, z^{\beta+p(k+l)}\rangle \neq 0\} = p_0 + 1$. It follows that $\omega_{\alpha+k+l} = \omega_{\beta+(p_0+1)(k+l)}$.

Given that $N \in \mathbb{N}$, suppose that $\max\{p \in \mathbb{Z}_+ : \langle Az^{\alpha+i(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\} = p_0 + i$ for every $i \leq N$. Set $Az^{\alpha+N(k+l)} = c_N z^{\beta+(p_0+N)(k+l)} + p_N(z) + h_N(z)$, where $c_N \neq 0$, $p_N(z) \in \overline{\text{span}}\{z^{\beta+p(k+l)} : p < p_0 + N, \beta + p(k+l) \succeq (0,0)\}$, and $h_N(z) \in H^0_{\beta}$. As above, $AT^*_{\varphi}T_{\varphi}z^{\alpha+N(k+l)} = T^*_{\varphi}T_{\varphi}Az^{\alpha+N(k+l)}$ implies that

$$\begin{aligned} &A(z^{\alpha+(N+1)(k+l)} + \lambda z^{\alpha+N(k+l)} + \mu z^{\alpha+(N-1)(k+l)}) \\ &\quad = c_N z^{\beta+(p_0+N+1)(k+l)} + P_N(z) + H_N(z), \end{aligned}$$

where $\lambda, \mu > 0$, $P_N(z) \in \overline{\text{span}}\{z^{\beta+p(k+l)} : p < p_0+N+1, \beta+p(k+l) \succeq (0,0)\}$, and $H_N(z) \in H^0_{\beta}$. Therefore, here $\max\{p \in \mathbb{Z}_+ : \langle Az^{\alpha+(N+1)(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\} = p_0 + N + 1$ and $\omega_{\alpha+(N+1)(k+l)} = \omega_{\beta+(p_0+N+1)(k+l)}$. By induction, we get the desired result. □

Lemma 2.2. *Let $\alpha, \beta \in \mathbb{Z}^2_+$, let $\alpha \not\succeq k + l$, and let $A \in \mathcal{V}^*(\varphi)$. If $Q_{\alpha}(\lambda) \not\equiv 0$ and $\Delta_1 = \{p \in \mathbb{Z}_+ : \langle Az^{\alpha}, z^{\beta+p(k+l)}\rangle \neq 0\}$ is finite, then* $\text{Card}\{p \in \mathbb{Z}_+ : \langle Az^{\alpha}, z^{\beta+p(k+l)}\rangle \neq 0\} \leq 1$.

Proof. Without loss of generality, we may assume that $\beta \not\succeq k + l$ and that Δ_1 is not empty. Since $Q_{\alpha}(\lambda) \not\equiv 0$, $Q_{\alpha}(\lambda) = c$ ($c \in \mathbb{R}$) has finite roots. It follows that the set

$$\big\{h \in \mathbb{Z}_+ : \langle Az^{\alpha+h(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\big\} \subseteq \{h \in \mathbb{Z}_+ : \omega_{\alpha+h(k+l)} = \omega_{\beta+p(k+l)}\}$$

is a finite set for every $p \in \mathbb{Z}_+$. Thus,

$$\Delta_2 = \bigcup_{0 \leq p \leq p_0} \big\{h \in \mathbb{Z}_+ : \langle Az^{\alpha+h(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\big\}$$

is finite, where $p_0 = \max \Delta_1$. Obviously, $\Delta_2 \neq \varnothing$ since $0 \in \Delta_2$.

Let $h_0 = \max\{h : h \in \Delta_2\}$. We will prove that

$$\omega_{\alpha+(h_0+h)(k+l)} = \omega_{\beta+(p_0+h)(k+l)}, \quad \forall h \in \mathbb{Z}_+. \tag{2.3}$$

Since $h_0 + 1 \notin \Delta_2$, set

$$Az^{\alpha+(h_0+1)(k+l)} = d_1 z^{\beta+(p_0+1)(k+l)} + f_1(z) + g_1(z),$$

where $f_1 \in \overline{\text{span}}\{z^{\beta+h(k+l)} : p_0+2 \le h \le p_0+h_0+1\}$ and $g_1 \in H^0_\beta$. Thus, $AT^*_\varphi T_\varphi z^{\alpha+(h_0+1)(k+l)} = T^*_\varphi T_\varphi A z^{\alpha+(h_0+1)(k+l)}$ gives

$$\begin{aligned} &A(z^{\alpha+(h_0+2)(k+l)} + \eta z^{\alpha+(h_0+1)(k+l)} + \rho z^{\alpha+h_0(k+l)}) \\ &\quad = d_1 \frac{\gamma_{\beta+(p_0+1)(k+l)}}{\gamma_{\beta+p_0(k+l)}} z^{\beta+p_0(k+l)} + F_1(z) + G_1(z), \end{aligned} \tag{2.4}$$

where $\eta, \rho > 0$, $F_1 \in \overline{\text{span}}\{z^{\beta+h(k+l)} : p_0+1 \le h \le p_0+h_0+2\}$, and $G_1 \in H^0_\beta$. Since $h_0+1, h_0+2 \notin \Delta_2$, we have

$$\rho\langle Az^{\alpha+h_0(k+l)}, z^{\beta+p_0(k+l)}\rangle = d_1\gamma_{\beta+(p_0+1)(k+l)}.$$

By the definition of h_0, there exists some $p \in [0, p_0]$ such that $\langle Az^{\alpha+h_0(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0$. Therefore, (2.4) shows that $d_1 \neq 0$. This means that $\min\{p \in \mathbb{Z}_+ : \langle Az^{\alpha+(h_0+1)(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\} = p_0+1$. Moreover, there is

$$\langle Az^{\alpha+(h_0+h)(k+l)}, z^{\beta+p_0(k+l)}\rangle = 0, \quad \forall h \in \mathbb{N}.$$

Assume that $N \in \mathbb{Z}_+$. For $i = N$, suppose that

$$\min\big\{p \in \mathbb{Z}_+ : \langle Az^{\alpha+(h_0+i+1)(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\big\} = p_0+i+1, \tag{2.5}$$

and that

$$\langle Az^{\alpha+(h_0+i+h)(k+l)}, z^{\beta+(p_0+i)(k+l)}\rangle = 0, \quad \forall h \in \mathbb{N}. \tag{2.6}$$

We may set

$$Az^{\alpha+(h_0+i+1+h)(k+l)} = d_{i+1+h} z^{\beta+(p_0+i+1)(k+l)} + f_{i+1+h}(z) + g_{i+1+h}(z),$$

where $f_{i+1+h} \in \overline{\text{span}}\{z^{\beta+h(k+l)} : p_0+i+2 \le h \le p_0+h_0+i+1+h\}$ and $g_{i+1+h} \in H^0_\beta$. Since $AT^*_\varphi T_\varphi = T^*_\varphi T_\varphi A$, a direct computation gives

$$\begin{aligned} &A(z^{\alpha+(h_0+i+2+h)(k+l)} + \eta' z^{\alpha+(h_0+i+1+h)(k+l)} + \rho' z^{\alpha+(h_0+i+h)(k+l)}) \\ &\quad = d_{i+1+h} \frac{\gamma_{\beta+(p_0+i+1)(k+l)}}{\gamma_{\beta+(p_0+i)(k+l)}} z^{\beta+(p_0+i)(k+l)} + F_{i+1+h}(z) + G_{i+1+h}(z), \end{aligned}$$

where $\eta', \rho' > 0$, $F_{i+1+h} \in \overline{\text{span}}\{z^{\beta+h(k+l)} : p_0+i+1 \le h \le p_0+h_0+i+h+2\}$, and $G_{i+1+h} \in H^0_\beta$. Note that (2.6) shows that $d_{i+1+h} = 0$ for $h \in \mathbb{N}$. In particular, for $h = 1$, let $Az^{\alpha+(h_0+i+2)(k+l)} = dz^{\beta+(p_0+i+2)(k+l)} + f(z) + g(z)$, where we have $f \in \overline{\text{span}}\{z^{\beta+h(k+l)} : p_0+i+3 \le h \le p_0+h_0+i+2\}$ and $g \in H^0_\beta$. Then

$$\begin{aligned} &A(z^{\alpha+(h_0+i+3)(k+l)} + \eta'' z^{\alpha+(h_0+i+2)(k+l)} + \rho'' z^{\alpha+(h_0+i+1)(k+l)}) \\ &\quad = d \frac{\gamma_{\beta+(p_0+i+2)(k+l)}}{\gamma_{\beta+(p_0+i+1)(k+l)}} z^{\beta+(p_0+i+1)(k+l)} + F(z) + G(z), \end{aligned}$$

where $F \in \overline{\text{span}}\{z^{\beta+h(k+l)} : p_0+i+2 \le h \le p_0+h_0+i+3\}$ and $G \in H^0_\beta$. Now, by equation (2.5), we have that $d \neq 0$. That is, the equalities (2.5) and (2.6) also hold for $i = N+1$. By induction, we get

$$\min\big\{p \in \mathbb{Z}_+ : \langle Az^{\alpha+(h_0+h)(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\big\} = p_0+h, \quad \forall h \in \mathbb{Z}_+. \tag{2.7}$$

Therefore, equation (2.3) holds.

Together with the equality $\omega_{\alpha+h(k+l)} = \omega_{\beta+(p_0+h)(k+l)}$, which comes from Lemma 2.1, we have

$$\omega_{\alpha+h(k+l)} = \omega_{\alpha+(h+h_0)(k+l)}, \quad \forall h \in \mathbb{Z}_+.$$

If $h_0 \in \mathbb{N}$, then $\omega_{\alpha+h_0(k+l)} = \omega_{\alpha+nh_0(k+l)} = \lim_{n\to\infty} \omega_{\alpha+n(k+l)} = 0$, which contradicts the assumption that $Q_\alpha(\lambda) \not\equiv 0$. So $h_0 = 0$, and the equation (2.7) implies that $\min\{p \in \mathbb{Z}_+ : \langle Az^\alpha, z^{\beta+p(k+l)}\rangle \neq 0\} = p_0 = \max\{p \in \mathbb{Z}_+ : \langle Az^\alpha, z^{\beta+p(k+l)}\rangle \neq 0\}$. Hence, $\mathrm{Card}\{p \in \mathbb{Z}_+ : \langle Az^\alpha, z^{\beta+p(k+l)}\rangle \neq 0\} \leq 1$. □

For convenience, we set

$$A_\alpha \triangleq (l_2 - k_2)(\alpha_1 + 1) + (l_1 - k_1)(\alpha_2 + 1), \quad \forall \alpha \in \mathbb{Z}_+^2.$$

Lemma 2.3. *Let $\alpha \in \mathbb{Z}_+^2$ and let $k \neq l$. Then $Q_\alpha(\lambda) \equiv 0$ if and only if $l_1 l_2 = k_1 k_2$ and $A_\alpha = 0$.*

Proof. By the definition of $Q_\alpha(\lambda)$, one can easily see that $Q_\alpha(\lambda) \equiv 0$ if and only if

$$\begin{aligned}
&\prod_{i=1}^{2}\big(\alpha_i + \lambda(k_i + l_i) + 1\big)^2 \big[(2\lambda + 1)(l_1 l_2 - k_1 k_2) + A_\alpha\big] \\
&\quad \equiv \prod_{i=1}^{2}\big(\alpha_i + \lambda(k_i + l_i) + k_i + 1\big)\big(\alpha_i + \lambda(k_i + l_i) + l_i + 1\big) \\
&\quad\quad \times \big[(2\lambda - 1)(l_1 l_2 - k_1 k_2) + A_\alpha\big].
\end{aligned} \tag{2.8}$$

By direct calculations, the coefficient of λ^5 is zero, and the coefficient of λ^4 is $2(l_1 l_2 - k_1 k_1)(k_1 + l_1)^2(k_2 + l_2)^2$. If $Q_\alpha(\lambda) \equiv 0$, then $l_1 l_2 = k_1 k_2$. Furthermore, we get

$$\prod_{i=1}^{2}(\alpha_i + 1)^2 A_\alpha = \prod_{i=1}^{2}(\alpha_i + k_i + 1)(\alpha_i + l_i + 1) A_\alpha,$$

which indicates that $A_\alpha = 0$. On the other side, by (2.8), $l_1 l_2 = k_1 k_2$ and $A_\alpha = 0$ imply immediately that $Q_\alpha(\lambda) \equiv 0$. □

Lemma 2.4. *Let $k \neq l$ and let $A \in \mathcal{V}^*(\varphi)$. If $\alpha \in \Omega_1$ such that $Q_\alpha(\lambda) \not\equiv 0$, then $\langle Az^\alpha, z^\beta\rangle = 0$ for every $\beta \in \Omega_4$.*

Proof. Suppose that there exists $\beta \in \Omega_4$ such that $\langle Az^\alpha, z^\beta\rangle \neq 0$. Then $\omega_\alpha = \omega_\beta$. We claim that $Q_\beta(\lambda) \not\equiv 0$. In fact, if we assume the contrary, that is, that $Q_\beta(\lambda) \equiv 0$, then Lemma 2.3 shows that $l_1 l_2 = k_1 k_2$. From (2.1) and (2.2), we have

$$\omega_\alpha = \omega_\beta = Q_\beta(0) = 0.$$

So $(\alpha_1 + k_1 + 1)(\alpha_2 + k_2 + 1) = (\alpha_1 + l_1 + 1)(\alpha_2 + l_2 + 1)$; that is, $A_\alpha = k_1 k_2 - l_1 l_2 = 0$. By Lemma 2.3 again, we get $Q_\alpha(\lambda) \equiv 0$, which is a contradiction. Therefore, we have that $\{p \in \mathbb{Z}_+ : \langle Az^\alpha, z^{\beta+p(k+l)}\rangle \neq 0\}$ is a finite set, which contains 0. Lemma 2.2 implies that $\{p \in \mathbb{Z}_+ : \langle Az^\alpha, z^{\beta+p(k+l)}\rangle \neq 0\} = \{0\}$. That is, $p_0 = 0$.

If $l_1 l_2 = k_1 k_2$, then we claim that $A_\alpha = 0$. In fact, Lemma 2.1 shows that $\omega_{\alpha+h(k+l)} = \omega_{\beta+h(k+l)}$, $\forall h \in \mathbb{Z}_+$. Therefore, $Q_\alpha(\lambda) \equiv Q_\beta(\lambda)$. Thus,

$$\omega_\alpha = \omega_\beta = Q_\beta(0) = Q_\alpha(0).$$

Since $\alpha \in \Omega_1$, we have $(\alpha_1 - l_1 + 1)(\alpha_2 - l_2 + 1) = (\alpha_1 - k_1 + 1)(\alpha_2 - k_2 + 1)$; that is, $A_\alpha = l_1 l_2 - k_1 k_2 = 0$. So $Q_\alpha(\lambda) \equiv 0$, which is a contradiction.

If $l_1 l_2 \neq k_1 k_2$, then Lemma 2.3 shows that $Q_n(\lambda) \not\equiv 0$ for any $n \in \mathbb{Z}_+^2$. Set

$$Az^\alpha = c_\beta z^\beta + q_1(z),$$

where $c_\beta \neq 0$ and $q_1(z) \in H_\beta^0$. Further, $A \in \mathcal{V}^*(\varphi)$ gives

$$Az^{\alpha+k} = T_\varphi A z^\alpha = c_\beta z^{\beta+k} + c_\beta \frac{\gamma_\beta}{\gamma_{\beta-l}} z^{\beta-l} + q_2(z),$$

where $q_2(z) \perp \{z^{\beta+k}, z^{\beta-l}\}$. Because $Q_{\alpha+k}(\lambda) \not\equiv 0$ and $\{p \in \mathbb{Z}_+ : \langle Az^{\alpha+k}, z^{\beta-l+p(k+l)}\rangle \neq 0\} = \{0,1\}$, again by Lemma 2.2, we get a contradiction. Thus we finish the proof. $\square$

Lemma 2.5. *Let $k \neq l$ and let $k_i^2 + l_i^2 \neq 0$ $(i = 1,2)$. Given that $\alpha, \beta \in \mathbb{Z}_+^2$, $\alpha \neq \beta$ such that $Q_\alpha(\lambda) \equiv Q_\beta(\lambda)$. Then if $Q_{\alpha+l}(\lambda) \equiv Q_{\beta+l}(\lambda)$, we have $Q_{\alpha+l}(\lambda) \not\equiv 0$, with $Q_\alpha(\lambda) \not\equiv 0$.*

Proof. If $k_1 k_2 \neq l_1 l_2$, then $Q_n(\lambda) \not\equiv 0$, $\forall n \in \mathbb{Z}_+$. Therefore, we obtain the desired result.

If $k_1 k_2 = l_1 l_2$, without loss of generality, we assume that $0 \leq k_1 < l_1$ and that $0 \leq l_2 < k_2$. Given $m, n \in \mathbb{Z}_+^2$, $m \neq n$. Let

$$\Delta_m \triangleq \bigcup_{i=1,2} \Big\{ \frac{m_i + 1}{k_i + l_i}, \frac{m_i + k_i + 1}{k_i + l_i}, \frac{m_i + l_i + 1}{k_i + l_i} \Big\},$$

$$\Delta_n \triangleq \bigcup_{i=1,2} \Big\{ \frac{n_i + 1}{k_i + l_i}, \frac{n_i + k_i + 1}{k_i + l_i}, \frac{n_i + l_i + 1}{k_i + l_i} \Big\}.$$

Clearly,

$$\lambda_{\min} \triangleq \min\Big\{ \frac{m_1 + 1}{k_1 + l_1}, \frac{m_2 + 1}{k_2 + l_2} \Big\} = \min \Delta_m,$$

$$\lambda_{\max} \triangleq \max\Big\{ \frac{m_1 + l_1 + 1}{k_1 + l_1}, \frac{m_2 + k_2 + 1}{k_2 + l_2} \Big\} = \max \Delta_m,$$

$$\mu_{\min} \triangleq \min\Big\{ \frac{n_1 + 1}{k_1 + l_1}, \frac{n_2 + 1}{k_2 + l_2} \Big\} = \min \Delta_n,$$

$$\mu_{\max} \triangleq \max\Big\{ \frac{n_1 + l_1 + 1}{k_1 + l_1}, \frac{n_2 + k_2 + 1}{k_2 + l_2} \Big\} = \max \Delta_n.$$

First, we claim that, if $Q_m(\lambda) \equiv Q_n(\lambda)$ and $Q_m(\lambda) \not\equiv 0$, then $\mu_{\max} = \lambda_{\max}$ and $\lambda_{\min} = \mu_{\min}$. In fact, $l_1 l_2 = k_1 k_2$ and $Q_m(\lambda) \equiv Q_n(\lambda)$ imply that

$$\begin{aligned} A_m \prod_{i=1}^{2} &\big(n_i + \lambda(k_i + l_i) + 1\big)\big(n_i + \lambda(k_i + l_i) + k_i + 1\big)\big(n_i + \lambda(k_i + l_i) + l_i + 1\big) \\ &\times \Big[\prod_{i=1}^{2} \big(m_i + \lambda(k_i + l_i) + 1\big)^2 - \prod_{i=1}^{2} \big(m_i + \lambda(k_i + l_i) + k_i + 1\big) \\ &\times \big(m_i + \lambda(k_i + l_i) + l_i + 1\big)\Big] \end{aligned}$$

$$
\begin{aligned}
&= A_n \prod_{i=1}^{2} \big(m_i + \lambda(k_i + l_i) + 1\big)\big(m_i + \lambda(k_i + l_i) + k_i + 1\big) \\
&\quad \times \big(m_i + \lambda(k_i + l_i) + l_i + 1\big) \\
&\quad \times \Big[\prod_{i=1}^{2} \big(n_i + \lambda(k_i + l_i) + 1\big)^2 - \prod_{i=1}^{2} \big(n_i + \lambda(k_i + l_i) + k_i + 1\big) \\
&\quad \times \big(n_i + \lambda(k_i + l_i) + l_i + 1\big)\Big]. \qquad (2.9)
\end{aligned}
$$

Since $Q_m(\lambda) \not\equiv 0$, there are $A_m \neq 0$ and $A_n \neq 0$. Put $\lambda = -\lambda_{\max}$ into (2.9), and then we have

$$
\prod_{i=1}^{2} \big(n_i - \lambda_{\max}(k_i + l_i) + 1\big)\big(n_i - \lambda_{\max}(k_i + l_i) + k_i + 1\big)\big(n_i - \lambda_{\max}(k_i + l_i) + l_i + 1\big) = 0.
$$

This gives $\lambda_{\max} \in \Delta_n$, and then $\lambda_{\max} \leq \mu_{\max}$. Similarly, putting $\lambda = -\mu_{\max}$ into (2.9), we have $\mu_{\max} \leq \lambda_{\max}$. Therefore, $\mu_{\max} = \lambda_{\max}$. By the same method, it is easy to get $\lambda_{\min} = \mu_{\min}$.

If $Q_\alpha(\lambda) \equiv 0$, then $l_1 l_2 = k_1 k_2$, $A_\alpha = A_\beta = 0$, and

$$
(l_1 - k_1)(\beta_2 - \alpha_2) + (l_2 - k_2)(\beta_1 - \alpha_1) = 0. \qquad (2.10)
$$

Thus $A_{\alpha+l} = A_{\beta+l} = (l_1 - k_1)(k_2 - l_2) \neq 0$. This indicates that $Q_{\alpha+l}(\lambda) \not\equiv 0$ and $Q_{\beta+l}(\lambda) \not\equiv 0$. Moreover, $\alpha_1 = \beta_1$ if and only if $\alpha_2 = \beta_2$. Without loss of generality, we may assume that $\alpha_1 < \beta_1$. Then $\alpha_2 \neq \beta_2$. The claim implies that

$$
\frac{\beta_2 + l_2 + 1}{k_2 + l_2} = \min\Big\{\frac{\alpha_1 + l_1 + 1}{k_1 + l_1}, \frac{\alpha_2 + l_2 + 1}{k_2 + l_2}\Big\}
$$

and that

$$
\frac{\alpha_2 + l_2 + k_2 + 1}{k_2 + l_2} = \max\Big\{\frac{\beta_1 + l_1 + k_1 + 1}{k_1 + l_1}, \frac{\beta_2 + l_2 + k_2 + 1}{k_2 + l_2}\Big\}.
$$

It follows that

$$
\frac{\beta_2 + l_2 + 1}{k_2 + l_2} = \frac{\alpha_1 + l_1 + 1}{k_1 + l_1}, \qquad \frac{\alpha_2 + l_2 + k_2 + 1}{k_2 + l_2} = \frac{\beta_1 + l_1 + k_1 + 1}{k_1 + l_1}.
$$

Therefore, $(k_1 + l_1)(\beta_2 - \alpha_2) + (k_2 + l_2)(\beta_1 - \alpha_1) = 0$. Together with (2.10), we obtain that $k_1 l_2 = k_2 l_1$. However, this is a contradiction to $k_1 k_2 = l_1 l_2$ and $k \neq l$. Hence, $Q_\alpha(\lambda) \not\equiv 0$.

If we suppose that $Q_{\alpha+l}(\lambda) \equiv 0$, then we similarly have $Q_\alpha(\lambda) \not\equiv 0$ and $Q_\beta(\lambda) \not\equiv 0$. Using the same method as above, we still get the contradiction. Therefore, $Q_{\alpha+l}(\lambda) \not\equiv 0$. $\square$

Remark 2.6. Because k and l are symmetric, we also have that if $Q_{\alpha+k}(\lambda) \equiv Q_{\beta+k}(\lambda)$, then $Q_{\alpha+k}(\lambda) \not\equiv 0$, $Q_\alpha(\lambda) \not\equiv 0$.

Lemma 2.7. *Let $k \neq l$, let $k_i^2 + l_i^2 \neq 0$ $(i = 1, 2)$, and let $A \in \mathcal{V}^*(\varphi)$. If $\alpha \in \Omega_1$ such that $Q_\alpha(\lambda) \not\equiv 0$, then $\langle Az^\alpha, z^\beta \rangle = 0$ for every $\beta \in \Omega_2 \cup \Omega_3$.*

Proof. Suppose that here exists $\beta \in \Omega_2$ such that $\langle Az^\alpha, z^\beta \rangle \neq 0$. By Lemma 2.4, we may set $Az^\alpha = c_\beta z^\beta + p(z)$, where $c_\beta \neq 0$ and $p(z) \in H^0_\beta$. By $T^*_\varphi Az^\alpha = AT^*_\varphi z^\alpha$, we get

$$Az^{\alpha+l} = c_\beta z^{\beta+l} + c_\beta \frac{\gamma_\beta}{\gamma_{\beta-k}} z^{\beta-k} + T^*_\varphi p(z). \tag{2.11}$$

Obviously, $E = \{p \in \mathbb{Z}_+ : \langle Az^{\alpha+l}, z^{\beta-k+p(k+l)} \rangle \neq 0\} = \{0,1\}$ is a finite set. And it is easy to see that $\{p \in \mathbb{Z}_+ : \langle Az^\alpha, z^{\beta+p(k+l)} \rangle \neq 0\} = \{0\}$ is also finite. Using Lemma 2.1, we have $\omega_{\alpha+l+h(k+l)} = \omega_{\beta+l+h(k+l)}$ and $\omega_{\alpha+h(k+l)} = \omega_{\beta+h(k+l)}$ for $h \in \mathbb{Z}_+$. Therefore, $Q_{\alpha+l}(\lambda) \equiv Q_{\beta+l}(\lambda)$ and $Q_\alpha(\lambda) \equiv Q_\beta(\lambda)$. It follows from Lemma 2.5 that $Q_{\alpha+l}(\lambda) \not\equiv 0$, and then Lemma 2.2 leads to $\operatorname{Card} E \leq 1$. This is a contradiction. Therefore, $\{\beta \in \Omega_2 : \langle Az^\alpha, z^\beta \rangle \neq 0\} = \varnothing$.

Substituting T^*_φ with T_φ, we get $\{\beta \in \Omega_3 : \langle Az^\alpha, z^\beta \rangle \neq 0\} = \varnothing$. So the desired result follows. □

In the following, we consider the case that $Q_\alpha(\lambda) \equiv 0$

Lemma 2.8. *Let $A \in \mathcal{V}^*(\varphi)$ and let $k \neq l$, $k_i^2 + l_i^2 \neq 0$ $(i = 1, 2)$. If $\alpha \in \Omega_1$ such that $Q_\alpha(\lambda) \equiv 0$, then $\Delta_1 = \{p \in \mathbb{Z} : \langle Az^\alpha, z^{\beta+p(k+l)} \rangle \neq 0, \beta + p(k+l) \succeq (0,0)\}$ is a finite set.*

Proof. Without loss of generality, suppose that $\beta \not\succeq k + l$. Suppose conversely that Δ_1 is an infinite set; then there exist $\{p_j : j \in \mathbb{N}\} \subseteq \Delta_1$ such that $p_j \to +\infty$ as $j \to +\infty$. Thus, $\omega_\alpha = \omega_{\beta+p_j(k+l)} = \lim_{i\to+\infty} \omega_{\beta+p_i(k+l)} = 0$, $\forall j \in \mathbb{N}$. Equally, $Q_\beta(\lambda) \equiv \omega_\alpha \equiv 0$. Lemma 2.3 shows that $l_1 l_2 = k_1 k_2$, $A_\alpha = (l_2 - k_2)(\alpha_1 + 1) + (l_1 - k_1)(\alpha_2 + 1) = 0$, and $A_\beta = (l_2 - k_2)(\beta_1 + 1) + (l_1 - k_1)(\beta_2 + 1) = 0$. Since $k \neq l$, we have $k_1 \neq l_1$ and $k_2 \neq l_2$. It follows that $A_{\alpha+l} = A_{\beta+l} = (l_1 - k_1)(l_2 - k_2) \neq 0$. Therefore, $Q_{\alpha+l}(\lambda) \not\equiv 0$, $Q_{\beta+l}(\lambda) \not\equiv 0$.

Now set

$$Az^\alpha = \sum_{p\in\mathbb{Z}_+} c_p z^{\beta+p(k+l)} + q(z),$$

where $c_p \in \mathbb{C}$ and $q(z) \in H^0_\beta$. Thus, $AT^*_\varphi z^\alpha = T^*_\varphi Az^\alpha$ shows that

$$Az^{\alpha+l} = cz^{\beta-k} + \sum_{p\in\mathbb{Z}_+} \Big(c_p + c_{p+1} \frac{\gamma_{\beta+(p+1)(k+l)}}{\gamma_{\beta-k+(p+1)(k+l)}} \Big) z^{\beta+l+p(k+l)} + T^*_\varphi q(z), \tag{2.12}$$

where $c = 0$ if $\beta \in \Omega_1 \cup \Omega_3$; $c = c_0$ if $\beta \in \Omega_2 \cup \Omega_4$. It is clear that $T^*_\varphi q(z) \perp \overline{\operatorname{span}}\{z^{\beta+l+p(k+l)} : p \in \mathbb{Z}_+\}$. Since $Q_{\beta+l}(\lambda) \not\equiv 0$, it will therefore now hold that $\{p \in \mathbb{Z}_+ : \langle Az^{\alpha+l}, z^{\beta+l+p(k+l)} \rangle \neq 0\}$ is a finite set. This means that there exists $N \in \mathbb{Z}_+$ such that

$$c_p + c_{p+1} \frac{\gamma_{\beta+(p+1)(k+l)}}{\gamma_{\beta-k+(p+1)(k+l)}} = 0, \quad p \geq N.$$

That is,

$$|c_{p+1}| = |c_p| \frac{\gamma_{\beta-k+(p+1)(k+l)}}{\gamma_{\beta+(p+1)(k+l)}}, \quad p \geq N. \tag{2.13}$$

Next, we prove that if $c_N \neq 0$, then $\sum_{p\in\mathbb{Z}_+} |c_p|^2\gamma_{\beta+p(k+l)}$ is divergent. In fact, if $c_N \neq 0$, by (2.13), we have that $c_p \neq 0$ as $p \geq N$ and that

$$p\Big(\frac{|c_p|^2\gamma_{\beta+p(k+l)}}{|c_{p+1}|^2\gamma_{\beta+(p+1)(k+l)}} - 1\Big) = \frac{pf(p)}{\prod_{i=1}^2(\beta_i + (p+1)(k_i+l_i)+1)(\beta_i + p(k_i+l_i)+1)},$$

where

$$f(p) = \prod_{i=1}^{2}\big(\beta_i + p(k_i+l_i)+k_i+1\big)^2 - \prod_{i=1}^{2}\big(\beta_i+(p+1)(k_i+l_i)+1\big)\big(\beta_i+p(k_i+l_i)+1\big).$$

Because $l_1l_2 = k_1k_2$, we can prove that $f(p)$ is a quadratic polynomial with respect to p. Thus,

$$\lim_{p\to+\infty} p\Big(\frac{|c_p|^2\gamma_{\beta+p(k+l)}}{|c_{p+1}|^2\gamma_{\beta+(p+1)(k+l)}} - 1\Big) = 0.$$

By Raabe's convergence test, $\sum_{p\in\mathbb{Z}_+} |c_p|^2\gamma_{\beta+p(k+l)}$ is divergent, which is a contradiction. So $\Delta_1 = \{p \in \mathbb{Z}_+ : \langle Az^\alpha, z^{\beta+p(k+l)}\rangle \neq 0\}$ is a finite set. □

Lemma 2.9. *If $k \neq l$, $k_i^2 + l_i^2 \neq 0$ $(i = 1, 2)$, and $A \in \mathcal{V}^*(\varphi)$, $\alpha \in \Omega_1$ such that $Q_\alpha(\lambda) \equiv 0$, then $Az^{\alpha+h(k+l)} = cz^{\alpha+h(k+l)}$ for some $c \in \mathbb{C}$ and $\forall h \in \mathbb{Z}_+$.*

Proof. By Lemma 2.8, $\Delta_1 = \{p \in \mathbb{Z}_+ : \langle Az^\alpha, z^{\beta+p(k+l)}\rangle \neq 0\}$ is a finite set with $\beta \not\succeq k + l$. If Δ_1 is not empty, let $p_0 = \max \Delta_1$. Lemma 2.1 implies that $\omega_{\alpha+h(k+l)} = \omega_{\beta+(p_0+h)(k+l)}$ for every $h \in \mathbb{Z}_+$. So $Q_\alpha(\lambda) \equiv Q_{\beta+p_0(k+l)}(\lambda) \equiv 0$.

On the other hand, let $Az^\alpha = c_{p_0}z^{\beta+p_0(k+l)} + g(z)$, where $c_{p_0} \neq 0$ and $g(z) \perp z^{\beta+p_0(k+l)}$. Thus, $A \in \mathcal{V}^*(\varphi)$ shows that

$$Az^{\alpha+l} = c_{p_0}z^{\beta+l+p_0(k+l)} + az^{\beta-k+p_0(k+l)} + T_\varphi^* g(z),$$

where $a = 0$ if $p_0 = 0$ and $\beta \in \Omega_1 \cup \Omega_3$. It is easy to see that $\max\{p \in \mathbb{Z}_+ : \langle Az^{\alpha+l}, z^{\beta+l+p(k+l)}\rangle\} = p_0$. Because $\alpha + l \not\succeq k + l$, again by Lemma 2.1, we have $\omega_{\alpha+l+h(k+l)} = \omega_{\beta+l+(p_0+h)(k+l)}$ for every $h \in \mathbb{Z}_+$. Thereby, $Q_{\alpha+l}(\lambda) \equiv Q_{\beta+l+p_0(k+l)}(\lambda)$. Then, Lemma 2.5 determines that $\alpha = \beta + p_0(k + l)$, indicating that $p_0 = 0$ and $\beta = \alpha$. This means that there exists $c \in \mathbb{C}$ such that $Az^\alpha = cz^\alpha$. Moreover, $AT_\varphi^*T_\varphi z^\alpha = T_\varphi^*T_\varphi Az^\alpha$ implies that $Az^{\alpha+k+l} = cz^{\alpha+k+l}$. By induction, we can get $Az^{\alpha+h(k+l)} = cz^{\alpha+h(k+l)}$ $(c \in \mathbb{C})$ for every $h \in \mathbb{Z}_+$. □

3. Reducing subspaces for T_φ

In this section, we mainly consider the reducing subspaces for T_φ with symbol $\varphi = z^k + \overline{z}^l$ ($k \neq l$, $k_i^2 + l_i^2 \neq 0$ for $i = 1, 2$). Note that the case $k_i^2 + l_i^2 = 0$ for some $i \in \{1, 2\}$ is left to be dealt with in the next section. It is known that T_φ and T_φ^* share the same reducing subspaces. Together with the symmetry of z_1 and z_2, in this section, we assume that

$$(1)\quad 0 \leq k_1 < l_1 \qquad \text{and} \qquad (2)\quad l_1l_2 \neq 0 \quad \text{if } k_1l_2 = k_2l_1.$$

For $m \in \mathbb{Z}_+^2$, let

$$L_m = \overline{\operatorname{span}}\big\{z^{m+uk+vl} : m + uk + vl \succeq (0, 0), u, v \in \mathbb{Z}\big\}.$$

Obviously, L_m are common reducing subspaces for T_{z^k} and T_{z^l} and surely the reducing subspaces for T_φ. Let

$$[m] = \{m + ik + jl : m + ik + jl \succeq (0,0), i, j \in \mathbb{Z}\},$$

and

$$\Delta = \begin{cases} \{(i,j) \in \mathbb{Z}_+^2 : i \in [0, s_1), j \in [0, \frac{|l_1k_2 - l_2k_1|}{s_1})\}, & k_1l_2 \neq k_2l_1, \\ \{(i,j) \in \mathbb{Z}_+^2 : i \in [0, s_1) \text{ or } j \in [0, s_2)\}, & k_1l_2 = k_2l_1, l_1l_2 \neq 0, \end{cases}$$

where

$$s_i = \begin{cases} \gcd\{k_i, l_i\}, & k_il_i \neq 0, \\ |l_i - k_i|, & k_il_i = 0, \end{cases} \quad i = 1, 2. \tag{3.1}$$

Then $\mathbb{Z}_+^2 = \bigcup_{m\in\Delta}[m]$. We show the details as follows.

Case one: $k_1l_2 \neq k_2l_1$. Since $k \neq l$, without loss of generality, suppose that $s_1 \neq 0$. For every $(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, there is $\alpha_1 = m_1 + qs_1$ with $0 \leq m_1 < s_1$ and $q \in \mathbb{Z}_+$. Moreover, there are $u, v \in \mathbb{Z}_+$ such that $vl_1 - uk_1 = s_1$. Set $\mathcal{F} = \{(\beta_1, \beta_2) \in \mathbb{Z}_+^2 : \beta_2l_1 - \beta_1k_1 = s_1\}$. By the assumption, there is a minimal element in $\mathcal{F}$ with respect to the partial order $\succeq$ denoted by (u, v). Notice that $\widetilde{u}k_2 - \widetilde{v}l_2 = \frac{|l_1k_2 - k_1l_2|}{s_1} = \min\{|t_1k_2 - t_2l_2| : t_1k_1 - t_2l_1 = 0\} > 0$. Then there is $\widetilde{q} \in \mathbb{Z}$ such that $m_2 = \alpha_2 - q(vl_2 - uk_2) - \widetilde{q}\frac{|l_1k_2 - k_1l_2|}{s_1} \in [0, \frac{|l_1k_2 - k_1l_2|}{s_1})$. Thus $(\alpha_1, \alpha_2) = (m_1, m_2) - (qu - \widetilde{q}\widetilde{u})k + (qv - \widetilde{q}\widetilde{v})l$.

Case two: $k_1l_2 = k_2l_1$, $l_1l_2 \neq 0$. For every $(\alpha_1, \alpha_2) \in \mathbb{Z}_+^2$, if $\frac{l_1}{l_2}\alpha_2 > \alpha_1$, then we choose m_1, q, u, v as in case one such that $\alpha_1 = m_1 + qs_1$ and $\alpha_2 = m_2 + q(vl_2 - uk_2)$. In this way, $m_2 = \alpha_2 - q(vl_2 - uk_2) = \alpha_2 - q(vl_1 - uk_1)\frac{l_2}{l_1} = \alpha_2 - (\alpha_1 - m_1)\frac{l_2}{l_1} \geq 0$ since $\frac{l_1}{l_2}\alpha_2 > \alpha_1$. If $\frac{l_1}{l_2}\alpha_2 < \alpha_1$, then we can prove that $(\alpha_1, \alpha_2) \in \{m + uk + vl : m + uk + vl \succeq (0,0), u, v \in \mathbb{Z}\}$ with $m_2 \in [0, s_2)$ and $m_1 \in \mathbb{Z}_+$. If $\frac{l_1}{l_2}\alpha_2 = \alpha_1$, then $(\alpha_1, \alpha_2) \in \{m + uk + vl : m + uk + vl \succeq (0,0), u, v \in \mathbb{Z}\}$ with $m_1s_2 = m_2s_1$ for $m_1 \in [0, s_1)$ and $m_2 \in [0, s_2)$.

Therefore, we have

$$L_a^2(\mathbb{D}^2) = \bigoplus_{m\in\Delta} L_m. \tag{3.2}$$

First, we prove that $L_m (m \in \Delta)$ are minimal reducing subspaces for T_φ.

Proposition 3.1. *Let $k \neq l$, let $k_i^2 + l_i^2 \neq 0$ $(i = 1, 2)$, let $A \in \mathcal{V}^*(\varphi)$, and let $m \in \Omega_1$ such that $Q_m(\lambda) \not\equiv 0$.*

(i) *If $l_1k_2 \neq l_2k_1$, then there is $c \in \mathbb{C}$ such that $Az^{m+h(k+l)} = cz^{m+h(k+l)}$ for any $h \in \mathbb{Z}_+$.*

(ii) *If $l_1k_2 = l_2k_1$ and $l_1l_2 \neq 0$, then $Az^m = az^m + bz^{m'}$, where $a, b \in \mathbb{C}$ and $m' = (\frac{l_1}{l_2}(m_2 + 1) - 1, \frac{l_2}{l_1}(m_1 + 1) - 1)$.*

Proof. By Lemmas 2.4 and 2.7, we may set

$$Az^m = \sum_{\beta\in\Omega_1} a_\beta z^\beta,$$

where $a_\beta \in \mathbb{C}$. Clearly, $\{p \in \mathbb{Z}_+ : \langle Az^m, z^{\beta+p(k+l)}\rangle \neq 0\} = \{0\}$. So, as in the proof of Lemmas 2.1 and 2.2, for every $h \in \mathbb{Z}_+$, we have

$$\begin{aligned}\max&\big\{p \in \mathbb{Z}_+ : \langle Az^{m+h(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\big\}\\ &= \min\big\{p \in \mathbb{Z}_+ : \langle Az^{m+h(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0\big\}\\ &= h.\end{aligned}$$

This means that

$$\text{if } \langle Az^{m+h(k+l)}, z^{\beta+p(k+l)}\rangle \neq 0, \quad \text{then } h = p. \tag{3.3}$$

Clearly, $T_\varphi^* T_\varphi A z^m = A T_\varphi^* T_\varphi z^m$ shows that

$$A\Big(z^{m+k+l} + \frac{\gamma_{m+k}}{\gamma_m} z^m\Big) = \sum_{\beta\in\Omega_1} a_\beta \Big(z^{\beta+k+l} + \frac{\gamma_{\beta+k}}{\gamma_\beta} z^\beta\Big).$$

If $\langle Az^m, z^\beta\rangle \neq 0$, from (3.3), then we get

$$\frac{\gamma_{m+k}}{\gamma_m} = \frac{\gamma_{\beta+k}}{\gamma_\beta}, \tag{3.4}$$

and

$$Az^{m+(k+l)} = \sum_{\beta\in\Omega_1} a_\beta z^{\beta+(k+l)}.$$

Suppose, for some $i \in \mathbb{Z}_+$, we have

$$Az^{m+i(k+l)} = \sum_{\beta\in\Omega_1} a_\beta z^{\beta+i(k+l)}.$$

By $T_\varphi^* T_\varphi A z^{m+i(k+l)} = A T_\varphi^* T_\varphi z^{m+i(k+l)}$, we get

$$\begin{aligned}A\Big(&z^{m+(i+1)(k+l)} + \lambda z^{m+i(k+l)} + \frac{\gamma_{m+i(k+l)}}{\gamma_{m+(i-1)(k+l)}} z^{m+(i-1)(k+l)}\Big)\\ &= \sum_{\beta\in\Omega_1} a_\beta \Big(z^{\beta+(i+1)(k+l)} + \mu z^{\beta+i(k+l)} + \frac{\gamma_{\beta+i(k+l)}}{\gamma_{\beta+(i-1)(k+l)}} z^{\beta+(i-1)(k+l)}\Big),\end{aligned} \tag{3.5}$$

where $\lambda, \mu > 0$. So (3.3) implies that

$$Az^{m+(i+1)(k+l)} = \sum_{\beta\in\Omega_1} a_\beta z^{\beta+(i+1)(k+l)}.$$

By induction, we have

$$Az^{m+p(k+l)} = \sum_{\beta\in\Omega_1} a_\beta z^{\beta+p(k+l)}, \quad \forall p \in \mathbb{Z}_+. \tag{3.6}$$

Moreover, $\langle Az^m, z^\beta\rangle \neq 0$ shows $a_\beta \neq 0$. It follows from (3.5) that

$$\frac{\gamma_{m+i(k+l)}}{\gamma_{m+(i-1)(k+l)}} = \frac{\gamma_{\beta+i(k+l)}}{\gamma_{\beta+(i-1)(k+l)}}, \quad \forall i \in \mathbb{N}.$$

Letting $i \to +\infty$, we have $\frac{\gamma_m}{\gamma_\beta} = \frac{\gamma_{m+i(k+l)}}{\gamma_{\beta+i(k+l)}} \to 1$. That is,

$$\gamma_{m+i(k+l)} = \gamma_{\beta+i(k+l)} \tag{3.7}$$

for all $i \in \mathbb{Z}_+$.

(i) Case $l_1k_2 \neq l_2k_1$. By $\gamma_m = \gamma_\beta$ and $\gamma_{m+(k+l)} = \gamma_{\beta+(k+l)}$, (3.4) shows that $\gamma_{m+k} = \gamma_{\beta+k}$. These give $k_1(m_2-\beta_2)+k_2(m_1-\beta_1)=0$ and $(k_1+l_1)(m_2-\beta_2)+(k_2+l_2)(m_1-\beta_1)=0$, which imply that $m=\beta$ as $l_1k_2 \neq l_2k_1$. Then (3.6) implies the desired result.

(ii) Case $l_1k_2 = l_2k_1$ and $l_1l_2 \neq 0$. From (3.7),

$$\big(m_1+1+(k_1+l_1)z\big)\big(m_2+1+(k_2+l_2)z\big) \\ -\big(\beta_1+1+(k_1+l_1)z\big)\big(\beta_2+1+(k_2+l_2)z\big)=0$$

is a quadratic equation with respect to variable z with infinite roots. Then we get

$$\big(m_1+1+(k_1+l_1)z\big)\big(m_2+1+(k_2+l_2)z\big) \\ \equiv \big(\beta_1+1+(k_1+l_1)z\big)\big(\beta_2+1+(k_2+l_2)z\big)$$

on $\mathbb{C}$. Through comparing the coefficients of 1, z, and z^2, we have $\beta = m$ or $\beta = (\frac{l_1}{l_2}(m_2+1)-1, \frac{l_2}{l_1}(m_1+1)-1)$. □

Proposition 3.2. *Let $k \neq l$, and let $k_i^2 + l_i^2 \neq 0$ $(i=1,2)$. Suppose that $\mathcal{M}$ is a reducing subspace for T_φ and that $\mathcal{M} \subseteq L_m$ where $m \in \Delta$. Then either $z^m \in \mathcal{M}$ or $z^m \in \mathcal{M}^\perp$.*

Proof. Let P be the orthogonal projection from $L_a^2(\mathbb{D}^2)$ onto $\mathcal{M}$. Since $\mathcal{M} \subseteq L_m$ and L_m is a reducing subspace for T_φ, we have $P|_{L_m} : L_m \to L_m$ and $P \in \mathcal{V}^*(\varphi)$. Notice that $m \in \Delta$ indicates $m \in \Omega_1$. If $k_1l_2 \neq k_2l_1$, by combining Proposition 3.1 and Lemma 2.9 we can easily get the desired result. If $k_1l_2 = k_2l_1$, we can easily see that $(\frac{l_1}{l_2}(m_2+1)-1, \frac{l_2}{l_1}(m_1+1)-1) \in \Delta$ as $m \in \Delta$. Then, Proposition 3.1 shows that $Pz^m = cz^m$, $c \in \mathbb{C}$. Thus, the desired conclusion follows. □

Theorem 3.3. *Let $k \neq l$ and let $k_i^2 + l_i^2 \neq 0$ $(i=1,2)$. Suppose that $m \in \Delta$. Then L_m is a minimal reducing subspace for T_φ.*

Proof. Assume that there is a nonzero reducing subspace $\mathcal{M} \subseteq L_m$. Then $L_m \ominus \mathcal{M}$ is also a reducing subspace. By Proposition 3.2, we may suppose that $z^m \in \mathcal{M}$. To get the desired result, we only need to prove that

$$z^{m+uk+vl} \in \mathcal{M} \tag{3.8}$$

for $u, v \in \mathbb{Z}$ and $m+uk+vl \succeq (0,0)$.

The proof is divided into two cases: $k_1k_2 \neq 0$ and $k_1k_2 = 0$.

Case one: $k_1k_2 \neq 0$. Write $c_i = \min\{c \in \mathbb{Z}_+ : m+ck \succeq il\}$, $i \in \mathbb{Z}_+$. If $m_1 < s_1$, then $m_1 < k_1 < l_1$. If $m_1 \geq s_1$, then $m_2 < s_2$ and $k_1l_2 = l_1k_2$, $l_1l_2 \neq 0$. By $k_1 < l_1$, we know that $m_2 < l_2 < k_2$. So it is easy to see that $c_i < c_{i+1}$. Note that

$$T_\varphi^j z^m = z^{m+jk} \in \mathcal{M}, \quad j \in (0, c_1] \cap \mathbb{N}, \tag{3.9}$$

and that

$$T_\varphi^* z^m = z^{m+l} \in \mathcal{M}.$$

Observe that

$$T_\varphi^{c_1+1} z^m = z^{m+(c_1+1)k} + \frac{\gamma_{m+c_1k}}{\gamma_{m+c_1k-l}} z^{m+c_1k-l} \in \mathcal{M}.$$

Then

$$T_\varphi^{c_1+1} z^m = T_\varphi^{c_1+1} P_{\mathcal{M}} z^m = P_{\mathcal{M}} z^{m+(c_1+1)k} + \frac{\gamma_{m+c_1 k}}{\gamma_{m+c_1 k-l}} P_{\mathcal{M}} z^{m+c_1 k-l} \in \mathcal{M}.$$

By the definition of c_1, along with $m + c_1 k - l \not\succeq l$, $m + c_1 k - l \not\succeq k$, and $m+(c_1+1)k \succeq k+l$, Lemmas 2.4 and 2.9 show that $P_{\mathcal{M}} z^{m+c_1 k-l} = z^{m+c_1 k-l}$ and that $P_{\mathcal{M}} z^{m+(c_1+1)k} = z^{m+(c_1+1)k}$. Thus we deduce that

$$z^{m+(c_1+1)k}, z^{m+c_1 k-l} \in \mathcal{M}. \tag{3.10}$$

By

$$T_\varphi^* z^{m+jk} = z^{m+jk+l} + \frac{\gamma_{m+jk}}{\gamma_{m+(j-1)k}} z^{m+(j-1)k} \in \mathcal{M}, \quad j \in [1, c_1] \cap \mathbb{N}$$

and (3.9), we see that $z^{m+jk+l} \in \mathcal{M}$, $j \in [1, c_1] \cap \mathbb{N}$. Furthermore,

$$T_\varphi^* z^{m+uk+vl} = z^{m+uk+(v+1)l} + \frac{\gamma_{m+uk+vl}}{\gamma_{m+(u-1)k+vl}} z^{m+(u-1)k+vl} \in \mathcal{M} \tag{3.11}$$

shows that (3.8) holds for $u, v \in \mathbb{Z}$, $1 \le u < c_1$, $v \ge 0$, and $u = c_1$, $v \ge -1$.

Suppose that the equality (3.8) holds for $u, v \in \mathbb{Z}$, $1 \le u \le c_n$ and $m+uk+vl \succeq (0,0)$. Then, the following statements hold:

(i) $T_\varphi^j z^{m+c_n k-nl} = z^{m+(c_n+j)k-nl} \in \mathcal{M}$, $j \in (0, c_{n+1} - c_n] \cap \mathbb{N}$;

(ii) $z^{m+c_{n+1}k-(n+1)l} \in \mathcal{M}$. The proof is similar to that of (3.10) by using Lemma 2.4, Lemma 2.9, and

$$T_\varphi z^{m+c_{n+1}k-nl} = z^{m+(c_{n+1}+1)k-nl} + \frac{\gamma_{m+c_{n+1}k-nl}}{\gamma_{m+c_{n+1}k-(n+1)l}} z^{m+c_{n+1}k-(n+1)l} \in \mathcal{M};$$

(iii) $z^{m+(c_n+j)k-vl} \in \mathcal{M}$ for $j \in [1, c_{n+1} - c_n] \cap \mathbb{N}$, $v \in (0, n-1] \cap \mathbb{N}$, since

$$z^{m+(c_n+j)k-vl} = T_\varphi^* z^{m+(c_n+j)k-(v+1)l} - \frac{\gamma_{m+(c_n+j)k-(v+1)l}}{\gamma_{m+(c_n+j-1)k-(v+1)l}} z^{m+(c_n+j-1)k-(v+1)l}.$$

Therefore, we have proved that (3.8) holds for $m+uk+vl \succeq (0,0)$, $v \in \mathbb{Z}$, $u \in \mathbb{N}$.

To end the proof of case one, we only need to prove that (3.8) holds for $m + vl - nk \succeq (0,0)$, $n \in \mathbb{Z}_+$, $v \in \mathbb{N}$. Write $d_i = \max\{d \in \mathbb{Z}_+ : m + il \succeq dk\}$, $i \in \mathbb{Z}_+$. Since we already have that $z^{m+vl+k} \in \mathcal{M}$ for $v \in \mathbb{Z}_+$.

$$T_\varphi^* z^{m+vl+k} = z^{m+(v+1)l+k} + \frac{\gamma_{m+vl+k}}{\gamma_{m+vl}} z^{m+vl}$$

implies that $z^{m+vl} \in \mathcal{M}$ for all $v \in \mathbb{Z}_+$. From

$$T_\varphi^* z^{m+vl-nk} = z^{m+(v+1)l-nk} + \frac{\gamma_{m+vl-nk}}{\gamma_{m+vl-(n+1)k}} z^{m+vl-(n+1)k},$$

we get that $z^{m+vl-nk} \in \mathcal{M}$ for all $v \in \mathbb{N}$ and $n \in [0, d_v] \cap \mathbb{N}$. Hence, (3.8) holds for $m + vl - nk \succeq (0,0)$, $n \in \mathbb{Z}_+$, $v \in \mathbb{N}$.

Case two: $k_1 k_2 = 0$. The proof of this part is trivial.

(1) If $k_2 = k_1 = 0$: Here, $T_\varphi^{*i} z^m = z^{m+il}, i \in \mathbb{Z}_+$ indicates that $z^{m+vl} \in \mathcal{M}$ for all $v \in \mathbb{Z}_+$.

(2) If $k_1 = 0$ and $k_2 \neq 0$: By the assumption we have $l_1 \neq 0$. From

$$T_\varphi^i z^m = z^{m+ik}, \quad i \in \mathbb{N},$$

we have $z^{m+uk} \in \mathcal{M}$ for $u \in \mathbb{Z}_+$.

(2.1) If $l_2 > k_2$: Here

$$T_\varphi^* z^{m+uk+nl} = z^{m+uk+(n+1)l} + \frac{\gamma_{m+uk+nl}}{\gamma_{m+(u-1)k+nl}} z^{m+(u-1)k+nl}, \quad n \in \mathbb{Z}_+$$

yields that $z^{m+uk+vl} \in \mathcal{M}$ for $u \in \mathbb{N}$ and $m + uk + vl \succeq (0,0)$. Using the same method as in case one, we can prove that (3.8) holds for $m + vl - nk \succeq (0,0)$, $n \in \mathbb{Z}_+$, $v \in \mathbb{N}$.

(2.2) If $k_2 \geq l_2$ and $l_2 \neq 0$: Let $c_i = \min\{c \in \mathbb{Z}_+ : m + cl \succeq ik\}$, $i \in \mathbb{Z}_+$. Exchanging T_φ and T_φ^*, k and l in the proof of case one, we get $z^{m+vl+uk} \in \mathcal{M}$ for $v \in \mathbb{N}$, $u \in \mathbb{Z}$ and $m + vl + uk \succeq (0,0)$.

(2.3) If $l_2 = 0$: Clearly, $T_\varphi^{*i} z^m = z^{m+il}$, $i \in \mathbb{Z}_+$ indicates that $z^{m+vl} \in \mathcal{M}$ for all $v \in \mathbb{Z}_+$.

(3) If $k_2 = 0$ and $k_1 \neq 0$: Here, $l_2 \neq 0$. Exchanging z_1 and z_2, we have the same conclusion as in (2).

Above all, $L_m \subseteq \mathcal{M}$. The theorem is proved. □

In particular, if $k_1 l_2 = k_2 l_1$, then there exist $p, q \in \mathbb{Z}$ such that $pk + ql = s$, where $s = (s_1, s_2)$ is defined by (3.1). By the proof in Theorem 3.3, we have the result as follows.

Corollary 3.4. *If $k \neq l$, $k_1 l_2 = k_2 l_1$, and $l_1 l_2 \neq 0$, then*

$$L_m = \overline{\operatorname{span}}\{z^{m+hs} : h = 0, 1, 2, \dots\}, \quad m \in \Delta$$

are minimal reducing subspaces for $T_{z^k + \bar{z}^l}$.

Second, we consider unitary equivalence between the reducing subspaces L_m (see [12]). Two reducing subspaces M_1 and M_2 of T_φ are called *unitarily equivalent* if there exists an operator U on $L_a^2(\mathbb{D}^2)$ such that $U|_{M_1}$ is unitary from M_1 onto M_2, $U|_{M_1^\perp} = 0$ and U commutes with both T_φ and T_φ^*.

Proposition 3.5. *Let $k \neq l$. Suppose that $m, m' \in \Delta$. Then the following statements hold:*

(i) *if $k_1 l_2 \neq k_2 l_1$, then L_m and $L_{m'}$ are unitarily equivalent if and only if $m = m'$;*

(ii) *if $k_1 l_2 = k_2 l_1$, $l_1 l_2 \neq 0$ and if $(\frac{l_1}{l_2}(m_2+1)-1, \frac{l_2}{l_1}(m_1+1)-1) \in \mathbb{Z}_+^2$, then L_m and $L_{m'}$ are unitarily equivalent if and only if $m' = m$ or $m' = (\frac{l_1}{l_2}(m_2 + 1)-1, \frac{l_2}{l_1}(m_1+1)-1)$. Otherwise, if $(\frac{l_1}{l_2}(m_2+1)-1, \frac{l_2}{l_1}(m_1+1)-1) \notin \mathbb{Z}_+^2$, then L_m and $L_{m'}$ are unitarily equivalent if and only if $m' = m$.*

Proof. It is sufficient to verify the necessity. Suppose that L_m and $L_{m'}$ are unitarily equivalent. Then there is an operator $U \in \mathcal{V}^*(\varphi)$ satisfying that $U|_{L_m} : L_m \to L_{m'}$ is unitary.

(i) Let $l_1k_2 \neq k_1l_2$. If $Q_m(\lambda) \not\equiv 0$, then Proposition 3.1 implies that $m = m'$ and $Uz^m = z^m$. On the other hand, if $Q_m(\lambda) \equiv 0$, then Lemma 2.9 shows the desired result. Therefore, statement (i) is set up.

(ii) If $l_1k_2 = k_1l_2$ and $l_1l_2 \neq 0$, then $l_1l_2 \neq k_1k_2$ and $Q_m(\lambda) \not\equiv 0$. Proposition 3.1 implies that

$$Uz^m = cz^{m'+ik+jl}, \tag{3.12}$$

where $c \in \mathbb{C}$ and one of the following statements holds:

(a) $m = m' + ik + jl$;

(b) $m' + ik + jl = (\frac{l_1}{l_2}(m_2+1) - 1, \frac{l_2}{l_1}(m_1+1) - 1)$, where $l_1l_2 \neq 0$ and $(\frac{l_1}{l_2}(m_2+1) - 1, \frac{l_2}{l_1}(m_1+1) - 1) \in \mathbb{Z}_+^2$.

Next, we prove that $i = j = 0$. Suppose on the contrary that $(i,j) \neq (0,0)$; then $|ik_p + jl_p| \geq s_p$ for $p = 1,2$. Since $k_1l_2 = k_2l_1$ and $l_1l_2 \neq 0$, there is $ik_1 + jl_1 = \frac{l_1}{l_2}(ik_2 + jl_2)$. Without loss of generality, suppose that $ik_1 + jl_1 > 0$. By statement (a), we have $m_p = m'_p + ik_p + jl_p \geq s_p$ $(p = 1,2)$, which contradicts the fact that $m \in \Delta$. By statement (b), we obtain that $\frac{l_1}{l_2}(m'_2+1) = m_1+1-(ik_1+jl_1)$ is clearly an integer and $\frac{l_1}{l_2}(m'_2+1) > 0$. It follows that

$$ik_1 + jl_1 = m_1 + 1 - \frac{l_1}{l_2}(m'_2+1) \leq m_1.$$

Likewise, we also get

$$ik_2 + jl_2 = m_2 + 1 - \frac{l_2}{l_1}(m'_1+1) \leq m_2.$$

Therefore, $s_p \leq ik_p + jl_p \leq m_p$ for $p = 1,2$, which contradicts the fact that $m \in \Delta$. So the proof is completed. □

Corollary 3.6. *If $k \neq l$, $k_1l_2 = k_2l_1$, $l_1l_2 \neq 0$, and $s_1 = s_2$, then L_m and $L_{m'}$ are unitarily equivalent if and only if $m' = m$ or $m' = (m_2, m_1)$.*

Theorem 3.7. *Let $k \neq l$, let $k_1l_2 = l_1k_2$, and let $l_1l_2 \neq 0$. Set*

$$\mathcal{M}_{m,a,b} = \overline{\operatorname{span}}\{(az^m + bz^{m'})z^{ik+jl} : i,j \in \mathbb{Z}, ik + jl + m \succeq (0,0)\},$$

where $m \in \Delta$ and $m' = (\frac{l_1}{l_2}(m_2+1)-1, \frac{l_2}{l_1}(m_1+1)-1) \in \mathbb{Z}_+^2$, $a, b \in \mathbb{C}$ and $ab \neq 0$. Then $\mathcal{M}_{m,a,b}$ is a minimal reducing subspace.

Proof. By the assumption, there exist $M, N \in \mathbb{Z}_+$ such that $k = (Ms_1, Ms_2)$, $l = (Ns_1, Ns_2)$ with $\gcd\{M,N\} = 1$. We first establish one claim: if $m \in \Delta$, then $m + ik + jl \succeq (0,0)$ if and only if $m' + ik + jl \succeq (0,0)$, where $m' = (\frac{l_1}{l_2}(m_2+1) - 1, \frac{l_2}{l_1}(m_1+1) - 1) \in \mathbb{Z}_+^2$.

Indeed, by the definition of Δ, we have $m = (m_1, m_2) \in \Delta$ if and only if $0 \leq m_1 < s_1$ or $0 \leq m_2 < s_2$. Without loss of generality, we will assume that $0 \leq m_1 < s_1$. Thus, $m'_2 = \frac{s_2}{s_1}(m_1+1) - 1 \leq s_2 - 1 < s_2$; that is, $m' \in \Delta$. Since $|ik_p + jl_p| \geq s_p$ $(p = 1,2)$ and $(ik_1 + jl_1)(ik_2 + jl_2) \geq 0$ as $k_1l_2 = l_1k_2$, then $m+ik+jl \succeq (0,0)$ if and only if $ik+jl \succeq (0,0)$ if and only if $m'+ik+jl \succeq (0,0)$, where $i,j \in \mathbb{Z}$.

To conclude, we only need to prove that L_m and $\mathcal{M}_{m,a,b}$ are unitarily equivalent. Let U be the linear map from L_m onto $\mathcal{M}_{m,a,b}$ defined by

$$U\Big(\frac{z^{m+ik+jl}}{\sqrt{\gamma_{m+ik+jl}}}\Big)=\frac{1}{\sqrt{|a|^2+|b|^2}}\Big(a\frac{z^m}{\sqrt{\gamma_{m+ik+jl}}}+b\frac{z^{m'}}{\sqrt{\gamma_{m'+ik+jl}}}\Big)z^{ik+jl}.$$

One can easily get $\gamma_{m+ik+jl}=\gamma_{m'+ik+jl}$ since $k_1l_2=l_1k_2$. Clearly, $z^{m+ik+jl}\perp z^{m'+ik+jl}$ whenever $m\neq m'$ and $m+ik+jl, m'+ik+jl\succeq(0,0)$. Fix $i,j,p,q\in\mathbb{Z}_+$. Write $(i,j)\sim(p,q)$ if $ik+jl=pk+ql$ and $ik+jl, pk+ql\succeq(0,0)$. Let $[(p,q)]$ denote the set of all $(i,j)\in\mathbb{Z}_+^2$ satisfying $(i,j)\sim(p,q)$. Thus,

$$\begin{aligned}
&\Big\|U\Big(\sum_{i,j\in\mathbb{Z}_+,m+ik+jl\succeq(0,0)}\Big(\sum_{(p,q)\in[(i,j)]}f_{pq}\Big)z^{m+ik+jl}\Big)\Big\|^2\\
&=\Big\|\frac{1}{\sqrt{|a|^2+|b|^2}}\Big(\sum_{i,j\in\mathbb{Z}_+,m+ik+jl\succeq(0,0)}\Big(\sum_{(p,q)\in[(i,j)]}f_{pq}\Big)(az^m+bz^{m'})z^{ik+jl}\Big)\Big\|^2\\
&=\sum_{i,j\in\mathbb{Z}_+,m+ik+jl\succeq(0,0)}\Big\|\Big(\sum_{(p,q)\in[(i,j)]}f_{pq}\Big)z^{m+ik+jl}\Big\|^2\\
&=\Big\|\sum_{i,j\in\mathbb{Z}_+,m+ik+jl\succeq(0,0)}\Big(\sum_{(p,q)\in[(i,j)]}f_{pq}\Big)z^{m+ik+jl}\Big\|^2,
\end{aligned}$$

where $f_{pq}\in\mathbb{C}$. That is, U is unitary. Since T is self-adjoint, then it remains to show that $UT_\varphi=T_\varphi U$. Before continuing, we observe that $m+ik+jl\succeq l$ if and only if $m'+ik+jl\succeq l$ by the claim. Hence, a direct computation leads to the fact that $UT_\varphi=T_\varphi U$. □

Next, we give a complete description of the reducing subspaces for T_φ and we show there is no other reducing subspaces for T_φ other than the common reducing subspaces for T_{z^k} and T_{z^l}.

Theorem 3.8. *Let $k\neq l$, let $k_i^2+l_i^2\neq 0$ $(i=1,2)$, and let $\mathcal{M}$ be a reducing subspace for T_φ. Then $\mathcal{M}$ is the orthogonal sum of some minimal reducing subspaces. Moreover, $\mathcal{M}$ is a minimal reducing subspace for T_φ if and only if $\mathcal{M}$ has the form as follows.*

(i) *If $l_1k_2\neq k_1l_2$, then $\mathcal{M}=L_m$ for some $m\in\Delta$.*
(ii) *If $l_1k_2=k_1l_2$ and $l_1l_2\neq 0$, then there exist $m\in\Delta$ and $a,b\in\mathbb{C}$ such that $\mathcal{M}=\mathcal{M}_{m,a,b}$ where $\mathcal{M}_{m,a,b}$ are defined by*

$$\mathcal{M}_{m,a,b}=\overline{\text{span}}\{(az^m+bz^{m'})z^{ik+jl}:i,j\in\mathbb{Z},ik+jl+m\succeq(0,0)\}$$

with $m'=(\frac{l_1}{l_2}(m_2+1)-1,\frac{l_2}{l_1}(m_1+1)-1)$. In particular, if $m'\notin\mathbb{Z}_+^2$, then $b=0$.

Proof. Notice that if $P_{\mathcal{M}}(z^m)=0$, then $L_m\perp\mathcal{M}$. By (3.2), there is $m\in\Delta$ such that $P_{\mathcal{M}}(z^m)\neq 0$. If $l_1k_2\neq k_1l_2$, then $P_{\mathcal{M}}(z^m)=cz^m\neq 0$. Therefore, $z^m\in\mathcal{M}$ and $L_m\subseteq\mathcal{M}$. If $l_1k_2=k_1l_2$ and $l_1l_2\neq 0$, then there are $a,b\in\mathbb{C}$ and m' defined as in condition (ii) such that $P_{\mathcal{M}}(z^m)=az^m+bz^{m'}\in\mathcal{M}$ by Proposition 3.1. Since $az^m+bz^{m'}\in\mathcal{M}_{m,a,b}$, which is a minimal reducing subspace for T_φ, then

$\mathcal{M}_{m,a,b} \subseteq \mathcal{M}$ and $\mathcal{M} \ominus \mathcal{M}_{m,a,b}$ is also a reducing subspace for T_φ. So we get the desired results. □

If $k_1 = k_2 = 0$, then we have the following corollary, which includes the results of $T_{z_1^N z_2^M}$ (see [18, Theorem 2.4] for the weighted case; see [16, Theorem 2.5] for the unweighted case).

Corollary 3.9. *Let $\mathcal{M}$ be a reducing subspace for T_{z^l} with $l = (l_1, l_2) \in \mathbb{N}^2$. Then there exist $m = (m_1, m_2) \in \mathbb{Z}_+^2$ satisfying $m_1 < l_1$ or $m_2 < l_2$, and $a, b \in \mathbb{C}$ such that*

$$\overline{\operatorname{span}}\{(az^m + bz^{m'})z^{jl} : j \in \mathbb{Z}_+\} \subseteq \mathcal{M},$$

where $m' = (\frac{l_1}{l_2}(m_2+1) - 1, \frac{l_2}{l_1}(m_1+1) - 1)$. Moreover, $\mathcal{M}$ is minimal if and only if

$$\mathcal{M} = \overline{\operatorname{span}}\{(az^m + bz^{m'})z^{jl} : j \in \mathbb{Z}_+\}.$$

In particular, if $m' \notin \mathbb{Z}_+^2$, then $b = 0$.

However, the case of $k = l$ is sharply different from that of $k \neq l$.

Theorem 3.10. *Let $k \in \mathbb{Z}_+^d$, and let $d \in \mathbb{N}$. Then $T_{z^k + \bar{z}^k}$ is completely reducible on $L_a^2(\mathbb{D}^d)$.*

Proof. By the spectral theorem of normal operators, a normal operator is completely reducible if and only if it has no eigenvalues. Thus, we only need to show that $T_{z^k+\bar{z}^k}$ has no eigenvalues on $L_a^2(\mathbb{D}^d)$. Recall that u is a pluriharmonic function on $\mathbb{D}^d$ if and only if $\frac{\partial^2}{\partial z_i \bar{z}_j} u \equiv 0$ for all $i, j = 0, 1, \ldots, d$. Thus, $f(z) = z^k + \bar{z}^k$ is surely a bounded real pluriharmonic function on $\mathbb{D}^d$ and continuous on $\overline{\mathbb{D}^d}$. McDonald and Sundberg [17] proved that if u is bounded real harmonic function on $\mathbb{D}$, then T_u has no (nonzero) eigenvectors. Next, we will prove a similar result over the polydisk. Suppose that u is a bounded real pluriharmonic function on $\mathbb{D}^d$ which is continuous on $\overline{\mathbb{D}^d}$. If $T_u f = 0$ for some $f \in L_a^2(\mathbb{D}^d)$, then we have $uf \in (L_a^2(\mathbb{D}^d))^\perp$. Therefore,

$$\begin{aligned} 0 &= \langle uf, fg \rangle \\ &= \int_{\mathbb{D}^d} u|f|^2 \bar{g} \, dA(z) \\ &= \overline{\int_{\mathbb{D}^d} u|f|^2 \bar{g} \, dA(z)} \\ &= \int_{\mathbb{D}^d} u|f|^2 g \, dA(z), \end{aligned}$$

for any $g \in H^\infty(\mathbb{D}^d)$. It follows that

$$\int_{\mathbb{D}^d} u|f|^2 \operatorname{Re}(g) \, dA(z) = 0.$$

By replacing $\operatorname{Re}(g)$ with u, we have

$$\int_{\mathbb{D}^d} u^2 |f|^2 \, dA(z) = 0,$$

which shows that $uf = 0$ almost everywhere on $\mathbb{D}^d$. Let $\mathcal{N}(u) = \{z \in \overline{\mathbb{D}^d} : u(z) = 0\}$. If $u(z)$ is not equal to zero identically, we must have $f(z) = 0$ on the nonempty open set $E = \mathcal{N}(u)^c \cap \mathbb{D}^d$, forcing $f \equiv 0$ on $\mathbb{D}^d$. Notice that $T_{z^k+\bar{z}^k}$ is self-adjoint, hence $\sigma(T_{z^k+\bar{z}^k}) \subseteq \mathbb{R}$. By the analysis above, $T_{z^k+\bar{z}^k}$ has no eigenvalues. Therefore, $T_{z^k+\bar{z}^k}$ is completely reducible on $L_a^2(\mathbb{D}^d)$. □

Now, we give two examples.

Example 3.11. Let $k = (2, 4)$, $l = (4, 2)$. It is easy to see that $k_1 l_2 \neq k_2 l_1$. Then $s = (2, 2)$ and

$$\Delta = \big\{(i, j) \in \mathbb{Z}_+^2 : i \in [0, 2), j \in [0, 6)\big\}.$$

Then, from Theorem 3.8, $L_m = \overline{\operatorname{span}}\{z^{m+u(2,4)+v(4,2)} : m + u(2, 4) + v(4, 2) \succeq (0, 0), u, v \in \mathbb{Z}\}$, $m \in \Delta$, are all the minimal reducing subspaces for $T_{z_1^2 z_2^4+\overline{z_1^4 z_2^2}}$.

Example 3.12. Let $k = (2, 4)$, $l = (3, 6)$. Obviously, $k_1 l_2 = k_2 l_1$. Then $s = (1, 2)$ and

$$\Delta = \big\{(i, j) \in \mathbb{Z}_+^2 : i \in [0, 1) \text{ or } j \in [0, 2)\big\}.$$

Corollary 3.4 indicates that $L_m = \overline{\operatorname{span}}\{z^{m+h(1,2)} : h \in \mathbb{Z}_+\}$, $m \in \Delta$, are minimal reducing subspaces for $T_{z_1^2 z_2^4+\overline{z_1^3 z_2^6}}$. But they are not all the minimal reducing subspaces for $T_{z_1^2 z_2^4+\overline{z_1^3 z_2^6}}$. For instance, by Theorem 3.8, let $m = (0, 3)$, $m' = (1, 1)$, we have $\mathcal{M}_{ab} = \overline{\operatorname{span}}\{(az_2^3 + bz_1 z_2)z^{ik+jl} : i, j \in \mathbb{Z}, ik + jl + (0, 3) \succeq (0, 0)\}$, $a, b \in \mathbb{C}$, is also minimal for $T_{z_1^2 z_2^4+\overline{z_1^3 z_2^6}}$.

4. Some results on the unit disk

Analogous to the proofs in Section 2 and Section 3, we can determine the reducing subspaces for $T_{z^k+\bar{z}^l}$ on the unit disk $\mathbb{D}$. If $k = l$, then Theorem 3.10 indicates that $T_{z^k+\bar{z}^l}$ is completely reducible on $L_a^2(\mathbb{D})$. If $k \neq l$, then we have the following results.

Theorem 4.1. *Let* $\varphi(z) = z^k + \bar{z}^l$, $z \in \mathbb{D}$ *with* $k, l \in \mathbb{Z}_+$ *and* $k \neq l$. *Let* $s = \gcd\{k, l\}$ *for* $kl \neq 0$; $s = |k - l|$ *for* $kl = 0$. *Then* $L_a = \overline{\operatorname{span}}\{z^{a+ns} : n \in \mathbb{Z}_+\}$ *(*$0 \leq a < s$*) are all the minimal reducing subspaces for* T_φ *and each reducing subspace is an orthogonal sum of some minimal reducing subspaces.*

Proof. Since $k \neq l$, we might as well assume that $0 \leq k < l$. Denote by $T = T_\varphi^* T_\varphi - T_\varphi T_\varphi^*$. Then, $Tz^n = \omega_n z^n$, where

$$\omega_n = \begin{cases} \frac{n+1}{n+k+1} - \frac{n+1}{n+l+1}, & 0 \leq n < k, \\ \frac{n+1}{n+k+1} - \frac{n-k+1}{n+1} - \frac{n+1}{n+l+1}, & k \leq n < l, \\ \frac{n+1}{n+k+1} - \frac{n-k+1}{n+1} - \frac{n+1}{n+l+1} + \frac{n-l+1}{n+1}, & n \geq l. \end{cases} \tag{4.1}$$

Clearly, the following statements hold:

(1) $\omega_{m_1} \neq \omega_{m_2}$ for $0 \leq m_1, m_2 < k$ and $m_1 \neq m_2$, since $f(x) = \frac{x}{x+k} - \frac{x}{x+l}$ is strictly increasing on $[0, k)$;

(2) $\omega_{m_1} \neq \omega_{m_2}$ for $k \leq m_1, m_2 < l$ and $m_1 \neq m_2$, since $h(x) = \frac{x}{x+k} - \frac{x-k}{x} - \frac{x}{x+l}$ is strictly decreasing on $[k, l)$;

(3) $\omega_{m_1} \neq \omega_{m_2}$ for $m_1, m_2 \geq l$ and $m_1 \neq m_2$, since $g(x) = \frac{x}{x+k} - \frac{x-k}{x} - \frac{x}{x+l} + \frac{x-l}{x}$ is a strictly decreasing function on $[l, +\infty)$;
(4) $\omega_n < 0$ for $n \geq l$, since $g_n(x) = \frac{n+1}{n+x+1} - \frac{n-x+1}{n+1}$ is strictly increasing on $[k, l]$.

Note that L_a is a reducing subspace for T_φ. We only need to prove the following statements:

(1′) L_a is minimal,
(2″) each nonzero reducing subspace contains L_{a_0} for some $0 \leq a_0 < s$.

By the definition of s, there exist integers M, N such that $k = Ms$ and $l = Ns$ with $0 \leq M < N$. Moreover, there exist nonnegative integers u, v such that $|uk - vl| = s$. Using the same method in Theorem 3.3, we can demonstrate that the reducing subspace generated by z^a equals to L_a.

Let $\mathcal{M}$ be a nonzero reducing subspace and let P be the orthogonal projection from $L_a^2(\mathbb{D})$ onto $\mathcal{M}$. By $PT = TP$, we have $Pz^a = \sum_{b=0}^{s-1} \sum_{n \in \mathbb{Z}_+} c_n z^{b+ns}$ with $\omega_{b+ns} = \omega_a$.

If $k = 0$, then $s = l$ and statements (1) and (4) show that $Pz^a = cz^a$, $c \in \mathbb{C}$.

If $k \neq 0$, then $0 < M < N$ and $\gcd\{M, N\} = 1$. The statements (1)–(4) show that $Pz^a = c_0 z^a + d_0 z^{a+n_0 s} + \sum_{b=0}^{s-1} c_b z^{b+n_b s}$ where $\omega_{a+n_0 s} = \omega_{b+n_b s} = \omega_a$ and $M \leq n_0, n_b < N$ for $0 \leq b \leq s-1$. By $PT_\varphi^* z^a = T_\varphi^* P z^a$, there is

$$Pz^{a+l} = c_0 z^{a+l} + d_0 z^{a+n_0 s+l} + d_0 \frac{\gamma_{a+n_0 s}}{\gamma_{a+n_0 s-k}} z^{a+n_0 s-k} + \sum_{b=0}^{s-1} \Big(c_b z^{b+n_b s+l} + c_b \frac{\gamma_{b+n_b s}}{\gamma_{b+n_b s-k}} z^{b+n_b s-k} \Big).$$

If $d_0 \neq 0$, then the fact $PT = TP$ implies that $\omega_{a+l} = \omega_{a+n_0 s+l}$, which contradicts (3). So $d_0 = 0$. Similarly, we have $c_b = 0$ for $0 \leq b \leq s-1$. Therefore, $Pz^a = cz^a$ for some $c \in \mathbb{C}$. This means that $P(z^a) \neq 0$ if and only if $L_a \subseteq \mathcal{M}$; $P(z^a) = 0$ if and only if $L_a \perp \mathcal{M}$. Let $\mathcal{M} \subseteq L_a$; then $\mathcal{M}$ equals either $\{0\}$ or L_a. Therefore, L_a is minimal. Let $\mathcal{M}$ be a nonzero reducing subspace. Since $L_a^2(\mathbb{D}) = \bigoplus_{a=0}^{s-1} L_a$, there exists $a_0 \in [0, s)$ such that $P(z^{a_0}) \neq 0$; that is, $L_{a_0} \subseteq \mathcal{M}$. Hence, (1′) and (2″) hold and we finish the proof. □

Letting $l = 0$ in Theorem 4.1, we get the following result, which corresponds somewhat to results given by Stessin and Zhu in [19].

Corollary 4.2. *The Toeplitz operator T_{z^k} on $L_a^2(\mathbb{D})$ has $2^k - 2$ nontrivial reducing subspaces. Moreover, $L_a = \overline{\text{span}}\{z^{a+nk} : n \in \mathbb{Z}_+\}$ $(0 \leq a < k)$ are all the minimal reducing subspaces for T_{z^k}, and each reducing subspace is an orthogonal sum of some minimal reducing subspaces.*

As an application of Theorem 4.1, we can also deal with the case $k_i^2 + l_i^2 = 0$ for some $i \in \{1, 2\}$ over the bidisk. Assume $k_2 = l_2 = 0$ (or $k_1 = l_1 = 0$); it is easy to prove the following result.

Corollary 4.3. *Let $N, M \in \mathbb{Z}_+$ with $N^2 + M^2 \neq 0$, and*

$$s = \begin{cases} \gcd\{N, M\}, & NM \neq 0, \\ |N - M|, & NM = 0. \end{cases}$$

Let $\mathcal{M}$ be a reducing subspace for $T_{z^N + \bar{z}^M}$ ($z \in \mathbb{D}$) on $L_a^2(\mathbb{D}^2)$. Then $\mathcal{M}$ is minimal if and only if there exist $0 \leq a < s$ and $f(w) \in L_a^2(\mathbb{D})$ such that $\mathcal{M} = f(w)\overline{\operatorname{span}}\{z^{a+ns} : n \in \mathbb{Z}_+\}$ with $w \in \mathbb{D}$.

Proof. The sufficiency is obvious. We only show the sketch of proof for necessity. If $\mathcal{M} \neq \{0\}$, then there is a nonzero function $h(z, w) = \sum_{k=0}^{\infty} h_k(w) z^k \in \mathcal{M}$, where $h_k \in L_a^2(\mathbb{D})$ and $\sum_{k=0}^{\infty} \|h_k(w)\|^2 \|z^k\|^2 < \infty$. As in Theorem 4.1, we can prove that there exists $0 \leq a < s$ such that $P(h_a(w) z^a) = c h_a(w) z^a \neq 0$. Let $f(w) = h_a(w)$. Therefore, $\mathcal{M} = f(w)\overline{\operatorname{span}}\{z^{a+ns} : n \in \mathbb{Z}_+\}$. □

5. The structure of $\mathcal{V}^*(\varphi)$

In this section, we consider the structure of $\mathcal{V}^*(\varphi)$ both over the bidisk and the unit disk, where $\varphi = z^k + \overline{z}^l$. Let $\mathcal{A}$ denote a von Neumann algebra. Then E is an Abelian projection if $E\mathcal{A}E$ is an Abelian algebra. We consider $\mathcal{A}$ to be homogeneous if there is a family of pairwise orthogonal Abelian projections that are mutually equivalent and whose sum is identity. As it is known, Conway [3] has characterized the structure of homogeneous von Neumann algebras. Recently, Guo and Huang [12] generalized this to the following.

Proposition 5.1 ([12, Corollary 8.2.6])**.** *Let ε denote the set of all minimal projections in a von Neumann algebra $\mathcal{A}$, and suppose that*

$$\bigvee_{E \in \varepsilon} E = I.$$

Then there is a family of $\{\Lambda_i\}$ of subsets of ε such that

(i) *each $\{\Lambda_i\}$ consists of pairwisely orthogonal, mutually equivalent projections in $\mathcal{A}$;*
(ii) *if E', E'' lie in different $\{\Lambda_i\}$, then E' is not equivalent to E'';*
(iii) $\sum_i \sum_{E \in \Lambda_i} = I$.

Consequently, the von Neumann algebra $\mathcal{A}$ is $*$- isomorphic to

$$\bigoplus_i M_{n_i}(\mathbb{C}),$$

where n_i denotes the cardinality of $\{\Lambda_i\}$, allowed to be infinite.

It is known that two reducing subspaces M_1 and M_2 for T_φ are unitarily equivalent if and only if P_{M_1} and P_{M_2} are equivalent in $\mathcal{V}^*(\varphi)$; that is, there is a partial isometry V in $\mathcal{V}^*(\varphi)$ such that

$$V^*V = P_{M_1}, \qquad VV^* = P_{M_2}.$$

Now, we are ready to give the main results in this section as follows:

Theorem 5.2. *Given that $\varphi(z) = z^k + \bar{z}^l$, where $k, l \in \mathbb{Z}_+^2$, $k \neq l$ and $k_i^2 + l_i^2 \neq 0$ $(i = 1, 2)$. Then $\mathcal{V}^*(\varphi)$ is a Type I von Neumann algebra. Furthermore, the following statements hold.*

(i) *If $k_1 l_2 \neq k_2 l_1$, then $\mathcal{V}^*(\varphi)$ is Abelian and is $*$-isomorphic to*

$$\bigoplus_{i=1}^{j} \mathbb{C},$$

where $j = |l_1 k_2 - l_2 k_1|$.

(ii) *If $k_1 l_2 = k_2 l_1$ and $s = (s_1, s_2)$ is defined as in (3.1), then $\mathcal{V}^*(\varphi) = \mathcal{V}^*(z^s)$ and $\mathcal{V}^*(\varphi)$ is never Abelian. Moreover, if $s_1 = s_2 = r$, then $\mathcal{V}^*(\varphi)$ is $*$-isomorphic to*

$$\bigoplus_{j=1}^{\infty} M_2(\mathbb{C}) \oplus \bigoplus_{i=1}^{r} \mathbb{C};$$

if $s_1 \neq s_2$, then $\mathcal{V}^*(\varphi)$ is $*$-isomorphic to the direct sum of countably many $M_2(\mathbb{C}) \oplus \mathbb{C}$.

Proof. Let P_m denote the orthogonal projection from $L_a^2(\mathbb{D}^2)$ onto L_m. By the definition of Δ, we have $\sum_{m \in \Delta} P_m = I$. Kadison and Ringrose [14] gave the fact that if E is a minimal projection in von Neumann algebra $\mathcal{R}$, then E is an Abelian projection in $\mathcal{R}$. Thus, every P_m is an Abelian projection. What's more, $\mathcal{V}^*(\varphi)$ is type I.

Let Λ_m denote the set of the orthogonal projections which are unitarily equivalent to P_m. If $k_1 l_2 \neq k_2 l_1$, Proposition 3.5(i) shows that $\Lambda_m = \{P_m\}$ for $m \in \Delta$ and $\operatorname{Card} \Delta = j$.

If $k_1 l_2 = k_2 l_1$ and $s_1 = s_2 = r$, then Corollary 3.6 shows that L_m is unitarily equivalent to $L_{m'}$ if and only if $m = m'$ whenever $m_1 = m_2$. That is, $\operatorname{Card} \Lambda_m = 1$ for $0 \leq m_1 = m_2 \leq r$, $\operatorname{Card} \Lambda_m = 2$ for $m \in \Delta$ and $m_1 \neq m_2$.

If $k_1 l_2 = k_2 l_1$ and $s_1 \neq s_2$, by Proposition 3.5(ii), we have $\operatorname{Card} \Lambda_m = 1$ for $m' \notin \mathbb{Z}_+^2$ and $\operatorname{Card} \Lambda_m = 2$ for $m' \in \mathbb{Z}_+^2$.

Therefore, by Proposition 5.1, we proved (ii). □

Remark 5.3. If $k_2 = l_1 = 0$, the statement (i) in Theorem 5.2 identifies with the case $\alpha = \beta$ of the main result in [5].

On the Bergman space over the unit disk, from Theorem 4.1 and Corollary 4.2, it is interesting to note that $\mathcal{V}^*(z^k + \bar{z}^l) = \mathcal{V}^*(z^s)$ for $k \neq l$. Moreover, in the proof of Theorem 4.1, we have proved that if $U \in \mathcal{V}^*(z^k + \bar{z}^l)$ satisfying $U|_{L_a} : L_a \to L_{a'}$ is unitary, then there is $c \in \mathbb{C}$ depending on U such that $Uz^a = cz^a$. That is, L_a and $L_{a'}$ are unitarily equivalent if and only if $a = a'$. Then we have the following result.

Theorem 5.4. *Given $\varphi(z) = z^k + \bar{z}^l$, $z \in \mathbb{D}$ with $k, l \in \mathbb{Z}_+$, $k \neq l$. Let $s = \gcd\{k, l\}$ for $kl \neq 0$; $s = |k - l|$ for $kl = 0$. Then $\mathcal{V}^*(\varphi)$ is an Abelian Type I von Neumann algebra, and it is $*$-isomorphic to*

$$\bigoplus_{i=1}^{s} \mathbb{C}.$$

For the case that $k = l$, since $T_{z^k+\bar{z}^k}$ is normal on $L_a^2(\mathbb{D}^d)$ ($d \in \mathbb{N}$), the von Neumann algebra $\mathcal{W}^*(z^k+\bar{z}^k)$ generated by $T_{z^k+\bar{z}^k}$ is Abelian. Thus, $\mathcal{W}^*(z^k+\bar{z}^k)$ is Type I. One of the main results in [14] asserts that, if $\mathcal{R}$ is a von Neumann algebra acting on a Hilbert space $\mathcal{H}$, then the commutant $\mathcal{R}'$ is of type I (or II, or III) when $\mathcal{R}$ has the same property. Therefore, the following result holds.

Theorem 5.5. *$\mathcal{V}^*(z^k+\bar{z}^k)$ on $L_a^2(\mathbb{D}^d)$ ($d \in \mathbb{N}$) is a Type I von Neumann algebra.*

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