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LINEAR DEPENDENCY OF TRANSLATIONS AND SQUARE-INTEGRABLE REPRESENTATIONS

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Abstract. Let G be a locally compact group. We examine the problem of determining when nonzero functions in $L^2(G)$ have linearly independent left translations. In particular, we establish some results for the case when G has an irreducible, square-integrable, unitary representation. We apply these results to the special cases of the affine group, the shearlet group, and the Weyl–Heisenberg group. We also investigate the case when G has an abelian, closed subgroup of finite index.

1. Introduction

Let G be a locally compact Hausdorff group with a fixed left Haar measure μ. Denote by $L^p(G)$ the set of complex-valued functions on G that are p-integrable with respect to μ, where $1 < p \in \mathbb{R}$. As usual, the functions are identified in $L^p(G)$ when they differ only on a set of μ-measure zero. We shall write $\| \cdot \|_p$ to indicate the usual L^p-norm on $L^p(G)$. The *regular representation* of G on $L^p(G)$ is given by $L(g)f(x) = f(g^{-1}x)$, where $g, x \in G$, and $f \in L^p(G)$. The function $L(g)f$ is known as the left translation of f by g (many papers use the word "translate" instead of "translation"). In [22], Rosenblatt investigated the problem of determining when the left translations of a nonzero function f in $L^2(G)$ are linearly independent. In other words, when can there be a nonzero function $f \in L^2(G)$, some nonzero complex constants c_k, and distinct elements

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$g_k \in G$, where $1 \le k \le n, k \in \mathbb{N}$ (the positive integers) such that

$$\sum_{k=1}^{n} c_k L(g_k) f = 0? \tag{1.1}$$

In the introduction of [22], it was shown that, if G has a nontrivial element of finite order, then there is a nonzero element in $L^2(G)$ that has a linear dependency among its left translations. Thus, when trying to find nontrivial functions that satisfy (1.1), it is more interesting to consider groups for which all nonidentity elements have infinite order. For $G = \mathbb{R}^n$, it is known that every nonzero function in $L^2(\mathbb{R}^n)$ has no linear dependency among its left translations. Rosenblatt attacked (1.1) by trying to determine if there is a relationship between the linear independence of the translations of functions in $L^2(G)$ and the linear independence of an element and its images under the action of G in an irreducible representation of G. To gain insights into possible connections between these concepts, he computed examples for specific groups, such as the Heisenberg group and the affine group. In fact, he showed that these groups have irreducible representations that are intimately related to a time-frequency equation. Recall that an equation of the form

$$\sum_{k=1}^{n} c_k \exp\bigl(ib_k h(t)\bigr) f(a_k + t) = 0 \tag{1.2}$$

is a time-frequency equation, where $a_k, b_k \in \mathbb{R}, f \in L^2(\mathbb{R})$, and $h : \mathbb{R} \to \mathbb{R}$ is a nontrivial function. The case $h(t) = t$ corresponds to the Heisenberg group, and $h(t) = e^t$ corresponds to the affine group.

Now, suppose that G is a group that has an irreducible representation related to (1.2). Rosenblatt wondered if there existed a nontrivial $f \in L^2(\mathbb{R})$ that would satisfy equation (1.2) such that f would produce a nonzero $F \in L^2(G)$ with a linear dependency among its left translations. He then showed that there exists a nonzero $f \in L^2(\mathbb{R})$ that satisfies the following time-frequency equation:

$$Cf(t) = f(t - \log 2) + \exp\Bigl(-\frac{i}{2}e^t\Bigr) f(t - \log 2), \tag{1.3}$$

where C is a constant (see [22, Proposition 3.1]). This time-frequency equation corresponds to the affine group A case since $h(t) = e^t$. This offers some hope that there might be a nonzero function in $L^2(A)$ that has a linear dependency among its left translations. However, there is no clear principle that can be used to show the existence of such a function given a nontrivial f that satisfies (1.3). Using the proof of the existence of f that satisfies (1.3) as a guide, a nonzero F in $L^2(A)$ with linearly dependent left translations was shown to exist (see [22, Proposition 3.2]).

Even less is known about the Heisenberg group $H_n, n \in \mathbb{N}$. The relevant time-frequency equation, which has been intensely studied in the context of Gabor analysis, is

$$\sum_{k=1}^{m} c_k e^{2\pi i b_k \cdot t} f(t + a_k) = 0, \tag{1.4}$$

where c_k are nonzero constants, $a_k, b_k \in \mathbb{R}^n$, and $f \in L^2(\mathbb{R}^n)$. Linnell showed that $f = 0$ is the only solution satisfying (1.4), when the subgroup generated by (a_k, b_k) $(k = 1, \ldots, m)$ is discrete. This gave a partial answer to a conjecture posed by Heil, Ramanathan, and Topiwala [11, p. 2790] that $f = 0$ is the only solution to (1.4) when $n = 1$. As far as we know, the conjecture is still open.

The motivation for this paper is to give a clearer picture of the link between the linear independence of an element and its images under the action of G in an irreducible representation of G and the linear independence of the left translations of a function in $L^2(G)$. In Section 2 we will prove the following result.

Proposition 1.1. *Let G be a locally compact group, and let π be an irreducible, unitary, square-integrable representation of G on a Hilbert space $\mathcal{H}_\pi$. If there exists a nonzero v in $\mathcal{H}_\pi$ such that*

$$\sum_{k=1}^{n} c_k \pi(g_k) v = 0$$

for some nonzero constants $c_k \in \mathbb{C}$ and $g_k \in G$, then there exists a nonzero $F \in L^2(G)$ that satisfies

$$\sum_{k=1}^{n} c_k L(g_k) F = 0.$$

In particular, if there exists a nonzero v in $\mathcal{H}_\pi$ with linearly dependent translations, then there exists a nonzero F in $L^2(G)$ with linearly dependent translations.

In Section 3 we will use Proposition 1.1 to construct explicit examples of nontrivial functions in $L^2(A)$, where A is the affine group and where they have a linear dependency among their left translations.

In Section 4 we investigate the case where G is a discrete group. We show that there is a connection between the linear independence of the left translations of a nonzero function in $\ell^2(G)$ and the strong Atiyah conjecture. In fact, we briefly review the strong Atiyah conjecture in Section 4.

After considering the discrete group case in Section 4, we shall return to studying the linear dependency problem for groups that satisfy our original hypotheses. Let K be a subgroup of a group G. If $k \in K, x \in G$, and $f \in L^2(G)$, then we say that

$$L(k)f(x) = f(k^{-1}x)$$

is a *left K-translation* of f. In Section 5, we prove the following main result.

Theorem 1.2. *Let G be a locally compact, σ-compact group, and let K be a torsion-free discrete subgroup of G. If K satisfies the strong Atiyah conjecture, then each nonzero function in $L^2(G)$ has linearly independent K-translations.*

In Section 6, we study the Weyl–Heisenberg group $\tilde{H}_n$, a variant of the Heisenberg group, H_n. The group $\tilde{H}_n$ is of interest to us because it has an irreducible unitary representation on $L^2(\mathbb{R}^n)$, the Schrödinger representation, which is square-integrable. Furthermore, the time-frequency equation (1.4) is related to the Schrödinger representation. Now, if K is a torsion-free discrete subgroup of $\tilde{H}_n$, then by Theorem 1.2 every nonzero element in $L^2(\tilde{H}_n)$ has linearly independent left K-translations. It will then follow from Proposition 1.1 that, if the subgroup of $\mathbb{R}^{2n}$ generated by $(a_k, b_k), 1 \leq k \leq m$ is discrete and the product $a_h \cdot b_k \in \mathbb{Q}$ for all h, k, then $f = 0$ is the only solution to (1.4). This gives a new proof of a special case of [16, Proposition 1.3] and sheds new insight on the problem.

In Section 7 we consider the problem of determining the linear independence of the left translations of a function in $L^2(S)$, where S is the shearlet group. By using Proposition 1.1, we show that this problem is related to the question of determining the linear independence of a shearlet system of a function in $L^2(\mathbb{R}^2)$, which was recently studied in [19].

In the last section of this paper we investigate the linear independence of left translations of functions in $L^p(G)$ for virtually abelian groups G with no nontrivial compact subgroups. In particular, we generalize [5, Theorem 1.2].

2. Proof of Proposition 1.1

In this section we will prove Proposition 1.1. First we prepare some necessary preliminaries. A *unitary representation* of G is a homomorphism π from G into the group $U(\mathcal{H}_\pi)$ of unitary operators on a nonzero Hilbert space $\mathcal{H}_\pi$ that is continuous with respect to the strong operator topology. This means that $\pi : G \to U(\mathcal{H}_\pi)$ satisfies $\pi(xy) = \pi(x)\pi(y), \pi(x^{-1}) = \pi(x)^{-1} = \pi(x)^*$, and $x \to \pi(x)u$ is continuous from G to $\mathcal{H}_\pi$ for each $u \in \mathcal{H}_\pi$. A closed subspace W of $\mathcal{H}_\pi$ is said to be *invariant* if $\pi(x)W \subseteq W$ for all $x \in G$. If the only invariant subspaces of $\mathcal{H}_\pi$ are $\mathcal{H}_\pi$ and 0, then π is said to be an *irreducible representation* of G. A representation is said to be *reducible* if it is not irreducible. If π_1 and π_2 are unitary representations of G, then an *intertwining operator* for π_1 and π_2 is a bounded linear map $T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ that satisfies $T\pi_1(g) = \pi_2(g)T$ for all $g \in G$. Throughout the paper, we assume that the inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}_\pi$ is conjugate-linear in the second component. If $u, v \in \mathcal{H}_\pi$, then a *matrix coefficient* of π is the function $F_{v,u} : G \to \mathbb{C}$ defined by

$$F_{v,u}(x) = \langle v, \pi(x)u \rangle.$$

We indicate $F_{u,u}$ by F_u. Moreover we say that u is *admissible* if $F_u \in L^2(G)$. An irreducible representation π is said to be *square-integrable* if there exists a nonzero $u \in \mathcal{H}_\pi$ such that u is admissible. The set of admissible elements in $\mathcal{H}_\pi$ will be denoted by $\mathrm{Ad}(\mathcal{H}_\pi)$. A consequence of π being irreducible is that, if there is a nonzero admissible element in $\mathcal{H}_\pi$, then $\mathrm{Ad}(\mathcal{H}_\pi)$ is dense in $\mathcal{H}_\pi$. In fact, $\mathrm{Ad}(\mathcal{H}_\pi) = \mathcal{H}_\pi$ if G is unimodular, in addition to $\mathrm{Ad}(\mathcal{H}_\pi)$ containing a nonzero element (see [23, Lemma 4.5.9.1]). According to [10, Theorem 3.1] there exists a self-adjoint positive operator $C : \mathrm{Ad}(\mathcal{H}_\pi) \to \mathcal{H}_\pi$ such that, if $u \in \mathrm{Ad}(\mathcal{H}_\pi)$, and $v \in \mathcal{H}_\pi$, then

$$\begin{aligned} \int_G \left|\langle v, \pi(x)u \rangle\right|^2 d\mu &= \int_G \langle v, \pi(x)u \rangle \overline{\langle v, \pi(x)u \rangle}\, d\mu \\ &= \|Cu\|^2 \|v\|^2, \end{aligned}$$

where $\|\cdot\|$ denotes the $\mathcal{H}_\pi$-norm. Thus, if $u \in \mathrm{Ad}(\mathcal{H}_\pi)$, then $F_{v,u} \in L^2(G)$ for all $v \in \mathcal{H}_\pi$.

We now prove Proposition 1.1. Suppose that there exists a nonzero $v \in \mathcal{H}_\pi$ for which there exists a linear dependency among some of the elements $\pi(g)v$, where $g \in G$. So there exist nonzero constants $c_1, c_2, \ldots, c_n$ and elements $g_1, g_2, \ldots, g_n$ in G with $\pi(g_j) \neq \pi(g_k)$ if $j \neq k$ such that

$$\sum_{k=1}^{n} c_k \pi(g_k) v = 0. \tag{2.1}$$

Let $u \in \mathrm{Ad}(\mathcal{H}_\pi)$. Then $0 \neq F_{v,u} \in L^2(G)$, and, since π is unitary, we have $\langle \pi(g)v, \pi(x)u\rangle = \langle v, \pi(g^{-1}x)u\rangle$ for all x and g in G. In other words, the continuous linear map $v \mapsto F_{v,u} : \mathcal{H} \to L^2(G)$ intertwines π with the regular representation L. Combining this observation with (2.1) yields that, for all $x \in G$,

$$\sum_{k=1}^{n} c_k L(g_k) F_{v,u}(x) = 0.$$

Thus, $F_{v,u}$ has linearly dependent left translations. The proof of Proposition 1.1 is now complete.

3. The affine group

In this section we give examples of nonzero functions in $L^2(G)$, where G is the affine group and where there is a linear dependency among some of its left translations. Let $\mathbb{R}$ denote the real numbers, and let $\mathbb{R}^*$ be the set $\mathbb{R} \setminus \{0\}$. Recall that $\mathbb{R}$ is a group under addition and that $\mathbb{R}^*$ is a group with respect to multiplication. The affine group, also known as the $ax + b$ group, is defined to be the semidirect product of $\mathbb{R}^*$ and $\mathbb{R}$; that is,

$$G = \mathbb{R}^* \rtimes \mathbb{R}.$$

Let (a, b) and (c, d) be elements of G. The group operation on G is given by $(a, b)(c, d) = (ac, b + ad)$. The identity element of G is $(1, 0)$, and $(a, b)^{-1} = (a^{-1}, -a^{-1}b)$. The left and right Haar measures on G are $d\mu = \frac{da\,db}{a^2}$ and $d\mu = \frac{da\,db}{|a|}$, respectively. Thus, G is a nonunimodular group because the right and left Haar measures do not agree. So $f \in L^2(G)$ if and only if

$$\int_{\mathbb{R}} \int_{\mathbb{R}^*} \left| f(a, b) \right|^2 \frac{da\,db}{a^2} < \infty.$$

An irreducible unitary representation of G can be defined on $L^2(\mathbb{R})$ by

$$\pi(a, b) f(x) = |a|^{-1/2} f\Big(\frac{x - b}{a}\Big),$$

where $(a, b) \in G$ and $f \in L^2(\mathbb{R})$. Before we show that π is square-integrable, we recall some facts from Fourier analysis.

If we let $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then the Fourier transform of f is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x}\, dx,$$

where $\xi \in \mathbb{R}$. The Fourier transform can be extended to a unitary operator on $L^2(\mathbb{R})$. For $y \in \mathbb{R}$ we also have the following unitary operators on $L^2(\mathbb{R})$:

$$T_y f(x) = f(x-y), \qquad E_y f(x) = e^{2\pi i y x} f(x)$$

and

$$D_y f(x) = |y|^{-1/2} f\Big(\frac{x}{y}\Big) \quad (y \neq 0).$$

Given $f, g \in L^2(\mathbb{R})$, the following relations are true: $\langle f, T_y g\rangle = \langle T_{-y} f, g\rangle$, $\langle f, E_y g\rangle = \langle E_{-y} f, g\rangle$, and $\langle f, D_y g\rangle = \langle D_{y^{-1}} f, g\rangle$. Furthermore, $\widehat{T_y f} = E_{-y}\widehat{f}$ and $\widehat{D_y f} = D_{y^{-1}}\widehat{f}$. Observe that, for $(a,b) \in G$ and $f \in L^2(\mathbb{R})$,

$$\pi(a,b)f(x) = T_b D_a f(x) = |a|^{-1/2} f\Big(\frac{x-b}{a}\Big).$$

Using the above relations it can be shown that, for $f \in L^2(\mathbb{R})$,

$$\int_G \langle f, \pi(a,b)f\rangle \, d\mu = \int_{\mathbb{R}} \int_{\mathbb{R}^*} \big|\langle f, T_b D_a f\rangle\big|^2 \frac{da\, db}{a^2} = \|f\|_2^2 \int_{\mathbb{R}^*} \frac{|\widehat{f}(\xi)|^2}{|\xi|}\, d\xi$$

(see [12, Theorem 3.3.5]). Thus, $f \in L^2(\mathbb{R})$ is admissible if $\int_{\mathbb{R}^*} \frac{|\widehat{f}(\xi)|^2}{|\xi|}\, d\xi < \infty$. Combining [20, Proposition 2.2.5] with [20, Example 2.2.7], the function $f(x) = \sqrt{2\pi} x e^{-\pi x^2}$ satisfies this criterion since $\widehat{f}(\xi) = -\sqrt{2\pi} i \xi e^{-\pi\xi^2}$. Hence, π is a square-integrable, irreducible unitary representation of the affine group G. We are now ready to construct a nonzero function in $L^2(G)$ that has linearly dependent left translations.

Let $\chi_{[0,1)}$ be the characteristic function on the interval $[0,1)$. It follows from the *refinement equation*

$$\chi_{[0,1)}(x) = \chi_{[0,1)}(2x) + \chi_{[0,1)}(2x-1)$$

that

$$\pi(1,0)\chi_{[0,1)}(x) = 2^{-1/2}\pi(2^{-1},0)\chi_{[0,1)}(x) + 2^{-1/2}\pi(2^{-1},2^{-1})\chi_{[0,1)}(x). \tag{3.1}$$

Thus, $\chi_{[0,1)}$ has a linear dependency among the $\pi(a,b)\chi_{[0,1)}$, where $(a,b) \in G$. We now use $\chi_{[0,1)}$ to construct a nontrivial function in $L^2(G)$ that has linearly dependent left translations. Let $f \in L^2(\mathbb{R})$ be an admissible function for π, and let $(a,b) \in G$. Then, the function

$$F(a,b) = \big\langle \chi_{[0,1)}, \pi(a,b)f\big\rangle = \int_0^1 |a|^{-1/2}\overline{f\Big(\frac{x-b}{a}\Big)}\, dx$$

belongs to $L^2(G)$. By Proposition 1.1, $F(a,b)$ has linearly dependent left translations. More specifically,

$$L(1,0)F(a,b) = 2^{-1/2}L(2^{-1},0)F(a,b) + 2^{-1/2}L(2^{-1},2^{-1})F(a,b),$$

which translates to

$$\int_0^1 |a|^{-1/2}\overline{f\Big(\frac{x-b}{a}\Big)}\, dx = \int_0^{1/2} |a|^{-1/2}\overline{f\Big(\frac{x-b}{a}\Big)}\, dx + \int_{1/2}^1 |a|^{-1/2}\overline{f\Big(\frac{x-b}{a}\Big)}\, dx.$$

Equation (3.1) above was used in the proof of [22, Proposition 3.1] to show that the time-frequency equation (1.3) with $C = \sqrt{2}$ has a nonzero solution. Basically, equation (1.3) is a reinterpretation of the above refinement equation where the representation π is replaced by an equivalent representation (see [22, Section 3]).

We now turn our attention to the subgroup K of the affine group G consisting of all $(a, b) \in G$ for which $a > 0$. This was the version of the affine group considered in [22]. Up to unitary equivalence there are two irreducible unitary infinite-dimensional representations of K (see [7, Section 6.7]). One of these representations is given by

$$\pi^+(a, b)f(x) = a^{1/2}e^{2\pi ibx}f(ax) = E_b D_{a^{-1}} f(x),$$

where $(a, b) \in K$ and $f \in L^2(0, \infty)$. The representation π^+ is square-integrable. We are now ready to produce a nontrivial function in $L^2(K)$ that has linearly dependent left translations. From (3.1), we have

$$\chi_{[0,1)} = 2^{-1/2} D_{2^{-1}} \chi_{[0,1)} + 2^{-1/2} T_{2^{-1}} D_{2^{-1}} \chi_{[0,1)}.$$

By taking Fourier transforms we obtain

$$\begin{aligned}\widehat{\chi}_{[0,1)}(\xi) &= 2^{-1/2} D_2 \widehat{\chi}_{[0,1)}(\xi) + 2^{-1/2} E_{-2^{-1}} D_2 \widehat{\chi}_{[0,1)}(\xi)\\ &= 2^{-1/2}\pi^+(2^{-1}, 0)\widehat{\chi}_{[0,1)}(\xi) + 2^{-1/2}\pi^+(2^{-1}, -2^{-1})\widehat{\chi}_{[0,1)}(\xi).\end{aligned}$$

Hence there is a linear dependency among the $\pi^+(a, b)\widehat{\chi}_{[0,1)}$, where $(a, b) \in K$. It follows from

$$\widehat{\chi}_{[0,1)}(\xi) = \frac{e^{-2\pi i\xi} - 1}{-2\pi i\xi}$$

that $\widehat{\chi}_{[0,1)} \in L^2(0, \infty)$. Pick an admissible function $f \in L^2(0, \infty)$ for π^+. Then, the function

$$F(a, b) = \left\langle \widehat{\chi}_{[0,1)}, \pi^+(a, b)f \right\rangle = \int_0^\infty \widehat{\chi}_{[0,1)}(\xi) a^{1/2} e^{-2\pi ib\xi}\overline{f(\xi)}\, d\xi$$

is a member of $L^2(K)$. Proposition 1.1 yields the following linear dependency in $L^2(K)$ among the left translations of $F(a, b)$:

$$F(a, b) = 2^{-1/2} L(2^{-1}, 0)F(a, b) + 2^{-1/2} L(2^{-1}, -2^{-1})F(a, b).$$

This equation can easily be verified using the relations

$$\widehat{\chi}_{[0,1)}\Big(\frac{\xi}{2}\Big)(1 + e^{-\pi i\xi}) = \frac{(e^{-\pi i\xi} - 1)(1 + e^{-\pi i\xi})}{-2\pi i\xi} = \widehat{\chi}_{[0,1)}(\xi).$$

4. Discrete groups and the Atiyah conjecture

In this section we connect the problem of linear independence of left translations of a function to the Atiyah conjecture. Unless otherwise stated, we make the assumption that all groups in this section are discrete. Let f be a complex-valued function on a group G. We will represent f as a formal sum $\sum_{g\in G} a_g g$, where $a_g \in \mathbb{C}$ and $f(g) = a_g$. Denote by $\ell^2(G)$ those formal sums for which $\sum_{g\in G} |a_g|^2 < \infty$, and $\mathbb{C}G$, the group ring of G over $\mathbb{C}$, will consist of all formal sums that satisfy $a_g = 0$ for all but finitely many g. The group ring $\mathbb{C}G$ can also be thought of as the set of all functions on G with compact support, and $\ell^2(G)$ is a Hilbert space with Hilbert basis $\{g \mid g \in G\}$. If $g \in G$ and $f = \sum_{x\in G} a_x x \in \ell^2(G)$, then the left translation of f by g is represented by the formal sum $\sum_{x\in G} a_{g^{-1}x} x$ since $L(g)f(x) = f(g^{-1}x)$. Suppose that $\alpha = \sum_{g\in G} a_g g \in \mathbb{C}G$ and that $f = \sum_{g\in G} b_g g \in \ell^2(G)$. We define a multiplication, known as convolution, $\mathbb{C}G \times \ell^2(G) \to \ell^2(G)$ by

$$\alpha * f = \sum_{g,h\in G} a_g b_h gh = \sum_{g\in G} \Big(\sum_{h\in G} a_{gh^{-1}} b_h \Big) g.$$

Sometimes, we will write αf instead of $\alpha * f$. Left multiplication by an element of $\mathbb{C}G$ is a bounded linear operator on $\ell^2(G)$. So $\mathbb{C}G$ can be considered as a subring of $\mathcal{B}(\ell^2(G))$, the space of bounded linear operators on $\ell^2(G)$. We say that G is *torsion-free* if the only element of finite order in G is the identity element of G. The strong Atiyah conjecture for the group G is concerned with the values that the L^2-Betti numbers can take, and it implies the following conjecture, which can also be considered as an analytic version of the zero divisor conjecture.

Conjecture 4.1. *Let G be a torsion-free group. If $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq f \in \ell^2(G)$, then $\alpha * f \neq 0$.*

The hypothesis that G is torsion-free is essential. Indeed, let 1 be the identity element of G, and let $g \in G$ such that $g \neq 1$, and $g^n = 1$ for some $n \in \mathbb{N}$. Then, $(1 + g + \cdots + g^{n-1}) * (1 - g) = 0$. The Atiyah conjecture is important in the study of von Neumann dimension (see [3], [14], [15], and [18, Section 10]). In particular, Conjecture 4.1 is known for free groups, left-ordered groups, and elementary amenable groups.

The following proposition gives the relation between zero divisors and the linear independence of left translations of a function.

Proposition 4.2. *Let G be a discrete group, and let $f \in \ell^2(G)$. Then, f has linearly independent left translations if and only if $\alpha * f \neq 0$ for all nonzero $\alpha \in \mathbb{C}G$.*

Proof. Let $g \in G$, and let $f = \sum_{x\in G} a_x x \in \ell^2(G)$. Then,

$$g * f = \sum_{x\in G} a_x gx = \sum_{x\in G} a_{g^{-1}x} x = L(g)f.$$

Consequently, if $g_1, \ldots, g_n \in G$ are distinct, and $c_1, \ldots, c_n$ are constants, then

$$\sum_{k=1}^n c_k L(g_k) f = \sum_{k=1}^n c_k g_k * f = \Big(\sum_{k=1}^n c_k g_k \Big) * f.$$

The proposition now follows since $\sum_{k=1}^n c_k g_k \in \mathbb{C}G$. $\square$

As we mentioned in Section 3, there are nontrivial, square-integrable functions on the affine group that have a linear dependency among their left translations. Since all nonidentity elements of the affine group have infinite order, by taking a discrete subgroup D of the affine group, such as $1 \rtimes \mathbb{Z}$, we might be able to construct a nonzero function in $\ell^2(D)$ that has a linear dependency among its left translations. It would then be an immediate consequence of Proposition 4.2 that

Conjecture 4.1 is false. However, for discrete subgroups D of the affine group, it is not true that there exists a nonzero function in $\ell^2(D)$ with a linear dependency among its left translations. Indeed, it is well known that the affine group with $a > 0$ is a connected solvable Lie group, and all discrete subgroups of connected solvable Lie groups are polycyclic (see [24, Proposition 4.1]). According to [14, Theorem 2], Conjecture 4.1 is true for torsion-free elementary amenable groups, a class of groups that contains all torsion-free polycyclic groups.

5. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. Let G be a σ-compact group, let $g_1, \dots, g_n$ be elements of G, and let $c_1, \dots, c_n \in \mathbb{C}$ be some constants. Set $\theta = \sum_{k=1}^n c_k L(g_k)$. Hence $\theta \in \mathcal{B}(L^2(G))$, the set of bounded linear operators on $L^2(G)$. Define

$$\mathbb{C}G = \Big\{ \sum_{g \in G} a_g L(g) \;\Big|\; a_g = 0 \text{ for all but finitely many } g \in G \Big\}.$$

Note that there exists a nonzero $f \in L^2(G)$ with linearly dependent left translations if and only if there exists a nonzero $\theta \in \mathbb{C}G$ with $\theta f = 0$.

For the rest of this section, H will denote a discrete subgroup of G. The subgroup H acts on G by left multiplication. By [1, Proposition B.2.4], there exists a Borel fundamental domain for this action of H on G. In fact, there exists a Borel subset B of G such that $hB \cap B = \emptyset$ for all $h \in H \setminus 1$ and $G = HB$ (thus, B is a system of right coset representatives of H in G which is also a Borel subset). If X is a Borel subset of G, then we identify $L^2(X)$ with the subspace of $L^2(G)$ consisting of all functions on G whose support is contained in X.

Let $\{q_i \mid i \in \mathcal{I}\}$ be a Hilbert basis for $L^2(B)$. We prove that $S := \{L(h)q_i \mid h \in H, i \in \mathcal{I}\}$ is a Hilbert basis for $L^2(G)$. First we show that S is orthonormal. Write hq_i for $L(h)q_i$. If $h \neq k$, then $\langle hq_i, kq_j\rangle = 0$ because the supports of hq_i and hq_j are contained in hB and kB, respectively, which are disjoint subsets of G. On the other hand, if $h = k$, then $\langle hq_i, hq_j\rangle = \langle q_i, q_j\rangle$. This proves that S is orthonormal. Finally we show that the closure of the linear span $\overline{S}$ of S is $L^2(G)$. Denote by χ_{hB} the characteristic function on hB. If $f \in L^2(G)$, then we may write $f = \sum_{h \in H} f_h$, where $f_h = \chi_{hB} f$ (hence, f_h has support contained in hB). Thus, it is sufficient to show that $L^2(hB) \subseteq \overline{S}$. Since $\overline{S}$ is invariant under H, it will be sufficient to show that $L^2(B) \subseteq \overline{S}$, which is obvious because the q_i form a Hilbert basis for $L^2(B)$.

For $i \in \mathcal{I}$, let $S_i = \{L(h)q_i \mid h \in H\}$, and let $\overline{S}_i$ denote the closure of the linear span of S_i. Now, $L^2(G) = \bigoplus_{i \in \mathcal{I}} \overline{S}_i$, where $\bigoplus$ indicates the Hilbert direct sum. The spaces $\overline{S}_i$ are isometric to $\ell^2(H)$. Indeed, define a map T_i from the Hilbert basis S_i of $\overline{S}_i$ to the Hilbert basis H of $\ell^2(H)$ via $L(h)q_i \mapsto h$. Extend T_i linearly to obtain an isometry $T_i : \overline{S}_i \to \ell^2(H)$. Moreover, the isometry T_i intertwines the natural left actions of H on $\overline{S}_i$ and $\ell^2(H)$. Also, let π_i denote the projection of $L^2(G)$ onto $\overline{S}_i$. Then, π_i also intertwines the natural left actions of H on $L^2(G)$ and $\overline{S}_i$. Now, suppose that there exists a nonzero $f \in L^2(G)$ and a nonzero $\theta \in \mathbb{C}H$ that satisfies $\theta f = 0$. Then, $k := T_i \pi_i f \neq 0$ for some i, and $\theta * k = \theta k = 0$ because $T_i \pi_i$ commutes with $\mathbb{C}H$. Furthermore, $k \in l^2(H)$. We can summarize the above as follows.

Proposition 5.1. *Let H be a discrete subgroup of the σ-compact locally compact group G, and let $\theta \in \mathbb{C}H$. If $\theta f = 0$ for some nonzero $f \in L^2(G)$, then $\theta * k = 0$ for some nonzero $k \in \ell^2(H)$.*

Now, let H be a torsion-free group which satisfies the strong Atiyah conjecture, for example, a torsion-free elementary amenable group. Then, for $0 \neq \theta \in \mathbb{C}H$, we know that $\theta * k \neq 0$ for all nonzero $k \in \ell^2(H)$. It follows from Proposition 5.1 that $\theta f \neq 0$ for all nonzero $f \in L^2(G)$; in other words, any nonzero element of $L^2(G)$ has linearly independent H-translations. The proof of Theorem 1.2 is now complete.

Similarly we have the following result.

Theorem 5.2. *Let G be a locally compact σ-compact group, and let H be an amenable discrete subgroup of G. If α is a non-zerodivisor in $\mathbb{C}H$, then $\alpha * f \neq 0$ for all nonzero $f \in L^2(G)$.*

Proof. Since $\alpha\beta \neq 0$ for all nonzero $\beta \in \mathbb{C}H$, then it follows that $\alpha\beta \neq 0$ for all nonzero $\beta \in \ell^2(H)$ by [6, Theorem] (see also [18, Theorem 6.37]). The result now follows from Proposition 5.1. □

We saw in Section 3 that for the affine group A there exists a nonzero function f in $L^2(A)$ with linearly dependent left translations. However, $\mathbb{Z}$ can be identified with the discrete subgroup $1 \rtimes \mathbb{Z}$ of A. A direct consequence of Theorem 1.2 is as follows.

Corollary 5.3. *Let A be the affine group. Then every nonzero f in $L^2(A)$ has linearly independent left $\mathbb{Z}$-translations.*

As noted in Section 4, if H is a discrete group, then we may regard $\mathbb{C}H$ as a subalgebra of $\mathcal{B}(\ell^2(H))$. Recall that the reduced group C^*-algebra of H, denoted by $\mathrm{C}^*_\mathrm{r}(H)$, is the operator norm closure of $\mathbb{C}H$ in $\mathcal{B}(\ell^2(H))$ and that the group von Neumann algebra of H, denoted by $\mathcal{N}(H)$, is the weak closure of $\mathbb{C}H$ in $\mathcal{B}(\ell^2(H))$. We can also identify the norm and weak closures of $\mathbb{C}H$ in $\mathcal{B}(L^2(G))$ with $\mathrm{C}^*_\mathrm{r}(H)$ and $\mathcal{N}(H)$, respectively. Though this is not needed in the sequel, we hope it may be useful to record this.

For $\theta \in \mathcal{B}(L^2(G))$ or $\mathcal{B}(\ell^2(H))$, let $\|\theta\|$ or $\|\theta\|'$, respectively, denote the corresponding operator norms. We retain the notation used in the proof of Proposition 5.1. Observe that we have a natural isomorphism $\mathcal{B}(\overline{S}_i) \to \mathcal{B}(\ell^2(H))$ induced by T_i. Furthermore, $L^2(G) = \bigoplus_{i \in \mathcal{I}} \overline{S}_i$, where $\bigoplus$ indicates the Hilbert direct sum, and this a decomposition of $L^2(G)$ as left $\mathbb{C}H$-modules. The following lemma will be required.

Lemma 5.4. *Let $\theta \in \mathbb{C}H$. Then $\|\theta\| = \|\theta\|'$.*

Proof. Note that θ can be considered as an operator on $L^2(G)$ or $\ell^2(H)$. If $u \in L^2(G)$, then we may write $u = \sum_{i\in\mathcal{I}} u_i$ with $u_i \in \overline{S}_i$; hence,

$$\begin{aligned}\|\theta\| &= \sup_{u\in L^2(G),\|u\|_2=1} \|\theta u\|_2 = \sup_{u\in L^2(G),\|u\|_2=1} \Big\|\theta \sum_{i\in\mathcal{I}} u_i\Big\|_2 \\ &\le \sup_{u\in L^2(G),\|u\|_2=1} \sqrt{\sum_{i\in\mathcal{I}} \|\theta\|'^2 \|u_i\|_2^2} = \|\theta\|'.\end{aligned}$$

Fix $\iota \in \mathcal{I}$. Then,

$$\|\theta\|' = \sup_{u\in\overline{S}_\iota,\|u\|_2=1} \|\theta u\|_2 \le \sup_{u\in L^2(G),\|u\|_2=1} \|\theta u\|_2 \le \|\theta\|.$$

Thus, $\|\theta\| = \|\theta\|'$. □

Denote the operator norm closure of $\mathbb{C}H$ in $\mathcal{B}(L^2(G))$ by $\mathcal{O}(H)$, and denote the weak closure of $\mathbb{C}H$ in $\mathcal{B}(L^2(G))$ by $\mathcal{W}(H)$. The space $\mathcal{W}(H)$ is a von Neumann algebra; by the double commutant theorem, it is equal to the strong closure of $\mathbb{C}H$ in $\mathcal{B}(L^2(G))$. Note that $\mathcal{O}(H) \subseteq \mathcal{W}(H)$ and that $\mathrm{C}^*_\mathrm{r}(H) \subseteq \mathcal{N}(H)$. We now relate these various algebras.

Proposition 5.5. *There is a $*$-isomorphism $\alpha : \mathcal{W}(H) \to \mathcal{N}(H)$. Moreover, α preserves the operator norm and maps $\mathcal{O}(H)$ onto $\mathrm{C}^*_\mathrm{r}(H)$.*

Proof. Recall that, for $u \in L^2(G)$, we can uniquely write $u = \sum_{i\in\mathcal{I}} u_i$ with $u_i \in \overline{S}_i$. Let $\theta \in \mathcal{W}(H)$. Then there exists a net (θ_i) in $\mathbb{C}H$ which converges strongly to θ. Thus, for every $u \in L^2(G)$, the net $(\theta_i u)$ is convergent in $L^2(G)$; consequently, the net $(\theta_i u_j)$ is convergent for every j. In particular, $(\theta_i f)$ is a Cauchy net in $\ell^2(H)$ for every $f \in \ell^2(H)$. We deduce that (θ_i) is a Cauchy net in $\mathcal{B}(\ell^2(H))$ (in the strong operator topology); hence, it converges to an operator $\theta' \in \mathcal{N}(H)$. We note that θ' does not depend on the choice of the net (θ_i); therefore, we have a well-defined map $\alpha : \mathcal{W}(H) \to \mathcal{N}(H)$, defined as $\alpha(\theta) = \theta'$, and α is the identity on $\mathbb{C}H$.

We now construct the inverse of α by reversing the above steps. Let $\phi \in \mathcal{N}(H)$. By the Kaplansky density theorem, there exists a net (θ_i) in $\mathbb{C}H$ which converges strongly to ϕ, and $\|\theta_i\|'$ is bounded. Thus, $\|\theta_i\|$ is bounded because $\|\theta_i\| = \|\theta_i\|'$ for each i by Lemma 5.4. Now let $u \in L^2(G)$. If $\mathcal{J}$ is a finite subset of $\mathcal{I}$, set $v_\mathcal{J} = \sum_{j\in\mathcal{J}} u_j$. Then, $(\theta_i v_j)$ converges in $L^2(G)$ for every $\mathcal{J}$. Since $\|\theta_i\|$ is bounded, it follows that $(\theta_i u)$ is convergent in $L^2(G)$, and we conclude that (θ_i) converges strongly to an operator $\tilde{\phi} \in \mathcal{B}(L^2(G))$. Thus, we obtain a well-defined map $\phi \to \tilde{\phi} : \mathcal{N}(H) \to \mathcal{W}(H)$, which is the inverse to α.

It is easily checked that α is a $*$-isomorphism and that it therefore is an isomorphism of C^*-algebras, in particular that it preserves the operator norm. We deduce that α maps $\mathcal{O}(H)$ onto $\mathrm{C}^*_\mathrm{r}(H)$. □

Remark 5.6. Proposition 5.5 can be used to give a different proof of Proposition 5.1 (see [21, Chapter 2.5]).

6. The Weyl–Heisenberg group

In this section we use techniques developed in this paper to determine when $f = 0$ is the only solution to the time-frequency equation (1.4). The relevant group here is the Weyl–Heisenberg group since it has an irreducible representation that is square-integrable.

Let $n \in \mathbb{N}$. The *Heisenberg group* H_n is the set of $(n+2) \times (n+2)$ matrices of the form

$$\begin{bmatrix} 1 & a & z \\ 0 & 1_n & b \\ 0 & 0 & 1 \end{bmatrix},$$

where a is a $(1 \times n)$-matrix, b is an $(n \times 1)$-matrix, the zero in the $(2,1)$ position is the $(n \times 1)$-zero matrix, the zero in the $(3,2)$ position is the $(1 \times n)$-zero matrix, and the 1_n in the $(2,2)$ position is the $(n \times n)$-identity matrix. Another way to represent H_n is as the product $\mathbb{R} \times \widehat{\mathbb{R}^n} \times \mathbb{R}^n$. Here, we view $\mathbb{R}^n$ as $(n \times 1)$-column matrices and $\widehat{\mathbb{R}^n}$ as $(1 \times n)$-row matrices. For $(z_1, a_1, b_1), (z_2, a_2, b_2) \in H_n$ the group law becomes $(z_1, a_1, b_1)(z_2, a_2, b_2) = (z_1 + z_2 + a_1 \cdot b_2, a_1 + a_2, b_1 + b_2)$. Thus, the identity element in H_n is $(0, 0, 0)$, and $(z, a, b)^{-1} = (a \cdot b - z, -a, -b)$. For $f \in L^2(\mathbb{R}^n)$ and $(z, a, b) \in H_n$, define

$$\pi(z, a, b) f(x) = e^{2\pi i z} e^{-2\pi i a \cdot b} e^{2\pi i a \cdot x} f(x - b).$$

It turns out that π is a representation of H_n on $L^2(\mathbb{R}^n)$. Indeed, let (z_1, a_1, b_1), $(z_2, a_2, b_2) \in H_n$. Then,

$$\begin{aligned}
&\pi(z_1, a_1, b_1)\big(\pi(z_2, a_2, b_2) f(x)\big) \\
&\quad = \pi(z_1, a_1, b_1)\big(e^{2\pi i z_2} e^{-2\pi i a_2 \cdot b_2} e^{2\pi i a_2 \cdot x} f(x - b_2)\big) \\
&\quad = e^{2\pi i z_1} e^{2\pi i z_2} e^{-2\pi i a_1 \cdot b_1} e^{-2\pi i a_2 \cdot b_2} e^{2\pi i a_1 \cdot x} e^{2\pi i a_2 \cdot (x - b_1)} f(x - b_2 - b_1) \\
&\quad = e^{2\pi i (z_1 + z_2)} e^{-2\pi i (a_1 \cdot b_1 + a_2 \cdot b_2)} e^{-2\pi i a_2 \cdot b_1} e^{2\pi i a_2 \cdot x} f\big(x - (b_1 + b_2)\big) \\
&\quad = e^{2\pi i (z_1 + z_2 + a_1 \cdot b_2)} e^{-2\pi i (a_1 + a_2) \cdot (b_1 + b_2)} e^{2\pi i (a_1 + a_2) \cdot x} f\big(x - (b_1 + b_2)\big) \\
&\quad = \big(\pi(z_1, a_1, b_1) \pi(z_2, a_2, b_2)\big) f(x).
\end{aligned}$$

Let $Z = \langle (2\pi, 0, 0) \rangle$, the subgroup of H_n generated by $(2\pi, 0, 0)$. Set $\tilde{H}_n = H_n / Z$. The group $\tilde{H}_n$ is known as the *Weyl–Heisenberg group*. Clearly, $Z = \ker \pi$; thus, π induces a representation $\tilde{\pi}$ on $\tilde{H}_n$. Observe that $\tilde{H}_n = \{(t, a, b) \mid t \in \mathbb{T}, a, b \in \mathbb{R}^n\}$ (here $\mathbb{T}$ is the unit circle $\{z \in \mathbb{Z} \mid |z| = 1\}$). The Lebesgue measure on $H_n = \mathbb{R} \times \widehat{\mathbb{R}^n} \times \mathbb{R}^n$ is the left and right Haar measure on H_n. Similarly, the Lebesgue measure on $\mathbb{T} \times \widehat{\mathbb{R}^n} \times \mathbb{R}^n$ is the left and right Haar measure on $\tilde{H}_n$ (here, the Lebesgue measure on $\mathbb{T}$ is normalized so that $\int_{\mathbb{T}} dt = 1$). The next result was proved in [12, Proposition 3.2.4] for the special case $n = 1$. By interchanging the roles of a and b, the proof given there applies equally to our case.

Proposition 6.1. *If $f, g \in L^2(\mathbb{R}^n)$, then*

$$\int_{\widehat{\mathbb{R}^n}} \int_{\mathbb{R}^n} \int_{\mathbb{T}} \left| \left\langle f, \tilde{\pi}(t, a, b) g \right\rangle \right|^2 dt\, db\, da = \|f\|_2^2 \|g\|_2^2.$$

Corollary 6.2. *The representation $\tilde{\pi}$ of $\tilde{H}_n$ on $L^2(\mathbb{R}^n)$ is irreducible and square-integrable, and every $g \in L^2(\mathbb{R}^n)$ is admissible.*

Proof. By taking $f = g$ in the above proposition, we see immediately that every element of $L^2(\mathbb{R}^n)$ is admissible. Suppose that $g \in L^2(\mathbb{R}^n) \setminus \{0\}$ is fixed, and assume that $f \in L^2(\mathbb{R}^n)$ satisfies $\langle f, \tilde{\pi}(t,a,b)g\rangle = 0$ for all $(t,a,b) \in \tilde{H}_n$. Then $\|f\|_2\|g\|_2 = 0$, and it follows that $f = 0$; hence, $\tilde{\pi}$ is irreducible as desired. □

Proposition 6.3. *Let $n \in \mathbb{N}$, let $(a_k, b_k) \in \mathbb{R}^{2n}$ be distinct nonzero elements such that (a_k, b_k) generate a discrete subgroup of $\mathbb{R}^{2n}$, and $a_h \cdot b_k \in \mathbb{Q}$ for all h, k (where $k, h \in \mathbb{N}$). If $r \in \mathbb{N}$, and*

$$\sum_{k=1}^{r} c_k e^{2\pi i b_k \cdot t} f(t + a_k) = 0$$

with $0 \neq c_k \in \mathbb{C}$ constants, then $f = 0$.

Proof. We have $\mathbb{R}^{2n} = \tilde{H}_n/\mathbb{T}$. Lift the (a_k, b_k) to the elements $g_k := (1, a_k, b_k) \in \tilde{H}_n$. Note that the hypothesis $a_h \cdot b_k \in \mathbb{Q}$ ensures that $\langle g_1, \ldots, g_r\rangle$ is a discrete subgroup of $\tilde{H}_n$. We claim that, if $0 \neq d_k \in \mathbb{C}$, then $\alpha := \sum_{k=1}^{r} d_k g_k$ is a non-zerodivisor in $\mathbb{C}\tilde{H}_n$. Indeed, if $0 \neq \beta \in \mathbb{C}\tilde{H}_n$ and $\alpha\beta = 0$, then let T be a transversal for $\mathbb{T}$ in $\tilde{H}_n$ containing $\{g_1, \ldots, g_r\}$, and write $\beta = \sum_{t \in T} \beta_t t$, where $\beta_t \in \mathbb{C}\mathbb{T}$. Since $\mathbb{R}^{2n}$ is an ordered group, we can apply a leading term argument: let k be such that $g_k \in T$ is largest, and let $s \in T$ be the largest element such that $\beta_s \neq 0$. Then, by considering $g_k s$, we see that $\alpha\beta \neq 0$ because $d_k\beta_s \neq 0$, which is a contradiction. The result now follows from Proposition 1.1, Corollary 6.2, and Theorem 5.2. □

7. Shearlet groups

We now investigate the problem of linear independence of left translations of functions in $L^2(S)$, where S denotes the shearlet group. This fits the theme of our paper since S has an irreducible, square-integrable representation on $L^2(\mathbb{R}^2)$. We begin by defining the shearlet group.

For $a \in \mathbb{R}^+$ (the positive real numbers) and $s \in \mathbb{R}$, let

$$A_a = \begin{bmatrix} a & 0 \\ 0 & \sqrt{a} \end{bmatrix},$$

let

$$S_s = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix},$$

and let $G = \{S_s A_a \mid a \in \mathbb{R}^+, s \in \mathbb{R}\}$. The *shearlet group* S is defined to be $S = G \ltimes \mathbb{R}^2$. The group multiplication for S is given by $(M,t)(M',t') = (MM', t + Mt')$, where $M \in G$ and $t \in \mathbb{R}^2$ (here, we are considering elements of $\mathbb{R}^2$ as column vectors). The left Haar measure for S is $\frac{da\,ds\,dt}{a^3}$, and the right Haar measure for S is $\frac{da\,ds\,dt}{a}$; hence, S is a nonunimodular group. A representation π of S on $L^2(\mathbb{R}^2)$ can be defined by

$$\pi(S_s A_a, t) f(x) = a^{-3/4} f\big((S_s A_a)^{-1}(x - t)\big).$$

The representation π is square-integrable and irreducible (see [4, Section 2]). We write f_{ast} to indicate $\pi(S_s A_a, t)f$. The function f_{ast} is also known as the *shearlet transform* of f. Since the shearlet transform is realized by an irreducible square-integrable representation of S on $L^2(\mathbb{R}^2)$, the question of linear independence of the left translations of a function in $L^2(S)$ is related to the question of the linear independence of the shearlets of a function in $L^2(\mathbb{R}^2)$. The question of linear independence of the shearlet transforms of f now becomes: Is $f = 0$ the only solution in $L^2(\mathbb{R}^2)$ that satisfies

$$\sum_{k=1}^{n} c_k f_{a_k s_k t_k} = 0, \tag{7.1}$$

where c_k are nonzero constants, and $(a_k, s_k, t_k) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^2$?

Proposition 7.1. *Let S be the shearlet group. There exists a nonzero function in $L^2(S)$ that has linearly dependent left translations.*

Proof. The proposition will follow immediately from Proposition 1.1 if we show that there exists a nonzero $f \in L^2(\mathbb{R}^2)$ that satisfies (7.1). Combining [13, Theorem 4.6] and [9, Example 5] we see that there exists a continuous nonzero $f \in L^2(\mathbb{R}^2)$ that satisfies

$$f(x) = \sum_{\beta \in \mathbb{Z}^2} a(\beta) f(A_4^{-1}x - \beta), \tag{7.2}$$

where $a(\beta) \in \mathbb{C}\mathbb{Z}^2$ and $x \in \mathbb{R}^2$. The function f is said to be *refinable*. In the literature, $a(\beta)$ is often referred to as a mask. The important thing here is that $a(\beta)$ has finite support. If

$$\beta = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

then set

$$\beta' = \begin{bmatrix} 4b_1 \\ 2b_2 \end{bmatrix}.$$

Using (7.2), we obtain

$$\begin{aligned} \pi(S_0 A_1, 0) f(x) &= \sum_{\beta \in \mathbb{Z}^2} a(\beta) 4^{3/4} 4^{-3/4} f(A_4^{-1}x - \beta) \\ &= \sum_{\beta \in \mathbb{Z}^2} a(\beta) 4^{3/4} 4^{-3/4} f(A_4^{-1}x - A_4^{-1}\beta') \\ &= \sum_{\beta \in \mathbb{Z}^2} 4^{3/4} a(\beta) \pi(S_0 A_4, \beta') f(x) \\ &= \sum_{\beta \in \mathbb{Z}^2} 4^{3/4} a(\beta) f_{4,0,\beta'}(x); \end{aligned}$$

hence, there is a linear dependency among the shearlet transforms of f. This proves the proposition. $\square$

Remark 7.2. The refinable function f used in the proof of the previous proposition has compact support since $a(\beta)$ has finite support (see [9, Theorem 5]). Compare this with [19, Theorem 4.3], where it was shown, in a slightly different setting, that a compactly supported separable shearlet system is linearly independent. Thus, it appears that in general the hypothesis of separability is important.

The next result gives a sufficient condition for linear independence of a shearlet system.

Proposition 7.3. *Let $0 \neq f \in L^2(\mathbb{R}^2)$. Then, $\{f_{1nt} \mid n \in \mathbb{Z}, t \in \mathbb{Z}^2\}$ is a linearly independent set.*

Proof. If we let $H = \{S_n A_1 \mid n \in \mathbb{Z}\}$, then $K = H \ltimes \mathbb{Z}^2$ is a torsion-free discrete subgroup of S. Because H and $\mathbb{Z}^2$ are solvable, K is solvable and thus satisfies the strong Atiyah conjecture. By Theorem 1.2, the K-left translations of a function in $L^2(S)$ are linearly independent. The proposition now follows from Proposition 1.1. □

The results obtained in this section are similar to the results from Section 3 for the affine group. This is not surprising since the shearlet transform involves a dilation and a translation.

8. Virtually abelian groups

In this section, we consider virtually abelian groups, that is, groups with an abelian subgroup of finite index.

Proposition 8.1. *Let G be a locally compact group which has an abelian closed subgroup A of finite index, and let $1 \leq p \in \mathbb{R}$. Assume that, if $0 \neq \phi \in \mathbb{C}A$ and $0 \neq f \in L^p(A)$, then $\phi f \neq 0$. Let $0 \neq f \in L^p(G)$, let $H \leq G$, and let $\theta \in \mathbb{C}H$.*

(a) *If θ is a non-zerodivisor in $\mathbb{C}H$, then $\theta f \neq 0$.*
(b) *If H is torsion-free and $\theta \neq 0$, then $\theta f \neq 0$.*

Proof. Note that $\mathbb{C}A$ is an integral domain. If we let B be the intersection of the conjugates of A in G, then B is a closed abelian normal subgroup of finite index in G. If we let $\{a_1, \ldots, a_m\}$ be a set of coset representatives for B in A, then $L^p(A) = \bigoplus_{i=1}^m L^p(B)a_i$, and we see that, if $0 \neq \phi \in \mathbb{C}B$ and $0 \neq f \in L^p(B)$, then $\phi f \neq 0$. If we let $\{g_1, \ldots, g_n\}$ be a set of coset representatives for B in G, then $L^p(G) = \bigoplus_{i=1}^n L^p(B)g_i$. We may view this as an isomorphism of $\mathbb{C}B$-modules. Set $S = \mathbb{C}B \setminus \{0\}$. Then, we may form the ring of fractions $S^{-1}\mathbb{C}G$. Since every element of S is a non-zerodivisor in $\mathbb{C}G$, then it follows that $S^{-1}\mathbb{C}G$ is a ring containing $\mathbb{C}G$. Furthermore, $S^{-1}\mathbb{C}B$ is a field, and $S^{-1}\mathbb{C}G$ has dimension n over this field. Thus, $S^{-1}\mathbb{C}G$ is an artinian ring, and, since $S^{-1}\mathbb{C}B$ is a field of characteristic zero, we see that $S^{-1}\mathbb{C}G$ is a semisimple artinian ring, by Maschke's theorem. We deduce that non-zerodivisors in $S^{-1}\mathbb{C}G$ are invertible. Using [8, Theorem 10.8], we may form the $S^{-1}\mathbb{C}G$-module $S^{-1}L^p(G)$.

(a) If θ is a non-zerodivisor in $\mathbb{C}H$, then θ is a non-zerodivisor in $\mathbb{C}G$; hence, it is invertible in $S^{-1}\mathbb{C}G$, and so θ^{-1} exists. We may regard f as an element of $S^{-1}L^p(G)$ because $S^{-1}L^p(G)$ contains $L^p(G)$. Thus, if $\theta f = 0$, then $\theta^{-1}\theta f = 0$; consequently, $f = 0$, and we have a contradiction.

(b) If H is torsion-free, then we know that every nonzero element of $\mathbb{C}H$ is a non-zerodivisor in $\mathbb{C}H$; this was first proved by K. A. Brown [2]. Thus, the result follows from (a). □

We now use the previous result to give the following generalization of [5, Theorem 1.2].

Theorem 8.2. *Let G be a locally compact group with no nontrivial compact subgroups, and suppose that G has an abelian closed subgroup of finite index. Then every nonzero element of $L^p(G)$, where $1 \leq p \leq 2$, has linearly independent translations.*

Proof. Since G has no nontrivial compact subgroups, it is torsion-free. Furthermore, for $1 \leq p \leq 2$, if $0 \neq \phi \in \mathbb{C}A$ and $0 \neq f \in L^p(A)$, then $\phi f \neq 0$ by [5, Theorem 1.2]. The result now follows from Proposition 8.1(b). □

Theorem 8.3. *Let G be a locally compact abelian group, let $n \in \mathbb{N}$, and let $1 \leq p \in \mathbb{R}$. Assume that $p \leq 2n/(n-1)$. Suppose that G has a closed subgroup of finite index isomorphic to $\mathbb{R}^n$ or $\mathbb{Z}^n$ as a locally compact abelian group. Let $H \leq G$, let $\theta \in \mathbb{C}H$, let $\theta \in \mathbb{C}G$, and let $0 \neq f \in L^p(G)$.*

(a) *If θ is a non-zerodivisor in $\mathbb{C}H$, then $\theta f \neq 0$.*
(b) *If H is torsion-free and $\theta \neq 0$, then $\theta f \neq 0$.*

Proof. We apply Proposition 8.1 with $A = \mathbb{R}^n$ or $\mathbb{Z}^n$. We need to check the hypothesis that, if $0 \neq \phi \in \mathbb{C}A$ and $0 \neq f \in L^p(A)$, then $\phi f \neq 0$. For the case $A = \mathbb{R}^n$, this follows from [22, Theorem 3], while for the case $A = \mathbb{Z}^n$, this follows from [17, Theorem 2.1]. □

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