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DUALITY PROPERTIES FOR GENERALIZED FRAMES

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ABSTRACT. We introduce the concept of Riesz-dual sequences for g-frames. In this paper we show that, for any sequence of operators, we can construct a corresponding sequence of operators with a kind of duality relation between them. This construction is used to prove duality principles in g-frame theory, which can be regarded as general versions of several well-known duality principles for frames. We also derive a simple characterization of a g-Riesz basic sequence as a g-R-dual sequence of a g-frame in the tight case.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper $\mathcal{H}$ and $\mathcal{K}$ are separable Hilbert spaces, and I denotes the countable (or finite) index set. Note that $\{V_i\}_{i\in I}$ and $\{W_j\}_{j\in I}$ are sequences of closed subspaces of $\mathcal{K}$ and that $B(\mathcal{H}, V_i)$ denotes the collection of all bounded linear operators from $\mathcal{H}$ into V_i.

Definition 1.1. A family $\Lambda = \{\Lambda_i \in B(\mathcal{H}, V_i) : i \in I\}$ is a generalized frame or simply a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ if there exist constants $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq D\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{1.1}$$

The constants C and D are called g-*frame bounds*. If only the right-hand inequality of (1.1) is required, we call it a g-*Bessel sequence*. We call $\{\Lambda_i\}_{i\in I}$ a *C-tight*

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g-*frame* if $C = D$, and we call it a *Parseval* g-*frame* if $C = D = 1$. We denote the representation space associated with a g-Bessel sequence $\{\Lambda_i\}_{i\in I}$ as follows:

$$\Big(\sum_{i\in I} \oplus V_i\Big)_{\ell^2} = \Big\{\{g_i'\}_{i\in I} \;\Big|\; g_i' \in V_i, \forall i \in I \text{ and } \sum_{i\in I} \|g_i'\|^2 < \infty\Big\}. \tag{1.2}$$

The analysis operator for a g-Bessel sequence $\Lambda = \{\Lambda_i\}_{i\in I}$ is defined as follows:

$$T_\Lambda : \mathcal{H} \to \Big(\sum_{i\in I} \oplus V_i\Big)_{\ell^2}, \qquad T_\Lambda f = \{\Lambda_i f\}_{i\in I} \quad \forall f \in \mathcal{H}, \tag{1.3}$$

and its adjoint operator, which is given by

$$T_\Lambda^* : \Big(\sum_{i\in I} \oplus V_i\Big)_{\ell^2} \to \mathcal{H}, \qquad T_\Lambda^*\big(\{g_i'\}_{i\in I}\big) = \sum_{i\in I} \Lambda_i^* g_i', \tag{1.4}$$

is called the *analysis operator* of Λ. By composing T_Λ and T_Λ^* we obtain the g-frame operator

$$S_\Lambda : \mathcal{H} \to \mathcal{H}, \qquad S_\Lambda f = T_\Lambda^* T_\Lambda f = \sum_{i\in I} \Lambda_i^* \Lambda_i f, \quad \forall f \in \mathcal{H}, \tag{1.5}$$

which is a positive, self-adjoint, and invertible operator, and $CI_\mathcal{H} \le S_\Lambda \le DI_\mathcal{H}$. The canonical dual g-frame for $\{\Lambda_i\}_{i\in I}$ is defined by $\{\widehat{\Lambda}_i\}_{i\in I}$, where $\widehat{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$, which is also a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ with $\frac{1}{D}$ and $\frac{1}{C}$ as its lower and upper g-frame bounds, respectively. Also we have

$$f = \sum_{i\in I} \Lambda_i^* \widehat{\Lambda}_i f = \sum_{i\in I} \widehat{\Lambda}_i^* \Lambda_i f, \quad \forall f \in \mathcal{H}.$$

(For more details about the theory of generalized frames, we refer the reader to the articles [14], [18],and [19]. For details about its applications, see [9] and [12]; for fusion frames, see [3].) Since almost all applications require a finite model for their numerical treatment, we restrict ourselves to a finite-dimensional space in the following examples.

Example 1.2. Let $\mathcal{H} = \mathbb{C}^N$, and let $V_1 = V_2 = \cdots = V_N = \mathbb{C}^{N+1}$. Define

$$\Lambda_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \qquad \Lambda_2 = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \qquad \dots,$$

$$\Lambda_N = \begin{bmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

Thus the set $\Lambda = \{\Lambda_i\}_{i=1}^N$ is a g-frame for $\mathbb{C}^N$ with respect to $\mathbb{C}^{N+1}$ with g-frame bounds $A = 2$ and $B = N+1$. To see this explicitly, note that, for any $f = \{z_i\}_{i=1}^N$ in $\mathbb{C}^N$, we have

$$\sum_{i=1}^{N} \|\Lambda_i f\|^2 = 2|z_1|^2 + 3|z_2|^2 + \cdots + (N+1)|z_N|^2.$$

From this, we have

$$2\|f\|^2 \leq \sum_{i=1}^{N} \|\Lambda_i f\|^2 \leq (N+1)\|f\|^2.$$

Example 1.3. Let $\mathcal{H} = \mathbb{C}^{N+1}$, and let $V_1 = V_2 = \cdots = V_{N+1} = \mathbb{C}^N$. Define

$$\Lambda_1 = \begin{bmatrix} -1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix}, \qquad \dots, \qquad \Lambda_N = \begin{bmatrix} 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & -1 & 0 \end{bmatrix},$$

and

$$\Lambda_{N+1} = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Thus the set $\{\Lambda_i\}_{i=1}^{N+1}$ is a N-tight g-frame for $\mathbb{C}^{N+1}$ with respect to $\mathbb{C}^N$. To see this explicitly, note that, for any $f = \{z_i\}_{i=1}^{N+1} \in \mathbb{C}^{N+1}$, we have

$$\sum_{i=1}^{N+1} \|\Lambda_i f\|^2 = N\big(|z_1|^2 + |z_2|^2 + \cdots + |z_{N+1}|^2\big) = N\|f\|^2.$$

Duality principles in Gabor theory such as the Ron–Shen duality principle [16] and the Wexler–Raz biorthogonality relations [20] play a fundamental role in analyzing Gabor systems. Casazza, Kutyniok, and Lammers introduced the concept of a Riesz-dual sequence ("R-dual sequence") in [4] and further considered it in [5]. In [4] Casazza et al. introduced a general approach to derive duality principles in abstract frame theory. For each sequence in a separable Hilbert space they defined an R-dual sequence dependent only on two orthonormal bases. They characterized exact properties of the first sequence in terms of the R-dual sequence, which yields duality relations for the frame setting.

Definition 1.4. Let $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ be orthonormal bases for a separable Hilbert space $\mathcal{H}$, and let $f = \{f_i\}_{i\in I}$ be any sequence in $\mathcal{H}$ for which

$$\sum_{i\in I} \big|\langle f_i, e_j\rangle\big|^2 < \infty \quad \forall j \in I.$$

Then the R-dual sequence of $\{f_i\}_{i\in I}$ with respect to $\{e_i\}_{i\in I}$ and $\{h_i\}_{i\in I}$ as the sequence $\{w_j^f\}_{j\in I}$ is given by

$$w_j^f = \sum_{i\in I}\langle f_i, e_j\rangle h_i, \quad \forall j \in I. \tag{1.6}$$

There exists a symmetric relation between the sequences $\{w_j^f\}_{j\in I}$ and $\{f_i\}_{i\in I}$ which is as follows:

$$f_i = \sum_{j\in I}\langle w_j^f, h_i\rangle e_j, \quad \forall i \in I. \tag{1.7}$$

In particular, this shows that $\{f_i\}_{i\in I}$ is the R-dual sequence for $\{w_j^f\}_{j\in I}$ with respect to $\{h_i\}_{i\in I}$ and $\{e_i\}_{i\in I}$. (We refer the reader to the articles [7], [8], [13], [17], and [21] for an introduction to the theory and applications of R-dual sequences.)

The structure of this paper is as follows. In the rest of this section we will briefly review the necessary parts from g-bases, g-orthonormal bases, and g-Riesz bases (for more information, see [1], [2], [6], [10], and [11]). Then we define the generalized R-dual sequence ("g-R-dual sequence") from a g-Bessel sequence with respect to a pair of g-orthonormal bases as a generalization of an R-dual sequence. We characterize the extent to which the g-R-dual sequence depends upon the chosen g-orthonormal bases. In Section 2, we obtain the g-frame conditions for a sequence of operators and its g-R-dual sequence. In Section 3, we characterize those pairs of g-frames and their g-R-dual sequences which are equivalent (unitarily equivalent). Finally, Section 4 deals with duality properties for g-frames by g-R-dual sequences; in it, we study properties of dual g-frames and canonical dual g-frames.

Definition 1.5. Let $\{\Xi_i \in B(\mathcal{H}, W_i) \mid i \in I\}$ be a sequence of operators. Then

(i) $\{\Xi_i\}_{i\in I}$ is a g-complete set for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ if $\mathcal{H} = \overline{\operatorname{Span}}\{\Xi_i^*(W_i)\}_{i\in I}$;

(ii) $\{\Xi_i\}_{i\in I}$ is a g-orthonormal system for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ if $\Xi_i\Xi_j^* = \delta_{ij}I_{W_j}$ for all $i, j \in I$;

(iii) a g-complete and g-orthonormal system $\{\Xi_i\}_{i\in I}$ is called a g-*orthonormal basis* for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$.

The following well-known characterization of g-orthonormal bases is sometimes more useful (see [2]).

Lemma 1.6. *Let $\Xi = \{\Xi_i\}_{i\in I}$ be a g-orthonormal system for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$. Then the following conditions are equivalent:*

(i) *Ξ is a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$,*

(ii) $\sum_{i\in I}\Xi_i^*\Xi_i = I_{\mathcal{H}}$,

(iii) $\|f\|^2 = \sum_{i\in I}\|\Xi_i f\|^2 \ \forall f \in \mathcal{H}$,

(iv) if $\Xi_i f = 0$ for all $i \in I$, then $f = 0$.

Let $\Xi = \{\Xi_i\}_{i\in I}$ be a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$. If $f = \sum_{i\in I}\Xi_i^* g_i$, then the coordinate representation of $f \in \mathcal{H}$ relative to the g-orthonormal basis Ξ is $[f]_\Xi = \{g_i\}_{i\in I}$. In this case $\{g_i\}_{i\in I} \in (\sum_{i\in I} \oplus W_i)_{\ell^2}$, and $\|f\| = \|[f]_\Xi\|_{\ell^2}$.

Let $\Xi = \{\Xi_i\}_{i\in I}$ and $\Xi' = \{\Xi'_i\}_{i\in I}$ be g-orthonormal bases for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$, respectively. The transition matrix from Ξ to Ξ' is the matrix $B = [B_{ij}]$ whose (i,j)-entry is $B_{ij} = \Xi'_i\Xi^*_j$ for all $i,j \in I$. Then we have $B[f]_\Xi = [f]_{\Xi'}$, where $[f]_\Xi$ is the coordinate representation of an arbitrary vector $f \in \mathcal{H}$ in the basis Ξ and similarly for Ξ'. The transition matrix from Ξ' to Ξ is $B^{-1} = B^*$. Thus, if $B^* = [B^*_{ij}]$, then $B^*_{ij} = (B_{ji})^* = \Xi_i\Xi'^*_j$ for all $i,j \in I$.

Example 1.7. Let $\{e_j\}_{j\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$, and let $\{W_j\}_{j\in\mathbb{N}}$ be a sequence of subspaces of $\mathcal{H}$ defined by

$$W_j = \mathrm{Span}\{e_{2j-1} + e_{2j}\} \qquad \text{and}$$
$$\Xi_j f = \frac{1}{2}\langle f, e_{2j-1} + e_{2j}\rangle(e_{2j-1} + e_{2j}) \quad \forall j \in \mathbb{N}.$$

A direct calculation shows that $\|\Xi_j\| = 1$ and that $\Xi_i\Xi^*_j g_j = \delta_{ij} g_j$ for all $1 \le i,j \le n$ and that $g_j \in W_j$. Since $\langle e_1 - e_2, e_{2j-1} + e_{2j}\rangle = 0$ for all $j \in \mathbb{N}$, then $\mathcal{H} \ne \overline{\mathrm{Span}}\{\Xi^*_j(W_j)\}_{j\in J}$. Thus $\{\Xi_j\}_{j\in\mathbb{N}}$ is a g-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$, but it is not a g-orthonormal basis for $\mathcal{H}$.

Example 1.8. Let $N \in \mathbb{N}, \mathcal{H} = \mathbb{C}^{N+1}$, and let $\{e_k\}_{k=1}^{N+1}$ be the standard orthonormal basis of $\mathcal{H}$. Define

$$W_j = \mathrm{Span}\Big\{\sum_{\substack{k=1\\k\ne j}}^{N+1} e_k\Big\}, \qquad \text{and} \qquad \Xi_j\big(\{c_i\}_{i=1}^{N+1}\big) = \frac{c_j}{\sqrt{N}}\sum_{\substack{k=1\\k\ne j}}^{N+1} e_k.$$

Then $\Xi^*_j(\lambda\sum_{\substack{k=1\\k\ne j}}^{N+1} e_k) = \sqrt{N}\lambda e_j$ for all $1 \le j \le N+1$. This shows that

$$\overline{\mathrm{Span}}\big\{\Xi^*_j(W_j)\big\}_{j=1}^{N+1} = \overline{\mathrm{Span}}\{e_j\}_{j=1}^{N+1} = \mathcal{H} \qquad \text{and that} \qquad \Xi_i\Xi^*_j = \delta_{ij}.$$

Hence $\{\Xi_j\}_{j\in\mathbb{N}}$ is a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_j\}_{j=1}^{N+1}$.

Example 1.9. Let $\mathcal{H} = \mathbb{C}^{2N}$, and let $W_1 = W_2 = \cdots = W_N = \mathbb{C}^2$. Define

$$\Xi_1 = \begin{bmatrix} 1 & 0 & \dots & 0 & 0\\ 0 & 1 & \dots & 0 & 0\end{bmatrix}, \qquad \dots, \qquad \Xi_N = \begin{bmatrix} 0 & 0 & \dots & 1 & 0\\ 0 & 0 & \dots & 0 & 1\end{bmatrix}.$$

A direct calculation shows that $\|\Xi_k\| = 1$ and that $\Xi_k\Xi^*_\ell = \delta_{k\ell}$ for any $1 \le k,\ell \le N$. We also have

$$\sum_{k=1}^{N}\|\Xi_k f\|^2 = \sum_{k=1}^{N}\big(|z_{2k-1}|^2 + |z_{2k}|^2\big) = \|f\|^2, \quad \forall f = \{z_i\}_{i=1}^{2N} \in \mathbb{C}^{2N}.$$

Thus $\Xi = \{\Xi_k\}_{k=1}^N$ is a g-orthonormal basis for $\mathbb{C}^{2N}$ with respect to $\mathbb{C}^2$. Similarly, the sequence $\Psi = \{\Psi_k\}_{k=1}^N$ defined by

$$\Psi_1 = \begin{bmatrix} 0 & 1 & \dots & 0 & 0\\ 1 & 0 & \dots & 0 & 0\end{bmatrix}, \qquad \dots, \qquad \Psi_N = \begin{bmatrix} 0 & 0 & \dots & 0 & 1\\ 0 & 0 & \dots & 1 & 0\end{bmatrix},$$

is also a g-orthonormal basis for $\mathbb{C}^{2N}$ with respect to $\mathbb{C}^2$ and the matrix

$$B = [\Psi_i \Xi_j^*]_{N\times N} = \begin{bmatrix} A & & \overline{0} \\ & \ddots & \\ \overline{0} & & A \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

is the transition matrix from Ξ to Ψ. Hence, for any $f \in \mathbb{C}^{2N}$, we have $B[f]_\Xi = [f]_\Psi$.

Definition 1.10. A sequence $\Gamma = \{\Gamma_j \in B(\mathcal{H}, W_j) \mid j \in I\}$ is called a g-*Riesz basis* for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ if $\{\Gamma_j\}_{j\in I}$ is a g-complete set for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ and there exist constants $0 < A \le B < \infty$ such that

$$A \sum_{j\in I} \|g_j\|^2 \le \left\| \sum_{j\in I} \Gamma_j^* g_j \right\|^2 \le B \sum_{j\in I} \|g_j\|^2 \tag{1.8}$$

for all sequences $\{g_j\}_{j\in I} \in (\sum_{j\in I} \oplus W_j)_{\ell^2}$. We define the g-Riesz basis bounds for $\{\Gamma_j\}_{j\in I}$ to be the largest number A and the smallest number B such that this inequality (1.8) holds. If $\{\Gamma_j\}_{j\in I}$ is a g-Riesz basis only for $\overline{\text{Span}}\{\Gamma_j^*(W_j)\}_{j\in I}$, then we call it is a g-*Riesz basic sequence* for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$.

The following result is a characterization of g-Riesz bases for $\mathcal{H}$ (for a proof of this standard result, see, e.g., [1, Theorem 3.17]).

Lemma 1.11. *Let $\{\Xi_j\}_{j\in I}$ be a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$. Then the following hold.*

(i) *Here $\Gamma = \{\Gamma_j \in B(\mathcal{H}, W_j) | j \in I\}$ is a g-Riesz basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ if and only if there exists a bounded bijective operator $U : \mathcal{H} \to \mathcal{H}$ such that $\Gamma_j = \Xi_j U^*$ for all $j \in I$.*

(ii) *Assume that $\overline{\text{Span}}\{\Gamma_j^*(W_j)\}_{j\in I} = \mathcal{H}$ and that $\| \sum_{j\in I} \Gamma_j^* g_j\|^2 = \sum_{j\in I} \|g_j\|^2$, for all sequences $\{g_j\}_{j\in I} \in (\sum_{j\in I} \oplus W_j)_{\ell^2}$. Then $\{\Gamma_j\}_{j\in I}$ is a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$.*

Example 1.12. Let $\mathcal{H} = \mathbb{C}^{2n}$, and let $W_1 = W_2 = \cdots = W_{2n} = \mathbb{C}^2$. Define

$$\Gamma_1 = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 2 & \dots & 0 & 0 \end{bmatrix}, \qquad \dots, \qquad \Gamma_n = \begin{bmatrix} 0 & 0 & \dots & 2n-1 & 0 \\ 0 & 0 & \dots & 0 & 2n \end{bmatrix}.$$

If $g_i = (z_{2i-1}, z_{2i}) \in \mathbb{C}^2$, then we have $\| \sum_{i=1}^n \Gamma_i^* g_i\|^2 = \sum_{i=1}^{2n} i^2 |z_i|^2$. Thus $\{\Gamma_i\}_{i=1}^n$ is a g-Riesz basis for $\mathbb{C}^{2n}$ with respect to $\mathbb{C}^2$ with g-Riesz bounds 1 and $4n^2$. Moreover, we can write $\{\Gamma_i\}_{i=1}^n = \{\Xi_i U^*\}_{i=1}^n$, where U is a bounded bijective operator defined by

$$U = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 2n \end{bmatrix},$$

and $\Xi = \{\Xi_k\}_{k=1}^n$ is the g-orthonormal basis defined in Example 1.9.

A g-R-dual sequence is a natural generalization of an R-dual sequence which provides a powerful tool in the analysis of duality relations in general g-frame theory. In the following, we define the generalized Riesz-dual sequence from a sequence of operators.

Definition 1.13. Let $\Xi = \{\Xi_i\}_{i\in I}$ and $\Psi = \{\Psi_i\}_{i\in I}$ be g-orthonormal bases for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$, respectively. Let $\Lambda = \{\Lambda_i : \mathcal{H} \to V_i \mid i \in I\}$ be such that the series $\sum_{i\in I} \Lambda_i^* g_i'$ is convergent for all $\{g_i'\}_{i\in I} \in (\sum_{i\in I} \oplus V_i)_{\ell^2}$. Define

$$\Gamma_j^\Lambda : \mathcal{H} \to W_j, \qquad \Gamma_j^\Lambda = \sum_{i\in I} \Xi_j \Lambda_i^* \Psi_i, \quad \forall j \in I. \tag{1.9}$$

Then $\{\Gamma_j^\Lambda\}_{j\in I}$ is the g-R-dual sequence for the sequence Λ with respect to (Ξ, Ψ).

The hypothesis that the series $\sum_{i\in I} \Lambda_i^* g_i'$ is convergent for all $\{g_i'\}_{i\in I} \in (\sum_{i\in I} \oplus V_i)_{\ell^2}$ is always fulfilled if the sequence $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-Bessel sequence with respect to $\{V_i\}_{i\in I}$.

Example 1.14. Let $\mathcal{H} = \mathbb{C}^{2N}$ and $\{\Xi_i\}_{i=1}^N, \{\Psi_i\}_{i=1}^N$ be the g-orthonormal bases for $\mathcal{H}$ with respect to $\mathbb{C}^2$ as defined in Example 1.9. Define

$$\Lambda_1 = \begin{bmatrix} 1 & 1 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \end{bmatrix}, \qquad \dots, \qquad \Lambda_N = \begin{bmatrix} 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Then $\Lambda = \{\Lambda_i\}_{i=1}^N$ is a g-Bessel sequence for $\mathcal{H}$ with respect to $\mathbb{C}^2$ with g-Bessel bound $B = 3$. The g-R-dual sequence for the sequence Λ with respect to (Ξ, Ψ) is defined as follows:

$$\Gamma_1^\Lambda = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 1 & 1 & \dots & 0 & 0 \end{bmatrix}, \qquad \dots, \qquad \Gamma_N^\Lambda = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 1 \end{bmatrix},$$

which is also a g-Bessel sequence for $\mathcal{H}$ with respect to $\mathbb{C}^2$ with g-Bessel bound $B = 3$.

Now we need an algorithm to invert the process and to calculate $\{\Lambda_i\}_{i\in I}$ from the sequence $\{\Gamma_j^\Lambda\}_{j\in I}$.

Theorem 1.15. *Let $\Xi = \{\Xi_i\}_{i\in I}$ and $\Psi = \{\Psi_i\}_{i\in I}$ be g-orthonormal bases for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$, respectively. Let $\{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then, for all $i \in I$,*

$$\Lambda_i = \sum_{j\in I} \Psi_i (\Gamma_j^\Lambda)^* \Xi_j. \tag{1.10}$$

In particular, this shows that $\{\Lambda_i\}_{i\in I}$ is the g-R-dual sequence for $\{\Gamma_j^\Lambda\}_{j\in I}$ with respect to (Ψ, Ξ).

Proof. The definition of $\{\Gamma_j^\Lambda\}_{j\in I}$ implies that, for every $i, j \in I$,

$$\begin{aligned} \Psi_i (\Gamma_j^\Lambda)^* &= \Psi_i \Big(\sum_{k\in I} \Xi_j \Lambda_k^* \Psi_k \Big)^* = \sum_{k\in I} \Psi_i \Psi_k^* \Lambda_k \Xi_j^* \\ &= \sum_{k\in I} \delta_{ik} \Lambda_k \Xi_j^* = \Lambda_i \Xi_j^*. \end{aligned}$$

Thus $\Psi_i(\Gamma_j^\Lambda)^* = \Lambda_i\Xi_j^*$. Now, by Lemma 1.6, we have

$$\Lambda_i = \Lambda_i I_{\mathcal{H}} = \Lambda_i\Big(\sum_{j\in I}\Xi_j^*\Xi_j\Big) = \sum_{j\in I}\Lambda_i\Xi_j^*\Xi_j = \sum_{j\in I}\Psi_i(\Gamma_j^\Lambda)^*\Xi_j. \qquad \square$$

In the following, we will characterize the extent to which the g-R-dual sequence of a g-Bessel sequence depends upon the chosen g-orthonormal bases.

Definition 1.16. Let $\Xi = \{\Xi_j\}_{j\in I}$ be a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$, and let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then the matrix $A = [A_{ij}]$ whose (i,j)-entry is $A_{ij} = \Lambda_i\Xi_j^*$ for all $i,j\in I$ is called the *analysis matrix* for Λ with respect to Ξ. A direct calculation shows that, for every $f\in\mathcal{H}$, we have $A[f]_\Xi = T_\Lambda f$, and $A^*A = S_\Lambda$.

The following result is a generalization of [4, Proposition 3] to g-frames concerning the dependence of the g-R-dual sequence $\{\Gamma_j^\Lambda\}_{j\in J}$ in choosing the g-orthonormal bases $\Xi = \{\Xi_i\}_{i\in I}$ and $\Psi = \{\Psi_i\}_{i\in I}$.

Theorem 1.17. *Let $\Xi = \{\Xi_j\}_{j\in I}$, $\Xi' = \{\Xi'_j\}_{j\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$, $\Psi' = \{\psi'_i\}_{i\in I}$ be g-orthonormal bases for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ and $\{V_i\}_{i\in I}$, and let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect $\{V_i\}_{i\in I}$. Denote the analysis matrix for Λ with respect to Ξ by A and the g-R-dual sequences of Λ with respect to (Ξ,Ψ) and (Ξ',Ψ') by $\{\Gamma_j^\Lambda\}_{j\in J}, \{\Gamma_j'^\Lambda\}_{j\in J}$, respectively. Then the following conditions are equivalent:*

(i) *$\Gamma_j^\Lambda = \Gamma_j'^\Lambda$ for all $j\in I$,*

(ii) *if B and C are the transition matrices from Ξ to Ξ' and Ψ to Ψ', respectively, then $AB^* = CA$.*

Proof. Let $B = [B_{ij}]$, and let $C = [C_{ij}]$. By the definition of $\{\Gamma_j^\Lambda\}_{j\in J}, \{\Gamma_j'^\Lambda\}_{j\in J}$ for every $i,j\in I$, we have $\Psi_i(\Gamma_j^\Lambda)^* = \Lambda_i\Xi_j^*$ and $\Psi'_i(\Gamma_j'^\Lambda)^* = \Lambda_i\Xi_j'^*$. Since

$$\begin{aligned}[AB^*]_{ij} &= \sum_{k\in I}A_{ik}B_{kj}^* = \sum_{k\in I}\Lambda_i\Xi_k^*\Xi_k\Xi_j'^* = \Lambda_i\Big(\sum_{k\in I}\Xi_k^*\Xi_k\Big)\Xi_j'^*\\ &= \Lambda_i\Xi_j'^* = \Psi'_i(\Gamma_j'^\Lambda)^*,\end{aligned}$$

and

$$\begin{aligned}[CA]_{ij} &= \sum_{k\in I}C_{ik}A_{kj} = \sum_{k\in I}\Psi'_i\Psi_k^*\Lambda_k\Xi_j^* = \sum_{k\in I}\Psi'_i\Psi_k^*\Psi_k(\Gamma_j^\Lambda)^*\\ &= \Psi'_i\Big(\sum_{k\in I}\Psi_k^*\Psi_k\Big)(\Gamma_j^\Lambda)^* = \Psi'_i(\Gamma_j^\Lambda)^*,\end{aligned}$$

and from this the claim follows immediately. $\square$

Corollary 1.18. *In addition to the hypothesis of Theorem 1.17, if $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ with g-frame operator S_Λ and $\{\Gamma_j^\Lambda\}_{j\in I} = \{\Gamma_j'^\Lambda\}_{j\in I}$, then $A^*C^*AS_\Lambda^{-1}B^* = I$, where I is the identity matrix.*

Proof. Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Definition 1.16 implies that $S_\Lambda^{-1}A^*A = I$. Thus, if $\Gamma_j^\Lambda = \Gamma_j'^\Lambda$ for all $j\in I$, then by Theorem 1.17,

$AB^* = CA$. This implies that $B^* = S_\Lambda^{-1}A^*CA$; however, B has to be unitary, which yields $A^*C^*AS_\Lambda^{-1}B^* = I$. □

2. Existence of g-frame bounds

In this section, we characterize all sequences with lower g-frame bounds, and we obtain the g-frame conditions for a sequence of operators and its g-R-dual sequence. Recall that a family $\{\Lambda_i\}_{i\in I}$ is a g-frame sequence with respect to $\{V_i\}_{i\in I}$ if it is a g-frame for $\overline{\mathrm{Span}}\{\Lambda_i^*(V_i)\}_{i\in I}$ with respect to $\{V_i\}_{i\in I}$. The next result gives a characterization of g-frame sequences which keeps the information about the g-frame bounds.

Proposition 2.1. *Let $\Lambda = \{\Lambda_i \in B(\mathcal{H}, V_i) : i \in I\}$. Then the following conditions are equivalent:*

(i) *$\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame sequence with respect to $\{V_i\}_{i\in I}$ with g-frame bounds A and B,*

(ii) *the synthesis operator T_Λ^* is well defined on $(\sum_{i\in I} \oplus V_i)_{\ell^2}$ such that*

$$A\|g'\|_{\ell^2}^2 \le \|T_\Lambda^* g'\|^2 \le B\|g'\|_{\ell^2}^2, \quad \forall g' \in (\ker_{T_\Lambda^*})^\perp.$$

Proof. We note that, if $f \in \overline{\mathrm{Span}}\{\Lambda_i^*(V_i)\}_{i\in I}^\perp$, then $\|\Lambda_i f\|^2 = \langle f, \Lambda_i^*\Lambda_i f\rangle = 0$ for all $i \in I$. This implies that the upper g-frame sequence condition with bound B is equivalent to the right-hand inequality in (ii). We therefore assume that $\{\Lambda_i\}_{i\in I}$ is a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$, and we prove the equivalence of the lower g-frame sequence condition with the left-hand inequality in (ii). First, assume that $\{\Lambda_i\}_{i\in I}$ satisfies the lower g-frame sequence condition with bound A. Then $\mathcal{R}_{T_\Lambda^*}$ is closed because $\mathcal{R}_{T_\Lambda}$ is closed. Hence $(\ker_{T_\Lambda^*})^\perp = \overline{\mathcal{R}_{T_\Lambda}} = \mathcal{R}_{T_\Lambda}$; that is, $(\ker_{T_\Lambda^*})^\perp = \{T_\Lambda f : f \in \mathcal{H}\}$. Now, for any $f \in \mathcal{H}$ we have

$$\begin{aligned}\|T_\Lambda f\|_{\ell^2}^4 &= \big|\langle T_\Lambda^* T_\Lambda f, f\rangle\big|^2 = \big|\langle S_\Lambda f, f\rangle\big|^2 \le \|S_\Lambda f\|^2\|f\|^2 \\ &\le \frac{1}{A}\|S_\Lambda f\|^2 \sum_{i\in I} \|\Lambda_i f\|^2 = \frac{1}{A}\|S_\Lambda f\|^2 \|T_\Lambda f\|_{\ell^2}^2.\end{aligned}$$

This implies that

$$A\|T_\Lambda f\|_{\ell^2}^2 \le \|S_\Lambda f\|^2 = \big\|T_\Lambda^*(T_\Lambda f)\big\|^2,$$

as desired. For the other implication, assume that the left-hand inequality in (ii) is satisfied. We prove that $\mathcal{R}_{T_\Lambda^*}$ is closed. Let $\{f_n\}_{n=1}^\infty \subset \mathcal{R}_{T_\Lambda^*}$, and let $\lim_{n\to\infty} f_n = f$ for some $f \in \mathcal{H}$. There exists a sequence $\{g_n'\}_{n=1}^\infty \subset (\ker_{T_\Lambda^*})^\perp$ such that $T_\Lambda^* g_n' = f_n$. Now (ii) implies that $\{g_n'\}_{n=1}^\infty$ is a Cauchy sequence. Therefore $\{g_n'\}_{n=1}^\infty$ converges to some $g' \in (\sum_{i\in I} \oplus V_i)_{\ell^2}$, which by continuity of T_Λ^* satisfies $T_\Lambda^* g' = f$. Thus $\mathcal{R}_{T_\Lambda^*}$ is closed. If we let $(T_\Lambda^*)^\dagger$ denote the pseudoinverse of T_Λ^*, then we have $T_\Lambda^*(T_\Lambda^*)^\dagger T_\Lambda^* = T_\Lambda^*$, and the operator $(T_\Lambda^*)^\dagger T_\Lambda^*$ is the orthogonal projection onto $(\ker_{T_\Lambda^*})^\perp$, and the operator $T_\Lambda^*(T_\Lambda^*)^\dagger$ is the orthogonal projection onto $\mathcal{R}_{T_\Lambda^*}$. Thus, for any $g' \in (\sum_{i\in I} \oplus V_i)_{\ell^2}$, the inequality (ii) implies that

$$A\big\|(T_\Lambda^*)^\dagger T_\Lambda^* g'\big\|^2 \le \big\|T_\Lambda^*(T_\Lambda^*)^\dagger T_\Lambda^* g'\big\|^2 = \|T_\Lambda^* g'\|^2.$$

Since $\ker_{(T_\Lambda^*)^\dagger} = \mathcal{R}_{T_\Lambda^*}^\perp$, then $\|(T_\Lambda^*)^\dagger\|^2 \leq A^{-1}$; however, $T_\Lambda^\dagger T_\Lambda$ is the orthogonal projection onto

$$\mathcal{R}_{T_\Lambda^\dagger} = (\ker_{(T_\Lambda^\dagger)^*})^\perp = (\ker_{(T_\Lambda^*)^\dagger})^\perp = \mathcal{R}_{T_\Lambda^*},$$

and thus, for all $f \in \overline{\mathrm{Span}}\{\Lambda_i^*(V_i)\}_{i\in I} = \mathcal{R}_{T_\Lambda^*}$, we obtain

$$\|f\|^2 = \|T_\Lambda^\dagger T_\Lambda f\|^2 \leq \frac{1}{A}\|T_\Lambda f\|^2 = \frac{1}{A}\sum_{i\in I}\|\Lambda_i f\|^2.$$

This shows that $\Lambda = \{\Lambda_i\}_{i\in I}$ satisfies in the lower g-frame sequence condition with bound A as desired. □

The next result shows a basic connection between a sequence of operators and its g-R-dual sequence.

Theorem 2.2. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect $\{V_i\}_{i\in I}$. Then for every $\{g_j\}_{j\in I} \in (\sum_{j\in I}\oplus W_j)_{\ell^2}$, $\{g_i'\}_{i\in I} \in (\sum_{i\in I}\oplus V_i)_{\ell^2}$ satisfying $f = \sum_{j\in I}\Xi_j^* g_j$ and $h = \sum_{i\in I}\Psi_i^* g_i'$, we have*

$$\Big\|\sum_{j\in I}(\Gamma_j^\Lambda)^* g_j\Big\|^2 = \sum_{i\in I}\|\Lambda_i f\|^2 \qquad \text{and} \qquad \Big\|\sum_{i\in I}\Lambda_i^* g_i'\Big\|^2 = \sum_{j\in I}\|\Gamma_j^\Lambda h\|^2.$$

Proof. It is easy to check that

$$\begin{aligned}\Big\|\sum_{j\in I}(\Gamma_j^\Lambda)^* g_j\Big\|^2 &= \Big\|\sum_{j\in I}\Big(\sum_{i\in I}\Xi_j\Lambda_i^*\Psi_i\Big)^* g_j\Big\|^2 = \Big\|\sum_{i\in I}\Psi_i^*\Lambda_i f\Big\|^2\\ &= \Big\langle\sum_{i\in I}\Psi_i^*\Lambda_i f, \sum_{j\in I}\Psi_j^*\Lambda_j f\Big\rangle = \sum_{i\in I}\sum_{j\in I}\langle\Lambda_i f, \Psi_i\Psi_j^*\Lambda_j f\rangle\\ &= \sum_{i\in I}\sum_{j\in I}\langle\Lambda_i f, \delta_{ij}\Lambda_j f\rangle = \sum_{i\in I}\|\Lambda_i f\|^2.\end{aligned}$$

Similarly, the second claim follows from Theorem 1.15. □

Corollary 2.3. *If we let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect $\{V_i\}_{i\in I}$, then*

$$\big\|T_{\Gamma^\Lambda}^*\big([f]_\Xi\big)\big\| = \|T_\Lambda f\|_{\ell^2}, \qquad \big\|T_\Lambda^*\big([f]_\Psi\big)\big\| = \|T_{\Gamma^\Lambda} f\|_{\ell^2}$$

for every $f \in \mathcal{H}$.

Proof. This follows immediately from Theorem 2.2. □

There exists an interesting relation between the synthesis operator of $\Lambda = \{\Lambda_i\}_{i\in I}$ and the span of $\{(\Gamma_j^\Lambda)^*(W_j)\}_{j\in I}$, which will turn out to be very useful in the sequel.

Theorem 2.4. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ with g-R-dual sequence $\{\Gamma_j^\Lambda\}_{j\in I}$ with respect to (Ξ, Ψ). Then the following statements hold.*

(i) $f \in (\overline{\mathrm{Span}}\{(\Gamma_j^\Lambda)^*(W_j)\}_{j\in I})^\perp$ *if and only if* $[f]_\Psi \in \ker T_\Lambda^*$.

(ii) $f \in (\overline{\mathrm{Span}}\{\Lambda_j^*(V_j)\}_{j\in I})^\perp$ *if and only if* $[f]_\Xi \in \ker T_{\Gamma^\Lambda}^*$.

Proof. Let $f \in \mathcal{H}$. First, for each $j \in J$ and $g_j \in W_j$, we observe that

$$\big\langle f, (\Gamma_j^\Lambda)^* g_j \big\rangle = \sum_{i\in J} \langle f, \Psi_i^* \Lambda_i \Xi_j^* g_j \rangle = \Big\langle \sum_{i\in J} \Lambda_i^* \Psi_i f, \Xi_j^* g_j \Big\rangle = \big\langle T_\Lambda^*\big([f]_\Psi\big), \Xi_j^* g_j \big\rangle.$$

Since $\Xi = \{\Xi_j\}_{j\in J}$ is a g-orthonormal basis for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$, then $\langle T_\Lambda^*([f]_\Psi), \Xi_j^* g_j\rangle = 0$ for all $j \in I$, and $g_j \in W_j$ if and only if $T_\Lambda^*([f]_\Psi) = 0$. Thus $f \in (\mathrm{Span}\{(\Gamma_j^\Lambda)^*(W_j)\}_{j\in I})^\perp$ is equivalent to $[f]_\Psi \in \ker T_\Lambda^*$. Similarly, the second claim follows from Theorem 1.15. □

Corollary 2.5. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ with g-R-dual sequence $\{\Gamma_j^\Lambda\}_{j\in I}$ with respect to (Ξ, Ψ). Then*

$$\dim\big(\overline{\mathrm{Span}}\big\{(\Gamma_j^\Lambda)^*(W_j)\big\}_{j\in I}\big)^\perp = \dim\ker T_\Lambda^*, \qquad \textit{and}$$
$$\dim\big(\overline{\mathrm{Span}}\big\{\Lambda_j^*(V_j)\big\}_{j\in I}\big)^\perp = \dim\ker T_{\Gamma^\Lambda}^*.$$

Proof. This follows immediately from the Theorem 2.4. □

The next result shows a kind of equilibrium between a sequence of operators and its R-dual sequence. It can be viewed as a general version of [4, Proposition 13].

Corollary 2.6. *The following conditions are equivalent.*

(i) *$\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame sequence with respect to $\{V_i\}_{i\in I}$ with g-frame bounds A, B.*
(ii) *$\{\Gamma_j^\Lambda\}_{j\in I}$ is a g-frame sequence with respect to $\{W_j\}_{j\in I}$ with g-frame bounds A, B.*
(iii) *$\{\Gamma_j^\Lambda\}_{j\in I}$ is a g-Riesz basic sequence with respect to $\{W_j\}_{j\in I}$ with g-frame bounds A, B.*

Proof. (i) ⇔ (ii) Proposition 2.1 and Theorem 2.4 conclude that $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame sequence with respect to $\{V_i\}_{i\in I}$ with g-frame bounds A, B if and only if

$$A\big\|[f]_\Psi\big\|_{\ell^2}^2 \le \big\|T_\Lambda^*\big([f]_\Psi\big)\big\|^2 \le B\big\|[f]_\Psi\big\|_{\ell^2}^2$$

for all $f \in \overline{\mathrm{Span}}\{(\Gamma_j^\Lambda)^*(W_j)\}_{j\in I}$. Now, Corollary 2.3 implies that

$$A\|f\|^2 \le \|T_{\Gamma^\Lambda} f\|_{\ell^2}^2 \le B\|f\|^2.$$

(i) ⇔ (iii) This equivalence follows immediately from Theorem 2.2. □

The dimension condition in Corollary 2.5 will play a crucial role for the g-R-dual sequence. Using Corollary 2.5 we can derive a simple characterization of an g-Riesz basic sequence being a g-R-dual sequence of a g-frame in the tight case.

Theorem 2.7. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a A-tight g-frames for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$, and let $\{\Gamma_j\}_{j\in I}$ be an A-tight g-Riesz basic sequence in $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$. Then $\{\Gamma_j\}_{j\in I}$ is a g-R-dual sequence of $\{\Lambda_i\}_{i\in I}$ with respect to (Ξ, Ψ) if and only if*

$$\dim\big(\overline{\mathrm{Span}}\big\{\Gamma_j^*(W_j)\big\}_{j\in I}\big)^\perp = \dim\ker T_\Lambda^*. \tag{2.1}$$

Proof. The necessity of the condition in (2.1) follows from Corollary 2.5. Now assume that (2.1) holds. Then, according to Lemma 1.11, the sequence $\{\frac{1}{\sqrt{A}}\Gamma_j\}_{j\in I}$ is a g-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$. Suppose that $\Xi = \{\Xi_j\}_{j\in I}$ and $\Psi = \{\Psi_i\}_{i\in I}$ are g-orthonormal bases for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ and $\{V_i\}_{i\in I}$, respectively. Consider the g-R-dual $\{\Theta_j\}_{j\in I}$ of $\Lambda = \{\Lambda_i\}_{i\in I}$ with respect to (Ξ, Ψ) (i.e., $\Theta_j = \sum_{i\in I}\Xi_j\Lambda_i^*\Psi_i, j \in I$). By Corollary 2.6, $\{\Theta_j\}_{j\in I}$ is an A-tight g-Riesz basic sequence with respect to $\{W_j\}_{j\in I}$; hence $\{\frac{1}{\sqrt{A}}\Theta_j\}_{j\in I}$ is also a g-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$. By Corollary 2.5 and (2.1),

$$\dim\big(\overline{\mathrm{Span}}\{\Theta_j^*(W_j)\}_{j\in I}\big)^{\perp} = \dim\ker T_\Lambda^* = \dim\big(\overline{\mathrm{Span}}\{\Gamma_j^*(W_j)\}_{j\in I}\big)^{\perp}. \tag{2.2}$$

In case $(\overline{\mathrm{Span}}\{\Theta_j^*(W_j)\}_{j\in I})^{\perp} = (\overline{\mathrm{Span}}\{\Gamma_j^*(W_j)\}_{j\in I})^{\perp} = \{0\}$, the g-orthonormality of the sequences $\{\frac{1}{\sqrt{A}}\Theta_i\}_{i\in I}$ and $\{\frac{1}{\sqrt{A}}\Gamma_i\}_{i\in I}$ implies that there exists unitary operator

$$U : \mathcal{H} \to \mathcal{H}, \quad \text{by } \Gamma_j = \Theta_j U^*, \quad \forall j \in I.$$

In case $(\overline{\mathrm{Span}}\{\Theta_j^*(W_j)\}_{j\in I})^{\perp} \neq \{0\}$, if we let $\{\Phi_j\}_{j\in I}$ and $\{\Omega_j\}_{j\in I}$ be g-orthonormal bases for

$$\big(\overline{\mathrm{Span}}\{\Theta_j^*(W_j)\}_{j\in I}\big)^{\perp} \qquad \text{and} \qquad \big(\overline{\mathrm{Span}}\{\Gamma_j^*(W_j)\}_{j\in I}\big)^{\perp},$$

respectively, with respect to $\{W_j\}_{j\in I}$, then (2.2) implies that there exists unitary operator

$$U : \mathcal{H} \to \mathcal{H}, \quad \text{by } \Gamma_j = \Theta_j U^*, \qquad \Omega_j = \Phi_j U^* \quad \forall j \in I.$$

In both cases, we have

$$\Gamma_j = \Theta_j U^* = \Big(\sum_{i\in I}\Xi_j\Lambda_i^*\Psi_i\Big)U^* = \sum_{i\in I}\Xi_j\Lambda_i^*\Psi_i U^*, \quad \forall j \in I,$$

which shows that $\{\Gamma_j\}_{j\in I}$ is a g-R-dual sequence of $\{\Lambda_i\}_{i\in I}$ with respect to $\{\Xi_j\}_{j\in I}$ and $\{\Psi_i U^*\}_{i\in I}$. □

3. Characterizations of equivalence by the g-R-dual sequence

In this section we characterize those pairs of g-frames which are equivalent (unitarily equivalent) by their g-R-dual sequences.

Definition 3.1. Two sequences $\{\Gamma_j \in B(\mathcal{H}, W_i) \mid j \in I\}$ and $\{\Gamma'_j \in B(\mathcal{H}, W_i) \mid j \in I\}$ are regarded as unitarily equivalent in $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ if there is a unitary $T : \mathcal{H} \to \mathcal{H}$ such that $T\Gamma_j^* = \Gamma_j'^*$ for all $j \in I$. We will say that they are equivalent if there is a bounded linear invertible operator $T : \mathcal{H} \to \mathcal{H}$ such that $T\Gamma_j^* = \Gamma_j'^*$ for all $j \in I$.

The following result is about different types of equivalence of g-frames, which is taken from [15, Proposition 4.2]. This result will then be employed in several proofs thereafter.

Proposition 3.2. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Lambda' = \{\Lambda'_i\}_{i\in I}$ be Parseval g-frames for $\mathcal{H}_1$ and $\mathcal{H}_2$ with respect to $\{V_i\}_{i\in I}$, respectively. Then Λ is unitarily equivalent to Λ' if and only if the analysis operators T_Λ and $T_{\Lambda'}$ have the same range. Likewise, two g-frames with respect to $\{V_i\}_{i\in I}$ are equivalent if and only if their analysis operators have the same range.*

Theorem 3.3. *Let $\{\Lambda_i\}_{i\in I}$ and $\{\Lambda'_i\}_{i\in I}$ be g-frames for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then*

(i) *$\{\Lambda_i\}_{i\in I}$ is equivalent to $\{\Lambda'_i\}_{i\in I}$ in $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ if and only if*

$$\overline{\text{Span}}\big\{(\Gamma_j^{\Lambda})^*(W_j)\big\}_{j\in I} = \overline{\text{Span}}\big\{(\Gamma_j^{\Lambda'})^*(W_j)\big\}_{j\in I},$$

(ii) *$\{\Lambda_i\}_{i\in I}$ is unitarily equivalent to $\{\Lambda'_i\}_{i\in I}$ in $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ if and only if $S_{\Gamma^\Lambda} = S_{\Gamma^{\Lambda'}}$,*

(iii) *$\{\Gamma_j^{\Lambda}\}_{j\in I}$ is unitarily equivalent to $\{\Gamma_j^{\Lambda'}\}_{j\in I}$ in $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$ if and only if $S_\Lambda = S_{\Lambda'}$.*

Proof. (i) By Proposition 3.2, $\{\Lambda_i\}_{i\in I}$ and $\{\Lambda'_i\}_{i\in I}$ are equivalent in $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ if and only if $\mathcal{R}_{T_\Lambda} = \mathcal{R}_{T_{\Lambda'}}$; hence $\ker T^*_\Lambda = \ker T^*_{\Lambda'}$. Now the claim follows from Theorem 2.4.

(ii) Using Propositions 2.1 and 3.2, $\{\Lambda_i\}_{i\in I}$ is unitarily equivalent to $\{\Lambda'_i\}_{i\in I}$ if and only if

$$\Big\|\sum_{i\in I} \Lambda_i^* g'_i\Big\|^2 = \Big\|\sum_{i\in I} \Lambda_i'^* g'_i\Big\|^2, \quad \forall \{g'_i\}_{i\in I} \in (\ker T^*_\Lambda)^\perp.$$

By Theorem 2.2, this is in turn equivalent to

$$\langle S_{\Gamma^\Lambda} f, f\rangle = \sum_{j\in I} \|\Gamma_j^\Lambda f\|^2 = \sum_{j\in I} \|\Gamma_j^{\Lambda'} f\|^2 = \langle S_{\Gamma^{\Lambda'}} f, f\rangle$$

for all $f \in \mathcal{H}$ and $g'_i = \Psi_i f (i \in I)$. It follows that $S_{\Gamma^\Lambda} = S_{\Gamma^{\Lambda'}}$, as required.

(iii) The proof follows immediately from (ii) and Theorem 1.15. □

Corollary 3.4. *Let $\{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then let*

$$\overline{\text{Span}}\big\{(\Gamma_j^{\Lambda})^*(W_j)\big\}_{j\in I} = \overline{\text{Span}}\big\{(\Gamma_j^{\widehat{\Lambda}})^*(W_j)\big\}_{j\in I},$$

where $\{\widehat{\Lambda}_i\}_{i\in I}$ is the canonical dual g-frame of $\{\Lambda_i\}_{i\in I}$.

Proof. Since $\{\widehat{\Lambda}_i\}_{i\in I}$ is equivalent to $\{\Lambda_i\}_{i\in I}$, this claim follows from Theorem 3.3. □

To have a better understanding of the different types of equivalence of the g-R-dual sequences, we prove the following characterization result.

Theorem 3.5. *Let $\Xi = \{\Xi_j\}_{j\in I}$, $\Xi' = \{\Xi'_j\}_{j\in I}$ and $\Psi = \{\psi_i\}_{i\in I}$, $\Psi' = \{\psi'_i\}_{i\in I}$ be g-orthonormal bases for $\mathcal{H}$ with respect $\{W_j\}_{j\in I}$ and $\{V_i\}_{i\in I}$, and let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Denote the analysis matrix for Λ with respect to Ξ by A and the g-R-dual sequences of Λ with respect (Ξ, Ψ) and (Ξ', Ψ') by $\{\Gamma_j^\Lambda\}_{j\in J}, \{\Gamma_j'^\Lambda\}_{j\in J}$, respectively. If $\Gamma = \{\Gamma_j^\Lambda\}_{j\in I}$ and $\Gamma' = \{\Gamma_j'^\Lambda\}_{j\in I}$ are g-frames for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$, then the following statements hold.*

(i) $\{\Gamma_j^\Lambda\}_{j\in I}$ *is equivalent to* $\{\Gamma_j'^\Lambda\}_{j\in I}$ *in* $\mathcal{H}$ *with respect to* $\{W_j\}_{j\in I}$ *if and only if* $\ker(A) = \ker(AB^*)$.

(ii) $\{\Gamma_j^\Lambda\}_{j\in I}$ *is unitarily equivalent to* $\{\Gamma_j'^\Lambda\}_{j\in I}$ *in* $\mathcal{H}$ *with respect to* $\{W_j\}_{j\in I}$, *if and only if*

$$A^*A = (AB^*)^*(AB^*).$$

Moreover, if $\Lambda = \{\Lambda_i\}_{i\in I}$ *is a g-frame for* $\mathcal{H}$ *with respect* $\{V_i\}_{i\in I}$ *with g-frame operator* S_Λ, *then the above is equivalent to* $S_\Lambda = BS_\Lambda B^*$.

Proof. (i) Let $g = \{g_j\}_{j\in I} \in (\sum_{j\in I} \oplus W_j)_{\ell^2}$ be arbitrary. First we observe that

$$\begin{aligned}\sum_{j\in I}(\Gamma_j'^\Lambda)^* g_j &= \sum_{k\in I}\sum_{j\in I} \Psi_k'^*\Lambda_k\Xi_j'^* g_j = \sum_{k\in I}\sum_{j\in I}\Psi_k'^*\Lambda_k\Big(\sum_{i\in I}\Xi_i^*\Xi_i\Xi_j'^* g_j\Big)\\ &= \sum_{k\in I}\sum_{j\in I}\sum_{i\in I}\Psi_k'^*\Lambda_k\Xi_i^*\Xi_i\Xi_j'^* g_j = \sum_{k\in I}\sum_{j\in I}\sum_{i\in I}\Psi_k'^* A_{ki}B_{ij}^* g_j\\ &= \sum_{k\in I}\Psi_k'^*\Big(\sum_{j\in I}[AB^*]_{kj}g_j\Big) = \sum_{k\in I}\Psi_k'^*(AB^*g)_k.\end{aligned}$$

This implies that

$$AB^*g = 0 \quad\Leftrightarrow\quad \sum_{j\in I}(\Gamma_j'^\Lambda)^* g_j = 0.$$

Next we have

$$\begin{aligned}\sum_{j\in I}(\Gamma_j^\Lambda)^* g_j &= \sum_{k\in I}\sum_{j\in I}\Psi_k^*\Lambda_k\Xi_j^* g_j = \sum_{k\in I}\sum_{j\in I}\Psi_k^* A_{kj} g_j\\ &= \sum_{k\in I}\Psi_k^*(Ag)_k;\end{aligned}$$

hence

$$Ag = 0 \quad\Leftrightarrow\quad \sum_{j\in I}(\Gamma_j^\Lambda)^* g_j = 0.$$

Now $\{\Gamma_j^\Lambda\}_{j\in I}$ is equivalent to $\{\Gamma_j'^\Lambda\}_{j\in I}$ if and only if there exists a bounded linear invertible operator $T : \mathcal{H} \to \mathcal{H}$ such that $T(\sum_{j\in I}(\Gamma_j^\Lambda)^* g_j) = \sum_{j\in I}(\Gamma_j'^\Lambda)^* g_j$ for all $\{g_j\}_{j\in I} \in (\sum_{j\in I}\oplus W_j)_{\ell^2}$. From this the claim follows immediately.

(ii) First, we prove that $[A^*A]_{ij} = \Gamma_i^\Lambda(\Gamma_j^\Lambda)^*$ and that $[(AB^*)^*(AB^*)]_{ij} = \Gamma_i'^\Lambda(\Gamma_j'^\Lambda)^*$. To see this, we have

$$\begin{aligned}\Gamma_i^\Lambda(\Gamma_j^\Lambda)^* &= \Big(\sum_{k\in I}\Xi_i\Lambda_k^*\Psi_k\Big)\Big(\sum_{m\in I}\Psi_m^*\Lambda_m\Xi_j^*\Big)\\ &= \sum_{k\in I}\sum_{m\in I}\delta_{km}\Xi_i\Lambda_k^*\Lambda_m\Xi_j^* = \sum_{k\in I}\Xi_i\Lambda_k^*\Lambda_k\Xi_j^*\\ &= \sum_{k\in I}A_{ik}^*A_{kj} = [A^*A]_{ij}.\end{aligned}$$

Then we obtain

$$\begin{aligned}
\Gamma_i'^{\Lambda}(\Gamma_j'^{\Lambda})^* &= \Big(\sum_{k\in I}\Xi_i'\Lambda_k^*\Psi_k'\Big)\Big(\sum_{m\in I}\Psi_m'^*\Lambda_m\Xi_j'^*\Big)\\
&= \sum_{k\in I}\sum_{m\in I}\delta_{km}\Xi_i'\Lambda_k^*\Lambda_m\Xi_j'^* = \sum_{k\in I}(\Lambda_k\Xi_i'^*)^*(\Lambda_k\Xi_j'^*)\\
&= \sum_{k\in I}\Big(\sum_{n\in I}\Lambda_k\Xi_n^*\Xi_n\Xi_i'^*\Big)^*\Big(\sum_{m\in I}\Lambda_k\Xi_m^*\Xi_m\Xi_j'^*\Big)\\
&= \sum_{k\in I}\Big(\sum_{n\in I}A_{kn}B_{ni}^*\Big)^*\Big(\sum_{m\in I}A_{km}B_{mj}^*\Big)\\
&= \sum_{k\in I}(AB^*)_{ik}^*(AB^*)_{kj} = \big[(AB^*)^*(AB^*)\big]_{ij}.
\end{aligned}$$

Now let $A^*A = (AB^*)^*(AB^*)$. Define the operator T as follows:

$$T : \mathrm{Span}\big\{(\Gamma_j^{\Lambda})^*(W_j)\big\}_{j\in I} \to \mathcal{H}, \qquad T\Big(\sum_{j\in J}(\Gamma_j^{\Lambda})^*g_j\Big) = \sum_{j\in J}(\Gamma_j'^{\Lambda})^*g_j$$

for all finite sequences $\{g_j : g_j \in W_j\}_{j\in J}$. If we let $f_1, f_2 \in \mathrm{Span}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I}$ as $f_1 = \sum_{j\in J_1}(\Gamma_j^{\Lambda})^*g_{1j}$ and we let $f_2 = \sum_{j\in J_2}(\Gamma_j^{\Lambda})^*g_{2j}$, then we have

$$\begin{aligned}
\langle Tf_1, Tf_2\rangle &= \Big\langle\sum_{j\in J_1}(\Gamma_j'^{\Lambda})^*g_{1j}, \sum_{k\in J_2}(\Gamma_k'^{\Lambda})^*g_{2k}\Big\rangle\\
&= \sum_{j\in J_1}\sum_{k\in J_2}\big\langle\Gamma_k'^{\Lambda}(\Gamma_j'^{\Lambda})^*g_{1j}, g_{2k}\big\rangle\\
&= \Big\langle\sum_{j\in J_1}(\Gamma_j^{\Lambda})^*g_{1j}, \sum_{k\in J_2}(\Gamma_k^{\Lambda})^*g_{2k}\Big\rangle\\
&= \langle f_1, f_2\rangle.
\end{aligned}$$

Thus the g-completeness of Γ for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$ implies that T has an extension isometry on $\mathcal{H}$ and that T is surjective. This makes sense because if $f \in \mathrm{Span}\{(\Gamma_j'^{\Lambda})^*(W_j)\}_{j\in I}$, then we can write

$$f = \sum_{j\in J}(\Gamma_j'^{\Lambda})^*g_j = T\Big(\sum_{j\in J}(\Gamma_j^{\Lambda})^*g_j\Big)$$

for some finite sequence $\{g_j : g_j \in W_j\}_{j\in J}$. Since Γ' is g-complete for $\mathcal{H}$ with respect to $\{W_i\}_{i\in I}$, then by the continuity of T it follows that T is surjective on $\mathcal{H}$ and that $T(\Gamma_j^{\Lambda})^* = (\Gamma_j'^{\Lambda})^*$ for all $j \in I$. This shows that Γ is unitarily equivalent to Γ' in $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$. The converse implication is obvious. Finally, if $\Lambda = \{\Lambda_i\}_{i\in I}$ is a g-frame for $\mathcal{H}$ with respect $\{V_i\}_{i\in I}$, then we have $A^*A = S_\Lambda$. Thus

$$S_\Lambda = A^*A = (AB^*)^*(AB^*) = BA^*AB^* = BS_\Lambda B^*. \qquad \square$$

4. Duality properties of the g-R-dual sequence

In this section we characterize all properties of a g-Bessel sequence in terms of properties of their g-R-dual sequence. We will study properties of dual g-frames and canonical dual g-frames. This is a general version of the duality principle for g-frames which follows from the duality relations in [4].

The next result gives an explicit form for g-R-dual sequences of the canonical dual g-frame.

Theorem 4.1. *Let $\{\Lambda_i\}_{i\in I}$ and $\{\Omega_i\}_{i\in I}$ be g-frames for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then $\{\Omega_i\}_{i\in I}$ is a dual g-frame of $\{\Lambda_i\}_{i\in I}$ if and only if g-R-dual sequences $\{\Gamma_j^{\Lambda}\}_{j\in I}$ and $\{\Gamma_j^{\Omega}\}_{j\in I}$ are g-biorthogonal; that is,*

$$\Gamma_i^{\Lambda}(\Gamma_j^{\Omega})^* g_j = \Gamma_i^{\Omega}(\Gamma_j^{\Lambda})^* g_j = \delta_{ij} g_j, \quad \forall i, j \in I, g_j \in W_j.$$

Proof. Let $\{\Omega_i\}_{i\in I}$ be a dual g-frame of $\{\Lambda_i\}_{i\in I}$. By definition of $\{\Gamma_j^{\Omega}\}_{j\in I}$ and $\{\Gamma_j^{\Lambda}\}_{j\in I}$, for every $i, j \in I$ and $g_j \in W_j$ we have

$$\begin{aligned}\Gamma_i^{\Lambda}(\Gamma_j^{\Omega})^* g_j &= \sum_{k\in I} \Xi_i \Lambda_k^* \Psi_k \Big(\sum_{m\in I} \Xi_j \Omega_m^* \Psi_m\Big)^* g_j = \sum_{k\in I}\sum_{m\in I} \Xi_i \Lambda_k^* \Psi_k \Psi_m^* \Omega_m \Xi_j^* g_j \\ &= \sum_{k\in I} \Xi_i \Lambda_k^* \Omega_k \Xi_j^* g_j = \Xi_i \Big(\sum_{k\in I} \Lambda_k^* \Omega_k \Xi_j^* g_j\Big) = \Xi_i \Xi_j^* g_j = \delta_{ij} g_j.\end{aligned}$$

The converse implication follows from Theorem 1.15. □

Corollary 4.2. *Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ with canonical dual g-frame denoted by $\{\widehat{\Lambda}_i\}_{i\in I}$. Then the g-R-dual sequences $\{\Gamma_j^{\Lambda}\}_{j\in I}$ and $\{\Gamma_j^{\widehat{\Lambda}}\}_{j\in I}$ are g-biorthogonal, that is,*

$$\Gamma_i^{\Lambda}(\Gamma_j^{\widehat{\Lambda}})^* g_j = \Gamma_i^{\widehat{\Lambda}}(\Gamma_j^{\Lambda})^* g_j = \delta_{ij} g_j$$

for all $i, j \in I$ and $g_j \in W_j$. Thus $\{\Gamma_j^{\widehat{\Lambda}}\}_{j\in I}$ is the dual g-Riesz basic sequence of $\{\Gamma_j^{\Lambda}\}_{j\in I}$.

The next result is a characterization of tight g-frames in terms of their g-R-dual sequences.

Corollary 4.3. *We have that $\{\Lambda_i\}_{i\in I}$ is an A-tight g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ if and only if g-R-dual sequence $\{\frac{1}{\sqrt{A}}\Gamma_j^{\Lambda}\}_{j\in I}$ is a g-orthonormal system for $\mathcal{H}$ with respect to $\{W_j\}_{j\in I}$. Thus the sequence $\{\Lambda_i\}_{i\in I}$ is a Parseval g-frame if and only if its g-R-dual sequence is an orthonormal system.*

Proof. This follows immediately from the Lemma 1.11, Corollary 2.6, and Theorem 4.2. □

Theorem 4.4. *Let $\{\Lambda_i\}_{i\in I}$ and $\{\Omega_i\}_{i\in I}$ be g-frames for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$. Then $\{\Omega_i\}_{i\in I}$ is a dual g-frame of $\{\Lambda_i\}_{i\in I}$ if and only if there exists a g-Bessel sequence $\{\Theta_j\}_{j\in I}$ for $(\overline{\mathrm{Span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I})^{\perp}$ with respect to $\{W_j\}_{j\in I}$ such that $\Gamma_j^{\Omega} = \Gamma_j^{\widehat{\Lambda}} + \Theta_j$ for all $j \in I$.*

Proof. Suppose that $\{\Omega_i\}_{i\in I}$ is a dual g-frame of $\{\Lambda_i\}_{i\in I}$. By Theorem 4.1, we have

$$\begin{aligned}\left\langle(\Gamma_i^{\Omega}-\Gamma_i^{\widehat{\Lambda}})^*g_i,(\Gamma_j^{\Lambda})^*g_j\right\rangle &= \left\langle g_i,(\Gamma_i^{\Omega}-\Gamma_i^{\widehat{\Lambda}})(\Gamma_j^{\Lambda})^*g_j\right\rangle\\ &= \left\langle g_i,\Gamma_i^{\Omega}(\Gamma_j^{\Lambda})^*g_j\right\rangle-\left\langle g_i,\Gamma_i^{\widehat{\Lambda}}(\Gamma_j^{\Lambda})^*g_j\right\rangle\\ &= \langle g_i,\delta_{ij}g_j\rangle-\langle g_i,\delta_{ij}g_j\rangle=0\end{aligned}$$

for all $i,j\in I$ and $g_i\in W_i, g_j\in W_j$. Thus Definition 1.13 implies that $\Theta_j=\Gamma_j^{\Omega}-\Gamma_j^{\widehat{\Lambda}}$ is a g-Bessel sequence for $(\overline{\text{Span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I})^{\perp}$ with respect to $\{W_j\}_{j\in I}$ and $\Gamma_j^{\Omega}=\Gamma_j^{\widehat{\Lambda}}+\Theta_j$. Now for the opposite implication, suppose that there exists a g-Bessel sequence $\{\Theta_j\}_{j\in I}$ for $(\overline{\text{Span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I})^{\perp}$ with respect to $\{W_j\}_{j\in I}$ such that $\Gamma_j^{\Omega}=\Gamma_j^{\widehat{\Lambda}}+\Theta_j$ for all $j\in I$. By Theorem 1.15, we have

$$\Omega_i=\widehat{\Lambda}_i+\sum_{j\in I}\Psi_i(\Theta_j)^*\Xi_j \quad \text{for all } i\in I$$

Hence, for each $f\in\mathcal{H}$, we have

$$\begin{aligned}\sum_{i\in I}\Lambda_i^*\Omega_i f &= \sum_{i\in I}\Lambda_i^*\Big(\widehat{\Lambda}_i+\sum_{j\in I}\Psi_i\Theta_j^*\Xi_j\Big)f\\ &= \sum_{i\in I}\Lambda_i^*\widehat{\Lambda}_i f+\sum_{i\in I}\sum_{j\in I}\Lambda_i^*\Psi_i\Theta_j^*\Xi_j f\\ &= f+\sum_{j\in I}\sum_{i\in I}\Lambda_i^*\Psi_i\Theta_j^*\Xi_j f.\end{aligned}$$

Since $\Theta_j^*\Xi_j f\in(\overline{\text{Span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I})^{\perp}$ for all $j\in I$. Theorem 2.4 implies that

$$\sum_{i\in I}\Lambda_i^*\Psi_i\Theta_j^*\Xi_j f=0.$$

This proves that $\{\Omega_i\}_{i\in I}$ is a dual g-frame of $\{\Lambda_i\}_{i\in I}$. □

Among the dual g-frames the canonical dual g-frame is distinguished by the following properties.

Theorem 4.5. *Let $\Lambda=\{\Lambda_i\}_{i\in I}$ be a g-frame for $\mathcal{H}$ with respect to $\{V_i\}_{i\in I}$ with canonical dual g-frame denoted by $\{\widehat{\Lambda}_i\}_{i\in I}$, and let $\{\Omega_i\}_{i\in I}$ be a dual g-frame of $\{\Lambda_i\}_{i\in I}$. Then*

$$\|\Gamma_j^{\widehat{\Lambda}}\|\leq\|\Gamma_j^{\Omega}\| \quad \textit{for all } j\in I$$

with equality if and only if $\{\Omega_j\}_{j\in I}=\{\widehat{\Lambda}_j\}_{j\in I}$.

Proof. By Theorem 4.4, $\{\Omega_i\}_{i\in I}$ is a dual g-frame of $\{\Lambda_i\}_{i\in I}$ if and only if $\Gamma_j^{\Omega}=\Gamma_j^{\widehat{\Lambda}}+\Theta_j$, where $(\Gamma_j^{\widehat{\Lambda}})^*g\in\overline{\text{Span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I}$, and $\Theta_j^*g\in(\overline{\text{Span}}\{(\Gamma_j^{\Lambda})^*(W_j)\}_{j\in I})^{\perp}$

for all $j \in I, g \in W_j$; hence

$$\begin{aligned}\|\Gamma_j^{\Omega}\|^2 = \left\|(\Gamma_j^{\Omega})^*\right\|^2 &= \sup_{\|g\|=1}\left\|(\Gamma_j^{\Omega})^* g\right\|^2 \\ &= \sup_{\|g\|=1}\left\|(\Gamma_j^{\widehat{\Lambda}})^* g\right\|^2 + \sup_{\|g\|=1}\|\Theta_j^* g\|^2 \\ &= \left\|(\Gamma_j^{\widehat{\Lambda}})^*\right\|^2 + \|\Theta_j^*\|^2 \\ &= \|\Gamma_j^{\widehat{\Lambda}}\|^2 + \|\Theta_j\|^2 \geq \|\Gamma_j^{\widehat{\Lambda}}\|^2\end{aligned}$$

with equality if and only if $\{\Omega_j\}_{j\in I} = \{\widehat{\Lambda}_j\}_{j\in I}$. □

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References

1. M. S. Asgari, *Operator-valued bases on Hilbert spaces*, J. Linear and Topological Algebra., **2** (2013), no. 4, 201–218, article available at jlta.iauctb.ac.ir/article_510022_6baf619a1cbe13440723e3b77b58e7c5.pdf. 883, 885
2. M. S. Asgari and H. Rahimi, *Generalized frames for operators in Hilbert spaces*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **17** (2014), no. 2, art. ID 1450013. Zbl 1294.41023. MR3212683. DOI 10.1142/S0219025714500131. 883
3. P. G. Casazza and G. Kutyniok, *Frames of subspaces*, in Wavelets, Frames and Operator Theory, Contemp. Math. **345** (2004) (Amer. Math. Soc.), 87–113. Zbl 1058.42019. MR2066823. DOI 10.1090/conm/345/06242. 881
4. P. G. Casazza, G. Kutyniok, and M. C. Lammers, *Duality principles in frame theory*, J. Fourier Anal. Appl., **10** (2004), no. 4, 383–408. Zbl 1058.42020. MR2078264. DOI 10.1007/s00041-004-3024-7. 882, 887, 890, 895
5. P. G. Casazza, G. Kutyniok, and M. C. Lammers, *Duality principles, localization of frames, and Gabor theory*, preprint, http://framerc.missouri.edu/pub/duality-principles-localization-frames-and-gabor-theory 882
6. O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser/Springer, Basel, 2016. Zbl 1348.42033. MR3495345. DOI 10.1007/978-3-319-25613-9. 883
7. O. Christensen, H. O. Kim, and R. Y. Kim, *On the duality principle by Casazza, Kutyniok, and Lammers*, J. Fourier Anal. Appl., **17** (2011), no. 4, 640–655. Zbl 1222.42031. MR2819169. DOI 10.1007/s00041-010-9151-4. 883
8. O. Christensen, X. C. Xiao, and Y. C. Zhu, *Characterizing R-duality in Banach spaces*, Acta Math. Sinica (Engl. Ser.) **29** (2013), no. 1, 75–84. Zbl 1268.42054. MR3001011. DOI 10.1007/s10114-012-1199-4. 883
9. L. Găvruţa and P. Găvruţa, *Some properties of operator-valued frames*, Acta Math. Sci. (Ser. B Engl. Ed.) **36** (2016), no. 2, 469–476. Zbl 1363.42054. MR3455102. DOI 10.1016/S0252-9602(16)30013-3. 881
10. X. Guo, *G-bases in Hilbert spaces*, Abstr. Appl. Anal. **2012,** art. ID 923729. Zbl 1266.46018. MR3004882. DOI 10.1155/2012/923729. 883
11. X. Guo, *Operator parameterizations of g-frames*, Taiwanese J. Math. **18** (2014) no. 1, 313–328. Zbl 1357.42031. MR3162128. DOI 10.11650/tjm.18.2014.3264. 883
12. D. Han, P. Li, B. Meng, and W. Tang, *Operator-valued frames and structured quantum channels*, Sci. China Math. **54** (2011), no. 11, 2361–2372. Zbl 1245.42024. MR2859699. DOI 10.1007/s11425-011-4292-8. 881

13. A. J. E. M. Janssen, *Duality and biorthogonality for Weyl–Heisenberg frames*, J. Fourier Anal. Appl., **1** (1995), no. 4, 403–437. Zbl 0887.42028. MR1350700. DOI 10.1007/s00041-001-4017-4. 883
14. V. Kaftal, D. Larson, and S. Zhang, *Operator-valued frames*, Trans. Amer. Math. Soc. **361** (2009), no. 12, 6349–6385 Zbl 1185.42032. MR2538596. DOI 10.1090/S0002-9947-09-04915-0. 881
15. A. Najati and A. Rahimi, *Generalized frames in Hilbert spaces*, Bull. Iranian Math. Soc. **35** (2009), no. 1, 97–109. Zbl 1180.41027. MR2547619. 891
16. A. Ron and Z. Shen, *Weyl–Heisenberg frames and Riesz bases in* $L^2(\mathbb{R}^d)$, Duke Math. J. **89** (1997), no. 2, 237–282. Zbl 0892.42017. MR1460623. DOI 10.1215/S0012-7094-97-08913-4. 882
17. D. T. Stoeva and O. Christensen, *On R-Duals and the Duality Principle in Gabor Analysis*, J. Fourier Anal. Appl., **21** (2015), no. 2, 383–400. Zbl 1312.42037. MR3319538. DOI 10.1007/s00041-014-9376-8. 883
18. W. Sun, *G-frames and g-Riesz bases*, J. Math. Anal. Appl. **322** (2006), no. 1, 437–452. Zbl 1129.42017. MR2239250. DOI 10.1016/j.jmaa.2005.09.039. 881
19. W. Sun, *Stability of g-frames*, J. Math. Anal. Appl. **326** (2007), no. 2, 858–868. Zbl 1130.42307. MR2280948. DOI 10.1016/j.jmaa.2006.03.043. 881
20. J. Wexler and S. Raz, *Discrete Gabor expansions*, preprint, `http://www.sciencedirect.com/science/article/pii/016516849090087F`. 882
21. X. M. Xian and Y. C. Zhu, *Duality principles of frames in Banach spaces*, Acta Math. Sci. (Ser. A Chin. Ed.) **29** (2009), no. 1, 94–102. MR2514792. 883

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