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ON BANACH SPACES OF VECTOR-VALUED RANDOM VARIABLES AND THEIR DUALS MOTIVATED BY RISK MEASURES

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Abstract. We introduce Banach spaces of vector-valued random variables motivated from mathematical finance. So-called *risk functionals* are defined in a natural way on these Banach spaces, and it is shown that these functionals are Lipschitz continuous. Since the risk functionals cannot be defined on strictly larger spaces of random variables, this creates an area of particular interest with regard to the spaces presented. We elaborate key properties of these Banach spaces and give representations of their dual spaces in terms of vector measures with values in the dual space of the state space.

1. Introduction

In the first part of this article, we introduce Banach spaces for vector-valued random variables. These spaces extend rearrangement spaces for functions in two ways. First, random variables are considered on a probability space, and second, we extend them to vector-valued (i.e., $\mathbb{R}^d$, or more general Banach space-valued) random variables.

It is natural to address differences and similarities between L^1- and L^p-spaces, and we elaborate on extensions in the second part of the article. We fully describe the duals of the new spaces. The duality theory for these spaces differs essentially from L^p-spaces. The new spaces are larger than L^∞, but are not L^p-spaces in

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general, and furthermore, their duals are not even similar to those of L^p-spaces. However, they are reflexive. The duality theory is particularly nice when the dual of the state space enjoys the Radon–Nikodým property.

An important motivation for considering these spaces derives from recent developments in mathematical finance. Vector-valued functions or portfolio vectors are naturally present in many real-life situations. To give one example, consider a portfolio with investments in d, say, different currencies. The random outcome is in $\mathbb{R}^d$ in this motivating example, and the related random variable is said to be *vector-valued*. Here, we consider more generally Banach space-valued random variables. The spaces can be associated with risk functionals, and we demonstrate that the spaces introduced are as large as possible such that the associated risk functionals remain continuous.

Rüschendorf [29] first introduced and considered vector-valued risk functionals. Svindland [30], Filipović and Svindland [14], Kupper and Svindland [18], and many other authors have considered and discussed different domain spaces for risk measures on portfolio vectors, for example, Orlicz spaces (as done in Cheridito and Li [6] and Bellini and Rosazza Gianin [4]). Ekeland and Schachermayer [13] considered the domain space L^∞ for these risk measures. Ekeland, Galichon, and Henry [12] provided the first multivariate generalization of a Kusuoka representation for risk measures on vector-valued random variables on L^2. In contrast, the present article extends these spaces and presents the largest possible Banach spaces for which those functionals remain continuous. The resulting spaces are neither Orlicz nor Lebesgue spaces, as considered in the earlier literature.

The spaces that we consider are in a way related to function spaces (rearrangement spaces) introduced by Lorentz [20], [21], following earlier results obtained by Halperin [16]. For unexplained notions from the theory of vector measures, we refer the reader to Diestel and Uhl [11].

Outline of the paper. The following section (Section 2) provides the mathematical setting including the relation to mathematical finance. The Banach spaces $L^p_\sigma(P, X)$ of X-valued random variables, introduced in Section 3, constitute the natural domains of risk functionals. We demonstrate that risk functionals are continuous with respect to the norm of the space introduced. In Section 4, we give a representation of the dual spaces of these Banach spaces in the scalar-valued case. This representation is used in Section 5 to derive representations of the duals in the general vector-valued case.

2. Mathematical setting and motivation

We consider a probability space $(\Omega, \mathcal{F}, P)$ and denote the *distribution function* of an $\mathbb{R}$-valued random variable Y by

$$F_Y(q) := P(Y \le q) = P\big(\{\omega : Y(\omega) \le q\}\big).$$

The *generalized inverse* is the nondecreasing and left-continuous function

$$F_Y^{-1}(\alpha) := \inf\big\{q : P(Y \le q) \ge \alpha\big\},$$

also called the *quantile* or *value-at-risk*.

We denote by $X = (X, \|\cdot\|)$ a Banach space, and we denote by X^* its continuous dual space. We use the notation $\langle \varphi, x\rangle$ for $\varphi(x)$, $\varphi \in X^*$ and $x \in X$. As usual, we denote for $p \in [1, \infty)$ by $L^p(P, X)$ the Bochner–Lebesgue space of p-Bochner integrable X-valued random variables Y on $(\Omega, \mathcal{F}, P)$ whose norm we denote by $\|\cdot\|_p$. Recall that for $Y \in L^p(P, X)$,

$$\|Y\|_p = \left(\int_0^1 F^{-1}_{\|Y\|}(u)^p \, du\right)^{1/p} = \left(\int_0^\infty p t^{p-1}\big(1 - F_{\|Y\|}(t)\big)\, dt\right)^{1/p}. \tag{1}$$

In this article, Banach spaces of vector-valued, strongly measurable random variables are introduced by weighting the quantiles in a different way than (1). The present results extend and generalize characterizations obtained in Pichler [26], where only real-valued random variables and $p = 1$ are considered (and elaborated in a context of insurance).

Remark 1. We will assume throughout the paper that the probability space $(\Omega, \mathcal{F}, P)$ is rich enough to carry a $[0, 1]$-valued, uniform distribution.[1] If this is not the case, then one may replace Ω by $\tilde{\Omega} := \Omega \times [0, 1]$ with the product measure $\tilde{P}(A \times B) := P(A) \cdot \text{Lebesgue measure}(B)$. Every random variable Y on Ω extends to $\tilde{\Omega}$ by $\tilde{Y}(\omega, u) := Y(\omega)$, and $U(\omega, u) := u$ is a uniform random variable, as $\tilde{P}(U \le u) = \tilde{P}(\Omega \times [0, u]) = u$. We denote the set of $[0, 1]$-valued uniform random variables on $(\Omega, \mathcal{F}, P)$ by $\mathscr{U}(0, 1)$.

With an $\mathbb{R}$-valued random variable Y one may further associate its *generalized quantile* transform

$$F(y, u) := (1 - u) \cdot \lim_{y' \uparrow y} F_Y(y') + u \cdot F_Y(y).$$

The random variable $F(Y, U)$ is uniformly distributed again and $F(Y, U)$ is coupled in a comonotone way with Y; that is, the inequality $(F(Y, U)(\omega) - F(Y, U)(\omega'))(Y(\omega) - Y(\omega')) \ge 0$ holds $P \otimes P$ almost everywhere (see, e.g., Pflug and Römisch [25, Proposition 1.3]).

Relation to mathematical finance: Risk measures and their continuity properties. Risk measures on $\mathbb{R}$-valued random variables have been introduced in the pioneering work by Artzner et al. [3]. An $\mathbb{R}$-valued random variable is typically associated with the total, or accumulated, return of a portfolio in mathematical finance. (The prevalent interpretation in insurance is the size of a claim, which happens with a probability specified by the probability measure P.)

The aggregated portfolio is composed of individual components such as stocks. From the perspective of comprehensive risk management it is desirable to understand not only the risk of the accumulated portfolio, but also its components. These more general risk measures on $\mathbb{R}^d$-valued random variables were considered first in Burgert and Rüschendorf [5], and further progress was made, for example, by Rüschendorf [29], Ekeland, Galichon, and Henry [12], and Ekeland and Schachermayer [13].

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[1] U is uniform, if $P(U \le u) = u$ for all $u \in [0, 1]$.

Ekeland and Schachermayer [13, Theorem 1.7] obtain a Kusuoka representation (see [19]) for risk measures based on $\mathbb{R}^d$-valued random variables. The risk functional identified there in the "regular case" for the homogeneous risk functional on random vectors is

$$\rho_Z(Y) := \sup\big\{\mathbb{E}\langle Z, Y'\rangle : Y' \sim Y\big\}, \tag{2}$$

where $Y \sim Y'$ indicates that Y and Y' enjoy the same law in $\mathbb{R}^d$.[2] The measure ρ_Z is called the *maximal correlation risk measure in direction* Z.

The rearrangement inequality (see, e.g., McNeil, Frey, and Embrechts [22, Theorem 5.25(2)], also known as *Chebyshev's sum inequality* (see Hardy, Littlewood, and Pólya [17, Section 2.17]), provides an upper bound for the natural linear form in (2) by

$$\big|\mathbb{E}\langle Z, Y\rangle\big| \le \mathbb{E}\|Z\|^* \cdot \|Y\| \le \mathbb{E}K \cdot \|Z\|_{\ell_1^d} \cdot \|Y\| \le K \cdot \int_0^1 F^{-1}_{\|Z\|_{\ell_1^d}}(u) \cdot F^{-1}_{\|Y\|}(u)\, du, \tag{3}$$

where the norms $\|\cdot\|$ and $\|\cdot\|^*$ are dual to each other on $\mathbb{R}^d$ (here, $K > 0$ is the constant linking the norms by $\|\cdot\|^* \le K \cdot \|\cdot\|_{\ell_1^d}$ on (the dual of) $\mathbb{R}^d$).

The maximal correlation risk measure (2) employs the linear form $\mathbb{E}\langle Z, Y\rangle$, which satisfies the bounds (3). This motivates fixing the function

$$\sigma(\cdot) := F^{-1}_{\|Z\|_{\ell_1^d}}(\cdot), \tag{4}$$

and to consider an appropriate vector space of random variables endowed with (see Pichler [27], [28] and Ahmadi-Javid and Pichler [2])

$$\|Y\|_\sigma := \int_0^1 \sigma(u) \cdot F^{-1}_{\|Y\|}(u)\, du.$$

It turns out that $\|\cdot\|_\sigma$ is a norm (Theorem 4 below) on this vector space of random variables and that the maximal correlation risk measure is continuous with respect to the norm (see Proposition 7).

3. The vector-valued Banach spaces $L^p_\sigma(P, X)$

Motivated by the observations made in the previous section, we introduce the following notions.

Definition 2. A nondecreasing, nonnegative function $\sigma : [0,1) \to [0,\infty)$, which is continuous from the left and normalized by $\int_0^1 \sigma(u)\, du = 1$, is called a *distortion function* (also occasionally referred to in the literature as a *spectrum function*; see Acerbi [1]).

Definition 3. For a distortion function σ, a Banach space $(X, \|\cdot\|)$, and a probability space $(\Omega, \mathcal{F}, P)$, we define for $p \in [1,\infty)$ and a strongly measurable X-valued random variable Y on $(\Omega, \mathcal{F}, P)$,

$$\|Y\|^p_{\sigma,p} := \sup_{U \text{ uniform}} \mathbb{E}\sigma(U)\|Y\|^p = \sup_{U \text{ uniform}} \int_\Omega \sigma\big(U(\omega)\big)\big\|Y(\omega)\big\|^p\, dP(\omega),$$

[2]That is, $P(Y_1 \le y_1, \dots, Y_d \le y_d) = P(Y'_1 \le y_1, \dots, Y'_d \le y_d)$ for all $(y_1, \dots, y_d) \in \mathbb{R}^d$.

where the supremum is taken over all $U \in \mathscr{U}(0,1)$, that is, over all $[0,1]$-valued, uniformly distributed random variables U on $(\Omega, \mathcal{F}, P)$. Moreover, we set

$$L^p_\sigma(P, X) := \big\{ Y : \Omega \to X \text{ strongly measurable and } \|Y\|^p_{\sigma,p} < \infty \big\},$$

where as usual we identify X-valued random variables which coincide P-almost everywhere.

Obviously, for $\sigma = 1$ one obtains the classical Bochner–Lebesgue spaces $L^p(P, X)$ which are well known to be Banach spaces.

Theorem 4. *We have that $L^p_\sigma(P, X)$ is a vector space and that $\|\cdot\|_{p,\sigma}$ is a norm on $L^p_\sigma(P, X)$ turning it into a Banach space which embeds contractively into $L^p(P, X)$.*

Moreover, for each X-valued, strongly measurable Y on $(\Omega, \mathcal{F}, P)$ and every $U \in \mathscr{U}(0,1)$ which is coupled in a comonotone way with $\|Y\|$, it follows that

$$\|Y\|^p_{\sigma,p} = \mathbb{E}\big(\sigma(U)\|Y\|^p\big) = \int_0^1 \sigma(u) F^{-1}_{\|Y\|}(u)^p \, du. \tag{5}$$

Proof. We denote the probability measure on $(\Omega, \mathcal{F})$ with P-density $\sigma \circ U$ for some $U \in \mathscr{U}(0,1)$ by $\sigma(U)P$, and we denote the expectation of a nonnegative random variable Z on $(\Omega, \mathcal{F}, \sigma(U)P)$ by $\mathbb{E}_U(Z)$. We obviously have

$$\|Y\|^p_{\sigma,p} = \sup_{U \in \mathscr{U}(0,1)} \mathbb{E}_U \|Y\|^p,$$

which implies that $L^p_\sigma(P, X)$ is a subspace of the intersection of Banach spaces $\bigcap_{U \in \mathscr{U}(0,1)} L^p(\sigma(U)P, X)$ and that $\|\cdot\|_{\sigma,p}$ is a seminorm on $L^p_\sigma(P, X)$.

By the rearrangement inequality (see, e.g., [22, Theorem 5.25(2)]) and the well-known fact that $F^{-1}_{\sigma(U)} = \sigma$ and $(F^{-1}_{\|Y\|})^p = F^{-1}_{\|Y\|^p}$, it follows that for every $U \in \mathscr{U}(0,1)$ and each X-valued, strongly measurable Y on $(\Omega, \mathcal{F}, P)$, we have

$$\mathbb{E}\big(\sigma(U)\|Y\|^p\big) \le \int_0^1 \sigma(u) F^{-1}_{\|Y\|}(u)^p \, du$$

so that

$$\|Y\|^p_{\sigma,p} \le \int_0^1 \sigma(u) F^{-1}_{\|Y\|}(u)^p \, du. \tag{6}$$

Moreover, if we fix for an X-valued, strongly measurable Y on $(\Omega, \mathcal{F}, P)$ some $U \in \mathscr{U}(0,1)$ such that U and $\|Y\|$ are coupled in a comonotone way (such U exists due to our general assumption on $(\Omega, \mathcal{F}, P)$ made in Remark 1), then (see Kusuoka [19])

$$\mathbb{E}\big(\sigma(U)\|Y\|^p\big) = \int_0^1 \sigma(u) F^{-1}_{\|Y\|}(u)^p \, du.$$

Together with (6) we obtain that for each X-valued, strongly measurable Y on $(\Omega, \mathcal{F}, P)$ there is $U \in \mathscr{U}(0,1)$ such that

$$\|Y\|^p_{\sigma,p} = \mathbb{E}\big(\sigma(U)\|Y\|^p\big) = \int_0^1 \sigma(u) F^{-1}_{\|Y\|}(u)^p \, du,$$

proving (5).

In order to see that the seminorm $\|\cdot\|_{\sigma,p}$ on $L^p_\sigma(P,X)$ is in fact a norm, we apply the continuous version of Chebyshev's inequality (see, e.g., Gradshteyn and Ryzhik [15, (12.314)]) to the nonnegative, nondecreasing functions σ and $(F^{-1}_{\|Y\|})^p$ on $[0,1)$ to obtain

$$\int_0^1 \sigma(u) F^{-1}_{\|Y\|}(u)^p\,du \geq \int_0^1 \sigma(u)\,du \cdot \int_0^1 F^{-1}_{\|Y\|}(u)^p\,du = \int_0^1 F^{-1}_{\|Y\|}(u)^p\,du = \mathbb{E}\big(\|Y\|^p\big),$$

where the last equality follows from $(F^{-1}_{\|Y\|})^p = F^{-1}_{\|Y\|^p}$. In particular, together with (5) we obtain for every X-valued, strongly measurable Y,

$$\mathbb{E}\big(\|Y\|^p\big) \leq \|Y\|^p_{\sigma,p},$$

which proves that $L^p_\sigma(P,X)$ embeds contractively into $L^p(P,X)$ and that $\|Y\|_{\sigma,p} = 0$ implies $Y=0$ so that $\|\cdot\|_{\sigma,p}$ is indeed a norm.

Finally, in order to prove that $L^p_\sigma(P,X)$ is a Banach space when equipped with the norm $\|\cdot\|_{\sigma,p}$, we first note that a Cauchy sequence $(Y_n)_{n\in\mathbb{N}}$ in $L^p_\sigma(P,X)$ is also a Cauchy sequence in $L^p(P,X)$ so that there is $Y \in L^p(P,X)$ with $Y = \lim_{n\to\infty} Y_n$ in $L^p(P,X)$. From this we conclude that $Y = \lim_{k\to\infty} Y_{n_k}$ P-almost everywhere on Ω for some subsequence $(Y_{n_k})_{k\in\mathbb{N}}$ of $(Y_n)_{n\in\mathbb{N}}$. Since for each $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that for all $U \in \mathscr{U}(0,1)$,

$$\varepsilon^p > \mathbb{E}\big(\sigma(U)\|Y_n - Y_m\|^p\big)$$

whenever $n,m \geq N$, it follows with Fatou's lemma that for every $U \in \mathscr{U}(0,1)$ and each $n \geq N$, we have

$$\mathbb{E}\big(\sigma(U)\|Y - Y_n\|^p\big) = \mathbb{E}\big(\lim_{k\to\infty} \sigma(U)\|Y_{n_k} - Y_n\|^p\big) \leq \liminf_{k\to\infty} \mathbb{E}\big(\sigma(U)\|Y_{n_k} - Y_n\|^p\big) \leq \varepsilon^p;$$

that is, $\|Y - Y_n\|_{\sigma,p} \leq \varepsilon^p$ for every $n \geq N$. Thus, we conclude that

$$Y = (Y - Y_N) + Y_N \in L^p_\sigma(P,X)$$

and that $(Y_n)_{n\in\mathbb{N}}$ converges to Y in $L^p_\sigma(P,X)$. $\square$

Remark 5. By (5), $L^p_\sigma(P,X)$-membership of Y only depends on the quantile function $F^{-1}_{\|Y\|}$ so that $L^p_\sigma(P,X)$ is invariant with respect to rearrangements. From the definition of $\|\cdot\|_{\sigma,p}$, it follows immediately that $L^p_\sigma(P,X)$ is an $L^\infty(P)$-module and that $\|\alpha Y\|_{\sigma,p} \leq \|\alpha\|_\infty \|Y\|_{\sigma,p}$ for all $\alpha \in L^\infty(P)$ and each $Y \in L^p_\sigma(P,X)$.

Next we show that the $L^p_\sigma(P,X)$-spaces behave like the classical Bochner–Lebesgue spaces $L^p(P,X)$ when one varies the exponent $p \in [1,\infty)$.

Proposition 6. *Let p, $p' \in [1,\infty)$ be such that $p < p'$.*

(i) $L^{p'}_\sigma(P,X) \subseteq L^p_\sigma(P,X)$ *and* $\|Y\|_{\sigma,p} \leq \|Y\|_{\sigma,p'}$ *for every* $Y \in L^{p'}_\sigma(P,X)$.

(ii) *If with* $r := p'/(p'-p)$ *the distortion function* σ *satisfies* $\int_0^1 \sigma^r(u)\,du < \infty$, *then even* $L^{p'}(P,X) \subseteq L^p_\sigma(P,X)$ *and* $\|Y\|_{\sigma,p} \leq \|Y\|_{p'}$ *for every* $Y \in L^{p'}(P,X)$.

Proof. Setting $r := p'/(p'-p)$, it follows from (5), $1/r+1/(p'/p) = 1$, and Hölder's inequality that for each X-valued, strongly measurable Y on $(\Omega,\mathcal{F},P)$,

$$\begin{aligned}\|Y\|_{\sigma,p}^p &= \int_0^1 \sigma^{\frac{1}{r}}(u)\sigma^{\frac{1}{p'/p}}(u)F_{\|Y\|}^{-1}(u)^p\,du\\ &\le \Big(\int_0^1 \sigma(u)\,du\Big)^{1/r}\Big(\int_0^1 \sigma(u)F_{\|Y\|}^{-1}(u)^{p'}\,du\Big)^{p/p'} = \|Y\|_{\sigma,p'}^p,\end{aligned}$$

which proves (i), while (ii) follows from (5), $1/r + 1/(p'/p) = 1$, and Hölder's inequality since

$$\begin{aligned}\|Y\|_{\sigma,p}^p = \int_0^1 \sigma(u)F_{\|Y\|}^{-1}(u)^p\,du &\le \Big(\int_0^1 \sigma^r(u)\,du\Big)^{1/r}\Big(\int_0^1 F_{\|Y\|}^{-1}(u)^{p\frac{p'}{p}}\,du\Big)^{p/p'}\\ &= \Big(\int_0^1 \sigma^r(u)\,du\Big)^{1/r}\big(\mathbb{E}\|Y\|^{p'}\big)^{p/p'}\end{aligned}$$

holds for each X-valued, strongly measurable Y on $(\Omega,\mathcal{F},P)$. □

For a Banach space X with (continuous) dual space X^* we write, as usual, $\langle x^*,x\rangle := x^*(x)$, $x\in X$, $x^*\in X^*$. The dual norm on X^* will also be denoted by $\|\cdot\|$. If Z is an X^*-valued, Bochner integrable random variable on $(\Omega,\mathcal{F},P)$ such that $\mathbb{E}\|Z\| = 1$, then $\sigma_Z := F_{\|Z\|}^{-1}$ is a distortion function. For two X-valued, strongly measurable Y_1, Y_2 on $(\Omega,\mathcal{F},P)$ we write $Y_1\sim Y_2$ if they have the same law, that is, if $P^{Y_1} = P^{Y_2}$.

Proposition 7. *Let X be a real Banach space, and let Z be an X^*-valued, Bochner integrable random variable on $(\Omega,\mathcal{F},P)$ such that $\mathbb{E}\|Z\| = 1$. Then, for every $p\in[1,\infty)$,*

$$\rho_Z : L^p_{\sigma_Z}(P,X)\to\mathbb{R},\quad Y\mapsto \sup\big\{\mathbb{E}\langle Z,Y'\rangle : Y\sim Y'\big\}$$

is a well-defined subadditive, convex functional. Moreover, for Y_1, $Y_2\in L^p_{\sigma_Z}(P,X)$ we have

$$\big|\rho_Z(Y_1)-\rho_Z(Y_2)\big| \le \|Y_1-Y_2\|_{\sigma,p}.$$

Proof. It follows from $Y\sim Y'$ that $F_{\|Y\|}^{-1} = F_{\|Y'\|}^{-1}$. Hence, $Y'\in L^p_{\sigma_Z}(P,X)$ whenever $Y\in L^p_{\sigma_Z}(P,X)$ by (5) in Theorem 4. From the strong measurability of Z and $Y\in L^p_{\sigma_Z}(P,X)$, it follows immediately that $\omega\mapsto\langle Z(\omega),Y(\omega)\rangle$ is an $\mathbb{R}$-valued random variable on $(\Omega,\mathcal{F},P)$. The rearrangement inequality, the definition of σ_Z, (5) in Theorem 4, and Proposition 6 imply that for $Y'\sim Y\in L^p_{\sigma_Z}(P,X)$,

$$\big|\mathbb{E}\langle Z,Y'\rangle\big| \le \mathbb{E}\big(\|Z\|\|Y'\|\big) \le \int_0^1 \sigma_Z(u)F_{\|Y\|}^{-1}(u)\,du = \|Y\|_{\sigma,1}\le\|Y\|_{\sigma,p},$$

which proves that ρ_Z is well defined and that

$$\big|\rho_Z(Y)\big| \le \|Y\|_{\sigma,p}. \tag{7}$$

Obviously, $\rho_Z(\lambda Y) = \lambda\rho_Z(Y)$ for all $\lambda>0$. Moreover, from the definition of ρ_Z and strong measurability, it follows immediately that ρ_Z is subadditive. Therefore,

$$\rho_Z(Y_1) = \rho(Y_2+Y_1-Y_2)\le\rho_Z(Y_2)+\rho_Z(Y_1-Y_2).$$

Interchanging the roles of Y_1, Y_2 in the above inequality gives

$$\big|\rho_Z(Y_1) - \rho_Z(Y_2)\big| \leq \rho_Z(Y_1 - Y_2),$$

which together with (7) proves $|\rho_Z(Y_1) - \rho_Z(Y_2)| \leq \|Y_1 - Y_2\|_{\sigma,p}$. □

In the remainder of this section, we will provide a closer look at the Banach spaces $L^p_\sigma(P, X)$.

Proposition 8. *Let $X \neq \{0\}$. Then the following are equivalent.*

(i) *For all $p \in [1, \infty)$, the spaces $L^p_\sigma(P, X)$ and $L^p(P, X)$ are isomorphic as Banach spaces.*
(ii) *There is $p \in [1, \infty)$ such that $L^p_\sigma(P, X) = L^p(P, X)$ as sets.*
(iii) *We have that σ is bounded.*

Proof. Obviously, (i) implies (ii). By Theorem 4, $L^p_\sigma(P, X)$ embeds contractively into $L^p(P, X)$. Thus, if (ii) holds, this embedding is onto so that by Banach's isomorphism theorem, there is $C > 0$ such that

$$\forall Y \in L^p(P, X)\colon \quad \sup_{U \in \mathscr{U}(0,1)} \int_\Omega \sigma\big(U(\omega)\big) \big\|Y(\omega)\big\|^p \, dP(\omega) \leq C \int_\Omega \big\|Y(\omega)\big\|^p \, dP(\omega),$$

where $\mathscr{U}(0,1)$ is defined as before. Choose $f \in L^1(P, \mathbb{R})$ and $x \in X$ with $\|x\| = 1$. Then $Y(\omega) := |f(\omega)|^{1/p} x$ defines an element of $L^p(P, X)$ so that for any $U \in \mathscr{U}(0,1)$, we have

$$\begin{aligned} \Big| \int_\Omega \sigma\big(U(\omega)\big) f(\omega) \, dP(\omega) \Big| &\leq \int_\Omega \sigma\big(U(\omega)\big) \big\|Y(\omega)\big\|^p \, dP(\omega) \\ &\leq C \int_\Omega \big\|Y(\omega)\big\|^p \, dP(\omega) \\ &= C \int_\Omega \big|f(\omega)\big| \, dP(\omega) < \infty. \end{aligned}$$

Since $f \in L^1(P, \mathbb{R})$ was chosen arbitrarily, it follows that $\sigma \circ U \in L^\infty(P, \mathbb{R})$, which by $U \in \mathscr{U}(0,1)$ and by the fact that σ is nondecreasing implies boundedness of σ. Thus, (iii) follows from (ii).

Finally, (iii) and the fact that $L^p_\sigma(P, X)$ embeds contractively into $L^p(P, X)$ for any $p \in [1, \infty)$ implies (i) by Theorem 4. □

Proposition 9. *We have the following.*

(i) *For every $p \in [1, \infty)$, $L^\infty(P, X)$ embeds contractively into $L^p_\sigma(P, X)$.*
(ii) *Simple functions are dense in $L^p_\sigma(P, X)$ for every $p \in [1, \infty)$.*

Proof. It follows from the definition of quantile functions that $0 \leq F^{-1}_{\|Y\|} \leq \|Y\|_\infty$ for every X-valued, strongly measurable Y on $(\Omega, \mathcal{F}, P)$, which implies by (5) in Theorem 4 that

$$\|Y\|^p_{\sigma,p} = \int_0^1 \sigma(u) F^{-1}_{\|Y\|}(u)^p \, du \leq \|Y\|^p_\infty,$$

proving (i).

In order to prove (ii), let $Y \in L^p_\sigma(P,X)$, and fix $\varepsilon \in (0,1)$. We choose $u_\varepsilon \in (0,1)$ such that $\int_{u_\varepsilon}^1 \sigma(u) F^{-1}_{\|Y\|}(u)^p\,du < \varepsilon^p$. By the strong measurability of Y there are $N \in \mathcal{F}$ with $P(N)=0$ and a separable, closed subspace X_1 of X such that $\mathbb{1}_{N^c}Y$ is X_1-valued. Let $\{x_j; j\in\mathbb{N}\}$ be a dense subset of X_1. Denoting the open ball about x_j with radius ε in X by $B_\varepsilon(x_j)$, we choose Borel subsets $E_j \subseteq B_\varepsilon(x_j)$ such that $X_1 \subseteq \bigcup_{j\in\mathbb{N}} E_j$ and such that the E_j's are pairwise disjoint. Then $((\mathbb{1}_{N^c}Y)^{-1}(E_j))_{j\in\mathbb{N}}$ is a pairwise disjoint sequence in $\mathcal{F}$ such that $P(\bigcup_{j\in\mathbb{N}}(\mathbb{1}_{N^c}Y)^{-1}(E_j))=1$. Let $n\in\mathbb{N}$ be such that

$$\sum_{j=1}^n P\big((\mathbb{1}_{N^c}Y)^{-1}(E_j)\big) > u_\varepsilon, \tag{8}$$

and set $E := \bigcup_{j=1}^n (\mathbb{1}_{N^c}Y)^{-1}(E_j)$.

Obviously, for $t \geq 0$ we have $\{\mathbb{1}_{E^c}\|Y\|^p \leq t\} \supseteq \{\|Y\|^p \leq t\}$ so that $F_{\mathbb{1}_{E^c}\|Y\|^p}(t) \geq F_{\|Y\|^p}(t)$. Therefore,

$$\forall u \in [0,1]: \quad \big\{t\geq 0; F_{\mathbb{1}_{E^c}\|Y\|^p}(t) \geq u\big\} \supseteq \big\{t \geq 0; F_{\|Y\|^p}(t) \geq u\big\},$$

which implies $F^{-1}_{\mathbb{1}_{E^c}\|Y\|^p} \leq F^{-1}_{\|Y\|^p}$. Furthermore,

$$F_{\mathbb{1}_{E^c}\|Y\|^p}(0) = P\big(\mathbb{1}_{E^c}\|Y\|^p = 0\big) \geq P(E)$$

so that $F^{-1}_{\mathbb{1}_{E^c}\|Y\|^p}(u) = 0$ for every $u \in [0, P(E)]$, which together with $F^{-1}_{\|Y\|^p} = (F^{-1}_{\|Y\|})^p$ and (8) yields for all $u \in [0,1]$,

$$F^{-1}_{\mathbb{1}_{E^c}\|Y\|^p}(u) \leq \mathbb{1}_{(P(E),1]}(u) F^{-1}_{\|Y\|}(u)^p \leq \mathbb{1}_{(u_\varepsilon,1]}(u) F^{-1}_{\|Y\|}(u)^p. \tag{9}$$

Defining $Y_\varepsilon := \sum_{j=1}^n \mathbb{1}_{(\mathbb{1}_{N^c}Y)^{-1}(E_j)} x_j$, it follows from the definition of E that $\|\mathbb{1}_{N^c}Y - Y_\varepsilon\| \leq \varepsilon$ on E while $Y_\varepsilon = 0$ on E^c. For every $U \in \mathscr{U}(0,1)$, we obtain

$$\begin{aligned}\int_\Omega \sigma(U)\|\mathbb{1}_{N^c}Y - Y_\varepsilon\|^p\,dP &= \int_\Omega \sigma(U)\mathbb{1}_E\|\mathbb{1}_{N^c}Y - Y_\varepsilon\|^p\,dP + \int_\Omega \sigma(U)\mathbb{1}_{E^c}\|Y\|^p\,dP \\ &\leq \varepsilon^p \int_\Omega \sigma(U)\,dP + \int_0^1 \sigma(u) F^{-1}_{\mathbb{1}_{E^c}\|Y\|^p}(u)\,du \\ &\leq \varepsilon^p + \int_{u_\varepsilon}^1 \sigma(u) F^{-1}_{\|Y\|^p}(u)^p\,du < 2\varepsilon^p,\end{aligned}$$

where we used the rearrangement inequality (see [22, Theorem 5.25(2)]) in the first inequality and (9) in the second one, while the last inequality follows from the choice of u_ε. Thus, $\|Y - Y_\varepsilon\| < \sqrt[p]{2}\varepsilon$, proving (ii). □

Theorem 10. *For $X \neq \{0\}$, the following are equivalent.*

(i) *$L^p_\sigma(P,X)$ is a Hilbert space.*
(ii) *X is a Hilbert space, $p=2$, and $\sigma = 1$ on $(0,1)$.*

Proof. Obviously, (ii) implies (i).

Let $E_\alpha \in \mathcal{F}$ be chosen with $P(E_\alpha) = \alpha$. For $\alpha \in (0,1)$ and $x \in X$, a straightforward calculation gives for $Y = \mathbb{1}_{E_\alpha}x$ that $F^{-1}_{\|Y\|} = \|x\|\mathbb{1}_{(1-\alpha,1)}$. Moreover, for $x_1,\ x_2 \in X$ with $\|x_1\| = \|x_2\| = 1$ and $Y_1 := \mathbb{1}_{E_\alpha}x_1, Y_2 := \mathbb{1}_{E_\alpha^c}x_2$, we have

$\|Y_1 \pm Y_2\|_{\sigma,p} = 1$. Thus, by the parallelogram identity and (5), we obtain for arbitrary $\alpha \in [0,1]$,

$$\begin{aligned} 1 &= \frac{1}{2}\|Y_1 - Y_2\|_{\sigma,p}^2 + \frac{1}{2}\|Y_1 + Y_2\|_{\sigma,p}^2 = \|Y_1\|_{\sigma,p}^2 + \|Y_2\|_{\sigma,p}^2 \\ &= \Big(\int_{1-\alpha}^1 \sigma(u)\,du\Big)^{2/p} + \Big(\int_\alpha^1 \sigma(u)\,du\Big)^{2/p}. \end{aligned} \tag{10}$$

Pick $\alpha \in (0,1)$ so that $0 < \int_0^\alpha \sigma(u)\,du < 1$. If $p > 2$, then

$$\begin{aligned} 1 &= \Big(\int_{1-\alpha}^1 \sigma(u)\,du\Big)^{2/p} + \Big(\int_\alpha^1 \sigma(u)\,du\Big)^{2/p} \\ &> \int_{1-\alpha}^1 \sigma(u)\,du + \int_\alpha^1 \sigma(u)\,du \\ &\geq \int_0^\alpha \sigma(u)\,du + \int_\alpha^1 \sigma(u)\,du = 1, \end{aligned}$$

as σ is nondecreasing. This is a contradiction, and hence $p \leq 2$.

Define the function

$$f(\alpha) := \Big(\int_\alpha^1 \sigma(u)\,du\Big)^{2/p}. \tag{11}$$

Since σ is continuous from the left, f is differentiable from the left with increasing left derivative; thus f is convex. Furthermore, we have $f(\alpha)+f(1-\alpha) = 1$ by (10) so that f is concave as well. Hence, f is affine, that is, $f(\alpha) = b + c\cdot\alpha$, and we deduce from $f(1) = 0, f(0) = 1$ and (11) the particular form

$$\sigma(u) = \frac{p}{2}(1-u)^{\frac{p}{2}-1}, \tag{12}$$

which implies $\int_{1-\alpha}^1 \sigma(u)\,du = \alpha^{p/2}$.

Next consider measurable sets A and B with $A \subseteq B$. The parallelogram law (10), applied to the random variables $Y_1 := \mathbb{1}_A x$ and $Y_2 := \mathbb{1}_B x$, reads

$$\frac{1}{2}\big(P(B) - P(A)\big) + \frac{1}{2}\big(P(B)^{p/2} - P(A)^{p/2} + 2^p P(A)^{p/2}\big)^{2/p} = P(A) + P(B),$$

that is,

$$P(B)^{p/2} + (2^p - 1)P(A)^{p/2} = \big(3P(A) + P(B)\big)^{p/2}.$$

We may specify the sets further by $P(B) = 4P(A)$. Then the latter equality reduces to

$$4^{p/2} + (2^p - 1) = 7^{p/2},$$

so that we are left with solving the equation

$$2x - 1 = x^{\frac{\log 7}{2\log 2}}$$

for $x = 2^p$.

The convex function $x^{\frac{\log 7}{\log 4}}$ does not have more than two intersections with the line $2x - 1$, and these are $x = 1$ and $x = 4$, that is, $p = 0$ and $p = 2$.

The intersection $p = 0$ does not qualify, and the distortion function for $p = 2$ is $\sigma(\cdot) = 1$, by (12). This concludes the proof. $\square$

4. The dual space in the scalar-valued case

In this section, we are going to determine the dual space of $L^p_\sigma := L^p_\sigma(P) := L^p_\sigma(P, \mathbb{K})$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. For $\varphi \in L^p_\sigma(P)^*$, we denote the dual norm of φ by $\|\varphi\|^*_{\sigma,p}$. Some of the results presented in this section are inspired by Lorentz [21].

Definition 11. As usual, we denote by $L^0(P)$ the set of $\mathbb{K}$-valued random variables on $(\Omega, \mathcal{F}, P)$, where random variables which coincide P-almost surely are identified. We define the *Köthe dual* of $L^p_\sigma(P)$ as

$$L^p_\sigma(P)^\times := \big\{Z \in L^0(P); \forall Y \in L^p_\sigma(P) : ZY \in L^1(P)\big\}.$$

Since $L^\infty(P) \subseteq L^p_\sigma(P)$ for all $p \in [1, \infty)$, it follows from taking $Y = \mathbb{1}_{\{Z \neq 0\}} \frac{\overline{Z}}{|Z|}$ that $Z \in L^1(P)$ whenever $Z \in L^p_\sigma(P)^\times$.

Proposition 12. *For every $Z \in L^p_\sigma(P)^\times$,*

$$\sup\big\{\big|\mathbb{E}(ZY)\big|; \|Y\|_{\sigma,p} \le 1\big\} < \infty.$$

Moreover,

$$\varphi_Z : L^p_\sigma(P) \to \mathbb{K}, \quad \varphi_Z(Y) = \mathbb{E}(ZY)$$

belongs to $L^p_\sigma(P)^*$ and

$$\Phi : L^p_\sigma(P)^\times \to L^p_\sigma(P)^*, \quad Z \mapsto \varphi_Z$$

is a linear isomorphism with

$$\forall Z \in L^p_\sigma(P)^\times\colon \quad \big\|\Phi(Z)\big\|^*_{\sigma,p} = \sup\big\{\big|\mathbb{E}(ZY)\big|; \|Y\|_{\sigma,p} \le 1\big\}. \tag{13}$$

Proof. Obviously, φ_Z is a well-defined linear functional on $L^p_\sigma(P)$ for every $Z \in L^p_\sigma(P)^\times$. The assumption

$$\infty = \sup\big\{\big|\mathbb{E}(ZY)\big|; \|Y\|_{\sigma,p} \le 1\big\}$$

implies the existence of a sequence $(Y_k)_{k\in\mathbb{N}}$ in the unit ball of $L^p_\sigma(P)$ such that

$$\forall k \in \mathbb{N}\colon \quad k^2 \le \big|\mathbb{E}(ZY_k)\big| \le \mathbb{E}\big(|ZY_k|\big).$$

Because $\tilde{Y}_k := \mathbb{1}_{\{ZY_k \neq 0\}} \frac{\overline{ZY_k}}{|Z\tilde{Y}_k|} Y_k$ belongs to the unit ball of $L^p_\sigma(P), k \in \mathbb{N}$, the completeness of $L^p_\sigma(P)$ implies that $(\sum_{k=1}^n \frac{1}{k^2}\tilde{Y}_k)_{n\in\mathbb{N}}$ converges in $L^p_\sigma(P)$ to some Y. As $Z \in L^p_\sigma(P)^\times$, it follows that $ZY \in L^1(P)$.

But on the other hand, since $L^p_\sigma(P)$ embeds contractively into $L^p(P)$ by Theorem 4, it follows that some subsequence $(\sum_{k=1}^{n_l} \frac{1}{k^2}\tilde{Y}_k)_{l\in\mathbb{N}}$ also converges P-almost surely to Y. Therefore, P-almost surely we have

$$ZY = Z \lim_{l\to\infty}\Big(\sum_{k=1}^{n_l} \frac{1}{k^2}\tilde{Y}_k\Big) = \lim_{l\to\infty} \sum_{k=1}^{n_l} \frac{1}{k^2} Z\tilde{Y}_k = \lim_{l\to\infty} \sum_{k=1}^{n_l} \frac{1}{k^2}|ZY_k|, \tag{14}$$

and by an application of the monotone convergence theorem, we conclude that

$$\mathbb{E}(ZY) = \mathbb{E}\Big(\lim_{l\to\infty}\sum_{k=1}^{n_l}\frac{1}{k^2}|ZY_k|\Big) = \lim_{l\to\infty}\sum_{k=1}^{n_l}\frac{1}{k^2}\mathbb{E}\big(|ZY_k|\big) \geq \lim_{l\to\infty}\sum_{k=1}^{n_l}1,$$

which contradicts $ZY \in L^1(P)$. Hence,

$$\infty > \sup\big\{\big|\mathbb{E}(ZY)\big|;\, \|Y\|_{\sigma,p} \leq 1\big\}$$

so that $\varphi_Z \in L^p_\sigma(P)^*$ with

$$\|\varphi_Z\|^*_{\sigma,p} = \sup\big\{\big|\mathbb{E}(ZY)\big|;\, \|Y\|_{\sigma,p} \leq 1\big\}. \tag{15}$$

This implies that Φ is a well-defined linear mapping which satisfies (13). In order to show that Φ is injective, choose $Z \in L^p_\sigma(P)^\times$ with $\Phi(Z) = 0$. We set $Y := \mathbb{1}_{\{Z\neq 0\}}\frac{\overline{Z}}{|Z|}$. Since simple functions belong to $L^p_\sigma(P)$, it follows easily that $Y \in L^p_\sigma(P)$. It follows that

$$0 = \Phi(Z)(Y) = \mathbb{E}\big(|Z|\big),$$

so that $Z = 0$.

In order to prove the surjectivity of Φ, let $\varphi \in L^p_\sigma(P)$. For $E \in \mathcal{F}$ and $Y = \mathbb{1}_E$, we have $F^{-1}_{|Y|} = \mathbb{1}_{(1-P(E),1]}$ so that by (5),

$$\big|\varphi(\mathbb{1}_E)\big| \leq \|\varphi^*_{\sigma,p}\|\|\mathbb{1}_E\|_{\sigma,p} = \|\varphi\|^*_{\sigma,p}\Big(\int_{1-P(E)}^1 \sigma(u)\,du\Big)^{1/p}.$$

Using this inequality, it is straightforward to show that

$$\mu : \mathcal{F} \to \mathbb{K}, \quad \mu(E) := \varphi(\mathbb{1}_E)$$

is a complex measure which is P-continuous; that is, $\mu(E) = 0$ whenever $P(E) = 0$. An application of the Radon–Nikodým theorem yields some $Z \in L^1(P)$ such that $\mu(E) = \int_\Omega \mathbb{1}_E Z\,dP = \mathbb{E}(Z\mathbb{1}_E)$ for all $E \in \mathcal{F}$. For simple functions Y, it follows that $\varphi(Y) = \mathbb{E}(ZY)$. As soon as we have shown that $Z \in L^p_\sigma(P)^\times$, it follows from the above and Theorem 9 that $\varphi = \Phi(Z)$.

In order to show that $Z \in L^p_\sigma(P)^\times$, we first observe that $\alpha Y \in L^p_\sigma(P)$ and $\|\alpha Y\|_{\sigma,p} \leq \|\alpha\|_\infty\|Y\|_{\sigma,p}$ for every $Y \in L^p_\sigma(P)$ and each $\alpha \in L^\infty(P)$. Therefore, by setting $E_n := \{|Z| \leq n\}, n \in \mathbb{N}$, we have $\|\mathbb{1}_{E_n}Y\|_{\sigma,p} \leq \|Y\|_{\sigma,p}$ for each $Y \in L^p_\sigma(P)$, which implies $\varphi_n \in L^p_\sigma(P)^*$ and $\|\varphi_n\|^*_{\sigma,p} \leq \|\varphi\|^*_{\sigma,p}$, where $\varphi_n(Y) := \varphi(\mathbb{1}_{E_n}Y)$. For simple functions Y, we have $\varphi_n(Y) = \mathbb{E}(Z\mathbb{1}_{E_n}Y)$. Additionally, by Hölder's inequality and Theorem 4, we obtain for arbitrary $Y \in L^p_\sigma(P)$,

$$\mathbb{E}\big(|Z\mathbb{1}_{E_n}Y|\big) \leq n\mathbb{E}\big(|Y|\big) \leq n\|Y\|_{\sigma,p},$$

so that $Z\mathbb{1}_{E_n} \in L^p_\sigma(P)^\times$. Because simple functions are dense in $L^p_\sigma(P)$ by Theorem 9, we conclude from the above that $\Phi(Z\mathbb{1}_{E_n}) = \varphi_n$. Finally, since

$$\begin{aligned}\mathbb{E}\big(|Z\mathbb{1}_{E_n}Y|\big) &= \Big|\varphi_n\Big(\mathbb{1}_{\{ZY\neq 0\}}\frac{\overline{ZY}}{|ZY|}Y\Big)\Big| \leq \|\varphi_n\|^*_{\sigma,p}\Big\|\mathbb{1}_{\{ZY\neq 0\}}\frac{\overline{ZY}}{|ZY|}Y\Big\|_{\sigma,p}\\ &\leq \|\varphi\|_{\sigma,p}\|Y\|_{\sigma,p},\end{aligned}$$

it follows with the aid of the monotone convergence theorem that

$$\mathbb{E}\big(|ZY|\big) = \lim_{n\to\infty} \mathbb{E}\big(|Z\mathbb{1}_{E_n}Y|\big) \leq \|\varphi\|_{\sigma,p}\|Y\|_{\sigma,p}$$

for each $Y \in L^p_\sigma(P)$ so that $Z \in L^p_\sigma(P)^\times$. □

Remark 13. Given the fact that for $\alpha \in L^\infty(P)$ the linear mapping $Y \mapsto \alpha Y$ is well defined and continuous from $L^p_\sigma(P)$ into itself, it is straightforward to see that $|Z| \in L^p_\sigma(P)^\times$ whenever $Z \in L^p_\sigma(P)^\times$ and that in this case $\|\varphi_Z\|^*_{\sigma,p} = \|\varphi_{|Z|}\|^*_{\sigma,p}$.

Our next aim is to give a representation of $L^p_\sigma(P)^\times$ and thus of the dual space of $L^p_\sigma(P)$. For this purpose, we introduce the following notion.

Definition 14. For a distortion function σ, we define

$$S_\sigma := S : [0,1] \to \mathbb{R}, \quad S_\sigma(\alpha) = \int_\alpha^1 \sigma(u)\, du.$$

Remark 15. Obviously, S is a continuous, nonincreasing function with $S(0) = 1$, $S(1) = 0$. If we set $u_0 := \inf\{u > 0; \sigma(u) > 0\}$, we have $u_0 < 1$, $S_{|[0,u_0]} = 1$, and $S_{|[u_0,1]}$ is an increasing bijection from $[u_0, 1]$ to $[0, 1]$. By abuse of notation, we denote the inverse of $S_{|[u_0,1]}$ by S^{-1}.

For α_1, $\alpha_2 \in [0,1]$, $\alpha_1 < \alpha_2$, and $\lambda \in (0,1)$, it follows from the fact that σ is nondecreasing that

$$\begin{aligned} S\big(\lambda\alpha_1 + (1-\lambda)\alpha_2\big) - S(\alpha_1) &= -\int_{\alpha_1}^{\lambda\alpha_1+(1-\lambda)\alpha_2} \sigma(u)\, du \\ &\geq -\sigma\big(\lambda\alpha_1 + (1-\lambda)\alpha_2\big)(1-\lambda)(\alpha_2 - \alpha_1) \end{aligned}$$

and

$$\begin{aligned} S(\alpha_2) - S\big(\lambda\alpha_1 + (1-\lambda)\alpha_2\big) &= -\int_{\lambda\alpha_1+(1-\lambda)\alpha_2}^{\alpha_2} \sigma(u)\, du \\ &\leq -\sigma\big(\lambda\alpha_1 + (1-\lambda)\alpha_2\big)\lambda(\alpha_2 - \alpha_1) \end{aligned}$$

so that

$$\begin{aligned} \frac{S(\lambda\alpha_1 + (1-\lambda)\alpha_2) - S(\alpha_1)}{(1-\lambda)(\alpha_2 - \alpha_1)} &\geq -\sigma\big(\lambda\alpha_1 + (1-\lambda)\alpha_2\big) \\ &\geq \frac{S(\alpha_2) - S(\lambda\alpha_1 + (1-\lambda)\alpha_2)}{\lambda(\alpha_2 - \alpha_1)}, \end{aligned}$$

which implies $S(\lambda\alpha_1 + (1-\lambda)\alpha_2) \geq \lambda S(\alpha_1) + (1-\lambda)S(\alpha_2)$; that is, S is concave. In particular, S is differentiable from the left and from the right on $(0,1]$ (resp., on $[0,1)$) and since σ is continuous from the left, it is straightforward to show that for the left derivative we have $S'_l(\alpha) = -\sigma(u), u \in (0,1]$.

Recall that for a nonnegative random variable Z the *average value-at-risk* of level $\alpha \in [0,1)$ is defined as $\mathsf{AV@R}_\alpha(Z) = \frac{1}{1-\alpha}\int_\alpha^1 F_Z^{-1}(u)\, du$.

Definition 16. For a distortion function σ, $Z \in L^0(P)$, and $\alpha \in [0,1)$, we define

$$|Z|^*_{\sigma,\infty} := \sup_{\alpha\in[0,1)} \frac{\mathsf{AV@R}_\alpha(|Z|)}{\frac{1}{1-\alpha} S_\sigma(\alpha)} \Big(= \sup_{\alpha\in[0,1)} \frac{\int_\alpha^1 F^{-1}_{|Z|}(u)\,du}{\int_\alpha^1 \sigma(u)\,du} \Big). \tag{16}$$

Moreover, we say that $Z' \in L^0(P)$ σ-*dominates* Z (in symbols $Z' \,{}_\sigma\!\succcurlyeq Z$) if there is a uniform random variable $U \in \mathscr{U}(0,1)$ such that

$$\mathsf{AV@R}_\alpha\big(\sigma(U)|Z'|\big) \geq \mathsf{AV@R}_\alpha\big(|Z|\big) \quad \text{for all } \alpha < 1. \tag{17}$$

Furthermore, we define, for $p \in (1,\infty)$,

$$|Z|^*_{\sigma,q} := \inf\big\{\|Z'\|_{\sigma,q} : Z' \,{}_\sigma\!\succcurlyeq Z\big\}, \tag{18}$$

where $q \in (1,\infty)$ is the conjugate exponent to p, that is, $1/p + 1/q = 1$, and where as usual $\inf\emptyset := \infty$.

Finally, for $p \in [1,\infty)$ with conjugate exponent q, that is, $1/p + 1/q = 1$, we set $L^*_{\sigma,q}(P) := \{Z \in L^0(P); |Z|^*_{\sigma,q} < \infty\}$ (and we identify random variables which coincide P-almost everywhere).

From the definition of quantile functions, it follows for Z_1, $Z_2 \in L^0(P)$ with $|Z_1| \leq |Z_2|$ that $F^{-1}_{|Z_1|} \leq F^{-1}_{|Z_2|}$, which implies $|Z_1|^*_{\sigma,q} \leq |Z_2|^*_{\sigma,q}$. Since also $F^{-1}_{|\alpha Z_1|} = |\alpha| F^{-1}_{|Z_1|}$ for $\alpha \in \mathbb{K}$, it follows also that $|\alpha Z_1|^*_{\sigma,q} = |\alpha|\,|Z_1|^*_{\sigma,q}$. Since $\mathsf{AV@R}_\alpha$ is subadditive (see Pflug and Römisch [25]), it follows easily that $L^*_{\sigma,q}(P)$ is a subspace of $L^0(P)$.

Remark 17 (Stochastic dominance of second order). The definition of $|\cdot|^*_{\sigma,\infty}$ reflects the duality of risk functionals. Indeed, the supremum (16) can be restated as

$$|Z|^*_{\sigma,\infty} = \inf\Big\{\eta \geq 0 : \mathsf{AV@R}_\alpha\big(|Z|\big) \leq \frac{\eta}{1-\alpha} \cdot \int_\alpha^1 \sigma(u)\,du \text{ for all } \alpha < 1\Big\}.$$

By the rearrangement inequality (see [22, Theorem 5.25(2)]), this equivalent formulation involves the statement

$$\mathsf{AV@R}_\alpha\big(|Z|\big) \leq \mathsf{AV@R}_\alpha\big(\eta\sigma(U)\big), \tag{19}$$

where $U \in \mathscr{U}(0,1)$. Choosing U to be coupled in a comonotone way with $|Z|$, it follows that

$$|Z|^*_{\sigma,\infty} = \inf\big\{\eta \geq 0, U \in \mathscr{U}(0,1) : \mathsf{AV@R}_\alpha\big(|Z|\big) \leq \mathsf{AV@R}_\alpha\big(\eta\sigma(U)\big) \text{ for all } \alpha < 1\big\}.$$

Following Ogryczak and Ruszczyński [23], (19) is equivalent to saying that $|Z|$ is dominated by $\|Z\|^*_\sigma \cdot \sigma(U)$ in second stochastic order.[3]

Remark 18.

(i) By the choice $\alpha = 0$ in (16), it follows that

$$|Z|^*_{\sigma,\infty} \geq \mathsf{AV@R}_0\big(|Z|\big) = \int_0^1 F^{-1}_{|Z|}(u)\,du = \mathbb{E}|Z| = \|Z\|_1, \tag{20}$$

so that $L^*_{\sigma,\infty}(P) \subseteq L^1(P)$.

[3]See Dentcheva and Ruszczyński [7]–[9] for stochastic dominance of second order.

(ii) Since for $Z, Z' \in L^0(P)$ with $Z' \,{}_\sigma\!\succcurlyeq Z$ we have, for $p \in [1,\infty)$ with Proposition 6,

$$\|Z\|_1 = \int_0^1 F_{|Z|}^{-1}(u)\,du \le \int_0^1 \sigma(u) F_{|Z'|}^{-1}(u)\,du = \|Z'\|_{\sigma,1} \le \|Z'\|_{\sigma,p},$$

it also follows that $L_{\sigma,q}^*(P) \subseteq L^1(P)$.

Proposition 19. *For $Z \in L^0(P)$, we have*

$$|Z|_{\sigma,\infty}^* = \inf\Big\{\eta \ge 0; \forall F : [0,1) \to [0,\infty) \text{ nondecreasing: } \int_0^1 F_{|Z|}^{-1}(u) F(u)\,du \le \eta \int_0^1 \sigma(u) F(u)\,du\Big\},$$

and for $p \in (1,\infty)$, we have

$$|Z|_{\sigma,q}^* = \inf\Big\{\|Z'\|_{\sigma,q}; Z' \in L^0(P) \text{ such that } \forall F : [0,1) \to [0,\infty) \text{ nondecreasing: } \int_0^1 F_{|Z|}^{-1}(u) F(u)\,du \le \int_0^1 \sigma(u) F_{|Z'|}^{-1}(u) F(u)\,du\Big\},$$

where as usual $1/p + 1/q = 1$.

Proof. Since $\mathbb{1}_{[\alpha,1]}$ is a nondecreasing, nonnegative function for every $\alpha \in [0,1)$, it follows that

$$\begin{aligned}&\Big\{\eta \ge 0; \forall F : [0,1) \to [0,\infty) \text{ nondecreasing: } \int_0^1 F_{|Z|}^{-1}(u) F(u)\,du \le \eta \int_0^1 \sigma(u) F(u)\,du\Big\}\\ &\subseteq \Big\{\eta \ge 0; \forall \alpha \in [0,1) : \int_\alpha^1 F_{|Z|}^{-1}(u)\,du \le \eta \int_\alpha^1 \sigma(u)\,du\Big\}.\end{aligned}$$

On the other hand, if for some $\eta \ge 0$ we have

$$\forall \alpha \in [0,1): \quad \int_\alpha^1 F_{|Z|}^{-1}(u)\,du \le \eta \int_\alpha^1 \sigma(u)\,du,$$

it follows for all $\gamma_1, \dots, \gamma_n \in [0,\infty)$ and every choice of $\alpha_1 < \cdots < \alpha_n \in [0,1)$ that

$$\int_0^1 F_{|Z|}^{-1}(u) \sum_{j=1}^n \gamma_j \mathbb{1}_{[\alpha_j,1]}(u)\,du \le \eta \int_0^1 \sigma(u) \sum_{j=1}^n \gamma_j \mathbb{1}_{[\alpha_j,1]}(u)\,du.$$

Since every nonnegative, nondecreasing function $F : [0,1) \to [0,\infty)$ is the pointwise limit of a nondecreasing sequence of such step functions $\sum_{j=1}^n \gamma_j \mathbb{1}_{[\alpha_j,1]}$, it follows from the monotone convergence theorem that

$$\int_0^1 F_{|Z|}^{-1}(u) F(u)\,du \le \eta \int_0^1 \sigma(u) F(u)\,du$$

for all such F. Hence, it also holds that

$$\Big\{\eta \geq 0; \forall F : [0,1) \to [0,\infty) \text{ nondecreasing:}$$
$$\int_0^1 F_{|Z|}^{-1}(u)F(u)\,du \leq \eta \int_0^1 \sigma(u)F(u)\,du\Big\}$$
$$\supseteq \Big\{\eta \geq 0; \forall \alpha \in [0,1) : \int_\alpha^1 F_{|Z|}^{-1}(u)\,du \leq \eta \int_\alpha^1 \sigma(u)\,du\Big\},$$

which proves the first claim. The rest of the proposition is proved mutatis mutandis. ☐

Proposition 20. *For a distortion function σ and $p \in [1,\infty)$, we have $L_{\sigma,q}^*(P) \subseteq L_\sigma^p(P)^\times$ and*

$$\sup\big\{\big|\mathbb{E}(ZY)\big|; Y \in L_\sigma^p(P), \|Y\|_{\sigma,p} \leq 1\big\} \leq |Z|_{\sigma,q}^* \tag{21}$$

for every $Z \in L_{\sigma,q}^*(P)$, where q is the conjugate exponent to p.

Proof. Let $p = 1$. For $Z \in L_{\sigma,\infty}^*(P)$ it follows for arbitrary $Y \in L_\sigma^1(P)$ from the rearrangement inequality combined with Proposition 19 that

$$\begin{aligned}\mathbb{E}\big(|ZY|\big) &\leq \int_0^1 F_{|Z|}^{-1}(u)F_{|Y|}^{-1}(u)\,du \leq |Z|_{\sigma,\infty}^* \int_0^1 \sigma(u)F_{|Y|}^{-1}(u)\,du \\ &= |Z|_{\sigma,\infty}^*\|Y\|_{\sigma,1}.\end{aligned}$$

Hence, $Z \in L_\sigma^1(P)^\times$ and the above inequality also implies that $|Z|_{\sigma,\infty}^*$ is an upper bound for $\sup\{|\mathbb{E}(ZY)|; Y \in L_\sigma^1(P), \|Y\|_{\sigma,1} \leq 1\}$.

Next let $p \in (1,\infty)$, and let q be the corresponding conjugate exponent. For $Z \in L_{\sigma,q}^*(P)$, let $Z' \in L_\sigma^q(P)$ with $Z' \,{}_\sigma\!\succcurlyeq Z$. For arbitrary $Y \in L_\sigma^1(P)$, it follows from the rearrangement inequality combined with Proposition 19 and Hölder's inequality that

$$\begin{aligned}\mathbb{E}\big(|ZY|\big) &\leq \int_0^1 F_{|Z|}^{-1}(u)F_{|Y|}^{-1}(u)\,du \leq \int_0^1 \sigma(u)F_{|Z'|}^{-1}(u)F_{|Y|}^{-1}(u)\,du \\ &\leq \|Z'\|_{\sigma,q}\|Y\|_{\sigma,p}.\end{aligned}$$

Thus, $Z \in L_\sigma^p(P)^\times$ and because $Z' \in L_\sigma^q(P)$ with $Z' \,{}_\sigma\!\succcurlyeq Z$ was chosen arbitrarily, it follows that $\sup\{|\mathbb{E}(ZY)|; Y \in L_\sigma^1(P), \|Y\|_{\sigma,p} \leq 1\}$ is bounded by $|Z|_{\sigma,q}^*$. ☐

In order to show that in fact $L_{\sigma,q}^*(P) = L_\sigma^p(P)^\times$ holds as well as equality in inequality (21), we have to distinguish the cases $p = 1$ and $p \in (1,\infty)$. We begin with the case $p = 1$.

Proposition 21. *For a $\mathbb{K}$-valued random variable Z on $(\Omega,\mathcal{F},P)$ and $\alpha \in [0,1)$, there is $E_\alpha \in \mathcal{F}$ such that $P(E_\alpha) = 1-\alpha$ and*

$$\mathsf{AV@R}_\alpha\big(|Z|\big) = \frac{1}{1-\alpha}\mathbb{E}\big(|Z|\mathbb{1}_{E_\alpha}\big).$$

Proof. Let $E \in \mathcal{F}$ with

$$\{|Z| > F_{|Z|}^{-1}(\alpha)\} \subseteq E \subseteq \{|Z| \geq F_{|Z|}(\alpha)\}$$

be arbitrary. Denoting the positive part of an $\mathbb{R}$-valued function f as usual by f_+, it follows that

$$\begin{aligned}
\mathsf{AV@R}_\alpha\big(|Z|\big) &= \frac{1}{1-\alpha}\int_\alpha^1 F_{|Z|}^{-1}(u)\,du \\
&= F_{|Z|}^{-1}(\alpha) + \frac{1}{1-\alpha}\int_0^1 \big(F_{|Z|}^{-1}(u) - F_{|Z|}^{-1}(\alpha)\big)_+\,du \\
&= F_{|Z|}^{-1}(\alpha) + \frac{1}{1-\alpha}\mathbb{E}\big(\big(|Z| - F_{|Z|}^{-1}(\alpha)\big)\mathbb{1}_E\big) \qquad (22) \\
&= F_{|Z|}^{-1}(\alpha) + \frac{1}{1-\alpha}\mathbb{E}\big(|Z|\mathbb{1}_E\big) - \frac{1}{1-\alpha}F_{|Z|}^{-1}(\alpha)P(E) \\
&= \frac{1}{1-\alpha}\mathbb{E}\big(|Z|\mathbb{1}_E\big) + \Big(1 - \frac{1}{1-\alpha}P(E)\Big)F_{|Z|}^{-1}(\alpha).
\end{aligned}$$

From the definition of $F_{|Z|}^{-1}$, it follows immediately that

$$P\big(|Z| < F_{|Z|}^{-1}(\alpha)\big) \leq \alpha \leq P\big(|Z| \leq F_{|Z|}^{-1}(\alpha)\big)$$

so that

$$P\big(|Z| > F_{|Z|}^{-1}(\alpha)\big) \leq 1-\alpha \qquad \text{and} \qquad P\big(|Z| \geq F_{|Z|}^{-1}(\alpha)\big) \geq 1-\alpha.$$

Let U be a $[0,1]$-valued, uniformly distributed random variable on $(\Omega, \mathcal{F}, P)$. We define

$$E_\beta := \{|Z| > F_{|Z|}^{-1}(\alpha)\} \cup \big(\{|Z| = F_{|Z|}^{-1}(\alpha)\} \cap \{U \in [0,\beta]\}\big)$$

for $\beta \in [0,1]$ and set

$$f : [0,1] \to [0,1], \quad f(\beta) := P(E_\beta).$$

From the properties of a probability measure, it follows easily that f is continuous as well as

$$f(0) = P\big(|Z| > F_{|Z|}^{-1}(\alpha)\big) \leq 1-\alpha \qquad \text{and} \qquad f(1) = P\big(|Z| \geq F_{|Z|}^{-1}(\alpha)\big) \geq 1-\alpha.$$

Hence, there is $\beta_0 \in [0,1]$ such that for E_{β_0} we have $P(E_{\beta_0}) = 1-\alpha$ and it follows from (22) that E_{β_0} has the desired property. □

For the case $p = 1$ we can now give the desired intrinsic description of $L_\sigma^1(P)^\times$.

Proposition 22. *For a distortion function σ, it holds that $L_\sigma^1(P)^\times = L_{\sigma,\infty}^*(P)$ and for every $Z \in L_\sigma^1(P)^\times$ we have*

$$|Z|_{\sigma,\infty}^* = \sup\big\{\big|\mathbb{E}(ZY)\big|; Y \in L_\sigma^1(P), \|Y\|_{\sigma,1} \leq 1\big\}.$$

Proof. Let $Z \in L^1_\sigma(P)^\times$. By Proposition 21, for any $\alpha \in [0,1)$ there is $E_\alpha \in \mathcal{F}$ such that $\mathsf{AV@R}_\alpha(|Z|) = \frac{1}{1-\alpha}\mathbb{E}(|Z|\mathbb{1}_{E_\alpha})$ and $P(E_\alpha) = 1-\alpha$. Employing the notation from Proposition 12, we obtain

$$\begin{aligned}\mathsf{AV@R}_\alpha\big(|Z|\big) &= \Big|\varphi_Z\Big(\frac{1}{1-\alpha}\mathbb{1}_{\{Z\neq 0\}}\frac{\overline{Z}}{|Z|}\mathbb{1}_{E_\alpha}\Big)\Big| \le \|\varphi_Z\|^*_{\sigma,1}\frac{1}{1-\alpha}\|\mathbb{1}_{E_\alpha}\|_{\sigma,1}\\ &= \|\varphi_Z\|^*_{\sigma,1}\frac{1}{1-\alpha}\int_0^1 \sigma(u)F^{-1}_{\mathbb{1}_{E_\alpha}}(u)\,du\\ &= \|\varphi_Z\|^*_{\sigma,1}\frac{1}{1-\alpha}\int_{1-P(E_\alpha)}^1 \sigma(u)\,du = \|\varphi_Z\|^*_{\sigma,1}\frac{1}{1-\alpha}S_\sigma(\alpha),\end{aligned}$$

so that $|Z|^*_{\sigma,\infty}$ is finite and bounded above by

$$\|\varphi_Z\|^*_{\sigma,1} = \sup\big\{\big|\mathbb{E}(ZY)\big|; Y \in L^1_\sigma(P), \|Y\|_{\sigma,1} \le 1\big\}.$$

Proposition 20 now yields the rest of the claim. □

Combining Propositions 12 and 22, we immediately derive the next result.

Theorem 23. *Let σ be a distortion function. Then $|\cdot|^*_{\sigma,\infty}$ is a norm on $L^*_{\sigma,\infty}(P)$ turning it into a Banach space. Moreover,*

$$\Phi : \big(L^*_{\sigma,\infty}(P), |\cdot|^*_{\sigma,\infty}\big) \to \big(L^1_\sigma(P)^*, \|\cdot\|^*_{\sigma,1}\big), \quad Z \mapsto \big(Y \mapsto \Phi(Z)(Y) := \mathbb{E}(ZY)\big)$$

is an isometric isomorphism.

In order to derive an analogous representation for the case $p \in (1,\infty)$, we need an equivalent result to Proposition 22 for this case. This requires some preparation. We begin by recalling a notion from Lorentz [21].

Definition 24. Let σ be a distortion function. A function $H : [0,1] \to \mathbb{R}$ is called *S_σ-concave* if whenever $y, b \in \mathbb{R}$ are such that $yS_\sigma(\alpha_1) + b = H(\alpha_1)$ and $yS_\sigma(\alpha_2) + b = H(\alpha_2)$ for some $\alpha_1 < \alpha_2 \in [0,1]$, then $H(\alpha) \ge yS_\sigma(\alpha) + b$ for each $\alpha \in [\alpha_1, \alpha_2]$.

The next proposition is essentially contained in Lorentz [21, Proof of Theorem 3.6.2]. Nevertheless, we include its proof for the reader's convenience.

Proposition 25. *Let σ be a distortion function, and let $u_0 := \inf\{u > 0; \sigma(u) > 0\}$. Moreover, let $\mathscr{H}$ be a set of S_σ-concave functions such that $H_{|[0,u_0]}$ is constant for every $H \in \mathscr{H}$. Assume that*

$$\forall \alpha \in [0,1]: \quad F(\alpha) := \inf\big\{H(\alpha); H \in \mathscr{H}\big\} > -\infty.$$

Then, F is S_σ-concave.

Proof. Let $y, b \in \mathbb{R}$, and let $\alpha_1 < \alpha_2 \in [0,1]$ be such that $yS_\sigma(\alpha_j) + b = F(\alpha_j)$, $j = 1,2$. Let $H \in \mathscr{H}$ be arbitrary. Since $H_{|[0,u_0]}$ is constant, there are $\bar{y}, \bar{b} \in \mathbb{R}$ such that $\bar{y}S_\sigma(\alpha_j) + \bar{b} = H(\alpha_j)$, $j = 1,2$.

In the case of $y - \bar{y} \ge 0$ it follows that $\bar{y}S_\sigma + \bar{b} - (yS_\sigma + b)$ is nonincreasing, while $\bar{y}S_\sigma + \bar{b} - (yS_\sigma + b)$ is nondecreasing in the case of $y - \bar{y} \le 0$. Therefore,

S_σ-concavity of H together with $yS_\sigma(\alpha_j)+b \le \bar{y}S_\sigma(\alpha_j)+\bar{b}, j=1,2$ implies

$$\forall \alpha \in [\alpha_1,\alpha_2]:\quad yS_\sigma(\alpha)+b \le \bar{y}S_\sigma(\alpha)+\bar{b} \le H(\alpha).$$

Since $H \in \mathscr{H}$ was arbitrary, we conclude that $F \ge yS_\sigma + b$ on $[\alpha_1,\alpha_2]$. □

Proposition 26. *Let σ be a distortion function, let $u_0 := \inf\{u>0; \sigma(u)>0\}$, and let $y_1,y_2,b_1,b_2 \in \mathbb{R}$. Moreover, let H be S_σ-concave, continuous from the right in u_0 such that $H_{|[0,u_0]}$ is constant.*

If for $y,b \in \mathbb{R}$ and $\alpha_1,\alpha_2 \in [0,1]$ with $\alpha_1<\alpha_2$ and $u_0<\alpha_2$ we have $H(\alpha_j) = yS_\sigma(\alpha_j)+b, j=1,2$, then it follows that $yS_\sigma+b \ge H$ on $[0,1]\setminus(\alpha_1,\alpha_2)$.

Proof. It is straightforward to show that if for $\alpha,\beta \in [0,1], \alpha<\beta$, it holds that $y_1S_\sigma(\alpha)+b_1 = y_2S_\sigma(\alpha)+b_2$ and $y_1S_\sigma(\beta)+b_1 > y_2S_\sigma(\beta)+b_2$, then $\beta>u_0$ and

$$\forall \gamma \in \big(\max\{\alpha,u_0\},\beta\big]:\quad y_1S_\sigma(\gamma)+b_1 > y_2S_\sigma(\gamma)+b_2, \tag{23}$$

while for $\alpha,\beta \in [u_0,1], \alpha<\beta$, the conditions $y_1S_\sigma(\alpha)+b_1 > y_2S_\sigma(\alpha)+b_2$ and $y_1S_\sigma(\beta)+b_1 = y_2S_\sigma(\beta)+b_2$ imply

$$\forall \gamma \in [\alpha,\beta):\quad y_1S_\sigma(\gamma)+b_1 > y_2S_\sigma(\gamma)+b_2. \tag{24}$$

In case of $\alpha_1 \le u_0$, it follows from the hypothesis that $H_{|[0,u_0]}$ is constant that trivially $yS+b \ge H$ on $[0,\alpha_1]$. Now let $u_0<\alpha_1$. We assume that $yS(\alpha)+b < H(\alpha)$ for some $\alpha \in (u_0,\alpha_1)$. Because S_σ is strictly decreasing in $[u_0,1]$, there are $\tilde{y},\tilde{b} \in \mathbb{R}$ such that

$$\tilde{y}S_\sigma(\alpha)+\tilde{b} = H(\alpha) \qquad \text{and} \qquad \tilde{y}S(\alpha_2)+\tilde{b} = H(\alpha_2).$$

The S_σ-concavity of H hence implies that $H \ge \tilde{y}S_\sigma+\tilde{b}$ on $[\alpha,\alpha_2]$. In particular,

$$yS_\sigma(\alpha_1)+b = H(\alpha_1) \ge \tilde{y}S_\sigma(\alpha_2)+\tilde{b}. \tag{25}$$

On the other hand,

$$\tilde{y}S_\sigma(\alpha)+\tilde{b} = H(\alpha) > yS_\sigma(\alpha)+b \qquad \text{and} \qquad \tilde{y}S_\sigma(\alpha_2)+\tilde{b} = H(\alpha_2) = yS_\sigma(\alpha_2)+b$$

so that by (24) we obtain $\tilde{y}S_\sigma+\tilde{b} > yS_\sigma+b$ on $[\alpha,\alpha_2)$, which contradicts (25). Therefore, $yS+b \ge H$ on (u_0,α_1). Since S_σ is continuous and H is continuous from the right in u_0, the same inequality holds on $[u_0,\alpha_1)$. Because S_σ and H are constant on $[0,u_0]$, we obtain $yS+b \ge H$ on $[0,\alpha_1]$.

It remains to show that $yS+b \ge H$ on $[\alpha_2,1]$ as well. Assume there is $\alpha \in [\alpha_2,1]$ with $yS_\sigma(\alpha)+b < H(\alpha)$. Since $\alpha_2>u_0$, there are again $\tilde{y},\tilde{b} \in \mathbb{R}$ such that

$$\tilde{y}S_\sigma(\alpha_1)+\tilde{b} = H(\alpha_1) \qquad \text{and} \qquad \tilde{y}S(\alpha)+\tilde{b} = H(\alpha).$$

Because H is S_σ-concave, this implies

$$\forall \beta \in [\alpha_1,\alpha]:\quad H(\beta) \ge \tilde{y}S_\sigma(\beta)+\tilde{b}. \tag{26}$$

On the other hand,

$$\tilde{y}S_\sigma(\alpha_1)+\tilde{b} = H(\alpha_1) = yS_\sigma(\alpha_1)+b \qquad \text{and} \qquad \tilde{y}S_\sigma(\alpha)+\tilde{b} = H(\alpha) > yS_\sigma(\alpha)+b.$$

By (23) it follows that $\tilde{y}S_\sigma + \tilde{b} > yS_\sigma + b$ on $(\max\{\alpha_1, u_0\}, \alpha]$. In particular,

$$\tilde{y}S_\sigma(\alpha_2) + \tilde{b} > yS_\sigma(\alpha_2) + b = H(\alpha_2),$$

which contradicts (26). □

Definition 27. Let σ be a distortion function. For a continuous function $G : [0,1] \to \mathbb{R}$, we define

$$G^*_\sigma : [0,\infty) \to \mathbb{R}, \quad G^*_\sigma(y) := \inf_{\alpha\in[0,1]} yS_\sigma(\alpha) - G(\alpha).$$

Then, $G(\alpha) + G^*_\sigma(y) \le yS_\sigma(\alpha)$ for all $\alpha \in [0,1], y \ge 0$ so that

$$G_\sigma : [0,1] \to \mathbb{R}, \quad G_\sigma(\alpha) := \inf_{y\ge 0} yS_\sigma(\alpha) - G^*_\sigma(y)$$

is well defined and satisfies $G_\sigma \ge G$.

Remark 28.

(i) Setting as before $u_0 := \inf\{u > 0; \sigma(u) > 0\}$, we have that $S_{\sigma|[u_0,1]}$ is a bijection from $[u_0, 1]$ onto $[0,1]$. Denoting by abuse of notation its inverse with S^{-1}, it follows that

$$\forall \alpha \in [0,1]\colon \quad G_\sigma\big(S^{-1}(\alpha)\big) = \inf_{y\ge 0} y\alpha - G^*_\sigma(y)$$

so that

$$\tilde{G}_\sigma : [0,1] \to \mathbb{R}, \quad \tilde{G}_\sigma(\alpha) := \inf_{y\ge 0} y\alpha - G^*_\sigma(y)$$

is well defined. Being the infimum of nondecreasing and concave functions, $\tilde{G}_\sigma$ is nondecreasing and concave too. Therefore, $\tilde{G}_\sigma$ is differentiable from the right on $[0,1)$ with nonnegative and nonincreasing right derivative. We obviously have $G_\sigma = \tilde{G}_\sigma \circ S_\sigma$ so that the concavity of S_σ implies that G_σ is concave as well. Denoting left and right derivatives by $\frac{d_l}{d\alpha}$ and $\frac{d_r}{d\alpha}$, respectively, an appropriate adaptation of your favorite proof of the chain rules yields

$$\forall \alpha \in (0,1]\colon \quad \frac{d_l}{d\alpha}G_\sigma(\alpha) = \frac{d_r}{d\alpha}\tilde{G}_\sigma\big(S_\sigma(\alpha)\big)\frac{d_l}{d\alpha}S_\sigma(\alpha) = -\frac{d_r}{d\alpha}\tilde{G}_\sigma\big(S_\sigma(\alpha)\big)\sigma(\alpha).$$

Combined with $G_\sigma(1) - G_\sigma(\alpha) = \int_\alpha^1 \frac{d_l}{d\alpha}G_\sigma(u)\,du$, we obtain

$$\forall \alpha \in [0,1]\colon \quad G_\sigma(\alpha) = G_\sigma(1) + \int_\alpha^1 H(u)\sigma(u)\,du \tag{27}$$

for a nonnegative, nondecreasing function H on $[0,1]$ which is continuous from the left.

(ii) For $y \ge 0$ the function $yS_\sigma - G^*_\sigma(y)$ is obviously S_σ-concave. It therefore follows from Proposition 25 that G_σ is S_σ-concave. Moreover, being the infimum of nonincreasing functions, G_σ is nonincreasing.

(iii) Because $S_{\sigma|[0,u_0]} = 1$, it follows that G_σ is constant on $[0, u_0]$. Hence, for all $\alpha_1, \alpha_2 \in [0,1], \alpha_1 < \alpha_2$, there are $y \geq 0, b \geq G^*_\sigma(y)$ with $G_\sigma(\alpha_j) = yS_\sigma(\alpha_j) - b$ for $j = 1, 2$. Indeed, if $\alpha_2 > u_0$, we choose $y = \frac{G_\sigma(\alpha_2) - G_\sigma(\alpha_1)}{S_\sigma(\alpha_2) - S_\sigma(\alpha_1)}$, which is well defined and nonnegative because S_σ is strictly decreasing on $[u_0, 1]$ and G_σ is nonincreasing. Then

$$
\begin{aligned}
b = yS_\sigma(\alpha_2) - G_\sigma(\alpha_2) &\geq \inf_{\alpha \in [0,1]} yS_\sigma(\alpha) - G_\sigma(\alpha) \\
&= \inf_{\alpha \in [0,1]} yS_\sigma(\alpha) - \Big(\inf_{\tilde{y} \geq 0} \tilde{y} S_\sigma(\alpha) - G^*_\sigma(\tilde{y})\Big) \geq G^*_\sigma(y).
\end{aligned}
$$

In case of $\alpha_2 \leq u_0$, we may choose $y = 0$ so that

$$
b = -G_\sigma(\alpha_2) = -G_\sigma(0) = -\inf_{y \geq 0} y - G^*_\sigma(y) \geq G^*_\sigma(0).
$$

If additionally G is nonincreasing, it holds that

$$
\begin{aligned}
G_\sigma(0) &= \inf_{y \geq 0} y - G^*_\sigma(y) \leq -G^*_\sigma(0) \\
&= -\inf_{\alpha \in [0,1]} \big(-G(\alpha)\big) = \sup_{\alpha \in [0,1]} G(\alpha) = G(0)
\end{aligned}
$$

so that, because $G \leq G_\sigma$, we conclude that

$$
b = -G_\sigma(0) = -G(0) = G^*_\sigma(0).
$$

Proposition 29. *Let $G : [0,1] \to \mathbb{R}$ be continuous and nonincreasing. If $\alpha \in (0,1)$ is such that $G_\sigma(\alpha) > G(\alpha)$, then there are $0 \leq \alpha_1 < \alpha < \alpha_2 \leq 1$ and $y \geq 0$ such that*

$$
\forall \beta \in (\alpha_1, \alpha_2): \quad G_\sigma(\beta) = yS_\sigma(\beta) - G^*_\sigma(y).
$$

Proof. By continuity of S_σ and G_σ, there are $0 \leq \alpha_1 < \alpha < \alpha_2 \leq 1$ such that $G_\sigma(\alpha_2) > G(\alpha_1)$. From Remark 28(iii), we conclude the existence of $y \geq 0$ and $b \geq G^*_\sigma(y)$ such that $G_\sigma(\alpha_j) = yS_\sigma(\alpha_j) - b, j = 1, 2$ and such that $y = 0$ in the case of $\alpha_2 \leq u_0$.

Because S_σ and G_σ are nonincreasing and $y \geq 0$, it follows from

$$
\inf_{\beta \in (\alpha_1, \alpha_2)} G_\sigma(\beta) = G_\sigma(\alpha_2) > G(\alpha_1) = \sup_{\beta \in (\alpha_1, \alpha_2)} G(\beta)
$$

that $yS_\sigma - b \geq G$ on (α_1, α_2).

If $\alpha_2 \leq u_0$, we have seen in Remark 28(iii) that without loss of generality we may assume $y = 0$ and $b = -G(0)$. Since G is nonincreasing, it thus follows that $yS_\sigma - b = G(0) \geq G$ on $[0,1]$.

If $\alpha_2 > u_0$, we apply Proposition 26 to G_σ to conclude $yS_\sigma - b \geq G_\sigma$ on $[0,1] \setminus (\alpha_1, \alpha_2)$. Since $G_\sigma \geq G$ and $yS_\sigma - b \geq G$ on (α_1, α_2), we obtain also in this case $yS_\sigma - b \geq G$ on $[0,1]$.

So in both cases $yS_\sigma - b \geq G$ or, equivalently, $yS_\sigma - G \geq b$ on $[0,1]$ so that $G^*_\sigma(y) = \inf_\alpha yS_\sigma(\alpha) - G(\alpha) \geq b$. Since also $b \geq G^*_\sigma(y)$, it follows $b = G^*_\sigma(y)$.

Finally, since G_σ is S_σ-concave and $G_\sigma(\alpha_j) = yS_\sigma(\alpha_j) - G_\sigma^*(\alpha_j)$ holds for $j = 1, 2$, it follows for $\beta \in (\alpha_1, \alpha_2)$ that

$$yS_\sigma(\beta) - G_\sigma^*(y) \le G_\sigma(\beta) = \inf_{\tilde{y}\ge 0} \tilde{y}S_\sigma(\beta) - G_\sigma^*(\tilde{y}) \le yS_\sigma(\beta) - G_\sigma^*(y),$$

which proves the claim. □

Proposition 30. *Assume that $G : [0,1] \to \mathbb{R}$ is continuous and nonincreasing and that $G(1) = 0$. Then $G_\sigma(0) = G(0)$ and $G_\sigma(1) = 0$.*

Proof. We have already observed in Remark 28(iii) that $G_\sigma(0) = G(0)$. Using the compactness of $[0,1]$, $G(1) = 0$, and that $S_\sigma(\alpha) = 0$ implies $\alpha = 1$, it follows that

$$\forall n \in \mathbb{N}\ \exists k_n \in \mathbb{N}\ \forall \alpha \in [0,1]: \quad G(\alpha) > k_n S_\sigma(\alpha) + \frac{1}{n}$$

which implies that for every $n \in \mathbb{N}$ there is $k_n \in \mathbb{N}$ with $-1/n \le G_\sigma^*(k_n)$. Because $S_\sigma(1) = 0$, we derive

$$G_\sigma(1) = \inf_{y\ge 0}\big(-G_\sigma^*(y)\big) \le \inf_{n\in\mathbb{N}}\big(-G_\sigma^*(k_n)\big) \le 0 = G(1) \le G_\sigma(1),$$

which gives $G_\sigma(1) = 0$. □

Combining Propositions 29 and 30, we immediately obtain the next result.

Proposition 31. *Let $G : [0,1] \to \mathbb{R}$ be continuous and nonincreasing such that $G(1) = 0$. If $\alpha \in (0,1)$ satisfies $G_\sigma(\alpha) > G(\alpha)$, there are $0 \le \alpha_1 < \alpha < \alpha_2 \le 1$ and $y \ge 0$ such that $G_\sigma = yS_\sigma - G_\sigma^*(y)$ on (α_1, α_2) and $G_\sigma(\alpha_j) = G(\alpha_j)$ for $j = 1, 2$.*

We now have everything at hand to derive the analogue of Proposition 22.

Lemma 32. *Let σ be a distortion function, and let $p \in (1,\infty)$ with conjugate exponent q. Then $L_\sigma^p(P)^\times = L_{\sigma,q}^*(P)$, for every $Z \in L_\sigma^p(P)^\times$ it holds that $|Z|_{\sigma,q}^* = \|\varphi_Z\|_{\sigma,p}^*$, and there is $Y \in L_\sigma^p(P)$ with $\|Y\|_{\sigma,p} = 1$ such that $\varphi_Z(Y) = \|\varphi_Z\|_{\sigma,p}^*$.*

Proof. By Proposition 20, we only have to show that $L_\sigma^p(P)^\times \subseteq L_{\sigma,q}^*(P)$ and that

$$\sup\big\{\big|\mathbb{E}(ZY)\big|; Y \in L_\sigma^p(P), \|Y\|_{\sigma,p} \le 1\big\}$$

is an upper bound for $|Z|_{\sigma,q}^*$ for any $Z \in L_\sigma^p(P)^\times$.

So we fix $Z \in L_\sigma^p(P)^\times$. By Remark 13, we also have $|Z| \in L_\sigma^p(P)^\times$. We define

$$G : [0,1] \to [0,\infty), \quad G(\alpha) := \int_\alpha^1 F_{|Z|}^{-1}(u)\, du$$

and observe that G is well defined by $|Z| \in L^1(P)$. The function G is obviously continuous, differentiable from the left, and nonincreasing with $G(1) = 0$. By Proposition 31 and Remark 28(i), there is a nonnegative, nondecreasing function H on $[0,1]$ which is continuous from that left such that $G_\sigma(\alpha) = \int_\alpha^1 H(u)\sigma(u)\, du$.

If there is $\alpha \in (0,1)$ with $G_\sigma(\alpha) > G(\alpha)$, it follows immediately from Proposition 31 that there are $0 \le \alpha_1 < \alpha < \alpha_2 \le 1$ and $y \ge 0$ such that $H(u) = y$ for $u \in (\alpha_1, \alpha_2)$ and $G_\sigma(\alpha_j) = G(\alpha_j)$, $j = 1, 2$ so that

$$\int_{\alpha_1}^{\alpha_2} H(u)^q \sigma(u)\, du = y^{q-1} \int_{\alpha_1}^{\alpha_2} H(u)\sigma(u)\, du = y^{q-1}\big(G_\sigma(\alpha_1) - G_\sigma(\alpha_2)\big)$$
$$= y^{q-1}\big(G(\alpha_1) - G(\alpha_2)\big) = \int_{\alpha_1}^{\alpha_2} H(u)^{q-1} F^{-1}_{|Z|}(u)\, du.$$

On the other hand, if $G_\sigma(\alpha) = G(\alpha)$ by continuity, there is a maximal closed interval $[\alpha_1, \alpha_2]$ containing α such that G and G_σ coincide on $[\alpha_1, \alpha_2]$. Thus, on (α_1, α_2) the left derivatives of G and G_σ coincide; that is, $F^{-1}_{|Z|} = H\sigma$ on (α_1, α_2), which implies again

$$\int_{\alpha_1}^{\alpha_2} H(u)^q \sigma(u)\, du = \int_{\alpha_1}^{\alpha_2} H(u)^{q-1} F^{-1}_{|Z|}(u)\, du.$$

Combining these arguments gives

$$\forall \alpha_1, \alpha_2 \in [0,1]\colon \quad \Big(G_\sigma(\alpha_1) = G(\alpha_1), G_\sigma(\alpha_2) = G(\alpha_2)$$
$$\Rightarrow \int_{\alpha_1}^{\alpha_2} H(u)^q \sigma(u)\, du = \int_{\alpha_1}^{\alpha_2} H(u)^{q-1} F^{-1}_{|Z|}(u)\, du\Big). \tag{28}$$

In order to proceed, we distinguish two cases. First, we assume that there is a strictly increasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $(0,1)$ converging to 1 such that $G(\alpha_n) = G_\sigma(\alpha_n)$. We define

$$Y_n := (\mathbb{1}_{[0,\alpha_n]} H^{q-1}) \circ U,$$

where $U \in \mathscr{U}(0,1)$ is coupled in a comonotone way with $|Z|$. From $1/p + 1/q = 1$, it follows that $|Y_n^p| = (\mathbb{1}_{[0,\alpha_n]} H^q) \circ U$, which implies

$$\|Y_n\|^p_{\sigma,p} = \int_0^1 F^{-1}_{|Y_n|}(u)^p \sigma(u)\, du = \int_0^{\alpha_n} H^q(u)\sigma(u)\, du < \infty \tag{29}$$

since H in nondecreasing and $\int_0^1 \sigma(u)\, du = 1$ so that $Y_n = |Y_n| \in L^p_\sigma(P)$. Using the notation from Proposition 12, because $|Z|$ and U are coupled in a comonotone way, we have by (29) and (28) applied to $\alpha_1 = 0$ and $\alpha_2 = \alpha_n$,

$$\int_0^{\alpha_n} H(u)^q \sigma(u)\, du = \int_0^{\alpha_n} H^{q-1}(u) F^{-1}_{|Z|}(u)\, du = \mathbb{E}\big(|Y_n||Z|\big)$$
$$\le \|\varphi_{|Z|}\|^*_{\sigma,p} \|Y_n\|_{\sigma,p} = \|\varphi_{|Z|}\|^*_{\sigma,p} \Big(\int_0^{\alpha_n} H(u)^q \sigma(u)\, du\Big)^{1/p},$$

which gives

$$\Big(\int_0^1 \mathbb{1}_{[0,\alpha_n]} H(u)^q \sigma(u)\, du\Big)^{1/q} \le \|\varphi_{|Z|}\|^*_{\sigma,p}$$

for all $n \in \mathbb{N}$. Using that $\lim_{n\to\infty} \alpha_n = 1$, an application of the monotone convergence theorem yields

$$\Big(\int_0^1 H(u)^q \sigma(u)\,du\Big)^{1/q} \le \|\varphi_{|Z|}\|^*_{\sigma,p} \tag{30}$$

so that $Z' := H \circ U$ belongs to $L^q_\sigma(P)$. Because $Z' = |Z'|$ and $F^{-1}_{|Z'|} = H$, it follows from

$$\forall \alpha \in [0,1]\colon \quad \int_\alpha^1 H(u)\sigma(u)\,du = G_\sigma(\alpha) \ge G(\alpha) = \int_\alpha^1 F^{-1}_{|Z|}(u)\,du$$

that $Z' \mathrel{{}_\sigma\succcurlyeq} Z$, which, combined with (30), yields $Z \in L^*_{\sigma,q}$ and

$$|Z|^*_{\sigma,q} \le \|\varphi_{|Z|}\|^*_{\sigma,p} = \|\varphi_Z\|^*_{\sigma,p},$$

where we have used Remark 13 in the last equality. Since also $\|Z\|^*_{\sigma,p} \le |Z|^*_{\sigma,q}$, we obtain from (30)

$$\|\varphi_Z\|^*_{\sigma,p} = \Big(\int_0^1 H(u)^q \sigma(u)\,du\Big)^{1/q} = \inf\big\{\|Z'\|_{\sigma,q};\, Z' \mathrel{{}_\sigma\succcurlyeq} Z\big\}. \tag{31}$$

Now we define

$$Y := \mathbb{1}_{\{Z\neq 0\}} \frac{\overline{Z}}{|Z|} H^{q-1}(U). \tag{32}$$

Then the same arguments used in deriving (29) combined with (30) show that $Y \in L^p_\sigma(P)$ and

$$\|Y\|_{\sigma,p} = \Big(\int_0^1 H^q(u)\sigma(u)\,du\Big)^{1/p}.$$

Moreover, using that $|Z|$ and U are coupled in a comonotone way, (28) applied to $\alpha_1 = 0$ and $\alpha_2 = 1$, and (31) give

$$\begin{aligned}
\varphi_Z(Y) &= \mathbb{E}(ZY) = \mathbb{E}\big(|ZY|\big) = \int_0^1 H(u)^{q-1} F^{-1}_{|Z|}(u)\,du \\
&= \int_0^1 H(u)^q \sigma(u)\,du = \Big(\int_0^1 H(u)^q\sigma(u)\,du\Big)^{1/q}\Big(\int_0^1 H(u)^q\sigma(u)\,du\Big)^{1/p} \\
&= \|\varphi_Z\|^*_{\sigma,p}\|Y\|_{\sigma,p}.
\end{aligned}$$

Next, if there is no strictly increasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $(0,1)$ converging to 1 such that $G(\alpha_n) = G_\sigma(\alpha_n)$, there is $\beta \in (0,1)$ such that $G(u) < G_\sigma(u)$ for all $\alpha \in (\beta,1)$ and such that $G(\beta) = G_\sigma(\beta)$. It therefore follows from Proposition 31 that there is $y \ge 0$ such that $H = y$ on $(\beta,1)$. Because H is nondecreasing, this implies that H is bounded so that trivially

$$\int_0^1 H(u)^q \sigma(u) < \infty.$$

By repeating the arguments from the first part of the proof it follows for $U \in \mathscr{U}(0,1)$ coupled in a comonotone way with $|Z|$ that $Z' := H \circ U$ satisfies $Z' \in L^q_\sigma(P)$ and $Z' \mathrel{{}_\sigma\succcurlyeq} Z$, which gives $Z \in L_{\sigma,q}(P)^*$ and $|Z|^*_{\sigma,q} = \|\varphi_Z\|^*_{\sigma,p} = \|Z'\|_{\sigma,q}$.

Defining Y as in (32) finally gives again $\varphi_Z(Y) = \|\varphi_Z\|^*_{\sigma,p}\|Y\|_{\sigma,p}$, which proves the claim. ☐

Combining Proposition 12 and Lemma 32, we immediately derive the next result.

Theorem 33. *Let σ be a distortion function, and let $p \in (1,\infty)$ with conjugate exponent q. Then $|\cdot|^*_{\sigma,q}$ is a norm on $L^*_{\sigma,q}(P)$ turning it into a Banach space. Moreover,*

$$\Phi : \left(L^*_{\sigma,q}(P), |\cdot|^*_{\sigma,q}\right) \to \left(L^p_\sigma(P)^*, \|\cdot\|^*_{\sigma,p}\right), \quad Z \mapsto \left(Y \mapsto \Phi(Z)(Y) := \mathbb{E}(ZY)\right)$$

is an isometric isomorphism. Furthermore, for every $\varphi \in L^p_\sigma(P)^*$ there is $Y \in L^p_\sigma(P)$ with $\|Y\|_{\sigma,p} = 1$ such that $\varphi(Y) = \|\varphi\|^*_{\sigma,p}$.

Corollary 34. *For a distortion function σ and $p \in (1,\infty)$, the Banach space $L^p_\sigma(P)$ is reflexive.*

Proof. This is an immediate consequence of James's theorem (see, e.g., Diestel [10, Theorem I.3]) and Theorem 33. ☐

Proposition 35. *Simple functions (and thus L^∞) are dense in $L^*_{\sigma,q}(P)$, whenever $q < \infty$.*

Proof. Let $\mathfrak{F}$ contain all *finite* sigma algebras $\mathcal{F}$ for which the measure P is defined. Note that $(\mathfrak{F}, \subseteq)$ is a filter, and the proof of Proposition 9 actually demonstrates that

$$\left\|\mathbb{E}(Y|\mathcal{F}) - Y\right\|_{\sigma,p} \underset{\mathfrak{F}}{\longrightarrow} 0$$

whenever $\mathcal{F} \in \mathfrak{F}$ increases.

Recall first that $\mathsf{AV@R}_\alpha(\mathbb{E}(Y|\mathcal{F})) \le \mathsf{AV@R}_\alpha(Y)$. Indeed, it follows from the conditional Jensen inequality (see Williams [31, Section 34]) that $(\mathbb{E}(Y|\mathcal{F})-q)_+ \le \mathbb{E}((Y-q)_+|\mathcal{F})$, and hence, using Pflug [24],

$$\begin{aligned}\mathsf{AV@R}_\alpha\big(\mathbb{E}(Y|\mathcal{F})\big) &= \min_{q\in\mathbb{R}} q + \frac{1}{1-\alpha}\mathbb{E}\big(\mathbb{E}(Y|\mathcal{F}) - q\big)_+ \\ &\le \min_{q\in\mathbb{R}} q + \frac{1}{1-\alpha}\mathbb{E}\mathbb{E}\big((Y-q)_+|\mathcal{F}\big) \\ &= \min_{q\in\mathbb{R}} q + \frac{1}{1-\alpha}\mathbb{E}\big((Y-q)_+|\mathcal{F}\big) = \mathsf{AV@R}_\alpha(Y).\end{aligned}$$

Suppose that $Z' \,{}_\sigma\!\succcurlyeq Z$. It follows that

$$\int_\alpha^1 \sigma(u) F^{-1}_{Z'}(u)\,du \ge \int_\alpha^1 F^{-1}_Z(u)\,du \ge \int_\alpha^1 F^{-1}_{\mathbb{E}(Z|\mathcal{F})}(u)\,du$$

for every $\alpha \le 1$, that is, $Z' \,{}_\sigma\!\succcurlyeq \mathbb{E}(Z|\mathcal{F})$, and thus $\|\mathbb{E}(Z|\mathcal{F})\|^*_{\sigma,q} \le \|Z\|^*_{\sigma,q}$. The assertion follows as $\{\mathbb{E}(Z|\mathcal{F}) : \mathcal{F} \in \mathfrak{F}\}$ is arbitrarily close to Z in the norm $\|\cdot\|_{\sigma,q}$ by Proposition 9. ☐

We close this section by taking a closer look at $L^1_\sigma(P)$ and its dual space.

Theorem 36. *The dual space of $L^1_\sigma(P)$ is not separable.*

Proof. It is enough to assume that σ is unbounded. As for bounded σ, we have that $L^1_\sigma(P)$ is isomorphic to $L^1(P)$ by Proposition 8 and its dual $L^\infty(P)$ is not separable.

For $\beta \in [0,1]$, consider the random variables

$$Z_\beta := \begin{cases} \sigma(U) & \text{if } U \leq \beta, \\ \sigma(1+\beta-U) & \text{if } U > \beta, \end{cases}$$

for a (fixed) uniform random variable $U \in \mathscr{U}(0,1)$. Note that $\|Z_\beta\|^*_{\sigma,1} = 1$, since Z_β is a rearrangement of $\sigma(U)$. Assume that $\beta < \gamma$, and observe that

$$\begin{aligned} Z_\gamma - Z_\beta &= \sigma(1+\gamma-U) - \sigma(1+\beta-U) \\ &\geq \sigma(1+\gamma-U) - \sigma(1+\beta-\gamma) \end{aligned}$$

whenever $U > \gamma$. Then it holds that

$$\begin{aligned} \|Z_\gamma - Z_\beta\|^*_{\sigma,1} &\geq \limsup_{\alpha\to 1} \frac{\mathsf{AV@R}_\alpha(Z_\gamma - Z_\beta)}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\,du} \\ &\geq \limsup_{\alpha\to 1} \frac{\mathsf{AV@R}_\alpha(\sigma(1+\beta-U) - \sigma(1+\beta-\gamma))}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\,du}. \end{aligned}$$

Now, as σ is unbounded, the denominator is unbounded as well (indeed, we have $\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\,du \geq \sigma(\alpha)$), and hence

$$\begin{aligned} \|Z_\gamma - Z_\beta\|^*_{\sigma,1} &\geq \limsup_{\alpha\to 1} \frac{\mathsf{AV@R}_\alpha(\sigma(1+\beta-U)) - \sigma(1+\beta-\gamma)}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\,du} \\ &= \lim_{\alpha\to 1} \frac{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\,du - \sigma(1+\beta-\gamma)}{\frac{1}{1-\alpha}\int_\alpha^1 \sigma(u)\,du} = 1. \end{aligned} \tag{33}$$

Suppose finally that there is a dense sequence $(D_k)_{k\in\mathbb{N}} \subset L^1_\sigma(P)^*$. For $\beta \in [0,1]$ fixed there is $k \in \mathbb{N}$ such that $\|Z_\beta - D_k\|^*_{\sigma,1} < \frac{1}{2}$. But $1 \leq \|Z_\beta - Z_\gamma\|^*_{\sigma,1} \leq \|Z_\beta - D_k\|^*_{\sigma,1} + \|D_k - Z_\gamma\|^*_{\sigma,1}$, from which follows that $\|D_k - Z_\gamma\|^*_{\sigma,1} > \frac{1}{2}$ whenever $\gamma \neq \beta$. Hence only *countably* many Z_β's can be approximated by the sequence $(D_k)_{k\in\mathbb{N}}$ with a distance $\|Z_\beta - D_k\|^*_\sigma < \frac{1}{2}$ and $(D_k)_{k\in\mathbb{N}}$ thus is not dense, giving the desired contradiction. □

5. The dual space in the vector-valued case

In this section, we determine the dual space of $L^p_\sigma(P,X)$ for arbitrary Banach spaces X over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. We denote the space of X-valued simple functions on $(\Omega,\mathcal{F},P)$ by $\mathcal{S}(X)$; that is,

$$\mathcal{S}(X) = \big\{Y : \Omega \to X; Y(\Omega) \text{ is finite and } \forall x \in X : Y^{-1}\big(\{x\}\big) \in \mathcal{F}\big\}.$$

Then it is straightforward to see and well known that

$$\big\{\varphi : \mathcal{S}(X) \to \mathbb{K}; \varphi \text{ linear and continuous with respect to } \|\cdot\|_\infty\big\}$$

and

$$\{\mu : \mathcal{F} \to X^*; \mu \text{ vector measure of bounded variation}\}$$

are isomorphic via the linear mapping

$$\Phi : \varphi \mapsto \big(\mu_\varphi(E)(x) := \varphi(\mathbb{1}_E x), x \in X, E \in \mathcal{F}\big). \tag{34}$$

For a vector measure μ, we denote by $|\mu|$ its variation.

Lemma 37. *For a linear mapping $\varphi : \mathcal{S}(X) \to \mathbb{K}$, we have*

$$\sup\Big\{\Big|\varphi\Big(\sum_{j=1}^n \mathbb{1}_{E_j} x_j\Big)\Big|; E_j \in \mathcal{F} \text{ partition of } \Omega, x_j \in X, \Big\|\sum_{j=1}^n \mathbb{1}_{E_j} x_j\Big\|_{\sigma,p} \le 1\Big\}$$
$$= \sup\Big\{\sum_{j=1}^n |\alpha_j| \big\|\mu_\varphi(E_j)\big\|; E_j \in \mathcal{F} \text{ partition of } \Omega, \alpha_j \in \mathbb{K}, \Big\|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big\|_{\sigma,p} \le 1\Big\}$$
$$= \sup\Big\{\int_\Omega \Big|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big|\, d|\mu_\varphi|;$$
$$E_j \in \mathcal{F} \text{ partition of } \Omega, \alpha_j \in \mathbb{K}, \Big\|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big\|_{\sigma,p} \le 1\Big\},$$

where μ_φ is defined as in (34).

Proof. For a partition $E_1, \dots, E_n \in \mathcal{F}$ of Ω, $\alpha_1, \dots, \alpha_n \in \mathbb{K}$, and $z_1, \dots, z_n \in X$ with $\|z_j\| = 1$, we have

$$\Big\|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j} z_j\Big\|_{\sigma,p}^p = \sup_{U \in \mathcal{U}(0,1)} \int_\Omega \sigma(U) \sum_{j=1}^n |\alpha_j|^p \mathbb{1}_{E_j}\, dP = \Big\|\sum_{j=1}^n \alpha_j, \mathbb{1}_{E_j}\Big\|_{\sigma,p}^p,$$

where the norm on the left-hand side is the one on $L^p_\sigma(P, X)$, while the norm on the right-hand side denotes the one on $L^p_\sigma(P)$. Therefore, we conclude that

$$\sup\Big\{\Big|\varphi\Big(\sum_{j=1}^n \mathbb{1}_{E_j} x_j\Big)\Big|; E_j \in \mathcal{F} \text{ partition of } \Omega, x_j \in X, \Big\|\sum_{j=1}^n \mathbb{1}_{E_j} x_j\Big\|_{\sigma,p} \le 1\Big\}$$
$$= \sup\Big\{\Big|\varphi\Big(\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j} z_j\Big)\Big|;$$
$$E_j \in \mathcal{F} \text{ partition of } \Omega, \alpha_j \ge 0, z_j \in X, \|z_j\| = 1 \text{ and } \Big\|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big\|_{\sigma,p} \le 1\Big\}$$
$$= \sup\Big\{\sum_{j=1}^n \big|\varphi(\alpha_j \mathbb{1}_{E_j} z_j)\big|;$$
$$E_j \in \mathcal{F} \text{ partition of } \Omega, \alpha_j \ge 0, z_j \in X, \|z_j\| = 1 \text{ and } \Big\|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big\|_{\sigma,p} \le 1\Big\}$$
$$= \sup\Big\{\sum_{j=1}^n |\alpha_j| \big\|\mu_\varphi(E_j)\big\|; E_j \in \mathcal{F} \text{ partition of } \Omega, \alpha_j \in \mathbb{K}, \Big\|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big\|_{\sigma,p} \le 1\Big\},$$

which gives the first equality. Using the definition of $|\mu_\varphi|$, we continue with

$$\sup\Big\{\sum_{j=1}^n |\alpha_j|\big\|\mu_\varphi(E_j)\big\|; E_j \in \mathcal{F} \text{ partition of } \Omega, \alpha_j \in \mathbb{K}, \Big\|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big\|_{\sigma,p} \leq 1\Big\}$$

$$= \sup\Big\{ \sum_{\substack{i=1\\ \alpha_i \text{ pairwise different}}}^n |\alpha_i| \sum_{j:\alpha_i=\alpha_j} \big\|\mu_\varphi(E_j)\big\|;$$

$$E_j \in \mathcal{F} \text{ partition of } \Omega, \alpha_j \in \mathbb{K} \text{ and } \Big\|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big\|_{\sigma,p} \leq 1\Big\}$$

$$= \sup\Big\{ \sum_{\substack{i=1\\ \alpha_i \text{ pairwise different}}}^n |\alpha_i| |\mu_\varphi|\Big(\bigcup_{j:\alpha_i=\alpha_j} E_j\Big);$$

$$E_j \in \mathcal{F} \text{ partition of } \Omega, \alpha_j \in \mathbb{K} \text{ and } \Big\|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big\|_{\sigma,p} \leq 1\Big\}$$

$$= \sup\Big\{\int_\Omega \Big|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big|\, d|\mu_\varphi|;$$

$$E_j \in \mathcal{F} \text{ partition of } \Omega, \alpha_j \in \mathbb{K}, \Big\|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big\|_{\sigma,p} \leq 1\Big\},$$

which proves the second equality. □

Definition 38. For a distortion function σ, $p \in [1,\infty)$ and a Banach space X, we define

$$\mathcal{L}_{\sigma,p}\big(\mathcal{S}(X)\big) := \big\{\varphi : \mathcal{S}(X) \to \mathbb{K}; \varphi \text{ linear and continuous with respect to } \|\cdot\|_{\sigma,p}\big\}.$$

Lemma 39. *Let Φ be the natural isomorphism from (34). Then $\Phi(\mathcal{L}_{\sigma,p}(\mathcal{S}(X)))$ coincides with the set*

$$\Big\{\mu : \mathcal{F} \to X^*; \mu \text{ is a } \sigma\text{-additive vector measure of bounded variation such that } |\mu| \ll P \text{ and } \frac{d|\mu|}{dP} \in L^*_{\sigma,q}(P)\Big\},$$

and

$$\forall \varphi \in \mathcal{L}_{\sigma,p}\big(\mathcal{S}(X)\big): \quad \|\varphi\| = \Big|\frac{d|\mu_\varphi|}{dP}\Big|^*_{\sigma,q},$$

where q is the conjugate exponent to p.

Proof. For $\varphi \in \mathcal{L}_{\sigma,p}(\mathcal{S}(X))$, it follows from the density of $\mathcal{S}(X)$ in $L^p_\sigma(P,X)$ that φ extends to a unique element of $L^p_\sigma(P,X)^\times$ which we still denote by φ. For a pairwise disjoint sequence $(E_j)_{j\in\mathbb{N}}$ in $\mathcal{F}$ and its union E, it follows for arbitrary

$x \in X$ that

$$\Big|\mu_\varphi(E)(x) - \mu_\varphi\Big(\bigcup_{j=1}^{m} E_j\Big)(x)\Big| = \Big|\mu_\varphi\Big(\bigcup_{j=m+1}^{\infty} E_j\Big)(x)\Big| = \big|\varphi(\mathbb{1}_{\bigcup_{j=m+1}^{\infty} E_j} x)\big|$$
$$\le \|\varphi\|_{\sigma,p}^* \|x\| \|\mathbb{1}_{\bigcup_{j=m+1}^{\infty} E_j}\|_{\sigma,p}$$
$$= \|\varphi\|_{\sigma,p}^* \|x\| \Big(\int_{1-\sum_{j=m+1}^{\infty} P(E_j)}^{1} \sigma(u)\,du\Big)^{1/p}.$$

With the aid of Lebesgue's dominated convergence theorem, it follows that

$$\Big\|\mu_\varphi(E) - \sum_{j=1}^{m} \mu_\varphi(E_j)\Big\| \le \|\varphi\|_{\sigma,p}^* \Big(\int_{1-\sum_{j=m+1}^{\infty} P(E_j)}^{1} \sigma(u)\,du\Big)^{1/p} \to_{m\to\infty} 0.$$

Thus $\Phi(\varphi) = \mu_\varphi$ is a σ-additive vector measure. Moreover, for every finite partition $E_1, \dots, E_n \in \mathcal{F}$ of Ω and $x_1, \dots, x_n \in X$ with $\|x_j\| \le 1$ we have

$$\sum_{j=1}^{n} \big|\mu_\varphi(E_j)(x_j)\big| = \sum_{j=1}^{n} \big|\varphi(\mathbb{1}_{E_j} x_j)\big| = \sum_{j=1}^{n} \operatorname{sign}\big(\varphi(\mathbb{1}_{E_j} x_j)\big)\varphi(\mathbb{1}_{E_j} x_j)$$
$$= \Big|\varphi\Big(\sum_{j=1}^{n} \operatorname{sign}\big(\varphi(\mathbb{1}_{E_j} x_j)\big)\mathbb{1}_{E_j} x_j\Big)\Big|$$
$$\le \|\varphi\|_{\sigma,p}^* \Big\|\sum_{j=1}^{n} \operatorname{sign}\big(\varphi(\mathbb{1}_{E_j} x_j)\big)\mathbb{1}_{E_j} x_j\Big\|_{\sigma,p}$$
$$\le \|\varphi\|_{\sigma,p}^* \Big(\sup_{U\in\mathscr{U}(0,1)} \int_\Omega \sum_{j=1}^{n} \operatorname{sign}\big(\varphi(\mathbb{1}_{E_j} x_j)\big)^p \mathbb{1}_{E_j} \|x_j\|^p \sigma(U)\,dP\Big)^{1/p}$$
$$\le \|\varphi\|_{\sigma,p}^* \Big(\sup_{U\in\mathscr{U}(0,1)} \int_\Omega \sigma(U)\,dP\Big)^{1/p} = \|\varphi\|_{\sigma,p}^*,$$

where, for a complex number α, as usual $\operatorname{sign}(\alpha) = \frac{\overline{\alpha}}{|\alpha|}$ in case $\alpha \ne 0$ (resp., $\operatorname{sign}(0) = 0$). Thus, for arbitrary $\varepsilon > 0$, it follows for suitable choices $x_j^\varepsilon \in X$ from the above inequality that

$$\sum_{j=1}^{n} \big\|\mu_\varphi(E_j)\big\| \le \sum_{j=1}^{n} \Big(\big|\mu_\varphi(E_j)(x_j^\varepsilon)\big| + \frac{\varepsilon}{n}\Big) \le \|\varphi\|_{\sigma,p}^* + \varepsilon,$$

that is, $\sum_{j=1}^{n} \|\mu_\varphi(E_j)\| \le \|\varphi\|_{\sigma,p}^*$, which in turn implies $|\mu_\varphi|(\Omega) \le \|\varphi\|_{\sigma,p}^*$. Hence, $\Phi(\varphi) = \mu_\varphi$ is of bounded variation.

Since μ_φ is σ-additive, the same holds for $|\mu_\varphi|$ (see Diestel and Uhl [11, Proposition I.1.9]); that is, $|\mu_\varphi|$ is a (finite) measure on $\mathcal{F}$. If $E \in \mathcal{F}$ satisfies $P(E) = 0$, it follows for $x \in X$ that

$$\|\mathbb{1}_E x\|_{\sigma,p} = \|x\| \Big(\sup_{u\in\mathscr{U}(0,1)} \int_E \sigma(U)\,dP\Big)^{1/p} = 0$$

and therefore $\|\mu_\varphi(E)\| = 0$. If $E_1, \dots, E_n \in \mathcal{F}$ is a partition of E, it follows that $P(E_j) = 0$ and thus $\sum_{j=1}^{n} \|\mu_\varphi(E_j)\| = 0$, which implies $|\mu_\varphi|(E) = 0$. By an application of the Radon–Nikodým theorem, we obtain $g_\varphi \in L^1(P), g_\varphi \geq 0$ such that

$$\forall E \in \mathcal{F}\colon \quad \int_E g_\varphi \, dP = |\mu_\varphi|(E).$$

From the fact that $\mathcal{S}(X)$ is dense in $L^p_\sigma(P, X)$ and $\mathcal{S}(\mathbb{K})$ is dense in $L^p_\sigma(P)$, it follows with Lemma 39 that

$$\begin{aligned}
\|\varphi\|^*_{\sigma,p} = {} & \sup\Big\{\Big|\varphi\Big(\sum_{j=1}^n \mathbb{1}_{E_j} x_j\Big)\Big|; \\
& \qquad E_j \in \mathcal{F} \text{ partition of } \Omega, x_j \in X, \Big\|\sum_{j=1}^n \mathbb{1}_{E_j} x_j\Big\|_{\sigma,p} \leq 1\Big\} \\
= {} & \sup\Big\{\int_\Omega \Big|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big| \, d|\mu_\varphi|; \\
& \qquad E_j \in \mathcal{F} \text{ partition of } \Omega, \alpha_j \in \mathbb{K} \text{ and } \Big\|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big\|_{\sigma,p} \leq 1\Big\} \\
= {} & \sup\Big\{\int_\Omega \Big|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big| g_\varphi \, dP; \\
& \qquad E_j \in \mathcal{F} \text{ partition of } \Omega, \alpha_j \in \mathbb{K} \text{ and } \Big\|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big\|_{\sigma,p} \leq 1\Big\} \\
= {} & \sup\Big\{\int_\Omega |f g_\varphi| \, dP; f \in L^p_\sigma(P), \|f\|_{\sigma,p} \leq 1\Big\},
\end{aligned}$$

so that in particular $g_\varphi \in L^p_\sigma(P)^\times = L^*_{\sigma,q}(P)$ and $\|\varphi\|^*_{\sigma,p} = |g_\varphi|^*_{\sigma,q}$. Since $\varphi \in \mathcal{L}_{\sigma,p}(\mathcal{S}(X))$ was chosen arbitrarily, this finally shows that $\Phi(\mathcal{L}_{\sigma,p}(\mathcal{S}(X)))$ is contained in the set of X^*-valued, σ-additive vector measures of bounded variation such that their bounded variation measure admits a P-density in $L^*_{\sigma,q}(P)$.

Next, let μ be such a measure, and set $\varphi := \Phi^{-1}(\mu)$. We have to show that φ belongs to $\mathcal{L}_{\sigma,p}(\mathcal{S}(X))$. But from the density of $\mathcal{S}(\mathbb{K})$ in $L^p_\sigma(P)$, it follows immediately together with Lemma 37 that

$$\begin{aligned}
& \sup\Big\{\Big|\varphi\Big(\sum_{j=1}^n \mathbb{1}_{E_j} x_j\Big)\Big|; E_j \in \mathcal{F} \text{ partition of } \Omega, x_j \in X, \Big\|\sum_{j=1}^n \mathbb{1}_{E_j} x_j\Big\|_{\sigma,p} \leq 1\Big\} \\
& \quad = \sup\Big\{\int_\Omega \Big|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big| \frac{d|\mu|}{dP} \, dP; \\
& \qquad E_j \in \mathcal{F} \text{ partition of } \Omega, \alpha_j \in \mathbb{K}, \Big\|\sum_{j=1}^n \alpha_j \mathbb{1}_{E_j}\Big\|_{\sigma,p} \leq 1\Big\} \\
& \quad = \Big|\frac{d|\mu|}{dP}\Big|^*_{\sigma,p} < \infty,
\end{aligned}$$

which shows $\varphi \in \mathcal{L}_{\sigma,p}(\mathcal{S}(X))$. □

Definition 40. Let X be a Banach space, let σ be a distortion function, and let $p \in [1, \infty)$ with conjugate exponent q. Then we define

$$L^*_{\sigma,q}(P, X^*) := \Big\{\mu : \mathcal{F} \to X^*; \mu \text{ is a } \sigma\text{-additive vector measure of bounded variation such that } |\mu| \ll P \text{ and } \frac{d|\mu|}{dP} \in L^*_{\sigma,q}(P)\Big\},$$

which is obviously a subspace of the space of all X^*-valued vector measures on $\mathcal{F}$. Moreover, for $\mu \in L^*_{\sigma,q}(P, X^*)$ we set $|\mu|^*_{\sigma,q} := |\frac{d|\mu|}{dP}|^*_{\sigma,q}$. Then, $|\cdot|^*_{\sigma,q}$ is obviously a norm on $L^*_{\sigma,q}(P, X^*)$.

Remark 41. For $\mu \in L^*_{\sigma,q}(P, X^*)$, it follows from Lemma 39 and the density of $\mathcal{S}(X)$ in $L^p_\sigma(P, X)$ that $\Phi^{-1}(\mu)$ can be extended in a unique way to a continuous linear functional on $L^p_\sigma(P, X)$, which we again denote by $\Phi^{-1}(\mu)$. For $Y \in L^p_\sigma(P, X)$, we also write for obvious reasons

$$\int_\Omega Y \, d\mu := \Phi^{-1}(\mu)(Y).$$

With this notation, the following theorem is an immediate consequence of Lemma 39, Proposition 22, and Lemma 32.

Theorem 42. *Let X be a Banach space, let σ be a distortion function, and let $p \in [1, \infty)$ with conjugate exponent q. Then $(L^*_{\sigma,q}(P, X^*), |\cdot|^*_{\sigma,q})$ is a Banach space and the mapping*

$$\Psi : \big(L^*_{\sigma,q}(P, X^*), |\cdot|^*_{\sigma,q}\big) \mapsto \big(L^p_\sigma(P, X)^*, \|\cdot\|^*_{\sigma,p}\big), \quad \mu \mapsto \Big(Y \mapsto \int_\Omega Y \, d\mu\Big)$$

is an isometric isomorphism.

Definition 43. For a Banach space X, $p \in [1, \infty)$ with conjugate exponent q, we define

$$L^{q*}_\sigma(P, X) := \big\{Z : \Omega \to X; Z \text{ strongly measurable}, \|Z\| \in L^*_{\sigma,q}(P)\big\},$$

and for $Z \in L^{q*}_\sigma(P, X)$, we set $|Z|^{q,*}_\sigma := |\|Z\||^*_{\sigma,q}$, where as usual we identify random variables which coincide P-almost everywhere. It follows easily that $L^{q*}_\sigma(P, X)$ is a vector space and that $|\cdot|^{q,*}_\sigma$ is a norm.

Remark 44. For $Z \in L^{q*}_\sigma(P, X^*)$, it follows from $\|Z\| \in L^*_{\sigma,q}(P) \subseteq L^1(P)$ that

$$\mu_Z : \mathcal{F} \to X^*, \quad \mu_Z(E) := \int_E Z \, dP$$

is a well-defined, σ-additive vector measure of bounded variation with $|\mu_Z|(E) = \int_E \|Z\| \, dP$ (see, e.g., [11, Theorem II.2.4]). A straightforward calculation gives for $Y \in \mathcal{S}(X)$

$$\int_\Omega Y \, d\mu_Z = \int_\Omega \big\langle Z(\omega), Y(\omega)\big\rangle \, dP(\omega).$$

Moreover, for $Z \in L_\sigma^{q*}(P, X^*)$ and $Y \in L_\sigma^p(P, X)$, it follows from $\|Z\| \in L_{\sigma,q}^*(P)$ and $\|Y\| \in L_\sigma^p(P)$ that

$$\int_\Omega \left|\langle Z(\omega), Y(\omega)\rangle\right| dP(\omega) \le \int_\Omega \|Z(\omega)\|\|Y(\omega)\| \, dP(\omega) \le \left|\|Z\|\right|_{\sigma,q}^* \left|\|Y\|\right|_{\sigma,p} = |Z|_\sigma^{q*}\|Y\|_{\sigma,p},$$

which implies that $\psi_Z : L_\sigma^p(P, X) \to \mathbb{K}, Y \mapsto \mathbb{E}(\langle Z, Y\rangle)$ is a well-defined continuous linear functional which coincides on the dense subspace $\mathcal{S}(X)$ with $\Psi(\mu_Z)$. Together with Theorem 42, this shows that

$$\iota : \left(L_\sigma^{q*}(P, X^*), |\cdot|_\sigma^{q,*}\right) \to L_\sigma^p(P, X)^*, \quad Z \mapsto \left(Y \mapsto \mathbb{E}\left(\langle Z, Y\rangle\right)\right)$$

is an isometry.

As in the case of Bochner–Lebesgue spaces, we have the following result.

Theorem 45. *For a Banach space X, a distortion function σ, and $p \in [1, \infty)$ with conjugate exponent q, the isometry*

$$\iota : \left(L_\sigma^{q*}(P, X^*), |\cdot|_\sigma^{q,*}\right) \to L_\sigma^p(P, X)^*, \quad Z \mapsto \left(Y \mapsto \mathbb{E}\left(\langle Z, Y\rangle\right)\right)$$

is an isomorphism if and only if X^* has the Radon–Nikodým property with respect to $(\Omega, \mathcal{F}, P)$.

Proof. Assume first that X^* has the Radon–Nikodým property with respect to $(\Omega, \mathcal{F}, P)$. By Remark 44 we only have to show surjectivity of ι. For an arbitrary $\varphi \in L_\sigma^p(P, X)^*$, there is by Theorem 42 a σ-additive X^*-valued vector measure of bounded variation such that $|\mu| \ll P$ and $\frac{d|\mu|}{dP} \in L_{\sigma,q}^*(P)$ with $\|\varphi\| = |\frac{d|\mu|}{dP}|_{\sigma,q}^*$. By the Radon–Nikodým property of X^*, it follows that there is $Z \in L^1(P, X^*)$ such that $\mu(E) = \int_E Z \, dP$ for all $E \in \mathcal{F}$. Since $|\mu|(E) = \int_E \|Z\| \, dP$ (see, e.g., [11, Theorem II.2.4]), it follows that $Z \in L_\sigma^{q*}(P, X^*)$ and $\iota(Z) = \mu$, showing the surjectivity of ι.

Now, let ι be an isometric isomorphism. The proof that X^* has the Radon–Nikodým property is along the same lines as the proof of the corresponding implication of [11, Theorem IV.1.1]. However, we include the proof for the reader's convenience. So, let $\mu : \mathcal{F} \to X^*$ be a P-continuous vector measure of bounded variation, and fix $E_0 \in \mathcal{F}$ such that $P(E_0) > 0$. By the Hahn decomposition theorem applied to the signed measure $kP - |\mu|$ for large enough $k > 0$ gives the existence of $B \in \mathcal{F}, B \subseteq E_0, P(B) > 0$ such that $|\mu|(E) \le kP(E)$ for all $E \in \mathcal{F}, E \subseteq B$. For $Y \in \mathcal{S}(X), Y = \sum_{j=1}^n \mathbb{1}_{E_j} x_j$ with pairwise disjoint $E_j \in \mathcal{F}$ and $x_j \in X$, we define

$$\varphi(Y) = \sum_{j=1}^n \mu(E_j \cap B)(x_j).$$

Denoting the norm in $L^1(P, X)$ as usual by $\|\cdot\|_1$, Theorem 4 then gives

$$\left|\varphi(Y)\right| \le \sum_{j=1}^n k\left\|\mu(E_j \cap B)(x_j)\right\| \le k\|Y\|_1 \le k\|Y\|_{\sigma,p}$$

so that the obviously linear mapping φ on $\mathcal{S}(X)$ is continuous with respect to $\|\cdot\|_{\sigma,p}$. By Proposition 9, φ extends (in a unique way) to an element of $L^p_\sigma(P,X)^*$ which we still denote by φ. Since ι is supposed to be surjective, there is $Z \in L^{q*}_\sigma(P,X^*) \subseteq L^1(P,X^*)$ such that

$$\forall Y \in L^p_\sigma(P,X): \quad \varphi(Y) = \mathbb{E}\big(\langle Z, Y\rangle\big).$$

Since $\mu(E\cap B)(x) = \varphi(\mathbb{1}_E x) = \int_E \langle Z(\omega), x\rangle\, dP(\omega) = \langle \int_E Z(\omega)\, dP(\omega), x\rangle$ for all $E \in \mathcal{F}, x \in X$, it follows that $\mu(E\cap B) = \int_E Z\, dP$.

Because $E_0 \in \mathcal{F}$ with $P(E_0) > 0$ was chosen arbitrarily, it follows from [11, Corollary III.2.5] that there is $Z \in L^1(P,X^*)$ such that $\mu(E) = \int_E Z\, dP$ for all $E \in \mathcal{F}$, which proves the Radon–Nikodým property of X^* with respect to $(\Omega,\mathcal{F},P)$. □

6. Summary

This article introduces Banach spaces, which naturally carry risk measures for vector-valued returns. Risk measures are continuous on these spaces, and the spaces are as large as possible. The spaces are built based on duality, and in this sense are natural for risk measures involving vector-valued returns. We provide a complete characterization of the topological dual, which essentially simplifies if the dual of the state space enjoys the Radon–Nikodým property.

It is a key property of these spaces that the corresponding risk functional is continuous (in fact, Lipschitz-continuous) with respect to any of the associated norms introduced, such that they all qualify as a domain space for the risk measure.

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