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POINT MULTIPLIERS AND THE GLEASON–KAHANE–ŻELAZKO THEOREM

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Abstract. Let A be a Banach algebra, and let $\mathcal{X}$ be a left Banach A-module. In this paper, using the notation of point multipliers on left Banach modules, we introduce a certain type of spectrum for the elements of $\mathcal{X}$ and we also introduce a certain subset of $\mathcal{X}$ which behaves as the set of invertible elements of a commutative unital Banach algebra. Among other things, we use these sets to give some Gleason–Kahane–Żelazko theorems for left Banach A-modules.

1. Introduction

By the classical Gleason–Kahane–Żelazko theorem, a linear functional Λ on a complex unital Banach algebra A, with unit element 1_A satisfying $\Lambda(1_A) = 1$ and $\Lambda(x) \neq 0$ for all invertible elements $x \in A$, is a character on A. There are several generalizations of this theorem, the most popular of which concerns the problem of characterizing spectrum-preserving maps. This problem has been studied by many authors; for surveys of many of these results and for some other generalizations, see, for example, [8], [14], [16], [23], [6], and [13]. Introducing different types of spectra, some authors study related problems for newly defined spectrums (see, e.g., example, [18] and the references therein).

For a compact Hausdorff space K and a Banach space E, let $C(K,E)$ be the space of E-valued continuous functions on K, which is obviously a Banach $C(K)$-module under the supremum norm $\|\cdot\|_K$. The problem of nonvanishing-

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preserving maps on $C(K, E)$ (maps preserving nonvanishing functions in both directions), which was studied in [21], is related. Such maps were also considered on the topological algebra $C(X)$ for a completely regular Hausdorff space X, rather than on Banach algebras (see [20]).

Common zero-preserving maps between spaces of functions send (in both directions) every pair of functions having common zeros to the functions with the same property. Since common zero-preserving maps between subspaces of continuous vector-valued functions may be compared with maps preserving joint spectrums in a commutative unital Banach algebra case, then the results concerning such maps can also be considered as generalizations of the Gleason–Kakane–Żelazko theorem. For some recent results on this topic, see [9], [12], and [19].

The recent work of Mashreghi and Ransford [22] is the only known reference in which the authors consider a similar problem in a module case. In [22], a generalization of the Gleason–Kahane–Żelazko theorem on modules with application to linear functionals on Hardy spaces has been obtained.

In this paper, we consider Banach (left) modules rather than Banach algebras, and we introduce zero sets and spectrum-like sets for the element of Banach modules. We also introduce a subset which behaves as the set of invertible elements of commutative unital Banach algebras, and we give some Gleason–Kahane–Żelazko-type theorems for Banach left modules. Our approach is based on the notion of point multipliers on Banach modules and their properties, which is given in [7].

2. Preliminaries

Let A be a Banach algebra with nonempty character space $\sigma(A)$, and let $\mathcal{X}$ be a left Banach A-module. For each $\varphi \in \sigma(A) \cup \{0\}$, a linear functional $\xi \in \mathcal{X}^*$ is called a *point multiplier* at φ if $\langle \xi, a \cdot x \rangle = \varphi(a)\langle \xi, x \rangle$ for all $a \in A$, $x \in \mathcal{X}$. The notation for right Banach A-modules is defined similarly. We denote the set of all nontrivial point multipliers on $\mathcal{X}$ in the unit ball of $\mathcal{X}^*$ by $\sigma_A(\mathcal{X})$. For some properties of point multipliers and other related notions which are used in this paper, we refer the reader to [7].

A submodule of $\mathcal{X}$ of codimension 1 is called a *hyper maximal submodule* of $\mathcal{X}$. Clearly the kernel of each nontrivial point multiplier on $\mathcal{X}$ is a hyper maximal submodule of $\mathcal{X}$, and it is easy to see that each closed hyper maximal submodule of $\mathcal{X}$ is the kernel of some point multiplier on $\mathcal{X}$ (see [7, p. 312]). The natural defined map $\nu_A^{\mathcal{X}} : \sigma_A(\mathcal{X}) \longrightarrow \sigma(A) \cup \{0\}$ associates to each point multiplier ξ on $\mathcal{X}$, with the unique point $\varphi \in \sigma(A) \cup \{0\}$ satisfying $\langle \xi, a \cdot x \rangle = \varphi(a)\langle \xi, x \rangle$ for all $a \in A$ and $x \in \mathcal{X}$. For a closed hyper maximal submodule P of $\mathcal{X}$ we may use the notation $\nu_A^{\mathcal{X}}(P)$ for $\nu_A^{\mathcal{X}}(\xi)$, where $\xi \in \sigma_A(\mathcal{X})$ such that $\ker(\xi) = P$. For the sake of simplicity we may also use ν_A instead of $\nu_A^{\mathcal{X}}$.

Clearly if $\xi \in \sigma_A(\mathcal{X})$, then $\nu_A^{\mathcal{X}}(\xi) = \nu_A^{\mathcal{X}}(\lambda\xi)$ for all complex scalars λ with $|\lambda| \leq 1$; that is, $\nu_A^{\mathcal{X}}$ is not injective. It should also be noted that the map $\nu_A^{\mathcal{X}}$ is not necessarily surjective (see [7, Proposition 3.6]). Consider the following equivalence relation on $\sigma_A(\mathcal{X})$:

$$\xi \sim \eta \quad \text{iff } \nu_A^{\mathcal{X}}(\xi) = \nu_A^{\mathcal{X}}(\eta).$$

Hence if $\xi \in \sigma_A(\mathcal{X})$, then $\xi \sim \lambda\xi$, where $|\lambda| \leq 1$. In general, however, there exist point multipliers in the same class with different kernels (see [7]).

We denote the set of all closed hyper maximal submodules of $\mathcal{X}$ by $\Delta_A(\mathcal{X})$, and we denote the set of elements in $\Delta_A(\mathcal{X})$ corresponding to $\varphi \in \sigma(A) \cup \{0\}$ by $\Delta_\varphi(\mathcal{X})$ which is clearly nonempty whenever φ is in the range of $\nu_A^{\mathcal{X}}$.

The *Gelfand radical* of $\mathcal{X}$ is defined by $\mathrm{Rad}_A(\mathcal{X}) = \bigcap_{P \in \Delta_A(X)} P$, and $\mathcal{X}$ is called *hyper semisimple* if $\mathrm{Rad}_A(\mathcal{X}) = \{0\}$.

Assume that A is unital. Then each nontrivial point multiplier on A, as a Banach module over itself, is of the form $\lambda\varphi$ for some nonzero scalar $\lambda \in \mathbb{C}$, and $\varphi \in \sigma(A)$. On the other hand, for each $\varphi \in \sigma(A)$, it holds that $\mathcal{X} = \ker(\varphi)$ is a Banach A-module such that each point multiplier at a point ψ distinct from φ is of the form $\lambda\psi|_{\mathcal{X}}$, $\lambda \in \mathbb{C}$, and a linear functional $\xi \in \mathcal{X}^*$ is a point multiplier at φ if and only if $\xi \in \overline{\mathcal{X}^2}^{\perp}$ (see [7, Example 2.2]).

Commutative semisimple Banach algebras A, as well as their maximal ideals, are hyper semisimple Banach A-modules.

Consider the following subset of $\Pi_{P \in \Delta_A(\mathcal{X})} \mathcal{X}/P$:

$$\underline{\mathcal{X}} = \big\{\underline{x} = (x_P + P)_{P \in \Delta_A(\mathcal{X})} : \sup_{P \in \Delta_A(\mathcal{X})} \|x_P + P\| < \infty\big\}.$$

Thus $\underline{\mathcal{X}}$ is a Banach space under the norm defined by $\|\underline{x}\| = \sup_{P \in \Delta_A(\mathcal{X})} \|x_P + P\|$, $\underline{x} = (x_P + P)_{P \in \Delta_A(\mathcal{X})} \in \underline{\mathcal{X}}$, which is actually a left Banach A-module in a natural way (see [7] for the case that A is unital). Furthermore, the map $G_{\mathcal{X}} : \mathcal{X} \longrightarrow \underline{\mathcal{X}}$, defined by $G_{\mathcal{X}}(x) = \widehat{x}$, is a norm-decreasing map which is injective if $\mathcal{X}$ is hyper semisimple. We should note that for each $x \in \mathcal{X}$, we may use the same notation $\hat{x}$ for the continuous function on the compact space $\sigma_A(\mathcal{X}) \cup \{0\}$ defined by $\hat{x}(\xi) = \langle \xi, x \rangle$, $\xi \in \sigma_A(\mathcal{X}) \cup \{0\}$. Here $\sigma_A(\mathcal{X}) \cup \{0\}$ is endowed with the relative weak-star topology.

The kernel of a nonempty subset S of $\Delta_A(\mathcal{X})$ is defined by $k_{\mathcal{X}}(S) = \bigcap_{P \in S} P$, and the hull of a submodule M of $\mathcal{X}$ is defined by $h_{\mathcal{X}}(M) = \{P \in \Delta_A(\mathcal{X}) : (M : \mathcal{X}) \subseteq (P : X)\}$, where for a submodule N of $\mathcal{X}$, $(N : \mathcal{X}) = \{a \in A : a \cdot \mathcal{X} \subseteq N\}$. The family $\{\Delta_A(\mathcal{X}) \backslash h_{\mathcal{X}}(M) : M \text{ is a submodule of } \mathcal{X}\}$ is a topology called *hull-kernel topology* on $\Delta_A(\mathcal{X})$.

We should note that when A is unital and $1_A \cdot x = x$ for all $x \in \mathcal{X}$ (that is, $\mathcal{X}$ is unital), there is no nontrivial point multiplier at zero. By [7, Proposition 3.5], the natural map $\nu_A : \Delta_A(\mathcal{X}) \longrightarrow \sigma(A) \cup \{0\}$ is continuous with respect to the hull-kernel topologies on both sides. Moreover, if $S \subseteq \Delta_A(X)$ is hull-kernel closed (open), then $\nu_A(S)$ is closed (open) in the relative hull-kernel topology on $\nu_A(\Delta_A(\mathcal{X})) \cup \{0\}$. We should note that in [7], by replacing A by its unitization A_1, since $\mathcal{X}$ is clearly a unital A_1-module, this proposition has been proved for the unital case (with the map ν_A whose values are in $\sigma(A)$); however the same is true for the nonunital case.

The notion of a *function algebra* on a compact Hausdorff space K is used for the closed subalgebras of $C(K)$ containing the constants and separating the points of K. The notion of a left *function module* was introduced in [4] and [3] as a left Banach A-module $\mathcal{X}$ such that there is a compact Hausdorff space K, a linear isometry $i : \mathcal{X} \to C(K)$, and a contractive unital homomorphism $\Theta : A \to C(K)$

such that $i(a\cdot x) = \Theta(a)i(x)$ for any $a \in A, x \in \mathcal{X}$. Indeed, a function A-module is a closed linear subspace of $C(K)$, for some compact Hausdorff space K, which is closed under multiplication by $\pi(A)$, for some contractive unital homomorphism $\pi : A \longrightarrow C(K)$.

For a Banach space $\mathcal{X}$, let $E_{\mathcal{X}}$ be the set of extreme points of the unit ball of $\mathcal{X}^*$ endowed with the relative weak-star topology, and let $j_{\mathcal{X}} : \mathcal{X} \longrightarrow C_b(E_{\mathcal{X}})$ be defined by $j_{\mathcal{X}}(x)(l) = l(x)$ for $x \in \mathcal{X}$ and $l \in E_{\mathcal{X}}$, where $C_b(E_{\mathcal{X}})$ is the space of all bounded continuous complex-valued functions on $E_{\mathcal{X}}$. By [4, Theorem 2.2], a left Banach A-module $\mathcal{X}$ is a left function A-module if and only if there exists a contractive unital homomorphism $\Theta : A \to C_b(E_{\mathcal{X}})$ such that

$$j_{\mathcal{X}}(a \cdot x) = \Theta(a)j_{\mathcal{X}}(x) \quad (a \in A, x \in \mathcal{X}).$$

For a Banach space $\mathcal{X}$, the multiplier algebra $\mathrm{Mult}(\mathcal{X})$ of $\mathcal{X}$ is defined by

$$\mathrm{Mult}(\mathcal{X}) = \big\{f \in C_b(E_{\mathcal{X}}) : f j_{\mathcal{X}}(\mathcal{X}) \subseteq j_{\mathcal{X}}(\mathcal{X})\big\},$$

which is a function algebra. By [4, Example 2.6], $\mathcal{X}$ is a $\mathrm{Mult}(\mathcal{X})$-function module and by [4, Theorem 2.2], for any unital Banach algebra A, there is a one-to-one correspondence between function A-module actions on $\mathcal{X}$ and contractive unital homomorphisms from A into $\mathrm{Mult}(\mathcal{X})$. If $\mathrm{Mult}(\mathcal{X}) = \mathbb{C}$, then, by [4, Example 2.6], any function module action on $\mathcal{X}$ arises from some $\varphi \in \sigma(A)$; that is, $a\cdot x = \varphi(a)x$ for all $a \in A$ and $x \in \mathcal{X}$. This holds, for instance, if $\mathcal{X}$ is strictly convex (see [15, Corollary 3]).

If $\mathcal{X}$ does not contain any isometric copy of c_0, then $\mathrm{Mult}(\mathcal{X})$ is finite-dimensional (see [15]). This holds, for instance, if $\mathcal{X}$ is reflexive. If n is the dimension of $\mathrm{Mult}(\mathcal{X})$, then there exist subspaces $\mathcal{X}_1, \dots, \mathcal{X}_n$ of $\mathcal{X}$ such that $\dim(\mathrm{Mult}(\mathcal{X}_j)) = 1$ for each $1 \le j \le n$, and $\mathcal{X}$ is equal to the l_∞ direct sum $\mathcal{X} = \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_n$ (see [2, Proposition 5.1]); furthermore, any function A-module action on $\mathcal{X}$ arises from n characters $\varphi_1, \dots, \varphi_n$ on A; that is, $a\cdot(x_1 + \cdots + x_n) = \varphi_1(a)x_1 + \cdots + \varphi_n(a)x_n$ for any $a \in A$, and $x_i \in \mathcal{X}_i$, $i = 1, \dots, n$ (see [4]).

3. Main results

In this section, we give some characterizations of point multipliers, we introduce the notion of zero sets for the elements of a left Banach A-module $\mathcal{X}$, and then we define an appropriate subset $\mathcal{X}_h^{-1}$ of $\mathcal{X}$ which behaves as the set of invertible elements of commutative unital Banach algebras. We also introduce some spectrum-like sets for elements of Banach modules. Then we give some Gleason–Kahane–Żelazko-type theorems for the Banach module case. Throughout this section, unless otherwise specified, A is a Banach algebra with nonempty character space, and $\mathcal{X}$ is a left Banach A-module with $\sigma_A(\mathcal{X}) \neq \emptyset$.

Theorem 3.1. *Assume that A is unital and that $\mathcal{X}$ is a unital left Banach A-module. Then we have the following.*

(i) *For a linear functional ξ on $\mathcal{X}$ (not assumed to be continuous),* $\ker(\xi)$ *is a submodule of $\mathcal{X}$ if and only if there exists $\varphi \in \sigma(A)$ such that*

$$\langle \xi, a \cdot x\rangle = \varphi(a)\langle \xi, x\rangle \quad (a \in A, x \in \mathcal{X}).$$

(ii) *A nonzero functional* $\xi \in \mathcal{X}^*$ *is a point multiplier on* $\mathcal{X}$ *if and only if*

$$\langle \xi, a \cdot x \rangle \neq 0 \quad \big(a \in A^{-1}, x \notin \ker(\xi)\big).$$

Proof. (i) In the case where $\xi = 0$ is trivial, since the "if" part is clear, we prove the "only if" part. Assume that ξ is a nonzero linear functional on $\mathcal{X}$ such that $\ker(\xi)$ is a submodule of $\mathcal{X}$. Let $s \in \mathcal{X} \setminus \ker(\xi)$. Then using the Gleason–Kahan–Żelazko theorem, the hypothesis easily implies that the functional $\varphi_s(a) = \frac{\langle \xi, a\cdot s\rangle}{\langle \xi, s\rangle}$ on A is an element of $\sigma(A)$. We can show that, for each $s_1, s_2 \in \mathcal{X} \setminus \ker(\xi)$, $\varphi_{s_1} = \varphi_{s_2}$. Given $s_1, s_2 \in \mathcal{X} \setminus \ker(\xi)$, let $\alpha \in \mathbb{C}$ such that $\langle \xi, s_1\rangle = \alpha\langle \xi, s_2\rangle$. Since $\ker(\xi)$ is a submodule of $\mathcal{X}$, then we have $a \cdot (s_1 - \alpha s_2) \in \ker(\xi)$ for each $a \in A$, and consequently

$$\varphi_{s_1}(a) = \frac{\langle \xi, a \cdot s_1\rangle}{\langle \xi, s_1\rangle} = \frac{\langle \xi, a \cdot s_2\rangle}{\langle \xi, s_2\rangle} = \varphi_{s_2}(a).$$

Hence for arbitrary $s_0 \in \mathcal{X} \setminus \ker(\xi)$, the functional $\varphi = \varphi_{s_0} \in \sigma(A)$ satisfies $\langle \xi, a \cdot s\rangle = \varphi(a)\langle \xi, s\rangle$ for all $a \in A$ and $s \in \mathcal{X} \setminus \ker(\xi)$. The proof is complete, since clearly $\langle \xi, a \cdot s\rangle = 0$ for each $a \in A$, and $s \in \ker(\xi)$.

(ii) If $\xi \in \mathcal{X}^*$ is a nonzero point multiplier, and since the corresponding complex homomorphism on A is obviously nonzero, then we see that $\langle \xi, a \cdot x\rangle \neq 0$ holds for all $a \in A^{-1}$ and $x \in \mathcal{X} \setminus \ker(\xi))$. Assume now that $\xi \in \mathcal{X}^*$ satisfies this property, and fix an element $x_0 \in \mathcal{X}$ with $\langle \xi, x_0\rangle \neq 0$. Then, using the Gleason–Kahane–Żelazko theorem, the hypothesis again implies that the functional $\varphi_{x_0}(a) = \frac{\langle \xi, a\cdot x_0\rangle}{\langle \xi, x_0\rangle}$ on A is an element of $\sigma(A)$. Thus

$$\langle \xi, ab \cdot x_0\rangle\langle \xi, x_0\rangle = \langle \xi, a \cdot x_0\rangle\langle \xi, b \cdot x_0\rangle$$

for all $a, b \in A$. Now for a point $x \in \ker(\xi)$ and $n \in \mathbb{N}$, set $x_n = x - \frac{x_0}{n}$, then $\{x_n\}$ is a sequence in $\mathcal{X} \setminus \ker(\xi)$ converging to x. By the above argument

$$\langle \xi, ab \cdot x_n\rangle\langle \xi, x_n\rangle = \langle \xi, a \cdot x_n\rangle\langle \xi, b \cdot x_n\rangle$$

for all $n \in \mathbb{N}$, $a, b \in A$. Thus

$$0 = \langle \xi, ab \cdot x\rangle\langle \xi, x\rangle = \langle \xi, a \cdot x\rangle\langle \xi, b \cdot x\rangle$$

for all $a, b \in A$ which, in particular, implies that $\langle \xi, a \cdot x\rangle = 0$ for all $x \in \ker(\xi)$ and $a \in A$. Therefore, $\ker(\xi)$ is an A-submodule of $\mathcal{X}$, and so, by (i), ξ is a point multiplier on $\mathcal{X}$. □

Corollary 3.2. *Under the assumption of the above theorem, if* $\mathcal{X}$ *is generated as a Banach* A*-module by an element* x_0*, then a nonzero linear functional* $\xi \in \mathcal{X}^*$ *is a point multiplier on* $\mathcal{X}$ *if and only if* $\langle \xi, a \cdot x_0\rangle \neq 0$ *for all* $a \in A^{-1}$.

Proof. Assume that $\xi \in \mathcal{X}^*$ satisfies the stated property. Then, as in the proof of the above theorem,

$$\langle \xi, ab \cdot x_0\rangle\langle \xi, x_0\rangle = \langle \xi, a \cdot x_0\rangle\langle \xi, b \cdot x_0\rangle \quad (a, b \in A).$$

Since by hypothesis $\{b \cdot x_0 : b \in A\}$ is dense in $\mathcal{X}$, it follows that $\langle \xi, a \cdot x\rangle = \varphi(a)\langle \xi, x\rangle$ for all $a \in A$ and $x \in \mathcal{X}$, where $\varphi \in \sigma(A)$ is defined by $\varphi(a) = \frac{\langle \xi, a\cdot x_0\rangle}{\langle \xi, x_0\rangle}$. □

Definition 3.3. For an element $x \in \mathcal{X}$, we set

$$\mathcal{Z}(x) = \big\{\varphi \in \nu_A\big(\sigma_A(\mathcal{X})\big)\backslash\{0\} : \langle \xi, x\rangle = 0 \text{ for all } \xi \in {\nu_A}^{-1}(\varphi)\big\},$$

and we call it the *zero set* of x. We also define

$$\mathcal{X}_h^{-1} = \big\{x \in \mathcal{X} : \mathcal{Z}(x) = \emptyset\big\},$$
$$\mathcal{X}_h^{-2} = \big\{(x,y) \in \mathcal{X} \times \mathcal{X} : \mathcal{Z}(x) \cap \mathcal{Z}(y) = \emptyset\big\}.$$

The above notation is defined similarly for right Banach A-modules.

We should note that, in general, the range of $\nu_A^{\mathcal{X}}$ is contained in $\sigma(A) \cup \{0\}$. If, however, $\overline{A \cdot \mathcal{X}} = \mathcal{X}$ (e.g., if $1 \cdot x = x$, for all $x \in \mathcal{X}$, when A is unital) then $0 \notin \nu_A^{\mathcal{X}}(\sigma_A(\mathcal{X}))$.

The definition of $\mathcal{Z}(x)$ shows that $x \in P$ for all $P \in \nu_A^{-1}(\mathcal{Z}(x))$. It is easy to see that, if A is commutative and unital, and if $\mathcal{X} = A$, then the zero set $\mathcal{Z}(a)$ of each $a \in A$ is the usual zero set $Z(\hat{a})$ of its Gelfand transformation, and $\mathcal{X}_h^{-1}$ is the same as the set A^{-1} of invertible elements of A. It is also easy to see that $\mathcal{Z}(a \cdot x) = \mathcal{Z}(a) \cup \mathcal{Z}(x)$ for all $a \in A$ and $x \in \mathcal{X}$ and that, in the unital case, $a \cdot x \in \mathcal{X}_h^{-1}$ for all $a \in A^{-1}$ and $x \in \mathcal{X}_h^{-1}$.

We recall that for a commutative unital Banach algebra A and for $n \in \mathbb{N}$, the joint spectrum $\sigma_A(x_1, \ldots, x_n)$ of $x_1, \ldots, x_n \in A$ is the set $\{\varphi(x_1), \ldots, \varphi(x_n) : \varphi \in \sigma(A)\}$. Hence, considering A as a Banach module over itself, for $\lambda_1, \lambda_2 \in \mathbb{C}$ and $x_1, x_2 \in A$ we have $(\lambda_1, \lambda_2) \notin \sigma_A(x_1, x_2)$ if and only if $\mathcal{Z}(\lambda_1 - x_1) \cap \mathcal{Z}(\lambda_2 - x_2) = \emptyset$; that is, $(\lambda_1 - x_1, \lambda_2 - x_2) \in A_h^{-2}$.

Here are some other simple examples.

Example 3.4. (i) For a Banach space E and for $\varphi \in \sigma(A)$, let E_φ be the Banach A-module E with the module action induced by φ. Then each element in E^* is a point multiplier on E_φ at φ. Hence in this case $\sigma_A(E_\varphi)$ is indeed the set of nonzero functionals in the unit ball of E^*, and the range of $\nu_A^{E_\varphi}$ is the singleton $\{\varphi\}$. Hence $(E_\varphi)_h^{-1} = E\backslash\{0\}$.

(ii) For $\mathcal{X} = Af_0$, where A is a function algebra on a compact Hausdorff space K, and f_0 is an invertible function in $C(K)$, since an element $\xi \in \mathcal{X}^*$ is a point multiplier on $\mathcal{X}$ if and only if there exists a character $\varphi \in \sigma(A)$ and scalar λ such that $\langle \xi, ff_0\rangle = \lambda\varphi(f)$ for all $f \in A$, we have $\mathcal{X}_h^{-1} = \{ff_0 : f \in A^{-1}\}$.

(iii) Let A be a commutative unital Banach algebra, and let I be a dense proper ideal of A which is a Banach algebra under some norm $\|\cdot\|'$ satisfying $\|ax\|' \leq \|a\|\|x\|'$ for all $a \in A$ and $x \in I$. Since each point multiplier on I is of the form $\alpha\varphi|_I$ for some $\varphi \in \sigma(A)$ and $\alpha \in \mathbb{C}$ (see [7]), it follows that $\mathcal{Z}(x) = Z(\hat{x})$ for $x \in I$ and that $I_h^{-1} = A^{-1} \cap I = \emptyset$.

(iv) Assume that $\mathcal{X} = A^*$, where A is amenable. Then, by [5, Proposition 4 in Section 43], for each $\varphi \in \sigma(A)$ there exists a point multiplier F on $\mathcal{X}^*$ at φ such that $F(\varphi) = 1$; hence $\sigma(A) \subseteq \mathcal{X}_h^{-1}$.

Example 3.5. Let B be a commutative regular unital Banach algebra, let F be a closed subset of $\sigma(B)$, and set $A = \{b \in B : \hat{b} = 0 \text{ on } F\}$. Then $\sigma(A) = \sigma(B)\backslash F$. Considering B as a Banach A-module, since B is unital, for each point multiplier ξ on B there exists scalar $\alpha \in \mathbb{C}$ and $\varphi \in \sigma(A)$ such that $\xi = \alpha\varphi$ on A. Clearly,

ν_A^B maps $\sigma_A(B)$ onto $(\sigma(B)\backslash F) \cup \{0\}$, and it is easy to see that, for each $b \in B$, we have $\mathcal{Z}(b) \subseteq Z(b)\backslash F$. Moreover, for each $a \in A$, we have $\mathcal{Z}(a) = Z(a)\backslash F$. Hence each element $a \in A$ with $Z(a) = F$ is in B_h^{-1}.

We do not know whether the defined zero set is a closed subset of $\nu_A(\sigma_A(\mathcal{X}))\backslash \{0\}$ or not; however, this holds in Banach multiplication A-modules whenever A is regular, as the next proposition shows. We recall that a left Banach A-module $\mathcal{X}$ over a commutative Banach algebra A is called a *Banach multiplication module* if for any closed submodule N of $\mathcal{X}$ there exists an ideal I of A such that $N = \overline{I \cdot \mathcal{X}}$, where $I \cdot \mathcal{X}$ is the linear span of $\{a \cdot x : a \in A, x \in \mathcal{X}\}$.

Clearly, every commutative Banach algebra with approximate identity is a Banach multiplication module over itself. Since each Segal algebra $S(G)$ on a locally compact abelian group G is dense in $L_1(G)$ and since for each $f \in S(G)$ and $\epsilon > 0$ there exists $g \in L_1(G)$ satisfying $\|g * f - f\| \le \epsilon$ (see [10]), then it follows that, for each closed $L_1(G)$-submodule M of $S(G)$, which is indeed an ideal of $L_1(G)$, $S(G) * M$ is dense in M; that is, every Segal algebra is a Banach multiplication $L_1(G)$-module.

Proposition 3.6. *Let A be commutative and unital, and let $\mathcal{X}$ be a Banach multiplication A-module. Then the zero set of each element of $\mathcal{X}$ is hull-kernel closed in $\nu_A(\sigma_A(\mathcal{X}))\backslash\{0\}$. In particular, if A is regular, then the zero sets are closed in $\nu_A(\sigma_A(\mathcal{X}))\backslash\{0\}$.*

Proof. Consider the natural map $\nu_A : \Delta_A(\mathcal{X}) \longrightarrow \sigma(A) \cup \{0\}$. As we mentioned before, ν_A sends hull-kernel closed (open) subsets of $\Delta_A(\mathcal{X})$ to hull-kernel closed (open) subsets of $\nu_A(\Delta_A(\mathcal{X}))$. We first note that, for each $x \in \mathcal{X}$, the complement of $\mathcal{Z}(x)$ in $\nu_A(\sigma_A(\mathcal{X}))\backslash\{0\}$ is just $\nu_A(\{P \in \Delta_A(\mathcal{X}) : x \notin P\})\backslash\{0\}$. Hence it suffices to show that the subset $\{P \in \Delta_A(\mathcal{X}) : x \notin P\}$ of $\Delta_A(\mathcal{X})$ is hull-kernel open in $\Delta_A(\mathcal{X})$ or equivalently that $E = \{P \in \Delta_A(\mathcal{X}) : x \in P\}$ is hull-kernel closed; that is, $h_{\mathcal{X}}k_{\mathcal{X}}(E) = E$. For this, assume that $P \in h_{\mathcal{X}}k_{\mathcal{X}}(E)$. Then $(k_{\mathcal{X}}(E) : \mathcal{X}) \subseteq (P : \mathcal{X})$, and, since $k_{\mathcal{X}}(E)$ is a closed submodule of $\mathcal{X}$, it follows from the assumption that there exists an ideal I in A such that $k_{\mathcal{X}}(E) = \overline{I \cdot \mathcal{X}}$. Hence $I \subseteq (k_{\mathcal{X}}(E) : \mathcal{X}) \subseteq (P : \mathcal{X})$, and consequently $I \cdot \mathcal{X} \subseteq P$; that is, $x \in k_{\mathcal{X}}(E) \subseteq P$. Thus $P \in E$, as desired. $\square$

Using the next proposition, we characterize the form of point multipliers on the Banach $C(K)$-module $C(K, E)$, where K is a compact Hausdorff space and E is a Banach space, and then we conclude that $C(K, E)_h^{-1} = \{F \in C(K, E) : F(x) \neq 0 \text{ for all } x \in K\}$.

For a Banach space E, let $A \otimes E$ be the algebraic tensor product of A and E, and, for a module norm $\|\cdot\|_\gamma$ on it, let $A \otimes_\gamma E$ be its completion. Then $A \otimes_\gamma E$ has a natural A-module structure making it a left Banach A-module.

Proposition 3.7. *Let A be unital with unit element 1_A, let E be a Banach space, and let $\|\cdot\|_\gamma$ be a cross module norm on $A \otimes E$ (that is, $\|a \otimes e\|_\gamma = \|a\|\|e\|$ for all $a \in A$ and $e \in E$) such that $\|\cdot\|_\gamma \ge \|\cdot\|_\varepsilon$, for the injective tensor norm $\|\cdot\|_\varepsilon$. Then $\sigma_A(A \otimes_\gamma E)$ is homeomorphic to $\sigma(A) \times (E_1^*\backslash\{0\})$, where E_1^* is the unit ball of E^* endowed with the relative weak-star topology.*

Proof. Given $\varphi \in \sigma(A)$ and $\Lambda \in E_1^*\backslash\{0\}$, by the universal property of tensor product, there exists a unique linear map $\omega : A \otimes E \to \mathbb{C}$ such that $\omega(a \otimes e) = \varphi(a)\Lambda(e)$ for each $a \in A$ and $e \in E$. By the definition of injective tensor norm, since $\|\cdot\|_\gamma \geq \|\cdot\|_\varepsilon$, it follows that for any $a_1, \ldots, a_n \in A$ and $e_1, \ldots, e_n \in E$, we have

$$\Big|\omega\Big(\sum_{j=1}^n a_j \otimes e_j\Big)\Big| = \Big|\sum_{j=1}^n \varphi(a_j)\Lambda(e_j)\Big| \leq \Big\|\sum_{j=1}^n a_j \otimes e_j\Big\|_\varepsilon \leq \Big\|\sum_{j=1}^n a_j \otimes e_j\Big\|_\gamma,$$

which shows that ω is continuous with respect to $\|\cdot\|_\gamma$. Clearly, $\omega(b \cdot (a \otimes e)) = \varphi(b)\omega(a \otimes e)$ holds for all $a, b \in A$ and $e \in E$. Hence, using the above inequality, we can extend ω to an element of $\sigma_A(A \otimes_\gamma E)$ denoted by $\varphi \otimes_\gamma \Lambda$. Now let the map $\Gamma : \sigma(A) \times (E_1^*\backslash\{0\}) \longrightarrow \sigma_A(A \otimes_\gamma E)$ be defined by $\Gamma(\varphi, \Lambda) = \varphi \otimes_\gamma \Lambda$. Then Γ is injective. Indeed, if $\varphi_1, \varphi_2 \in \sigma(A)$, and $\Lambda_1, \Lambda_2 \in E_1^*\backslash\{0\}$ such that $\varphi_1 \otimes_\gamma \Lambda_1 = \varphi_2 \otimes_\gamma \Lambda_2$, then fixing $e \in E$ with $\Lambda_1(e) = 1$ we have

$$\varphi_1(x) = \varphi_1(x)\Lambda_1(e) = (\varphi_1 \otimes_\gamma \Lambda_1)(x \otimes e) = (\varphi_2 \otimes_\gamma \Lambda_2)(x \otimes e) = \varphi_2(x)\Lambda_2(e)$$

for all $x \in A$. Thus $\ker(\varphi_2) \subseteq \ker(\varphi_1)$, that is, $\varphi_1 = \varphi_2$, and consequently $\Lambda_1 = \Lambda_2$. To prove that Γ is surjective, assume that $\rho \in \sigma_A(A \otimes_\gamma E)$ is a point multiplier on $A \otimes_\gamma E$ at some $\varphi \in \sigma(A)$. Then considering $\Lambda \in E^*$ defined by $\Lambda(y) = \rho(1_A \otimes y)$, $y \in E$, it is easy to see that $\Lambda \in E_1^*\backslash\{0\}$ and that $\rho(x \otimes y) = \varphi(x)\Lambda(y)$ for all $x \in A$ and $y \in E$; that is, $\rho = \Gamma(\varphi, \Lambda)$.

The continuity of Γ (and its inverse) is easily verified. □

For a compact Hausdorff space K and a Banach space E, by [11, Lemma 1] the Banach space $C(K, E)$ of all continuous E-valued functions on K, which is clearly a Banach $C(K)$-module, is indeed the closed linear span of $\{fe : f \in C(K), e \in E\}$. We should note that in [11, Lemma 1] this result has been proven for the case in which E is a Banach algebra, while the algebraic structure of E has no role in the proof. Hence, by using either the same argument as in the above proposition or the same proof of [17, Proposition 1.5.6] to show that the supremum norm on $C(K, E)$ is the same as the injective tensor norm, we conclude the following corollary.

Corollary 3.8. *Let K be a compact Hausdorff space, and let E be a Banach space. Then each point multiplier on $C(K, E)$ is of the form $\Lambda \circ \varphi_x$ for some $\Lambda \in E^*$ and $x \in K$, where $\varphi_x : C(K, E) \longrightarrow E$ is defined by $\varphi_x(F) = F(x)$.*

The above proposition also shows that, when $E \neq \{0\}$, the natural defined map $\nu_{C(K)}$ for $\mathcal{X} = C(K, E)$ is onto $\sigma(C(K)) = K$; thus we get the next corollary.

Corollary 3.9. *Under the above assumptions, $C(K, E)_h^{-1} = \{F \in C(K, E) : F(x) \neq 0$ for all $x \in K\}$.*

Clearly, if E is a commutative unital Banach algebra and if $C(K, E)$ is considered a Banach module over itself, then $C(K, E)_h^{-1}$ is the same as the set of invertible elements of $C(K, E)$, that is, the set $\{F \in C(K, E) : F(x) \in E^{-1}$ for all $x \in K\}$.

Remark. We should note that, in spite of the Banach algebra case, in general a point multiplier ξ on $\mathcal{X}$ need not satisfy $\langle \xi, x\rangle \neq 0$ for all $x \in \mathcal{X}_h^{-1}$. For example, in the Banach A-module E_φ, as we mentioned before, $(E_\varphi)_h^{-1} = E\backslash\{0\}$, and there exists no point multiplier on E_φ satisfying this condition unless $\dim(E) = 1$. The same is true for the Banach $C(K)$-module $C(K,E)$.

It is however clear that, for each point multiplier ξ on $\mathcal{X}$ and $x \in \mathcal{X}_h^{-1}$, there exists a $\xi' \in [\xi]$ satisfying $\langle \xi', x\rangle \neq 0$. Hence, if $\mathcal{X}$ has this additional property that for each $\varphi \in \sigma(A)$, $\Delta_\varphi(\mathcal{X})$ is either empty or a singleton (that is, $[\xi] = \{\lambda\xi : |\lambda| \leq 1\}$), then clearly each point multiplier satisfies the above-mentioned property. For an example of such Banach modules, we can refer to right Banach $C(K)$-module $\mathcal{X} = C(K)^*$ for a compact Hausdorff space K (see [7, p. 317]).

Now we introduce appropriate spectrum sets for Banach module elements, and we give a result concerning linear maps preserving such spectrums.

Definition 3.10. Given a subset $\mathcal{F}$ of $\sigma_A(\mathcal{X})$, for each $x \in \mathcal{X}$, we set $\sigma_h^{\mathcal{F}}(x) = \{\xi(x) : \xi \in \mathcal{F}\}$. We also use $\sigma_h(x)$ for $\sigma_h^{\mathcal{F}}(x)$ whenever $\mathcal{F} = \sigma_A(\mathcal{X})$, that is $\sigma_h(x) = \{\xi(x) : \xi \in \sigma_A(\mathcal{X})\}$.

Clearly, in the commutative and unital case, considering A as a Banach module over itself for $\mathcal{F} = \sigma(A)$, the set $\sigma_h^{\mathcal{F}}(x)$ is the usual spectrum of an element $x \in A$. In particular, for a compact Hausdorff space K and a commutative unital Banach algebra E, considering $C(K,E)$ as a module over itself, for each $F \in C(K,E)$, setting $\mathcal{F} = \{\varphi \circ \varphi_x : \varphi \in \sigma(E), x \in K\}$, $\sigma_h^{\mathcal{F}}(F)$ is the same as the spectrum of F in the Banach algebra $C(K,E)$. Meanwhile, the same conclusion holds if we consider $C(K,E)$ as a $C(K)$-module, for the same family $\mathcal{F}$.

By the definition of $\mathcal{X}_h^{-1}$, it is easy to see that, for an element $x \in \mathcal{X}$, we have $x \in \mathcal{X}_h^{-1}$ if and only if there exists a subset $\mathcal{F}$ of $\sigma_A(\mathcal{X})$ containing at least one point of each equivalence class such that $0 \notin \sigma_h^{\mathcal{F}}(x)$.

Given a subset $\mathcal{F}$ of $\sigma_A(\mathcal{X})$, set

$$S_{\mathcal{F}}(\mathcal{X}) = \big\{\xi \in \mathcal{X}^* : \xi(x) \in \ \mathrm{co}\big(\sigma_h^{\mathcal{F}}(x)\big) \text{ for all } x \in \mathcal{X}\big\}.$$

Then, since $S_{\mathcal{F}}(\mathcal{X})$ contains the weak-star closure of $\mathcal{F}$, one can apply a similar argument as in [1, Lemma 4.1.16] to get the next lemma.

Lemma 3.11. *If $\mathcal{F}$ is compact (with respect to the relative weak-star topology), then each extreme point of $S_{\mathcal{F}}(\mathcal{X})$ is an element of $\mathcal{F}$. In particular, each extreme point of $S_{\mathcal{F}}(\mathcal{X})$ is a point multiplier on $\mathcal{X}$.*

If $T : \mathcal{X} \longrightarrow \mathcal{Y}$ is a bijective continuous multiplier between left Banach A-modules $\mathcal{X}$ and $\mathcal{Y}$, and if $T_1 = \frac{T}{\|T\|}$, since for each point multiplier $\xi \in \sigma_A(\mathcal{Y})$ clearly $\xi \circ T_1 \in \sigma_A(\mathcal{X})$, then it follows that $\sigma_h(x) = \sigma_h(T_1(x))$ holds for all $x \in \mathcal{X}$. Meanwhile, $\sigma_h^{\mathcal{F}}(x) = \sigma_h^{\mathcal{F}'}(T_1(x))$ also holds for all $x \in \mathcal{X}$ and subsets $\mathcal{F}$ of $\sigma_A(\mathcal{X})$, where $\mathcal{F}' = \{\xi \circ T_1^{-1} : \xi \in \mathcal{F}\}$.

A similar conclusion holds whenever $T : \mathcal{X} \longrightarrow \mathcal{Y}$ is a continuous bijective linear map satisfying $T(a \cdot x) = \Theta(a)T(x)$, $a \in A$, $x \in \mathcal{X}$, where Θ is an algebra isomorphism on A. In particular, assume that K is a compact Hausdorff space, that E is a strictly convex Banach space, and that $T : C(K, E) \longrightarrow C(K, E)$ is a surjective isometry. Then it is well known that T is of the form

$$T(F)(t) = V_t\big(f\big(\phi(t)\big)\big) \quad \big(t \in K, F \in C(K, E)\big),$$

where ϕ is a homeomorphism on K and that, for each $t \in K$, V_t is a surjective isometry on E. Thus $T(fF) = \Theta(f)T(F)$ holds for all $f \in C(K)$ and $F \in C(K, E)$, where Θ is the algebra isomorphism on $C(K)$ defined by $\Theta(f) = f \circ \phi$, $f \in C(K)$. Hence, by the above argument, T preserves $\sigma_h(\cdot)$.

Motivated by the fact that each bijective spectrum-preserving map $T : A \longrightarrow B$ between commutative semisimple unital Banach algebras A and B is an algebra isomorphism and consequently that it can be stated as a weighted composition operator on the Gelfand transformation of A and B, we state the next theorem.

Theorem 3.12. *Let A be unital, let $\mathcal{X}$ and $\mathcal{Y}$ be unital left Banach A-modules and let $\mathcal{F}$ and $\mathcal{F}'$ be compact subsets of $\sigma_A(\mathcal{X})$ and $\sigma_A(\mathcal{Y})$, respectively, such that $\bigcap_{\eta\in\mathcal{F}} \ker(\eta) = \{0\}$ and $\bigcap_{\xi\in\mathcal{F}'} \ker(\xi) = \{0\}$. Suppose that $T : \mathcal{X} \longrightarrow \mathcal{Y}$ is a surjective bounded linear map satisfying $\sigma_h^{\mathcal{F}'}(Tx) = \sigma_h^{\mathcal{F}}(x)$ for all $x \in A$. Then there are subsets E_0 and F_0 of $\Delta_A(\mathcal{X})$ and $\Delta_A(\mathcal{Y})$, respectively, such that $\sigma_{\mathcal{A}}(\mathcal{X}) \subseteq co\{\eta : \ker(\eta) \in E_0\}$ and $\sigma_{\mathcal{A}}(\mathcal{Y}) \subseteq co\{\xi : \ker(\xi) \in F_0\}$, and T is a local weighted composition operator of the form*

$$\widehat{T(x)}(P) = J_P\big(\hat{x}\big(h(P)\big)\big) \quad (x \in \mathcal{X}, P \in F_0),$$

where $h : F_0 \longrightarrow E_0$ is a bijective map and $J_P : \mathcal{X}/h(p) \longrightarrow \mathcal{Y}/P$ is a bijective linear map.

Proof. We first note that T is injective. Indeed, if $x \in \mathcal{X}$ such that $Tx = 0$, then by hypothesis $\sigma_h^{\mathcal{F}}(x) = \{0\}$, and since $\bigcap_{\eta\in\mathcal{F}} \ker(\eta) = \{0\}$, it follows that $x = 0$.

We now show that $T^*(\text{ext}(S_{\mathcal{F}'}(\mathcal{Y}))) = \text{ext}(S_{\mathcal{F}}(\mathcal{X}))$, where $\text{ext}(\cdot)$ denotes the set of extreme points. Let $\xi \in \mathcal{Y}^*$ be an extreme point of $S_{\mathcal{F}'}(\mathcal{Y})$, and set $\eta = \xi \circ T$. Next, by the above lemma, $\xi \in \mathcal{F}'$, and we have $\eta(x) = \xi \circ T(x) \in \sigma_h^{\mathcal{F}'}(Tx) = \sigma_h^{\mathcal{F}}(x)$ for all $x \in \mathcal{X}$. In particular, $\eta \in S_{\mathcal{F}}(\mathcal{X})$, and it is easy to see that η is an extreme point of $S_{\mathcal{F}}(\mathcal{X})$. Hence $T^*(\text{ext}(S_{\mathcal{F}'}(\mathcal{Y}))) \subseteq \text{ext}(S_{\mathcal{F}}(\mathcal{X}))$. The other inclusion is similar.

We note that, by the above lemma, for each $\xi \in \text{ext}(S_{\mathcal{F}'}(\mathcal{Y}))$, the functional $\eta = \xi \circ T$ is indeed a point multiplier on $\mathcal{X}$. Consider the following subsets of $\Delta_A(\mathcal{X})$ and $\Delta_A(\mathcal{Y})$,

$$E_0 = \big\{\ker(\eta) : \eta \in \text{ext}\big(S_{\mathcal{F}}(\mathcal{X})\big)\big\}, \qquad F_0 = \big\{\ker(\xi) : \xi \in \text{ext}\big(S_{\mathcal{F}'}(\mathcal{Y})\big)\big\}.$$

The above argument shows that we can define a bijective map h from F_0 onto E_0. Since for each $\xi \in \text{ext}(S_{\mathcal{F}'}(\mathcal{Y}))$, $\eta = \xi \circ T$ is a point multiplier on $\mathcal{X}$, we have

$$\langle \eta, a \cdot x\rangle = \psi(a)\langle \eta, x\rangle = \psi(a)\langle \xi, Tx\rangle \quad (a \in A, x \in \mathcal{X})$$

for some $\psi \in \sigma(A)$. In particular, $\langle \eta, a \cdot x\rangle = 0 = \langle \xi, a \cdot Tx\rangle$ for all $a \in A$, and $x \in \mathcal{X}$ with $Tx \in \ker(\xi)$. Assuming that $P = \ker(\xi)$ and that $Q = \ker(\eta)$, it follows that, for each $x \in \mathcal{X}$, $Tx \in P$ implies that $a \cdot x \in Q$ for all $a \in A$. Hence the map $K_P : \mathcal{Y}/P \longrightarrow \mathcal{X}/h(P)$ defined by $K_P(Tx + P) = x + h(P)$, for $y \in \mathcal{Y}$ and $Tx = y$, is a well-defined map which is clearly linear. Since T^{-1} has the same properties as T, it is easy to see that K_P is bijective. Let J_P be the inverse of K_P. Then clearly,

$$\widehat{Tx}(P) = Tx + P = J_P\big(x + h(P)\big) = J_P\big(\hat{x}\big(h(P)\big)\big) \quad (P \in E_0, x \in \mathcal{X}). \qquad \square$$

Remark. (i) Since each J_P is indeed a linear map on $\mathbb{C}$, it can be considered a multiplication map by some nonzero scalar $\lambda_P \in \mathbb{C}$.

(ii) Clearly if, in the above theorem, $\mathcal{X}$ and $\mathcal{Y}$ are hyper semisimples and if equivalence relations on $\sigma_A(\mathcal{X})$ and $\sigma_A(\mathcal{Y})$ are trivial (that is $[\xi] = \{\lambda\xi : |\lambda| \le 1\}$, then all subsets $\mathcal{F}$ and $\mathcal{F}'$ of $\sigma_A(\mathcal{X})$ and $\sigma_A(\mathcal{Y})$ containing at least one element of each equivalence class satisfy $\bigcap_{\eta\in\mathcal{F}} \ker(\eta) = \{0\}$ and $\bigcap_{\xi\in\mathcal{F}'} \ker(\xi) = \{0\}$.

The next proposition gives a natural characterization of a function module.

Proposition 3.13. *Let A be unital, and let $\mathcal{X}$ be a unital left Banach A-module. Then*

(i) *$\mathcal{X}$ is a (left) function A-module if and only if the map $x \to \hat{x}$ from $\mathcal{X}$ into $C(\sigma_A(\mathcal{X}) \cup \{0\})$ is a linear isometry,*

(ii) *if $\mathcal{X}$ is a left function A-module, then each extreme point of the unit ball of $\mathcal{X}^*$ is a point multiplier on $\mathcal{X}$.*

Proof. (i) Assume first that $x \mapsto \hat{x}$ is an isometry. For each $a \in A$, let $\pi(a) : \sigma_A(\mathcal{X}) \longrightarrow \mathbb{C}$ be defined by $\pi(a)(\xi) = \nu_A(\xi)(a)$, $\xi \in \sigma_A(\mathcal{X})$. Then it is easy to see that $\pi(a)$ is continuous with respect to the relative weak-star topology on $\sigma_A(\mathcal{X})$. Indeed, assume that $\{\xi_\alpha\}$ is a net in $\sigma_A(\mathcal{X})$ converging to $\xi \in \sigma_A(\mathcal{X})$, and put $\varphi_\alpha = \nu_A(\xi_\alpha)$ for each α and $\varphi = \nu_A(\xi)$. Then, as it passes through a subnet, since $\sigma(A)$ is compact, we can assume that $\varphi_\alpha \to \varphi_0$ for some $\varphi_0 \in \sigma(A)$. Thus, since for each α,

$$\langle \xi_\alpha, a \cdot x\rangle = \varphi_\alpha(a)\langle \xi_\alpha, x\rangle \quad (x \in \mathcal{X}),$$

it follows that $\langle \xi, a \cdot x\rangle = \varphi_0(a)\langle \xi, x\rangle$ for all $x \in \mathcal{X}$, which shows that $\varphi_0 = \nu_A(\xi)$. This argument easily implies that $\varphi_\alpha \to \varphi$, that is, that $\pi(a)$ is continuous. Now it is clear that the map $\pi : A \to C_b(\sigma_A(\mathcal{X}))$, $a \mapsto \pi(a)$, is a contractive unital homomorphism. Moreover, for each $a \in A$, $x \in \mathcal{X}$ and $\xi \in \sigma_A(\mathcal{X})$, we have

$$\widehat{a \cdot x}(\xi) = \langle \xi, a \cdot x\rangle = \nu_A(\xi)(a)\langle \xi, x\rangle = \pi(a)(\xi)\langle \xi, x\rangle = \pi(a)(\xi)\hat{x}(\xi) = \big(\pi(a)\hat{x}\big)(\xi),$$

which shows that $\widehat{a \cdot x} = \pi(a)\hat{x}$.

We should note that $\sigma_A(\mathcal{X})$ is locally compact (since its union with $\{0\}$ is compact); hence it is a completely regular space. If we consider K the Stone–Čech compactification of $\sigma_A(\mathcal{X})$, then the above argument shows that there exists a linear isometry $i : \mathcal{X} \longrightarrow C(K)$ and a contractive unital homomorphism $\pi : A \longrightarrow C(K)$ such that $i(a \cdot x) = \pi(a)i(x)$ holds for all $a \in A$ and $x \in \mathcal{X}$; that is, $\mathcal{X}$ is a left function A-module.

Now suppose that $\mathcal{X}$ is a closed linear subspace of $C(K)$ (for some compact Hausdorff space K) which is closed under the multiplication induced by some contractive unital homomorphism $\pi : A \to C(K)$. We note that for each $t \in K$, φ_t is indeed a point multiplier on $\mathcal{X}$ at the character $a \to \pi(a)(t)$ on A, since

$$\varphi_t(a \cdot g) = \big(\pi(a)g\big)(t) = \pi(a)(t)g(t) \quad (a \in A, g \in \mathcal{X}).$$

Hence for each $g \in \mathcal{X}$

$$\|g\|_K \geq \|\hat{g}\|_{\sigma_A(\mathcal{X})\cup\{0\}} = \sup_{\xi\in\sigma_A(\mathcal{X})\cup\{0\}} \big|\langle \xi, g\rangle\big| \geq \sup_{t\in K}\big|\hat{g}(\varphi_t)\big| = \|g\|_K;$$

that is, $\|\hat{g}\|_{\sigma_A(\mathcal{X})\cup\{0\}} = \|g\|_K$ for all $g \in \mathcal{X}$, which shows that the linear map $x \mapsto \hat{x}$ is an isometry.

(ii) This follows from (i) since, for each subspace M of $C(K)$, for some compact Hausdorff K, each extreme point of the unit ball of M^* is of the form $\alpha\varphi_t$ for some $\alpha \in \mathbb{T}$ and $t \in K$. $\square$

Remark. It should be noted that, in the above theorem, if (i) holds, then the corresponding map $x \mapsto \hat{x}$ from $\mathcal{X}$ into $\Pi_{P\in\Delta_A(\mathcal{X})}\mathcal{X}/P$ is also an isometry with respect to the defined norm on $\underline{\mathcal{X}}$. To see this, note that for each $P \in \Delta_A(\mathcal{X})$ and $\xi \in \sigma_A(\mathcal{X})$ with $\ker(\xi) = P$, we have $|\langle\xi, x\rangle| = |\langle\xi, x - y\rangle|$ for all $y \in P$, and consequently $|\langle\xi, x\rangle| \leq \inf_{y\in P}\|x + y\| = \|x + P\|$.

For a character $\varphi \in \sigma(A)$, set $\mathrm{Ch}_\varphi(\mathcal{X}) = \{x \in \mathcal{X} : a\cdot x = \varphi(a)x \text{ for all } a \in A\}$. Then clearly $\mathrm{Ch}_\varphi(\mathcal{X})$ is a closed submodule of $\mathcal{X}$, and, for distinct $\varphi, \psi \in \sigma(A)$, we have $\mathrm{Ch}_\varphi(\mathcal{X}) \cap \mathrm{Ch}_\psi(\mathcal{X}) = \{0\}$.

Theorem 3.14. *Let $\mathcal{X}$ be a left Banach A-module such that $\mathcal{X} = \sum_{i=1}^n \mathcal{X}_i$, where, for each $i = 1, \ldots, n$, $\mathcal{X}_i = \mathrm{Ch}_{\varphi_i}(\mathcal{X})$ for some distinct $\varphi_1, \ldots, \varphi_n \in \sigma(A)$. Then the set of point multipliers on $\mathcal{X}$ can be identified with the disjoint union $\mathcal{X}_1^* \cup \cdots \cup \mathcal{X}_n^*$.*

Proof. Since, $\varphi_1, \varphi_2, \ldots, \varphi_n$ are distinct, the hypothesis implies that $\mathcal{X}$ is indeed the direct sum $\bigoplus_{i=1}^n \mathcal{X}_i$ of its closed submodules $\mathcal{X}_i$, $i = 1, \ldots, n$, and that, for each $x \in \mathcal{X}$,

$$a\cdot x = \varphi_1(a)x_1 + \cdots + \varphi_n(a)x_n \quad (a \in A),$$

where for $i = 1, \ldots, n$, $x_i \in \mathcal{X}_i$ with $x = x_1 + \cdots + x_n$. Let $\pi_i : \mathcal{X} \longrightarrow \mathcal{X}_i$, $i = 1, \ldots, n$, be the projection maps. Clearly for each $i = 1, \ldots, n$ and $\eta \in \mathcal{X}_i^*$, the continuous linear functional ξ on $\mathcal{X}$ defined by $\xi = \eta \circ \pi_i$ is a point multiplier on $\mathcal{X}$ at φ_i. Assume now that ξ is a nontrivial point multiplier on $\mathcal{X}$ at a point $\varphi \in \sigma(A) \cup \{0\}$, and set $\xi_i = \xi \circ \pi_i$, $i = 1, \ldots, n$. Then since $\xi = \xi_1 + \cdots + \xi_n$, functionals $\xi_1, \ldots, \xi_n$ are not zero at the same time. We now show that, if $\xi_1 \neq 0$, then $\xi_i = 0$ for $i \neq 1$. Clearly we can choose $x \in \mathcal{X}_1$ such that $\langle\xi_1, x\rangle = 1$. Then $\langle\xi, x\rangle = \langle\xi_1, x\rangle = 1$, $\langle\xi_i, x\rangle = 0$ for $i = 2, \ldots, n$, and for any $a \in A$,

$$\varphi(a) = \varphi(a)\langle\xi, x\rangle = \langle\xi, a\cdot x\rangle = \big\langle\xi, \varphi_1(a)x\big\rangle = \varphi_1(a);$$

that is, $\varphi = \varphi_1$. Thus, for each $a \in A$ and $y \in \mathcal{X}$, we have

$$\langle\xi, a\cdot y\rangle = \varphi(a)\langle\xi, y\rangle = \varphi_1(a)\langle\xi_1, y\rangle + \cdots + \varphi_1(a)\langle\xi_n, y\rangle.$$

On the other hand,

$$\begin{aligned}\langle\xi, a\cdot y\rangle &= \big\langle\xi, \varphi_1(a)y_1 + \cdots + \varphi_n(a)y_n\big\rangle = \varphi_1(a)\langle\xi, y_1\rangle + \cdots + \varphi_n(a)\langle\xi, y_n\rangle \\ &= \varphi_1(a)\langle\xi_1, y\rangle + \cdots + \varphi_n(a)\langle\xi_n, y\rangle.\end{aligned}$$

These equalities imply that, for all $a \in A$ and $y \in \mathcal{X}$,

$$\varphi_1(a)\langle\xi_2, y\rangle + \cdots + \varphi_1(a)\langle\xi_n, y\rangle = \varphi_2(a)\langle\xi_2, y\rangle + \cdots + \varphi_n(a)\langle\xi_n, y\rangle.$$

Since distinct homomorphisms on A are linearly independent, the above equality implies that $\langle \xi_i, y\rangle = 0$ for each $i = 2, \ldots, n$, as desired. $\square$

Corollary 3.15. *If A is unital and $\mathcal{X}$ is a unital left function A-module which does not contain an isometric copy of c_0, then the set of point multipliers on $\mathcal{X}$ is a disjoint union $\mathcal{X}_1^* \cup \cdots \cup \mathcal{X}_n^*$ for some closed subspaces $\mathcal{X}_1, \ldots, \mathcal{X}_n$ of $\mathcal{X}$.*

The next theorem gives a module version of a known result concerning common zero-preserving maps for certain subspaces of vector-valued continuous functions (see [9], [12], and [19]). Before stating the theorem we introduce the notion of Z-regularity of Banach modules.

Definition 3.16. For a left (right) Banach A-module $\mathcal{X}$, we call a nonzero point $\varphi \in \nu_A(\sigma_A(\mathcal{X}))$ a *Z-regular point* if for each neighborhood U of φ in $\sigma(A)$ there exists $x \in \mathcal{X}$ such that $\varphi \in \mathcal{Z}(x) \subseteq U$. We denote the set of Z-regular points of $\mathcal{X}$ by $\Theta(\mathcal{X})$, and we say that $\mathcal{X}$ is *Z-regular* if $\Theta(\mathcal{X}) = \nu_A(\sigma_A(\mathcal{X}))\backslash\{0\}$. We also say that a nonzero point $\varphi \in \nu_A(\sigma_A(\mathcal{X}))$ is a *zero point* for $\mathcal{X}$ if there exists $x \in \mathcal{X}$ with $\mathcal{Z}(x) = \{\varphi\}$.

Clearly, all zero points are Z-regular points.

Example 3.17. (i) Clearly, any regular commutative unital Banach algebra A is a Z-regular Banach module over itself; and all zero sets are closed in $\sigma(A)$. The disk algebra $A(\mathbb{D})$, however, is a nonregular Banach algebra such that all points in the closed unit ball $\mathbb{D}$ are zero points of $A(\mathbb{D})$ as a module over itself; that is, $A(\mathbb{D})$ is Z-regular. On the other hand, since, in a function algebra A on a compact Hausdorff space K, for each point t in the Choquet boundary of A and neighborhood U of t, there exists $f \in A$ with $f(t) = 1 = \|f\|_K$ and $|f| < 1$ on $K\backslash U$, then it follows that all points in the Choquet boundary of A are Z-regular points for A as a module over itself.

(ii) For a compact Hausdorff space K and a Banach space E, $C(K, E)$ is a Z-regular Banach $C(K)$-module with closed zero sets.

(iii) Let K be a compact Hausdorff space, and let F be a closed subset of K whose complement is finite. Then clearly F is a clopen subset of K. Considering the Banach A-module $\mathcal{X} = C(K)$ over the Banach algebra $A = \{f \in C(K) : f|_F = 0\}$, it follows from Example 3.5 that the range of $\nu_A^{\mathcal{X}}$ is the same as $K\backslash F \cup \{0\}$ and that, for each $a \in A$, $\mathcal{Z}(a) = Z(a)\backslash F$. Now, for each $t \in K\backslash F$ and neighborhood U of t, assume without loss of generality that $U \cap F = \emptyset$. Then there exists $a \in C(K)$ with $a = 0$ on $F \cup \{t\}$ and $a = 1$ on $K\backslash(U \cup F)$. In particular, $a \in A$, and consequently $\mathcal{Z}(a) = Z(a)\backslash F \subseteq U$; that is, $\mathcal{X}$ is Z-regular. Clearly, the zero sets of all elements are closed in $K\backslash F$.

Now we state the theorem. For simplicity's sake, for each $x \in \mathcal{X}$ and $Q_0 \in \Delta_{\mathcal{A}}(\mathcal{X})$, we denote $(x + Q)_{Q\in[Q_0]}$ as the element $(x_Q + Q)_{Q\in\Delta_{\mathcal{A}}(\mathcal{X})}$ of $\underline{\mathcal{X}}$, where $x_Q = x$ for all $Q \in [Q_0]$, and $x_Q = 0$ for the other points $Q \in \Delta_{\mathcal{A}}(\mathcal{X})$.

Theorem 3.18. *Let A and B be Banach algebras, and let $\mathcal{X}, \mathcal{Y}$ be left Banach modules over A and B, respectively, with $\Theta(\mathcal{X}) \neq \emptyset$ and $\Theta(\mathcal{Y}) \neq \emptyset$ such that, for each $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, $\mathcal{Z}(x)$ and $\mathcal{Z}(y)$ are closed in $\nu_A(\sigma_A(\mathcal{X}))\backslash\{0\}$, and*

$\nu_B(\sigma_B(\mathcal{Y}))\backslash\{0\}$. *Then for each surjective linear map* $T:\mathcal{X}\to\mathcal{Y}$ *satisfying*

$$(x,y)\in\mathcal{X}_h^{-2} \quad \text{if and only if} \quad (Tx,Ty)\in\mathcal{Y}_h^{-2}$$

there exist subsets E_0 *and* F_0 *of* $\Delta_A(\mathcal{X})$ *and* $\Delta_B(\mathcal{Y})$, *respectively (whose images under the natural maps* ν_A *and* ν_B *contain zero points), and a bijection* $\tilde{h}$: $F_0/\sim\longrightarrow E_0/\sim$, *and, for each* $P\in F_0$ *submodules* $\mathcal{M}_P,\mathcal{N}_P$ *of* $\underline{\mathcal{X}}$ *and* $\underline{\mathcal{Y}}$, *respectively, and also linear bijections* $J_P:\mathcal{M}_P\longrightarrow\mathcal{N}_P$, *such that*

$$(Tx+P')_{P'\in[P]}=J_P\big((x+Q)_{Q\in\tilde{h}([P])}\big)\quad(x\in\mathcal{X},P\in F_0).$$

Proof. The proof is a modification of Theorem 4.5 in [12]. For each $\varphi\in\Theta(\mathcal{X})$, we set $\mathcal{I}_\varphi:=\bigcap_{\varphi\in\mathcal{Z}(x)}\mathcal{Z}(Tx)\cap\Theta(\mathcal{Y})$. We first show that $\mathcal{I}_\varphi$ has at most one element. Assume on the contrary that ψ_1,ψ_2 are distinct points in $\mathcal{I}_\varphi$. Next we choose disjoint neighborhoods V_1,V_2 of ψ_1, and ψ_2 in $\sigma(B)$. Since ψ_1,ψ_2 are Z-regular points, we can find elements $y_1,y_2\in\mathcal{Y}$ such that $\psi_1\in\mathcal{Z}(y_1)\subseteq V_1$, and $\psi_2\in\mathcal{Z}(y_2)\subseteq V_2$. In particular, $\mathcal{Z}(y_1)\cap\mathcal{Z}(y_2)=\emptyset$; that is, $(y_1,y_2)\in\mathcal{Y}_h^{-2}$. Hence $(x_1,x_2)\in\mathcal{X}_h^{-2}$, where $Tx_1=y_1$, and $Tx_2=y_2$. Given a neighborhood U of φ in $\sigma(A)$, let $x\in\mathcal{X}$ be such that $\varphi\in\mathcal{Z}(x)\subseteq U$. Then, since $\psi_2\in\mathcal{Z}(y_2)\cap\mathcal{Z}(Tx)$, it follows that $\mathcal{Z}(x_2)\cap\mathcal{Z}(x)\neq\emptyset$. Thus

$$\mathcal{Z}(x_2)\cap U\cap\big(\nu_A\big(\sigma_A(\mathcal{X})\big)\backslash\{0\}\big)=\mathcal{Z}(x_2)\cap U\neq\emptyset,$$

and, since U is arbitrary, the closedness of $\mathcal{Z}(x_2)$ in $\nu_A(\sigma_A(\mathcal{X}))\backslash\{0\}$ implies that $\varphi\in\mathcal{Z}(x_2)$. A similar discussion shows that $\varphi\in\mathcal{Z}(x_1)$, which is a contradiction.

Now we consider the subset $\sigma_0(A):=\{\varphi\in\Theta(\mathcal{X}):\mathcal{I}_\varphi\neq\emptyset\}$ of $\sigma(A)$. By the above argument, for each $\varphi\in\sigma_0(A)$ there exists a unique point $\psi\in\Theta(\mathcal{Y})$ such that $\mathcal{I}_\varphi=\{\psi\}$. This allows us to define a function $k:\sigma_0(A)\to\sigma(B)$ such that for each $\varphi\in\sigma_0(A)$, $k(\varphi)$ is the unique point in $\mathcal{I}_\varphi$; that is, $\mathcal{I}_\varphi=\{k(\varphi)\}$. Similarly, for each $\psi\in\Theta(\mathcal{Y})$, setting $\mathcal{J}_\psi:=\bigcap_{\psi\in\mathcal{Z}(Tx)}\mathcal{Z}(x)\cap\Theta(\mathcal{X})$, we can consider the subset $\sigma_0(B):=\{\psi\in\Theta(\mathcal{Y}):\mathcal{J}_\psi\neq\emptyset\}$ of $\sigma(B)$, and we can define a function $h:\sigma_0(B)\to\sigma(A)$ such that for each $\psi\in\sigma_0(B)$, $\mathcal{J}_\psi=\{h(\psi)\}$.

We note that $\sigma_0(A)$ and $\sigma_0(B)$ contain all zero points of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Indeed if $\varphi_0\in\nu_A(\sigma_A(\mathcal{X}))\backslash\{0\}$ is a zero point, then $\mathcal{Z}(x_0)=\{\varphi_0\}$ for some $x_0\in\mathcal{X}$, and, by hypothesis, $\mathcal{Z}(Tx_0)\neq\emptyset$. A similar argument shows that $\mathcal{Z}(Tx_0)$ has at most one element, and consequently that $\mathcal{Z}(Tx_0)=\{\psi_0\}$ for some zero point ψ_0. Since for each $x\in\mathcal{X}$ with $\varphi_0\in\mathcal{Z}(x)$, we have $\mathcal{Z}(x)\cap\mathcal{Z}(x_0)\neq\emptyset$, it follows that $\mathcal{Z}(Tx)\cap\mathcal{Z}(Tx_0)\neq\emptyset$; that is, $\psi_0\in\mathcal{Z}(Tx)$. This implies that $\psi_0\in\mathcal{I}_{\varphi_0}$, and so $\varphi_0\in\sigma_0(A)$, as desired. Similar reasoning may be applied to $\sigma_0(B)$.

Now we show that k is a bijective map from $\sigma_0(A)$ onto $\sigma_0(B)$ and that $h=k^{-1}$. For $\varphi\in\sigma_0(A)$, it suffices to show that $\mathcal{J}_{k(\varphi)}=\{\varphi\}$, which clearly implies that $k(\varphi)\in\sigma_0(B)$ and that $h(k(\varphi))=\varphi$. For this, suppose that $x\in\mathcal{X}$ and that $k(\varphi)\in\mathcal{Z}(Tx)$. Let U be an arbitrary neighborhood of φ in $\sigma(A)$, and choose, by the Z-regularity property, $x_0\in\mathcal{X}$ such that $\varphi\in\mathcal{Z}(x_0)\subseteq U$. Since $k(\varphi)\in\mathcal{Z}(Tx)\cap\mathcal{Z}(Tx_0)$, it follows that $\mathcal{Z}(x)\cap\mathcal{Z}(x_0)\neq\emptyset$, and so $\mathcal{Z}(x)\cap U\cap(\nu_A(\sigma_A(\mathcal{X}))\backslash\{0\})=\mathcal{Z}(x)\cap U\neq\emptyset$. Hence $\varphi\in\mathcal{Z}(x)$, by the closedness of $\mathcal{Z}(x)$. Thus $\mathcal{J}_{k(\varphi)}=\{\varphi\}$, and consequently $k(\varphi)\in\sigma_0(B)$, and $h(k(\varphi))=\varphi$.

We now set $E_0 := \{\ker(\xi) : \xi \in {\nu_A}^{-1}(\sigma_0(A))\}$, and we similarly set $F_0 = \{\ker(\eta) : \eta \in {\nu_B}^{-1}(\sigma_0(B))\}$. Clearly, E_0 contains all hyper maximal submodules of $\mathcal{X}$ whose images under ν_A are zero points, and a similar assertion holds for F_0. Consider the map $\tilde{h} : F_0/\sim \longrightarrow E_0/\sim$ defined by $\tilde{h}([P]) = [Q]$, where for each $P \in F_0$, $Q \in E_0$ is an arbitrary element in ${\nu_A}^{-1}(h(\nu_B(P)))$. Clearly, $\tilde{h}$ is well defined and bijective. Also for each $P \in F_0$ consider the following subsets of $\underline{\mathcal{X}}$ and $\underline{\mathcal{Y}}$, respectively:

$$\mathcal{M}_P = \{(x+Q)_{Q\in\tilde{h}([P])}, x \in \mathcal{X}\}, \qquad \mathcal{N}_P = \{(y+P')_{P'\in[P]}, y \in \mathcal{Y}\}.$$

Then clearly $\mathcal{M}_P$ and $\mathcal{N}_P$ are submodules of $\underline{\mathcal{X}}$ and $\underline{\mathcal{Y}}$, respectively.

For each $P \in F_0$, let $J_P : \mathcal{M}_P \longrightarrow \mathcal{N}_P$ be defined by $J_P((x+Q)_{Q\in\tilde{h}([P])}) = (Tx+P')_{P'\in[P]}$. We note that J_P is well defined, indeed, if $x - x' \in Q$ for all $Q \in \tilde{h}[P]$, then by letting $\varphi = \nu_A(Q)$, the definition of h shows that $\varphi = h(\nu_B(P))$, and so $k(\varphi) = \nu_B(P)$. Since $\varphi \in \mathcal{Z}(x - x')$, it follows that $\nu_B(P) = k(\varphi) \in \mathcal{Z}(Tx - Tx')$; thus $Tx - Tx' \in P'$ for all $P' \in [P]$, as desired. Clearly, for each $P \in F_0$, J_P is linear, and J_P is also a bijection. If we assume that $J_P((x+Q)_{Q\in\tilde{h}[P]}) = 0$ for some $P \in F_0$, then $Tx \in P'$ for all $P' \in [P]$. Hence $\nu_B(P) \in \mathcal{Z}(Tx)$, and so $h(\nu_B(P)) \in \mathcal{Z}(x)$. Hence $x \in Q$ for all $Q \in \tilde{h}[P]$; that is, $(x+Q)_{Q\in\tilde{h}[P]} = 0$ (i.e., J_P is injective). It is clear that J_P is surjective. □

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