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# SINE AND COSINE EQUATIONS ON HYPERGROUPS

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Abstract. This article deals with trigonometric functional equations on hypergroups. We describe the general continuous solution of sine and cosine addition formulas and a so-called *sine-cosine functional equation* on a locally compact hypergroup in terms of exponential functions, sine functions, and second-order generalized moment functions.

## 1. Introduction

In this paper, $\mathbb{C}$ denotes the set of complex numbers. A *hypergroup* is a locally compact Hausdorff space $K$ equipped with an involution and a convolution operation defined on the space of all bounded complex regular measures on $K$. (For the formal definition, historical background, and some basic facts about hypergroups, see [1], [4], [5], [10].) In the present article, $K$ denotes a locally compact hypergroup with identity element $e$, involution $^\vee$, and convolution $*$. In fact, the quadruple $(K, e, ^\vee, *)$ is what, for all intents and purposes, we should call a "hypergroup," but for the sake of simplicity we will reserve the term for $K$.

The idea of investigating functional equations on hypergroups relies on using a generalized translation structure defined by the convolution of measures instead of using a classical group operation. For any elements $x, y$ in $K$, we consider the point masses $\delta_x$ and $\delta_y$, which are probability measures on $K$. For a continuous


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function $h\colon K \to \mathbb{C}$, the symbol $h(x*y)$ is defined in the following way:

$$h(x*y) := \int_K h(t)\, d(\delta_x * \delta_y)(t),$$

whenever $x, y$ are in $K$. The continuity assumption guarantees the existence of the integral with respect to the compactly supported measure $\delta_x * \delta_y$. The detailed study of functional equations on hypergroups started with the [14] and [15], and a comprehensive monograph on the subject is [20]. (For further results and references on this topic, see [12], [13], [17], [21], and [22]; for similar trigonometric-type functional equations, see [2], [3], [8], [9], [16], [18].)

In the following we study the *sine functional equation*

$$f(x*y) = f(x)g(y) + f(y)g(x), \tag{1.1}$$

the *cosine functional equation*

$$g(x*y) = g(x)g(y) - f(x)f(y), \tag{1.2}$$

and the *sine-cosine functional equation*

$$f(x*y) = f(x)g(y) + f(y)g(x) + h(x)h(y) \tag{1.3}$$

on an arbitrary hypergroup $K$. In these three equations we will always assume that $f, g, h : K \to \mathbb{C}$ are continuous functions.

We note that these equations are fundamental in the theory of functional equations. Clearly, equation (1.3) is a common generalization of the sine equation (1.1) and the cosine equation (1.2). In [24] the authors solve equations (1.1) and (1.2) under the assumption that the functions $f, g$ are continuous bounded functions (see [24, Corollaries 26.1, 26.2]). In this article, we do not impose any condition on the unknown functions except continuity, which is always satisfied if we consider the discrete topology on $K$. Our method, which is completely different from that of [24], is based on Cauchy differences in the case of the sine equation, on modified Cauchy differences in the case of the cosine equation, and on the exponential matrix equation in the case of the sine-cosine equation. We note that in different particular cases of (1.3), we obtain important functional equations; for example, if in (1.3) we have $g = 1$ and $h = 0$, then $f$ is an *additive function*, that is,

$$f(x*y) = f(x) + f(y);$$

if we have $g = f$ and $h = if$, then $f$ is an *exponential*, that is,

$$f(x*y) = f(x)f(y).$$

We emphasize that in the above equations, the expressions on the left-hand sides are integrals with respect to compactly support measures. In fact, those equations are integral equations.

We will see that in the case of the sine equation (1.1), the situation is very sophisticated if $g = m$ is an exponential. In the group case, exponentials are never zero, and hence we can divide by $m$ and deduce immediately that $f$ has the form $f = a{\cdot}m$, where $a$ is additive. It turns out that on a hypergroup the solutions $f$ of (1.1) with an exponential $g = m$ produce a new basic function class, which cannot be described directly using exponentials and additive functions. This is a new feature provided by the delicate structure of hypergroups and it seems reasonable to introduce the following definition: if $K$ is a hypergroup and $m$ is an exponential on $K$, then the function $f : K \to \mathbb{C}$ will be called an *$m$-sine function*, if it satisfies

$$f(x * y) = f(x)m(y) + f(y)m(x)$$

for each $x, y$ in $K$. We call $f$ a *sine function* if it is an $m$-sine function for some exponential $m$ (see [6] and [23]). We note that, in [24], bounded functions with this property are said to be *associated with* $m$. Obviously, we have $f(e) = 0$ for every sine function $f$. Additive functions on $K$ are exactly the 1-sine functions. If $K = G$ is a group, then for a given exponential $m$ the $m$-sine functions are exactly the functions of the form $f = a \cdot m$, where $a$ is additive. But this is not the case on hypergroups. For instance, let $K$ be the polynomial hypergroup generated by the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$ (see [1]). It is known that all exponential functions on $K$ have the form $n \mapsto P_n(\lambda)$ with some complex number $\lambda$, and all additive functions have the form $n \mapsto cP_n'(0)$ with some complex number $c$. Further, if we define

$$f(n) = P_n'(\lambda), \qquad m(n) = P_n(\lambda)$$

for each $n$ in $\mathbb{N}$ with some complex number $\lambda$, then it is easy to check that the following equation holds for each $m, n$ in $\mathbb{N}$:

$$f(n * k) = f(n)m(k) + f(k)m(n);$$

that is, $f$ is an $m$-sine function. On the other hand, it is easy to see that it does not have the form $n \mapsto cP_n'(0)P_n(\lambda)$ for any complex $c$.

We will see that we have to face a similar problem in case of the functional equation (1.3) if $g = m$ is an exponential. As a motivation, we present the following result which follows from [20, Theorem 2.5].

**Theorem 1.1.** *Let $K$ be the polynomial hypergroup associated with the sequence of polynomials $(P_n)_{n\in\mathbb{N}}$, let $g = m : K \to \mathbb{C}$ be an exponential, and let $h : K \to \mathbb{C}$ be an $m$-sine function. Then the functions $f, g, h$ satisfy (1.3) if and only if*

$$f(n) = cP_n'(\lambda) + \frac{d^2}{2}P_n''(\lambda), \qquad h(n) = dP_n'(\lambda)$$

*holds for each $n$ in $\mathbb{N}$, where $\lambda, c, d$ are complex numbers with $m(n) = P_n(\lambda)$.*

In fact, the cited theorem is about the general description of generalized moment function sequences on polynomial hypergroups. This result shows that in the case of equation (1.3), the general solution includes functions which cannot be expressed in terms of exponential and additive functions, even on commutative hypergroups. We recall that the sequence $\varphi : K \to \mathbb{C}$ of continuous functions is called a *generalized moment function sequence* of order $N$ if

$$\varphi_k(x * y) = \sum_{j=0}^{k} \binom{k}{j} \varphi_j(x)\varphi_{k-j}(y)$$

holds for each $x, y$ in $K$ and for $k = 0, 1, \ldots, N$ (see [20]). In particular, in any generalized moment function sequence, $\varphi_0$ is an exponential and $\varphi_1$ is a $\varphi_0$-sine function. The function $\varphi_k$ is said to be a *generalized moment function of order* $k$ associated to the exponential $\varphi_0$. Using this terminology, we can say that if in (1.3) the function $g$ is an exponential and $h$ is a $g$-sine function, then $g, \sqrt{2}h/2, f$ form a generalized moment function sequence of order 2. No further information can be expected unless we specialize the hypergroup $K$. Consequently, it seems to be reasonable to consider generalized moment functions as basic functions on hypergroups, and we should accept those functions as fundamental as exponentials and additive functions—which are special generalized moment functions, as well. Another motivation to include these functions as basic functions is their close connection with exponential monomials on commutative hypergroups, which serve as basic building blocks of spectral synthesis.

In the rest of the current article, we will describe the general continuous solution of the functional equations (1.1), (1.2), and (1.3) in terms of exponential functions, sine functions, and second-order generalized moment functions. The methods we use can be applied in wide classes of functional equations; in particular, the method used for (1.3) can be extended for more general Levi-Civitá-type functional equations. We will discuss these problems elsewhere.

## 2. Sine functional equations on hypergroups

In this section, we describe the nonzero solutions of the sine functional equation (1.1) on hypergroups.

**Theorem 2.1.** *Let $K$ be a hypergroup, and let $f, g : K \to \mathbb{C}$ be continuous functions satisfying (1.1) for each $x, y$ in $K$. If $f, g$ are nonidentically zero, then there exists a complex number $c \neq 0$ and there are continuous exponentials $M, N : K \to \mathbb{C}$ such that we have one the following possibilities:*

(i) *$g(x) = M(x)$, and $f$ is an $M$-sine function.*

(ii)
$$f(x) = \frac{1}{2c}M(x), \qquad g(x) = \frac{1}{2}M(x)$$
*for each $x$ in $K$;*

(iii)
$$f(x) = \frac{1}{2c}\big[M(x) - N(x)\big], \qquad g(x) = \frac{1}{2}\big[M(x) + N(x)\big]$$
*for each $x$ in $K$.*

*If $f$ is zero, then $g$ is arbitrary, and if $g$ is zero, then $f$ is zero. Conversely, the functions $f, g$ given above are continuous solutions of (1.1) for every nonzero complex number $c$ and continuous complex exponentials $M, N$.*

*Proof.* Clearly, we may suppose that $f$ and $g$ are nonidentically zero. As the case (i) obviously describes a possible solution, hence we will suppose that $g$ is not an exponential. Suppose first that $g(e) \neq 1$. By substitution $y = e$ into (1.1), we get
$$f(x)\big(1 - g(e)\big) = f(e)g(x);$$

that is, $f(x) = \frac{1}{c}g(x)$ with some complex number $c \neq 0$. It follows from (1.1) that

$$2g(x * y) = 2g(x)2g(y)$$

and hence $g = \frac{1}{2}m$ and $f = \frac{1}{2c}m$, where $m$ is an exponential, which is given in (ii) with $M = m$.

Now we assume that $g(e) = 1$. By substituting $y = e$ into (1.1), we get

$$f(x) = f(x)g(e) + f(e)g(x) = f(x) + f(e)g(x),$$

which implies that $f(e)g(x) = 0$, and hence that $f(e) = 0$.

We introduce the Cauchy difference: for each $x, y$ in $K$ we define

$$F(x, y) = f(x * y) - f(x) - f(y)$$

which can be written as

$$F(x, y) = f(x)\big[g(y) - 1\big] + f(y)\big[g(x) - 1\big].$$

Obviously, $F$ satisfies

$$F(x, y) + F(x * y, z) = F(x, y * z) + F(y, z)$$

for each $x, y, z$ in $K$, by the associativity of the hypergroup operation. After substitution and simplification we get the equation

$$f(z)\big[g(x * y) - g(x)g(y)\big] = f(x)\big[g(y * z) - g(y)g(z)\big]$$

for each $x, y, z$ in $K$. We let $\Gamma(x, y) = g(x * y) - g(x)g(y)$ for each $x, y$ in $K$ the modified Cauchy difference, then we have the identity

$$f(x)\Gamma(y, z) = f(z)\Gamma(x, y)$$

for each $x, y, z$ in $K$. Putting $x$ for $z$, $y$ for $x$ and $z$ for $y$, we have

$$f(y)\Gamma(z, x) = f(x)\Gamma(y, z),$$

which yields

$$f(z)\Gamma(x, y) = f(y)\Gamma(z, x).$$

Then we can write

$$f(z)^2\Gamma(x, y) = f(z)f(y)\Gamma(z, x) = f(y)f(z)\Gamma(z, x) = f(y)f(x)\Gamma(z, z);$$

so, choosing $z_0$ with $f(z_0) \neq 0$, we obtain

$$g(x * y) - g(x)g(y) = \Gamma(x, y) = \frac{\Gamma(z_0, z_0)}{f(z_0)^2} f(x)f(y).$$

With the notation $\frac{\Gamma(z_0,z_0)}{f(z_0)^2} = -d^2$ we infer that

$$g(x * y) = g(x)g(y) - df(x)df(y),$$

or, writing $h = df$, we obtain (1.2) for $g$ and $h$. Here $d \neq 0$, as otherwise $g$ is an exponential and we have (i).

Then we multiply (1.1) by $d$ and we have the system for the pair $g, h$:

$$h(x * y) = h(x)g(y) + h(y)g(x),$$
$$g(x * y) = g(x)g(y) - h(x)h(y)$$

for each $x, y$ in $K$. For each $x$ in $K$, let

$$M(x) = g(x) + ih(x) \qquad \text{and} \qquad N(x) = g(x) - ih(x).$$

We have for each $x, y$ in $K$

$$\begin{aligned} M(x * y) &= g(x * y) + ih(x * y) = g(x)g(y) - h(x)h(y) + ig(x)h(y) + ig(y)h(x) \\ &= \big(g(x) + ih(x)\big)\big(g(y) + ih(y)\big) = M(x)M(y) \end{aligned}$$

and

$$\begin{aligned} N(x * y) &= g(x * y) - ih(x * y) = g(x)g(y) - h(x)h(y) - ig(x)h(y) - ig(y)h(x) \\ &= \big(g(x) - ih(x)\big)\big(g(y) - ih(y)\big) = N(x)N(y). \end{aligned}$$

This means that $M, N : K \to \mathbb{C}$ are exponentials. On the other hand, we have

$$g = \frac{1}{2}(M + N), \qquad h = \frac{1}{2i}(M - N).$$

It follows $f = \frac{1}{2di}(M - N)$, and we have (iii) with $c = di$.

The converse statement can be verified easily by direct computation. □

## 3. Cosine functional equations on hypergroups

In this section we describe the nonzero solutions of the cosine functional equation (1.2) on hypergroups.

**Theorem 3.1.** *Let $K$ be a hypergroup, and let $f, g : K \to \mathbb{C}$ be continuous functions satisfying (1.2) for each $x, y$ in $K$. If $f$ and $g$ are nonidentically zero, then there exist complex numbers $c \neq \pm 1$ and $d \neq 0$, and there are continuous exponentials $M, N : K \to \mathbb{C}$ such that we have one of the following possibilities:*

(i)

$$f(x) = \frac{c}{1 - c^2}M(x), \qquad g(x) = \frac{1}{1 - c^2}M(x)$$

*for each $x$ in $K$;*

(ii) *$f$ is an $M$-sine function, and $g(x) = M(x) \pm f(x)$ for each $x$ in $K$;*

(iii)

$$f(x) = \pm\frac{1}{2di}\big[M(x) - N(x)\big], \qquad g(x) = \pm\frac{\pm di - \lambda}{2di}M(x) \pm \frac{\pm di + \lambda}{2di}N(x)$$

*for each $x$ in $K$, where $\lambda^2 = 1 - d^2$, and we choose $+$ or $-$ at each place in the same way.*

*If $f$ is zero, then $g$ is an arbitrary exponential. If $g$ is zero, then $f$ is zero. Conversely, the functions $f, g$ given above are continuous solutions of equation (1.2) for any nonzero complex numbers $c, d$, $c \neq \pm 1$ and continuous complex exponentials $M, N$.*

*Proof.* Clearly, we may suppose that $f$ and $g$ are nonidentically zero. Then $g$ is not an exponential. Substituting $y = e$ in (1.2) we have $g(x)(1 - g(e)) = -f(x)f(e)$, hence if $g(e) \neq 1$, then $g(x) = \frac{1}{c}f(x)$ with some complex number $c \neq 0$. We also have $c \neq \pm 1$, otherwise $g = \pm f$, and substituting into (1.2) gives $f = g = 0$. It follows from (1.2) that

$$\frac{1-c^2}{c}f(x*y) = \frac{1-c^2}{c}f(x)\frac{1-c^2}{c}f(y)$$

which implies that$f(x) = \frac{c}{1-c^2}m(x)$ and that $g(x) = \frac{1}{1-c^2}m(x)$ with some exponential $m$, which is (i) with $M = m$.

Now we assume $g(e) = 1$, and in this case (1.2) implies that $f(e) = 0$. We define a modified Cauchy difference $G(x, y) = g(x*y) - g(x)g(y)$ such that we have

$$g(z)G(x, y) + G(x*y, z) = G(x, y*z) + g(x)G(y, z)$$

for each $x, y, z$ in $K$. Then, by (1.2) $G(x, y) = -f(x)f(y)$, and it follows that

$$f(x)\big[f(y*z) - f(y)g(z)\big] = f(z)\big[f(x*y) - g(x)f(y)\big]. \tag{3.1}$$

As $f \neq 0$, this implies that

$$f(x*y) = f(x)\varphi(y) + f(y)g(x)$$

with some continuous function $\varphi : K \to \mathbb{C}$. Substituting into (3.1), we obtain

$$f(x)f(z)\big[\varphi(y) - g(y)\big] = f(y)f(z)\big[\varphi(x) - g(x)\big]$$

for each $x, y, z$ in $K$. As $f \neq 0$ this implies that

$$\varphi(x) = g(x) + 2\lambda f(x)$$

with some complex number $\lambda$. If $\lambda = 0$, then we have $\varphi = g$, and the pair $f, g$ satisfies the sine equation (1.1). Case (i) in Theorem 2.1 cannot occur. Case (ii) in Theorem 2.1 gives $c = \pm i$, which is included in (i) above with $c = \pm i$. Finally, case (iii) in Theorem 2.1 gives $c = \pm i$ which is included in case (i) above with $c = \pm i$.

Now we assume that $\lambda \neq 0$, then we have

$$f(x*y) = f(x)g(y) + 2\lambda f(x)f(y) + f(y)g(x).$$

We introduce the function

$$h(x) = g(x) + \lambda f(x),$$

then a simple calculation shows that

$$f(x*y) = f(x)h(y) + f(y)h(x) \tag{3.2}$$

and

$$h(x*y) = h(x)h(y) - (1-\lambda^2)f(x)f(y). \tag{3.3}$$

Equation (3.2) shows that $f$ and $h$ satisfy the sine equation (1.1), and hence we have the description of the solutions, we just have to extract the solutions of (1.2). But we also have to consider equation (3.3) which depends on $\lambda$. If $\lambda^2 = 1$, then $h = m$ is an exponential and $f$ is an $m$-sine function. In this case we have

$g = m \pm f$ and substitution into (1.2) gives that this is a solution indeed, which is covered by case (ii) above.

Finally we suppose that $\lambda^2 \neq 1$. We take $d \neq 0$ with $d^2 = 1 - \lambda^2$, then we have by (3.3)

$$h(x * y) = h(x)h(y) - df(x)df(y)$$

and, multiplying (3.2) by $d$ gives

$$df(x * y) = df(x)h(y) + df(y)h(x)$$

for each $x, y$ in $K$. This means that the pair $df, h$ satisfies the sine and the cosine functional equations simultaneously, and in this case $h$ is not an exponential, hence we have to consider cases (ii) and (iii) only, in Theorem 2.1. In case (ii) we get $c = \pm i$ and, by the definition of $h$

$$f(x) = \pm\frac{1}{2di}M(x), \qquad g(x) + \lambda f(x) = \frac{1}{2}M(x)$$

which implies that $f$ and $g$ are constant multiples of each other, hence we have case (i) above. In case (iii) of Theorem 2.1 we obtain

$$f(x) = \frac{1}{2cd}\big[M(x) - N(x)\big], \qquad g(x) = \frac{cd - \lambda}{2cd}M(x) + \frac{cd + \lambda}{2cd}N(x).$$

Substituting into (1.2) gives $c = \pm i$, and we have

$$f(x) = \pm\frac{1}{2di}\big[M(x) - N(x)\big], \qquad g(x) = \pm\frac{\pm di - \lambda}{2di}M(x) \pm \frac{\pm di + \lambda}{2di}N(x),$$

where $d^2 = 1 - \lambda^2$, and we choose $+$ or $-$ at each place in the same way, as it is given in case (iii) above. The converse statement can be verified easily by direct computation. $\square$

Using results concerning the form of exponentials on some particular hypergroups discussed in [20], one can obtain explicit forms of sine functions on certain hypergroups.

## 4. Sine-cosine functional equations on hypergroups

In this section, we will first consider the matrix equation

$$L(x * y) = L(x)L(y) = L(y)L(x) \tag{4.1}$$

on the hypergroup $K$, where $L : K \to \mathcal{L}(\mathbb{C}^n)$ is a continuous mapping and the equation is supposed to hold for each $x, y$ in $K$. Here $\mathcal{L}(\mathbb{C}^n)$ denotes the space of all linear operators on $\mathbb{C}^n$, which is identified with the space of all $n \times n$ complex matrices. This equation has been studied on Abelian groups, even on commutative semigroups (see, e.g. [19] and further references given therein). In those cases the assumption on the commuting property of the matrices $L(x)$ is unnecessary. Here we will not assume the commutativity of the semigroup $K$, just the property of $L$ that its values form a commuting family. We will apply our results on the equations in the previous paragraphs.

We will use the following result (see [7, Chapter IV], [11]).

**Theorem 4.1.** *Let $\mathcal{S}$ be a family of commuting linear operators in $\mathcal{L}(\mathbb{C}^n)$. Then $\mathbb{C}^n$ decomposes into a direct sum of linear subspaces $X_j$ such that each $X_j$ is a minimal invariant subspace under the operators in $\mathcal{S}$. Further, $\mathbb{C}^n$ has a basis in which every operator in $\mathcal{S}$ is represented by an upper triangular matrix.*

In other words, there exist positive integers $k, n_1, n_2, \dots, n_k$ with the property $n_1+n_2+\dots+n_k = n$, and there exists a regular matrix $C$ such that every matrix $L$ in $\mathcal{S}$ has the form

$$L = C^{-1}\,\mathrm{diag}\{L_1, L_2, \dots, L_k\}C,$$

where $L_j$ is upper triangular for $j = 1, 2, \dots, k$. Here $\mathrm{diag}\{L_1, L_2, \dots, L_k\}$ denotes the block matrix with blocks $L_1, L_2, \dots, L_k$ along the main diagonal, and all diagonal elements of the block $L_j$ are the same. Using this result we have the following theorem.

**Theorem 4.2.** *Let $K$ be a hypergroup and let $L : K \to \mathcal{L}(\mathbb{C}^n)$ be a continuous mapping satisfying (4.1) for each $x, h$ in $K$. Then there exist positive integers $k, n_1, n_2, \dots, n_k$ with the property $n_1 + n_2 + \dots + n_k = n$, and there exists a regular matrix $C$ such that*

$$L(x) = C^{-1}\,\mathrm{diag}\big\{L_1(x), L_2(x), \dots, L_k(x)\big\}C \tag{4.2}$$

*for each $x$ in $K$, where $L_j : K \to \mathcal{L}(\mathbb{C}^{n_j})$ is upper triangular, all diagonal elements of it are the same, and satisfies (4.1) for each $x, y$ in $K$ and for every $j = 1, 2, \dots, l$.*

Now we apply this theorem for the sine-cosine functional equation:

$$f(x * y) = f(x)g(y) + f(y)g(x) + h(x)h(y),$$

where $f, g, h : K \to \mathbb{C}$ are continuous functions and the equation holds for each $x, y$ in $K$.

**Theorem 4.3.** *Let $f, g, h$ satisfy (1.3), with $f, h$ are linearly dependent. If $f \neq 0$ then we have one of the following cases*

- (i) *There is an exponential $M : K \to \mathbb{C}$ and a complex number $\lambda$ such that $f$ is an $M$-sine function, $h = \lambda f$ and $g = M - \frac{1}{2}\lambda^2 f$.*
- (ii) *There is an exponential $M : K \to \mathbb{C}$ and complex numbers $c, \lambda$ with $c \neq 0$ such that $f = \frac{1}{2c}M$, $g = \frac{2c-\lambda^2}{4c}M$, and $h = \frac{\lambda}{2c}M$.*
- (iii) *There are exponentials $M, N : K \to \mathbb{C}$ and complex numbers $c, \lambda$ with $c \neq 0$ such that $f = \frac{1}{2c}[M - N]$, $g = (\frac{1}{2} - \frac{\lambda^2}{4c})M + (\frac{1}{2} + \frac{\lambda^2}{4c})N$, and $h = \frac{\lambda}{2c}[M - N]$.*

*If $f = 0$, then $h = 0$ and $g$ is arbitrary.*

*Proof.* Obviously, we may suppose that $f \neq 0$. Let $h = \lambda f$, then we have $\lambda \neq 0$ and

$$f(x * y) = f(x)k(y) + f(y)k(x)$$

for each $x, y$ in $K$, where $k = g + \frac{1}{2}\lambda^2 f$. This is the sine equation (1.1) for $f, k$ and we extract the solutions of (1.3) form Theorem 2.1 as given above. □

**Lemma 4.4.** *Let $f, g, h$ satisfy (1.3), with $f, h$ are linearly independent. Then we have that $f, g, h$ are constant on the set $\{x * y, y * x\}$ for each $x, y$ in $K$ and $f$ is constant on the set $\{x * y * z, y * x * z, x * z * y\}$ for each $x, y, z$ in $K$. If $f, g, h$ are linearly independent, then this latter holds for $g, h$, too.*

*Proof.* We denote $\tau_u f(x) = f(x*u)$ for each $u, x$ in $K$. Obviously $f(x*y) = f(y*x)$ for each $x, y$ in $K$. By repeating the calculation in [3, pp. 267–268], we obtain the same for the functions $g, h$. Now we have from (1.3)

$$\begin{aligned} f(x * y * z) &= f(x)g(y * z) + f(y * z)g(x) + h(x)h(y * z) \\ &= f(x)g(z * y) + f(z * y)g(x) + h(x)h(z * y) = f(x * z * y) \end{aligned}$$

and

$$f(x * y * z) = f(z * x * y) = f(z * y * x) = f(y * x * z),$$

which proves our statement for $f$. It follows that the translate $\tau_u f$ has the same property for each $u$ in $K$. If the functions $f, g, h$ are linearly independent, then the functions $g, h$ are in the linear space spanned by the translates $\tau_u f$ of $f$, hence they satisfy the same property. □

**Theorem 4.5.** *Let $K$ be a hypergroup and let $f, g, h : K \to \mathbb{C}$ be linearly independent continuous functions satisfying the sine-cosine functional equation (1.3). Then we have one of the following cases:*

(i) *There exist exponential functions $M_1, M_2, M_3 : K \to \mathbb{C}$ and complex numbers $\alpha_i, \beta_i, \gamma_i$ for $i = 1, 2, 3$ such that*

$$\begin{aligned} f(x) &= \alpha_1 M_1(x) + \beta_1 M_2(x) + \gamma_1 M_3(x), \\ g(x) &= \alpha_2 M_1(x) + \beta_2 M_2(x) + \gamma_2 M_3(x), \\ h(x) &= \alpha_3 M_1(x) + \beta_3 M_2(x) + \gamma_3 M_3(x) \end{aligned}$$

*holds for each $x$ in $K$, further we have*

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \cdot \begin{bmatrix} \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \beta_1 & 0 \\ 0 & 0 & \gamma_1 \end{bmatrix}.$$

(ii) *There exist exponential functions $M_1, M_2 : K \to \mathbb{C}$, an $M_1$-sine function $S_1 : K \to \mathbb{C}$ and complex numbers $\alpha_i, \beta_i, \gamma_i$ for $i = 1, 2, 3$ such that*

$$\begin{aligned} f(x) &= \alpha_1 M_1(x) + \beta_1 S_1(x) + \gamma_1 M_2(x), \\ g(x) &= \alpha_2 M_1(x) + \beta_2 S_1(x) + \gamma_2 M_2(x), \\ h(x) &= \alpha_3 M_1(x) + \beta_3 S_1(x) + \gamma_3 M_2(x) \end{aligned}$$

*holds for each $x$ in $K$, further we have*

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \cdot \begin{bmatrix} \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 & 0 \\ \beta_1 & 0 & 0 \\ 0 & 0 & \gamma_1 \end{bmatrix}.$$

(iii) *There exists an exponential function $M : K \to \mathbb{C}$, an $M$-sine function $S : K \to \mathbb{C}$ and a function $T : K \to \mathbb{C}$ such that $M, S, T$ form a generalized moment sequence of order 2, and complex numbers $\alpha_i, \beta_i, \gamma_i$ for $i = 1, 2, 3$ such that*

$$\begin{aligned} f(x) &= \alpha_1 M(x) + \beta_1 S(x) + \gamma_1 T(x), \\ g(x) &= \alpha_2 M(x) + \beta_2 S(x) + \gamma_2 T(x), \\ h(x) &= \alpha_3 M(x) + \beta_3 S(x) + \gamma_3 T(x) \end{aligned}$$

*holds for each $x$ in $K$, further we have*

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \cdot \begin{bmatrix} \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \beta_1 & 2\gamma_2 & 0 \\ \gamma_1 & 0 & 0 \end{bmatrix}.$$

*Proof.* By the linear independence of the functions $f, g, h$ there exist elements $y_1, y_2, y_3$ in $K$ such that the matrix

$$\begin{bmatrix} g(y_1) & f(y_1) & h(y_1) \\ g(y_2) & f(y_2) & h(y_2) \\ g(y_3) & f(y_3) & h(y_3) \end{bmatrix}$$

is regular. We introduce the notation

$$U(x) = \begin{bmatrix} g(x*y_1) & f(x*y_1) & h(x*y_1) \\ g(x*y_2) & f(x*y_2) & h(x*y_2) \\ g(x*y_3) & f(x*y_3) & h(x*y_3) \end{bmatrix},$$

further

$$\widetilde{f}(x) = \begin{bmatrix} f(x*y_1) \\ f(x*y_2) \\ f(x*y_3) \end{bmatrix} \qquad \text{and} \qquad \widetilde{\varphi}(x) = \begin{bmatrix} f(x) \\ g(x) \\ h(x) \end{bmatrix}$$

for each $x$ in $K$. Using this notation we have, by equation (1.3),

$$\widetilde{f}(x*y) = U(y)\widetilde{\varphi}(x)$$

for each $x, y$ in $K$. We let $V(y) = U(e)^{-1}U(y)$, then $U(e)^{-1}\widetilde{f}(x) = \widetilde{\varphi}(x)$, hence

$$\widetilde{\varphi}(x*y) = V(y)\widetilde{\varphi}(x)$$

holds for each $x, y$ in $K$. From this relation we have two different expressions for $\widetilde{\varphi}(x*y*z)$ which are equal, by associativity, and we infer

$$V(y*z)\widetilde{\varphi}(x) = V(y)V(z)\widetilde{\varphi}(x)$$

for each $x, y, z$ in $K$. By the linear independence of $f, g, h$ the vectors $\widetilde{\varphi}(x)$ span $\mathbb{C}^n$ and we conclude

$$V(y*z) = V(y)V(z)$$

for each $y, z$ in $K$. On the other hand, the entries of $V(y*z)$ are constant on the set $\{y*z, z*y\}$ for each $y, z$ in $K$, by Lemma 4.4. It follows that the family of matrices $V(y)$ for $y$ in $K$ is commuting and we can apply Theorem 4.2. For the decomposition of $\mathbb{C}^3$ into invariant subspaces we have three possibilities.

In the first case, $\mathbb{C}^3$ decomposes into the sum of three one dimensional invariant subspaces, in which case the decomposition of $V(x)$ corresponding to (4.2) gives the following:

$$V(x) = C^{-1}\operatorname{diag}\big\{M_1(x), M_2(x), M_3(x)\big\}C,$$

where the $1\times 1$ matrices $M_1, M_2, M_3$ satisfy (4.1), that is, they are exponentials. In this case, in view of the definition of $V$, we have case (i) in the statement of the theorem. The given relation between the constants $\alpha_i, \beta_i, \gamma_i$ can be obtained by simple substitution, using the linear independence of the functions $M_1, M_2, M_3$.

In the second case, $\mathbb{C}^3$ decomposes into the sum of a two dimensional and a one dimensional invariant subspace, in which case the decomposition of $V(x)$ corresponding to (4.2) gives the following:

$$V(x) = C^{-1}\operatorname{diag}\big\{W(x), M_2(x)\big\}C,$$

where $W : K \to \mathcal{L}(\mathbb{C}^2)$ is a solution of (4.1) and $W(x)$ is upper triangular for each $x$ in $K$, further $M_2 : K \to \mathbb{C}$ is an exponential. Writing

$$W(x) = \begin{bmatrix} W_{11}(x) & W_{12}(x) \\ 0 & W_{22}(x) \end{bmatrix}$$

with $W_{11} = W_{22} = M_1$, an exponential, by (4.1); further

$$W_{12}(x * y) = M_1(x)W_{12}(y) + M_1(y)W_{12}(x);$$

that is, $W_{12} = S_1$ is an $M_1$-sine function, and we have case (ii) above. Again, the given relation between the constants $\alpha_i, \beta_i, \gamma_i$ can be obtained by simple substitution, using the linear independence of the functions $M_1, S_1, M_2$.

Finally, in the third case there is no nonzero proper invariant subspace in $\mathbb{C}^3$, hence $V(x)$ has the form

$$V(x) = C^{-1}F(x)C,$$

where $F : K \to \mathcal{L}(\mathbb{C}^3)$ satisfies equation (4.1), and $F(x)$ is upper triangular for each $x$ in $K$. We write

$$F(x) = \begin{bmatrix} \varphi_0(x) & \varphi_1(x) & \varphi_2(x) \\ 0 & \psi_0(x) & \psi_1(x) \\ 0 & 0 & \chi_0(x) \end{bmatrix},$$

where $\varphi_0 = \psi_0 = \chi_0 = M$ is an exponential, and from (4.1) we infer that

$$\begin{aligned} \varphi_1(x * y) &= M(x)\varphi_1(y) + M(y)\varphi_1(x), \\ \psi_1(x * y) &= M(x)\psi_1(y) + M(y)\psi_1(x), \\ \varphi_2(x * y) &= M(x)\varphi_2(y) + \varphi_1(x)\psi_1(y) + \varphi_2(x)M(y). \end{aligned}$$

It follows that $\varphi_1(x)\psi_1(y) = \varphi_1(y)\psi_1(x)$ holds for each $x, y$ in $K$. If $\varphi_1 \neq 0$ and $\psi_1 \neq 0$, then there is a nonzero complex number $\lambda$ such that

$$\varphi_1(x) = \lambda\sqrt{2}S(x), \qquad \psi_1(x) = \sqrt{2}S(x), \qquad \varphi_2(x) = \lambda T(x),$$

where $M, S, T$ is a generalized moment function sequence of order 2. In this case $f, g, h$ are linear combinations of $M, S, T$ and, by substitution, we obtain case (iii) above.

If $\varphi_1 = 0$, or $\psi_1 = 0$, then $\varphi_1, \psi_1, \varphi_2$ are $M$-sine functions corresponding to the exponential $M = \varphi_0$. As at most two of the three functions $\varphi_1, \psi_1, \varphi_2$ are linearly independent, hence in this case we have

$$\begin{aligned} f(x) &= \alpha_1 M(x) + \beta_1 S_1(x) + \gamma_1 S_2(x), \\ g(x) &= \alpha_2 M(x) + \beta_2 S_1(x) + \gamma_2 S_2(x), \\ h(x) &= \alpha_3 M(x) + \beta_3 S_1(x) + \gamma_3 S_2(x), \end{aligned}$$

where $S_1, S_2$ are $M$-sine functions. Substitution into (1.3) gives the following condition for the constants:

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix} \cdot \begin{bmatrix} \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_3 & \beta_3 & \gamma_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_2 \\ \beta_1 & 0 & 0 \\ \gamma_1 & 0 & 0 \end{bmatrix}.$$

As the matrix on the right hand side has determinant zero, it follows that $f, g, h$ are linearly independent, which is not the case, by assumption. The proof is complete. □

**Theorem 4.6.** *Let $K$ be a hypergroup and let $f, g, h : K \to \mathbb{C}$ be continuous functions satisfying the sine-cosine functional equation (1.3). Suppose that $f, h$ are linearly independent and $f, g, h$ are linearly dependent. Then we have one of the following cases:*

(i) *There exist exponential functions $M_1, M_2 : K \to \mathbb{C}$ and complex numbers $\alpha, \beta, \alpha_i, \beta_i$, for $i = 1, 2$ such that*

$$\begin{aligned} f(x) &= \alpha_1 M_1(x) + \beta_1 M_2(x), \\ g(x) &= (\alpha\alpha_1 + \beta\alpha_2) M_1(x) + (\alpha\beta_1 + \beta\beta_2) M_2(x), \\ h(x) &= \alpha_2 M_1(x) + \beta_2 M_2(x) \end{aligned}$$

*holds for each $x$ in $K$, further we have*

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} \cdot \begin{bmatrix} 2\alpha & \beta \\ \beta & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \beta_1 \end{bmatrix}.$$

(ii) *There exists an exponential function $M : K \to \mathbb{C}$, an $M$-sine function $S : K \to \mathbb{C}$ and complex numbers $\alpha, \beta, \alpha_i, \beta_i, \gamma_i$ for $i = 1, 2$ such that*

$$\begin{aligned} f(x) &= \alpha_1 M(x) + \beta_1 S(x), \\ g(x) &= (\alpha\alpha_1 + \beta\alpha_2) M(x) + (\alpha\beta_1 + \beta\beta_2) S(x), \\ h(x) &= \alpha_2 M(x) + \beta_2 S(x) \end{aligned}$$

*holds for each $x$ in $K$, further we have*

$$\begin{bmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{bmatrix} \cdot \begin{bmatrix} 2\alpha & \beta \\ \beta & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_1 & 0 \end{bmatrix}.$$

*Proof.* As $f$ and $h$ are linearly independent we have, by Lemma 4.4, that $f, g, h$ are constant on the set $\{x * y, y * x\}$ for each $x, y$ in $K$ and $f$ is constant on the set $\{x*y*z, y*x*z, x*z*y\}$ for each $x,y,z$ in $K$. We let $g=\alpha f+\beta h$ with some complex numbers $\alpha,\beta$. Substituting into (1.3) we have

$$f(x*y)=2\alpha f(x)f(y)+\beta\big[f(x)h(y)+f(y)h(x)\big]+h(x)h(y) \tag{4.3}$$

for each $x,y$ in $K$. Using associativity we can write $f(x*y*z)$ in two different ways, and after substituting $f(x*y)$ and $f(y*z)$ from (4.3) we have

$$\begin{aligned} f(x)\big[(\beta^2-2\alpha)h(y)h(z)-\beta h(y*z)\big]+h(x)\big[(2\alpha-\beta^2)h(y)f(z)-h(y*z)\big]\\ =-\big[\beta f(z)+h(z)\big]h(x*y) \end{aligned} \tag{4.4}$$

for each $x,y,z$ in $K$. As $f$ and $h$ are linearly independent, hence there is a $z$ in $K$ such that $\beta f(z)+h(z)\neq 0$, and we have

$$h(x*y)=f(x)k(y)+h(x)l(y) \tag{4.5}$$

with some continuous functions $k,l:K\to\mathbb{C}$ for each $x,y$ in $K$. Substituting into (4.4) and using the linear independence of $f$ and $h$ again we obtain that

$$k(y)=af(y)+bh(y),\qquad l(y)=cf(y)+dh(y)$$

holds for each $y$ in $K$ with some complex numbers $a,b,c,d$. Then we have from (4.5)

$$h(x*y)=af(x)f(y)+ch(x)f(y)+bf(x)h(y)+dh(x)h(y).$$

As $h(x*y)=h(y*x)$ this implies $b=c$, hence we conclude

$$h(x*y)=af(x)f(y)+b\big[h(x)f(y)+f(x)h(y)\big]+dh(x)h(y)$$

for each $x,y$ in $K$. Then it follows

$$\begin{aligned} h(x*y*z)&=af(x)f(y*z)+b\big[h(x)f(y*z)+f(x)h(y*z)\big]+dh(x)h(y*z)\\ &=af(x)f(z*y)+b\big[h(x)f(z*y)+f(x)h(z*y)\big]+dh(x)h(z*y)\\ &=h(x*z*y) \end{aligned}$$

and

$$h(x*y*z)=h(z*x*y)=h(z*y*x)=h(y*x*z),$$

that is, $h$ is constant on the set $\{x*y*z, x*z*y, y*x*z\}$, as well. Further, we have the functional equation

$$f(x*y)=f(x)\big[2\alpha f(y)+\beta h(y)\big]+h(x)\big[\beta f(y)+h(y)\big] \tag{4.6}$$

for each $x,y$ in $K$. First we assume that the functions

$$k(y)=2\alpha f(y)+\beta h(y),\qquad l(y)=\beta f(y)+h(y)$$

are linearly independent. In this case we can follow the ideas in the proof of Theorem 4.5. Indeed, we choose elements $y_1,y_2$ in $K$ such that the matrix

$$\begin{bmatrix} k(y_1) & l(y_1)\\ k(y_2) & l(y_2)\end{bmatrix}$$

is regular. We introduce the notation

$$U(x)=\begin{bmatrix} k(x*y_1) & l(x*y_1)\\ k(x*y_2) & l(x*y_2)\end{bmatrix},$$

further

$$\widetilde{f}(x) = \begin{bmatrix} f(x*y_1) \\ f(x*y_2) \end{bmatrix} \qquad \text{and} \qquad \widetilde{\varphi}(x) = \begin{bmatrix} f(x) \\ h(x) \end{bmatrix}$$

for each $x$ in $K$.

Then, exactly in the same manner as we did in the proof of Theorem 4.5, we infer that the matrix function $V : K \to \mathcal{L}(\mathbb{C}^2)$ given by $V(y) = U(e)^{-1}U(y)$ satisfies the equation

$$V(y*z) = V(y)V(z) = V(z)V(y)$$

for each $y, z$ in $K$. As the matrices $V(y)$ commute for $y$ in $K$, we can apply Theorem 4.2 about the decomposition of $\mathbb{C}^2$ into the direct sum of invariant subspaces. Here we have two possibilities.

In the first case $\mathbb{C}^2$ decomposes into the sum of two one dimensional invariant subspaces, in which case the decomposition of $V(x)$ corresponding to (4.2) gives the following:

$$V(x) = C^{-1}\,\mathrm{diag}\big\{M_1(x), M_2(x)\big\}C,$$

where the $1 \times 1$ matrices $M_1, M_2$ satisfy (4.1), that is, they are exponentials. In this case, in view of the definition of $V$, we have case (i) in the statement of the theorem. The given relation between the constants $\alpha, \beta, \alpha_i, \beta_i$ can be obtained by simple substitution into (1.3), using the linear independence of the functions $M_1, M_2$.

In the second case, $\mathbb{C}^2$ has no proper invariant subspace in $\mathbb{C}^2$, hence $V(x)$ has the form

$$V(x) = C^{-1}F(x)C,$$

where $F : K \to \mathcal{L}(\mathbb{C}^2)$ satisfies equation (4.1), and $F(x)$ is upper triangular with equal diagonal elements for each $x$ in $K$. We write

$$F(x) = \begin{bmatrix} \varphi_0(x) & \varphi_1(x) \\ 0 & \psi_0(x) \end{bmatrix},$$

where $\varphi_0 = \psi_0 = M$ is an exponential, and from (4.1) we infer that $\varphi_1 = S$ is an $M$-sine function. In this case, in view of the definition of $V$, we have case (ii) in the statement of the theorem. Again, the given relation between the constants $\alpha, \beta, \alpha_i, \beta_i$ can be obtained by simple substitution into (1.3), using the linear independence of the functions $M, S$.

There is one case left: this is when the functions $2\alpha f + \beta h$ and $\beta f + h$ in (4.6) are linearly dependent. That means $l(x) = \lambda k(x)$ holds for each $x$ in $K$ with some complex number $\lambda$. Then we have

$$f(x*y) = f(x)k(y) + \lambda h(x)k(y) = \big(f(x) + \lambda h(x)\big)k(y)$$

for each $x, y$ in $K$. In this case $f, f + \lambda h, k$ satisfy a Pexider equation, which implies that they are all multiples of a single exponential. As $f, h$ are linearly independent, this cannot happen. The proof is complete. □

We can summarize the above result with the assertion that—apart from trivial cases—our theorems describe all continuous solutions of the sine-cosine functional equation (1.3) on any hypergroup.

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