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ANALYTIC FOURIER–FEYNMAN TRANSFORMS AND CONVOLUTION PRODUCTS ASSOCIATED WITH GAUSSIAN PROCESSES ON WIENER SPACE

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ABSTRACT. Using Gaussian processes, we define a very general convolution product of functionals on Wiener space and we investigate fundamental relationships between the generalized Fourier–Feynman transforms and the generalized convolution products. Using two rotation theorems of Gaussian processes, we establish that both of the generalized Fourier–Feynman transform of the generalized convolution product and the generalized convolution product of the generalized Fourier–Feynman transforms of functionals on Wiener space are represented as products of the generalized Fourier–Feynman transforms of each functional, with examples.

1. INTRODUCTION

Let $C_0[0,T]$ denote 1-parameter Wiener space, that is, the space of all real-valued continuous functions x on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$, and let $\mathfrak{m}$ denote the Wiener measure. Then, as is well known, $(C_0[0,T],\mathcal{M},\mathfrak{m})$ is a complete measure space. Throughout this article, we will denote the Wiener integral of a Wiener measurable functional

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$$E[F] \equiv E_x\big[F(x)\big] = \int_{C_0[0,T]} F(x)\, d\mathfrak{m}(x).$$

A subset B of $C_0[0,T]$ is said to be *scale-invariant measurable* (s.i.m.) provided $\rho B \in \mathcal{M}$ for all $\rho > 0$, and a scale-invariant measurable set N is said to be s*cale-invariant null* provided that $\mathfrak{m}(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold *scale-invariant almost everywhere* (s-a.e.). If two functionals F and G are equal s-a.e., we write $F \approx G$. Note that the relation $\approx$ is an equivalence relation.

The concept of the analytic *Fourier–Feynman transform* (FFT) on the Wiener space $C_0[0,T]$, initiated by Brue [2], has been developed further in the literature. This transform and its properties are similar in many respects to the ordinary Fourier transform (for an elementary introduction to the analytic FFT, see [21] and the references cited therein). First, we refer to [21] for the precise definitions and the notations of the analytic FFT and of the *convolution product* (CP) of functionals on the Wiener space $C_0[0,T]$. In [10], Huffman, Park, and Skoug defined a CP for functionals on $C_0[0,T]$ and obtained various results for the analytic FFT and the CP (see [11]–[13]). In previous research involving [10]–[13], the authors have established the relationship between the analytic FFT and the CP of functionals F and G on $C_0[0,T]$, in the form

$$T_q\big((F*G)_q\big)(y) = T_q(F)\Big(\frac{y}{\sqrt{2}}\Big)T_q(G)\Big(\frac{y}{\sqrt{2}}\Big) \tag{1.1}$$

for scale-almost every $y \in C_0[0,T]$.

An essential structure hidden in the proof of equation (1.1) is based on the fact that the Gaussian processes

$$\mathfrak{Z}_+ \equiv \Big\{\frac{x_1 + x_2}{\sqrt{2}} : x_1, x_2 \in C_0[0,T]\Big\} \qquad \text{and}$$
$$\mathfrak{Z}_- \equiv \Big\{\frac{x_1 - x_2}{\sqrt{2}} : x_1, x_2 \in C_0[0,T]\Big\}$$

are independent, and the processes $\mathfrak{Z}_+$ and $\mathfrak{Z}_-$ are equivalent to the standard Wiener process. More precisely, the product Wiener measure $\mathfrak{m} \times \mathfrak{m}$ is rotation-invariant in $C_0^2[0,T]$ (see [4, Lemmas 1 and 2]). As discussed in [4], those rotation-invariant properties of $\mathfrak{m} \times \mathfrak{m}$ were concretely realized by Bearman [1].

Recently in [8], the present authors and Skoug used another rotation form of Wiener measure $\mathfrak{m}$ to define a multiple *generalized FFT associated with Gaussian processes* $\mathcal{Z}_h$ ($\mathcal{Z}_h$-GFFT) on $C_0[0,T]$. The rotation form used in [8] is a generalization of Bearman's celebrated result and is intended to interpret behaviors of nonstationary Gaussian processes on $C_0[0,T]$. The authors also investigated various relationships which exist between the multiple GFFT and the corresponding CP associated with the Gaussian processes on $C_0[0,T]$. In this paper, motivated by the results in [7], [10]–[13], we study the fundamental relationships between the analytic $\mathcal{Z}_h$-GFFTs and the *generalized CPs* (GCP) on Wiener space.

This paper is organized as follows. In Section 2, we briefly recall well-known results for Gaussian processes on Wiener space and introduce the concepts of the $\mathcal{Z}_h$-GFFT and the GCP of functionals on Wiener space. In Section 3, we emphasize the main purpose of this paper via specific examples. To do this, we introduce the partially exponential-type functionals on $C_0[0,T]$. In Section 4, as preliminary results, we investigate rotation properties of the Gaussian processes on product Wiener spaces. In Section 5, we also investigate fundamental relationships between the analytic $\mathcal{Z}_h$-GFFTs and the GCPs. Finally, in Section 6, we present examples which shed light upon the conditions in our two main assertions, and which also illustrate that the conclusions of the main assertions are valid.

2. Preliminaries

In this section, we first present the brief backgrounds which are needed in the following sections. For each $v\in L_2[0,T]$ and $x\in C_0[0,T]$, we let $\langle v,x\rangle=\int_0^T v(t)\,dx(t)$ denote the Paley–Wiener–Zygmund stochastic integral (see [8], [15]–[17]). It is known that for each $v\in L_2[0,T]$, the Paley–Wiener–Zygmund stochastic integral $\langle v,x\rangle$ exists for s-a.e. $x\in C_0[0,T]$, and it is a Gaussian random variable with mean zero and variance $\|v\|_2^2$, where $\|\cdot\|_2$ denotes the $L_2[0,T]$-norm. Thus, using the change of variable formula, one can establish the integration formula on $C_0[0,T]$:

$$E_x\big[\exp\big\{\langle v,\rho x\rangle\big\}\big]=\exp\Big\{\frac{\rho^2}{2}\|v\|_2^2\Big\}\tag{2.1}$$

for every $v\in L_2[0,T]$ and $\rho\in\mathbb{R}\setminus\{0\}$.

For any $h\in L_2[0,T]$ with $\|h\|_2>0$, let $\mathcal{Z}_h$ be the stochastic process (see [9], [13], [18], [20]) on $C_0[0,T]\times[0,T]$ given by

$$\mathcal{Z}_h(x,t)=\int_0^t h(s)\,dx(s)=\langle h\chi_{[0,t]},x\rangle.\tag{2.2}$$

Of course, if $h(t)\equiv 1$ on $[0,T]$, then $\mathcal{Z}_h(x,t)=x(t)$ is an ordinary Wiener process.

Given $h\in L_2[0,T]$ with $\|h\|_2>0$, let $\beta_h(t)=\int_0^t h^2(u)\,du$. It is easy to see that $\mathcal{Z}_h$ is a Gaussian process with mean zero and covariance function

$$E_x\big[\mathcal{Z}_h(x,s)\mathcal{Z}_h(x,t)\big]=\int_0^{\min\{s,t\}}h^2(u)\,du=\beta\big(\min\{s,t\}\big).$$

In addition, $\mathcal{Z}_h(\cdot,t)$ is stochastically continuous in t on $[0,T]$ and for any $h_1,h_2\in L_2[0,T]$,

$$E_x\big[\mathcal{Z}_{h_1}(x,s)\mathcal{Z}_{h_2}(x,t)\big]=\int_0^{\min\{s,t\}}h_1(u)h_2(u)\,du.\tag{2.3}$$

It is known (see [9]) that, for $v\in L_2[0,T]$ and $h\in L_\infty[0,T]$,

$$\big\langle v,\mathcal{Z}_h(x,\cdot)\big\rangle=\langle vh,x\rangle\tag{2.4}$$

for s-a.e. $x\in C_0[0,T]$.

Throughout this paper, we will assume that each functional F (or G) we consider satisfies the conditions:

$$F:C_0[0,T]\to\mathbb{C}\quad\text{is s-a.e. defined and s.i.m.,}\tag{2.5}$$

and for all $h \in L_2[0,T]$ and each $\rho > 0$,

$$E_x\big[\big|F\big(\rho\mathcal{Z}_h(x,\cdot)\big)\big|\big] < +\infty. \tag{2.6}$$

Next, let $BV[0,T]$ denote the space of functions of bounded variation on $[0,T]$. Also, let $\mathbb{C}$, $\mathbb{C}_+$, and $\widetilde{\mathbb{C}}_+$ denote the set of complex numbers, complex numbers with positive real part, and nonzero complex numbers with nonnegative real part, respectively. For each $\lambda \in \widetilde{\mathbb{C}}_+$, $\lambda^{1/2}$ denotes the principal square root of λ (i.e., $\lambda^{1/2}$ is always chosen to have positive real part so that $\lambda^{-1/2} = (\lambda^{-1})^{1/2}$ is in $\mathbb{C}_+$ for all $\lambda \in \widetilde{\mathbb{C}}_+$).

Definition 2.1. Let F satisfy conditions (2.5) and (2.6) above. Let $\mathcal{Z}_h$ be the Gaussian process given by (2.2), and for $\lambda > 0$ let $J(h;\lambda) = E_x[F(\lambda^{-1/2}\mathcal{Z}_h(x,\cdot))]$. If there exists a function $J^*(h;\cdot)$ analytic on $\mathbb{C}_+$ such that $J^*(h;\lambda) = J(h;\lambda)$ for all $\lambda > 0$, then $J^*(h;\lambda)$ is defined to be the analytic $\mathcal{Z}_h$-Wiener integral (namely, the generalized analytic Wiener integral associated with the Gaussian paths $\mathcal{Z}_h(x,\cdot)$) of F over $C_0[0,T]$ with parameter λ. In this case, for $\lambda \in \mathbb{C}_+$ we write

$$E_x^{\mathrm{an}\,w_\lambda}\big[F\big(\mathcal{Z}_h(x,\cdot)\big)\big] = J^*(h;\lambda).$$

Let $q \neq 0$ be a real number, and let F be a functional such that the analytic $\mathcal{Z}_h$-Wiener integral $E_x^{\mathrm{an}\,w_\lambda}[F(\mathcal{Z}_h(x,\cdot))]$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the *analytic $\mathcal{Z}_h$-Feynman integral* (namely, the generalized analytic Feynman integral associated with the Gaussian paths $\mathcal{Z}_h(x,\cdot)$) of F with parameter q, and we write

$$E_x^{\mathrm{an}\,f_q}\big[F\big(\mathcal{Z}_h(x,\cdot)\big)\big] = \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} E_x^{\mathrm{an}\,w_\lambda}\big[F\big(\mathcal{Z}_h(x,\cdot)\big)\big]. \tag{2.7}$$

Note that if $h \equiv 1$ on $[0,T]$, then these definitions agree with the previous definitions of the analytic Wiener integral and the analytic Feynman integral (see [3], [5], [10]–[12], [14], [19]).

Next (see [7], [8], [13], [20]), we state the definition of the GFFT.

Definition 2.2. For $k \in L_2[0,T] \setminus \{0\}$, $\lambda \in \mathbb{C}_+$, and $y \in C_0[0,T]$, let

$$T_{\lambda,k}(F)(y) = E_x^{\mathrm{an}\,w_\lambda}\big[F\big(y + \mathcal{Z}_k(x,\cdot)\big)\big]. \tag{2.8}$$

We define the L_1 analytic $\mathcal{Z}_k$-GFFT (namely, the GFFT associated with the Gaussian paths $\mathcal{Z}_k(x,\cdot)$), $T_{q,k}^{(1)}(F)$ of F, by the formula

$$T_{q,k}^{(1)}(F)(y) = \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} T_{\lambda,k}(F)(y),$$

for s-a.e. $y \in C_0[0,T]$ whenever this limit exists.

We note that if $T_{q,k}^{(1)}(F)$ exists and if $F \approx G$, then $T_{q,k}^{(1)}(G)$ exists and $T_{q,k}^{(1)}(G) \approx T_{q,k}^{(1)}(F)$. One can see that, for each $k \in L_2[0,T] \setminus \{0\}$, $T_{q,k}^{(1)}(F) \approx T_{q,-k}^{(1)}(F)$ since

$$\int_{C_0[0,T]} F(x)\,dm(x) = \int_{C_0[0,T]} F(-x)\,dm(x).$$

By Definitions 2.1 and 2.2, it is easy to see that for a nonzero real number q,

$$T_{q,k}^{(1)}(F)(y) = E_x^{\mathrm{an}\, f_q}\big[F\big(y + \mathcal{Z}_k(x,\cdot)\big)\big] \tag{2.9}$$

for s-a.e. $y \in C_0[0,T]$ if both sides exist. We now present the definition of the GCP, which involves various kinds of CPs studied in previous research.

Definition 2.3. Let F and G be functionals on $C_0[0,T]$. For $\lambda \in \widetilde{\mathbb{C}}_+$, $g_1, g_2 \in BV[0,T]$ and $h_1, h_2 \in L_2[0,T] \setminus \{0\}$, we define their GCP with respect to $\{\mathcal{Z}_{g_1}, \mathcal{Z}_{g_2}, \mathcal{Z}_{h_1}, \mathcal{Z}_{h_2}\}$ (if it exists) by

$$\begin{aligned}(F*G)_\lambda^{(g_1,g_2;h_1,h_2)}&(y)\\ &= \begin{cases} E_x^{\mathrm{an}\, w_\lambda}[F(\mathcal{Z}_{g_1}(y,\cdot) + \mathcal{Z}_{h_1}(x,\cdot))G(\mathcal{Z}_{g_2}(y,\cdot) + \mathcal{Z}_{h_2}(x,\cdot))], \\ \quad \lambda \in \mathbb{C}_+ \\ E_x^{\mathrm{an}\, f_q}[F(\mathcal{Z}_{g_1}(y,\cdot) + \mathcal{Z}_{h_1}(x,\cdot))G(\mathcal{Z}_{g_2}(y,\cdot) + \mathcal{Z}_{h_2}(x,\cdot))], \\ \quad \lambda = -iq, q \in \mathbb{R}, q \neq 0. \end{cases}\end{aligned} \tag{2.10}$$

When $\lambda = -iq$, we will denote $(F*G)_\lambda^{(g_1,g_2;h_1,h_2)}$ by $(F*G)_q^{(g_1,g_2;h_1,h_2)}$.

Remark 2.4.

(i) Given a function h in $L_2[0,T]$ with $\|h\|_2 > 0$, letting $h_1 = -h_2 = h/\sqrt{2}$ and $g_1 = g_2 \equiv 1/\sqrt{2}$, equation (2.10) yields the CP studied in [7], [8], [13], [20]:

$$(F*G)_q^{(g_1,g_2;h_1,h_2)}(y) = E_x^{\mathrm{an}\, f_q}\Big[F\Big(\frac{y + \mathcal{Z}_h(x,\cdot)}{\sqrt{2}}\Big)G\Big(\frac{y - \mathcal{Z}_h(x,\cdot)}{\sqrt{2}}\Big)\Big].$$

(ii) Choosing $h_1 = -h_2 = g_1 = g_2 \equiv 1/\sqrt{2}$, equation (2.10) yields the CP studied in [10]–[12]:

$$(F*G)_q^{(g_1,g_2;h_1,h_2)}(y) = E_x^{\mathrm{an}\, f_q}\Big[F\Big(\frac{y+x}{\sqrt{2}}\Big)G\Big(\frac{y-x}{\sqrt{2}}\Big)\Big].$$

(iii) Choosing $h_1 = h_2 = g_1 = -g_2 = 1/\sqrt{2}$ and $\lambda = 1$, equation (2.10) yields the CP studied in [22]:

$$(F*G)^{(g_1,g_2;h_1,h_2)}(y) = E_x\Big[F\Big(\frac{y+x}{\sqrt{2}}\Big)G\Big(\frac{-y+x}{\sqrt{2}}\Big)\Big].$$

If h is of bounded variation on $[0,T]$, then $\mathcal{Z}_h(x,t)$ is continuous in t for all $x \in C_0[0,T]$ (i.e., $\mathcal{Z}_h(\cdot,\cdot)$ is a continuous process on $C_0[0,T] \times [0,T]$). Thus throughout the remainder of this paper, we require h (or k) to be in $BV[0,T]$ and with $\|h\|_2 > 0$ for each process $\mathcal{Z}_h$.

3. Observations on the class $\mathcal{E}(C_0[0,T])$

Let $\mathcal{E}$ be the class of all functionals having the form

$$\Psi_u(x) = \exp\big\{\langle u, x\rangle\big\} \tag{3.1}$$

for $\mathfrak{m}$-a.e. $x \in C_0[0,T]$, where $u \in L_2[0,T]$, and given $q \in \mathbb{R}\setminus\{0\}$, $v \in L_2[0,T]$, and $k \in BV[0,T]$, let $\mathcal{E}_{q,v,k}$ be the class of all functionals having the form

$$\Psi_u^{q,v,k}(x) = \Psi_u(x)\exp\Big\{\frac{i}{2q}\|vk\|_2^2\Big\} \tag{3.2}$$

for $\mathfrak{m}$-a.e. $x \in C_0[0,T]$, where Ψ_u is given by equation (3.1). For notational convenience, let $\Psi_u^{0,v,k}(x) = \Psi_u(x)$ and let $\mathcal{E}_{0,v,k} = \mathcal{E}$.

The functionals given by equation (3.2) and linear combinations (with complex coefficients) of the $\Psi_u^{q,v,k}$'s are called the *partially exponential-type functionals* on $C_0[0,T]$. The functionals given by (3.1) are also partially exponential-type functionals because $\Psi_u^{q,v,0}(x) = \Psi_u(x)$ for $\mathfrak{m}$-a.e. $x \in C_0[0,T]$.

For each $(q,v,k) \in \mathbb{R}\times L_2[0,T]\times BV[0,T]$, the class $\mathcal{E}_{q,v,k}$ is dense in $L_2(C_0[0,T])$. Furthermore, $\operatorname{Span}\mathcal{E}_{q,v,k}$, the linear manifold generated by $\mathcal{E}_{q,v,k}$ in $L_2(C_0[0,T])$, is closed under the ordinary multiplication because

$$\Psi_{u_1}^{q,v,k}(x)\Psi_{u_2}^{q,v,k}(x) = \alpha\exp\Big\{\langle u_1+u_2, x\rangle + \frac{i}{2q}\|vk\|_2^2\Big\} = \alpha\Psi_{u_1+u_2}^{q,v,k}(x)$$

for $\mathfrak{m}$-a.e. $y \in C_0[0,T]$, where the complex coefficient α is given by $\exp\{(i/2q)\|vk\|_2^2\}$. Thus the class $\operatorname{Span}\mathcal{E}_{q,v,k}$ is a commutative algebra over the complex field $\mathbb{C}$.

In fact, using the fact that

$$\Psi_{u_1}^{q_1,v_1,k_1}(x)\Psi_{u_2}^{q_2,v_2,k_2}(x) = \beta\exp\big\{\langle u_1+u_2, x\rangle\big\} = \beta\Psi_{u_1+u_2}(x)$$

with

$$\beta = \exp\Big\{\frac{i}{2q_1}\|v_1k_1\|_2^2 + \frac{i}{2q_2}\|v_2k_2\|_2^2\Big\},$$

one can see that

$$\operatorname{Span}\Big(\bigcup_{\substack{q\in\mathbb{R}\\ v\in L_2[0,T]\\ k\in BV[0,T]}} \mathcal{E}_{q,v,k}\Big) = \operatorname{Span}\mathcal{E}.$$

We denote the set of all partially exponential-type functionals on $C_0[0,T]$ by $\mathcal{E}(C_0[0,T])$ (i.e., $\mathcal{E}(C_0[0,T]) = \operatorname{Span}\mathcal{E}$).

Note that every partially exponential-type functional is scale-invariant measurable. Since we will identify functionals which coincide s-a.e. on $C_0[0,T]$, $\mathcal{E}(C_0[0,T])$ can be regarded as the space of all s-equivalence classes of partially exponential-type functionals. Throughout this article, we let equation (3.1) holds for s-a.e. $x \in C_0[0,T]$. Strictly speaking, the quotient space $\mathcal{E}(C_0[0,T])/\approx$ is again denoted by the same symbol $\mathcal{E}(C_0[0,T])$ in the rest of this paper.

From now on, we reveal the main purpose of this paper via specific examples. First, using (2.7) with F replaced with Ψ_u, (2.4), (2.1), it follows that for all $k \in BV[0,T]\setminus\{0\}$,

$$E_x^{\mathrm{an}\,f_q}\big[\Psi_u\big(\mathcal{Z}_k(x,\cdot)\big)\big] = \exp\Big\{\frac{i}{2q}\|uk\|_2^2\Big\}. \tag{3.3}$$

Thus, using equations (2.9), (3.3), and (3.2), we see that the L_1 analytic $\mathcal{Z}_k$-GFFT, $T^{(1)}_{q,k}(\Psi_u)$ of Ψ_u, exists for all $q \in \mathbb{R}\setminus\{0\}$, and it is given by

$$T^{(1)}_{q,k}(\Psi_u)(y) = \Psi_u(y) E^{\mathrm{an}\, f_q}_x\big[\Psi_u\big(\mathcal{Z}_k(x,\cdot)\big)\big] = \Psi^{q,u,k}_u(y) \tag{3.4}$$

for s-a.e. $y \in C_0[0,T]$. From equation (3.4), we also see that $T^{(1)}_{q,k} : \mathcal{E}(C_0[0,T]) \to \mathcal{E}(C_0[0,T])$ is well defined.

Next, using (2.10) with F and G replaced with Ψ_u and Ψ_v, it follows that for all real $q \in \mathbb{R}\setminus\{0\}$ and $g_1, g_2, h_3, h_4 \in BV[0,T]$, the GCP of Ψ_u and Ψ_v, $(\Psi_u * \Psi_v)^{(g_1,g_2;h_3,h_4)}_q$, exists and is given by

$$(\Psi_u * \Psi_v)^{(g_1,g_2;h_1,h_2)}_q(y) = \exp\Big\{\langle ug_1 + vg_2, y\rangle + \frac{i}{2q}\|uh_1 + vh_2\|_2^2\Big\} \tag{3.5}$$

for s-a.e. $y \in C_0[0,T]$. Also, the functional $(\Psi_u * \Psi_v)^{(g_1,g_2;h_3,h_4)}_q \equiv \Psi^{q,uh_1+vh_2,1}_{ug_1+vg_2}$ is an element of $\mathcal{E}(C_0[0,T])$.

Using (3.5) and applying the techniques similar to those used in the calculation of (3.4), one can see that, for s-a.e. $y \in C_0[0,T]$,

$$\begin{aligned}
&T^{(1)}_{q,k}\big((\Psi_u * \Psi_v)^{(g_1,g_2;h_1,h_2)}_q\big)(y)\\
&\quad= \exp\Big\{\langle ug_1 + vg_2, y\rangle + \frac{i}{2q}\|uh_1 + vh_2\|_2^2 + \frac{i}{2q}\big\|(ug_1 + vg_2)k\big\|_2^2\Big\}\\
&\quad= \exp\Big\{\langle ug_1 + vg_2, y\rangle + \frac{i}{q}\int_0^T u(t)v(t)\big(h_1(t)h_2(t) + g_1(t)g_2(t)k^2(t)\big)\,dt\\
&\qquad+ \frac{i}{2q}\int_0^T u^2(t)\big(h_1^2(t) + g_1^2(t)k^2(t)\big)\,dt + \frac{i}{2q}\int_0^T v^2(t)\big(h_2^2(t) + g_2^2(t)k^2(t)\big)\,dt\Big\}.
\end{aligned}$$

In order to obtain an equation similar to (1.1), one may put the condition that

$$h_1(t)h_2(t) + g_1(t)g_2(t)k^2(t) = 0 \quad m_L\text{-a.e. on } [0,T], \tag{3.6}$$

where m_L denotes the Lebesgue measure on $[0,T]$. Then we can expect the following equation:

$$T^{(1)}_{q,k}\big((\Psi_u * \Psi_v)^{(g_1,g_2;h_1,h_2)}_q\big)(y) = T^{(1)}_{q,\mathbf{s}_1}(\Psi_u)\big(\mathcal{Z}_{g_1}(y,\cdot)\big)T^{(1)}_{q,\mathbf{s}_2}(\Psi_v)\big(\mathcal{Z}_{g_2}(y,\cdot)\big) \tag{3.7}$$

for s-a.e. $y \in C_0[0,T]$, where $\mathbf{s}_i$ $(i = 1,2)$ is the function of bounded variation on $[0,T]$ such that $\mathbf{s}_i^2(t) = g_i^2(t)k^2(t) + h_i^2(t)$ for m_L-a.e. on $[0,T]$. On the other hand, using (3.4) and (2.10), we also obtain that for s-a.e. $y \in C_0[0,T]$,

$$\begin{aligned}
&\big(T^{(1)}_{q,k_1}(\Psi_u) * T^{(1)}_{q,k_2}(\Psi_v)\big)^{(g_1,g_2;h_3,h_4)}_q(y)\\
&\quad= \exp\Big\{\langle ug_1 + vg_2, y\rangle + \frac{i}{q}\int_0^T u(t)v(t)h_3(t)h_4(t)\,dt\\
&\qquad+ \frac{i}{2q}\int_0^T u^2(t)\big(h_3^2(t) + k_1^2(t)\big)\,dt + \frac{i}{2q}\int_0^T v^2(t)\big(h_4^2(t) + k_2^2(t)\big)\,dt\Big\},
\end{aligned}$$

and under the condition

$$m_L\big(\mathrm{supp}(h_3) \cap \mathrm{supp}(h_4)\big) = 0 \tag{3.8}$$

it follows that

$$\begin{aligned}
&\big(T_{q,k_1}^{(1)}(\Psi_u) * T_{q,k_2}^{(1)}(\Psi_v)\big)_q^{(g_1,g_2;h_3,h_4)}(y) \\
&\quad = T_{q,\mathbf{s}_3}^{(1)}(\Psi_u)\big(\mathcal{Z}_{g_1}(y,\cdot)\big) T_{q,\mathbf{s}_4}^{(1)}(\Psi_v)\big(\mathcal{Z}_{g_2}(y,\cdot)\big)
\end{aligned} \tag{3.9}$$

for s-a.e. $y \in C_0[0,T]$, where $\mathbf{s}_3$ and $\mathbf{s}_4$ are the functions of bounded variation on $[0,T]$ such that $\mathbf{s}_3^2(t) = h_3^2(t) + k_1^2(t)$ and $\mathbf{s}_4^2(t) = h_4^2(t) + k_2^2(t)$.

In Section 5 below, we establish the relationships appearing in (3.7) and (3.9) for general functionals F and G on Wiener space. Equations (3.6) and (3.8) play key roles in our main theorems (see Theorems 5.3 and 5.5 below).

4. Rotation properties of Gaussian processes

The essential purpose of this section is to establish two rotation properties of Gaussian processes on the product Wiener spaces $C_0^2[0,T]$ and $C_0^3[0,T]$. For nonzero functions h_1 and h_2 in $BV[0,T]$, let $\mathcal{Z}_{h_1}$ and $\mathcal{Z}_{h_2}$ be the Gaussian processes given by (2.2) with h replaced with h_1 and h_2, respectively. Then the process

$$\mathfrak{Z}_{h_1,h_2} : C_0[0,T] \times C_0[0,T] \times [0,T] \to \mathbb{R}$$

given by

$$\mathfrak{Z}_{h_1,h_2}(x_1,x_2,t) = \mathcal{Z}_{h_1}(x_1,t) + \mathcal{Z}_{h_2}(x_2,t)$$

is also a Gaussian process with mean zero and covariance function

$$\begin{aligned}
\mathfrak{v}_{h_1,h_2}\big(\min\{s,t\}\big) &\equiv E_{x_1}\big[E_{x_2}\big[\mathfrak{Z}_{h_1,h_2}(x_1,x_2,s)\mathfrak{Z}_{h_1,h_2}(x_1,x_2,t)\big]\big] \\
&= \beta_{h_1}\big(\min\{s,t\}\big) + \beta_{h_2}\big(\min\{s,t\}\big).
\end{aligned}$$

On the other hand, let h_1 and h_2 be nonzero functions of $BV[0,T]$. Then there exists a nonzero function $\mathbf{s} \in BV[0,T]$ such that

$$\mathbf{s}^2(t) = h_1^2(t) + h_2^2(t) \tag{4.1}$$

for m_L-a.e. $t \in [0,T]$. Note that the function $\mathbf{s}$ satisfying (4.1) is not unique. We will use the symbol $\mathbf{s}(h_1,h_2)$ for the functions $\mathbf{s}$ that satisfy (4.1) above.

Given nonzero functions h_1 and h_2 in $BV[0,T]$, we consider the stochastic process $\mathcal{Z}_{\mathbf{s}(h_1,h_2)}$. Then $\mathcal{Z}_{\mathbf{s}(h_1,h_2)}$ is a Gaussian process with mean zero and covariance

$$\begin{aligned}
&E_x\big[\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,s)\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,t)\big] \\
&\quad = \int_0^{\min\{s,t\}} \mathbf{s}^2(h_1,h_2)(u)\, db(u) \\
&\quad = \int_0^{\min\{s,t\}} \big(h_1^2(u) + h_2^2(u)\big)\, db(u) \\
&\quad = \beta_{h_1}\big(\min\{s,t\}\big) + \beta_{h_2}\big(\min\{s,t\}\big) \\
&\quad = \mathfrak{v}_{h_1,h_2}\big(\min\{s,t\}\big).
\end{aligned}$$

From these facts, one can see that $\mathfrak{Z}_{h_1,h_2}$ and $\mathcal{Z}_{\mathbf{s}(h_1,h_2)}$ have the same distribution and that for any random variable F on $C_0[0,T]$,

$$E_{x_1}\big[E_{x_2}\big[F\big(\mathcal{Z}_{h_1}(x_1,\cdot) + \mathcal{Z}_{h_2}(x_2,\cdot)\big)\big]\big] \overset{*}{=} E_x\big[F\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot)\big)\big], \tag{4.2}$$

where by $\overset{*}{=}$ we mean that if either side exists, then both sides exist and equality holds.

4.1. **A rotation property of Gaussian processes on $C_0^2[0,T]$.** The following lemma will be very useful in the proof of our main theorem in this subsection.

Lemma 4.1. *Given nonzero functions h_1, h_2, h_3, and h_4 in $BV[0,T]$, let the two stochastic processes $\mathfrak{Z}_{h_1,h_2}$ and $\mathfrak{Z}_{h_3,h_4}$ on $C_0^2[0,T]\times[0,T]$ be given by*

$$\mathfrak{Z}_{h_1,h_2}(x_1,x_2,t)=\mathcal{Z}_{h_1}(x_1,t)+\mathcal{Z}_{h_2}(x_2,t) \tag{4.3}$$

and

$$\mathfrak{Z}_{h_3,h_4}(x_1,x_2,t)=\mathcal{Z}_{h_3}(x_1,t)+\mathcal{Z}_{h_4}(x_2,t), \tag{4.4}$$

respectively. Then the following assertions are equivalent:

(i) *$\mathfrak{Z}_{h_1,h_2}$ and $\mathfrak{Z}_{h_3,h_4}$ are independent processes,*
(ii) *$h_1h_3+h_2h_4=0$.*

Proof. Since the processes $\mathfrak{Z}_{h_1,h_2}$ and $\mathfrak{Z}_{h_3,h_4}$ are Gaussian, we know that $\mathfrak{Z}_{h_1,h_2}$ and $\mathfrak{Z}_{h_3,h_4}$ are independent if and only if

$$E_{x_2}\big[E_{x_1}\big[\mathfrak{Z}_{h_1,h_2}(x_1,x_2,s)\mathfrak{Z}_{h_3,h_4}(x_1,x_2,t)\big]\big]=0$$

for all $s,t\in[0,T]$. But using the Fubini theorem and equation (2.3), we have

$$\begin{aligned}
&E_{x_2}\big[E_{x_1}\big[\mathfrak{Z}_{h_1,h_2}(x_1,x_2,s)\mathfrak{Z}_{h_3,h_4}(x_1,x_2,t)\big]\big]\\
&\quad=E_{x_2}\big[E_{x_1}\big[\big(\mathcal{Z}_{h_1}(x_1,s)\mathcal{Z}_{h_3}(x_1,t)+\mathcal{Z}_{h_1}(x_1,s)\mathcal{Z}_{h_4}(x_2,t)\\
&\qquad+\mathcal{Z}_{h_2}(x_2,s)\mathcal{Z}_{h_3}(x_1,t)+\mathcal{Z}_{h_2}(x_2,s)\mathcal{Z}_{h_4}(x_2,t)\big)\big]\big]\\
&\quad=\int_0^{\min\{s,t\}}h_1(u)h_3(u)\,du+\int_0^{\min\{s,t\}}h_2(u)h_4(u)\,du\\
&\quad=\int_0^{\min\{s,t\}}\big(h_1(u)h_3(u)+h_2(u)h_4(u)\big)\,du.
\end{aligned}$$

From this we can obtain the desired result. □

Theorem 4.2. *Let h_1, h_2, h_3, and h_4 be nonzero functions in $BV[0,T]$ with*

$$h_1h_3+h_2h_4=0,$$

and let $\mathbf{F}:C_0^2[0,T]\to\mathbb{C}$ be a $\mathfrak{m}\times\mathfrak{m}$-integrable functional. Then

$$\begin{aligned}
&E_{x_1}\big[E_{x_2}\big[\mathbf{F}\big(\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot),\mathcal{Z}_{h_3}(x_1,\cdot)+\mathcal{Z}_{h_4}(x_2,\cdot)\big)\big]\big]\\
&\quad=E_y\big[E_x\big[\mathbf{F}\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot),\mathcal{Z}_{\mathbf{s}(h_3,h_4)}(y,\cdot)\big)\big]\big].
\end{aligned}$$

Proof. Let the processes $\mathfrak{Z}_{h_1,h_2},\mathfrak{Z}_{h_3,h_4}:C_0^2[0,T]\times[0,T]\to\mathbb{R}$ be given by equation (4.3) and (4.4) respectively. Since h_i's are functions of bounded variation, for all $(x_1,x_2)\in C_0^2[0,T]$ the sample paths $\mathfrak{Z}_{h_1,h_2}(x_1,x_2,\cdot)$ and $\mathfrak{Z}_{h_3,h_4}(x_1,x_2,\cdot)$ of the processes are continuous functions on $[0,T]$. Let X_{h_1,h_2} and X_{h_3,h_4} be measurable transforms from $C_0^2[0,T]$ into $C_0^2[0,T]$ given by

$$X_{h_1,h_2}(x_1,x_2)=\mathfrak{Z}_{h_1,h_2}(x_1,x_2,\cdot)$$

and

$$X_{h_3,h_4}(x_1,x_2) = \mathfrak{Z}_{h_3,h_4}(x_1,x_2,\cdot),$$

respectively. Also let $P \equiv X_{h_1,h_2}(C_0^2[0,T])$ and $Q \equiv X_{h_3,h_4}(C_0^2[0,T])$ be the image spaces of the measurable transforms X_{h_1,h_2} and X_{h_3,h_4} respectively. For simplicity, let $\mathfrak{m}^2$ denote the product Wiener measure $\mathfrak{m} \times \mathfrak{m}$ on $C_0^2[0,T]$.

By Lemma 4.1, we see that $\mathfrak{Z}_{h_1,h_2}$ and $\mathfrak{Z}_{h_3,h_4}$ are independent processes on $C_0^2[0,T]$ and so X_{h_1,h_2} and X_{h_3,h_4} are independent measurable transforms. Thus, by the change of variables formula, the Fubini theorem and (4.2), it follows that

$$\begin{aligned}
&E_{x_2}\big[E_{x_1}\big[\mathbf{F}\big(\mathfrak{Z}_{h_1,h_2}(x_1,x_2,\cdot),\mathfrak{Z}_{h_3,h_4}(x_1,x_2,\cdot)\big)\big]\big]\\
&\quad= \int_{C_0^2[0,T]} \mathbf{F}\big(X_{h_1,h_2}(x_1,x_2),X_{h_3,h_4}(x_1,x_2)\big)\, d\mathfrak{m}^2(x_1,x_2)\\
&\quad= \int_{P\times Q} \mathbf{F}(z_1,z_2)\, d\big[(\mathfrak{m}^2\circ X_{h_1,h_2}^{-1})\times(\mathfrak{m}^2\circ X_{h_3,h_4}^{-1})\big](z_1,z_2)\\
&\quad= \int_Q \Big[\int_P \mathbf{F}(z_1,z_2)\, d(\mathfrak{m}^2\circ X_{h_1,h_2}^{-1})(z_1)\Big]\, d(\mathfrak{m}^2\circ X_{h_3,h_4}^{-1})(z_2)\\
&\quad= \int_Q \Big[\int_{C_0^2[0,T]} \mathbf{F}\big(X_{h_1,h_2}(x_1,x_2),z_2\big)\, d\mathfrak{m}^2(x_1,x_2)\Big]\, d(\mathfrak{m}^2\circ X_{h_3,h_4}^{-1})(z_2)\\
&\quad= \int_Q \Big[\int_{C_0^2[0,T]} \mathbf{F}\big(\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot),z_2\big)\, d\mathfrak{m}^2(x_1,x_2)\Big]\, d(\mathfrak{m}^2\circ X_{h_3,h_4}^{-1})(z_2)\\
&\quad= \int_Q \Big[\int_{C_0[0,T]} \mathbf{F}\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot),z_2\big)\, d\mathfrak{m}(x)\Big]\, d(\mathfrak{m}^2\circ X_{h_3,h_4}^{-1})(z_2)\\
&\quad= \int_{C_0[0,T]} \Big[\int_Q \mathbf{F}\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot),z_2\big)\, d(\mathfrak{m}^2\circ X_{h_3,h_4}^{-1})(z_2)\Big]\, d\mathfrak{m}(x)\\
&\quad= \int_{C_0[0,T]} \Big[\int_{C_0^2[0,T]} \mathbf{F}\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot),X_{h_3,h_4}(x_1,x_2)\big)\, d\mathfrak{m}^2(x_1,x_2)\Big]\, d\mathfrak{m}(x)\\
&\quad= \int_{C_0[0,T]} \Big[\int_{C_0^2[0,T]} \mathbf{F}\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot),\mathcal{Z}_{h_3}(x_1,\cdot)+\mathcal{Z}_{h_4}(x_2,\cdot)\big)\, d\mathfrak{m}^2(x_1,x_2)\Big]\, d\mathfrak{m}(x)\\
&\quad= \int_{C_0[0,T]} \Big[\int_{C_0[0,T]} \mathbf{F}\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot),\mathcal{Z}_{\mathbf{s}(h_3,h_4)}(y,\cdot)\big)\, d\mathfrak{m}(y)\Big]\, d\mathfrak{m}(x).
\end{aligned}$$

Thus we obtain the desired result. □

The following corollaries are very simple consequences of Theorem 4.2.

Corollary 4.3. *Let h_1 and h_2 be nonzero functions in $BV[0,T]$, and let $\mathbf{F}: C_0^2[0,T] \to \mathbb{C}$ be a $\mathfrak{m}^2$-integrable functional. Then*

$$\begin{aligned}
&E_{x_2}\big[E_{x_1}\big[\mathbf{F}\big(\mathcal{Z}_{h_1}(x_1,\cdot)-\mathcal{Z}_{h_2}(x_2,\cdot),\mathcal{Z}_{h_2}(x_1,\cdot)+\mathcal{Z}_{h_1}(x_2,\cdot)\big)\big]\big]\\
&\quad= E_y\big[E_x\big[\mathbf{F}\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot),\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(y,\cdot)\big)\big]\big].
\end{aligned} \tag{4.5}$$

Remark 4.4. For any function $\theta(\cdot)$ of bounded variation, choosing $h_1(t) = \cos\theta(t)$ and $h_2(t) = \sin\theta(t)$ on $[0,T]$ in equation (4.5) yields the main result in [1, p. 130].

Corollary 4.5. *Let h_1, h_2, h_3, and h_4 be as in Theorem 4.2. Let F and G be $\mathfrak{m}$-integrable functionals. Then*

$$\begin{aligned}
&E_{x_1}\big[E_{x_2}\big[F\big(\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot)\big)G\big(\mathcal{Z}_{h_3}(x_1,\cdot)+\mathcal{Z}_{h_4}(x_2,\cdot)\big)\big]\big] \\
&\quad = E_x\big[F\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot)\big)\big]E_x\big[G\big(\mathcal{Z}_{\mathbf{s}(h_3,h_4)}(x,\cdot)\big)\big].
\end{aligned} \tag{4.6}$$

4.2. A rotation property of Gaussian processes on $C_0^3[0,T]$.

Lemma 4.6. *Given nonzero functions h_1, h_2, h_3, and h_4 in $BV[0,T]$, let the two stochastic processes $\mathfrak{Z}_{h_1,h_2,0}$ and $\mathfrak{Z}_{h_3,0,h_4}$ on $C_0^3[0,T]\times[0,T]$ be given by*

$$\mathfrak{W}_{h_1,h_2,0}(x_1,x_2,x_3,t)=\mathcal{Z}_{h_1}(x_1,t)+\mathcal{Z}_{h_2}(x_2,t) \tag{4.7}$$

and

$$\mathfrak{W}_{h_3,0,h_4}(x_1,x_2,x_3,t)=\mathcal{Z}_{h_3}(x_1,t)+\mathcal{Z}_{h_4}(x_3,t), \tag{4.8}$$

respectively. If $m_L(\mathrm{supp}(h_1)\cap\mathrm{supp}(h_3))=0$, then $\mathfrak{W}_{h_1,h_2,0}$ and $\mathfrak{W}_{h_3,0,h_4}$ are independent processes.

Remark 4.7. By the consistency property, the processes $\mathfrak{W}_{h_1,h_2,0}$ and $\mathfrak{W}_{h_1,0,h_3}$ can be considered as processes on $C_0^2[0,T]\times[0,T]$.

Proof of Lemma 4.6. Since the processes $\mathfrak{W}_{h_1,h_2,0}$ and $\mathfrak{W}_{h_3,h_4}$ are Gaussian, we know that $\mathfrak{W}_{h_1,h_2,0}$ and $\mathfrak{W}_{h_3,0,h_4}$ are independent if and only if

$$E_{x_3}\big[E_{x_2}\big[E_{x_1}\big[\mathfrak{W}_{h_1,h_2,0}(x_1,x_2,x_3,s)\mathfrak{W}_{h_3,0,h_4}(x_1,x_2,x_3,t)\big]\big]\big]=0$$

for all $s,t\in[0,T]$. But using the Fubini theorem and equation (2.3), we have

$$\begin{aligned}
&E_{x_3}\big[E_{x_2}\big[E_{x_1}\big[\mathfrak{W}_{h_1,h_2,0}(x_1,x_2,x_3,s)\mathfrak{W}_{h_3,0,h_4}(x_1,x_2,x_3,t)\big]\big]\big] \\
&\quad = E_{x_3}\big[E_{x_2}\big[E_{x_1}\big[\mathcal{Z}_{h_1}(x_1,s)\mathcal{Z}_{h_3}(x_1,t)+\mathcal{Z}_{h_1}(x_1,s)\mathcal{Z}_{h_4}(x_3,t) \\
&\qquad + \mathcal{Z}_{h_2}(x_2,s)\mathcal{Z}_{h_3}(x_1,t)+\mathcal{Z}_{h_2}(x_2,s)\mathcal{Z}_{h_4}(x_3,t)\big]\big]\big] \\
&\quad = \int_0^{\min\{s,t\}} h_1(u)h_3(u)\,du.
\end{aligned}$$

From this we can obtain the desired result. □

Theorem 4.8. *Let h_1, h_2, h_3, and h_4 be nonzero functions in $BV[0,T]$ with*

$$m_L\big(\mathrm{supp}(h_1)\cap\mathrm{supp}(h_3)\big)=0,$$

and let $\mathbf{F}: C_0^2[0,T]\to\mathbb{C}$ be a $\mathfrak{m}^2$-integrable functional. Then

$$\begin{aligned}
&E_{x_3}\big[E_{x_2}\big[E_{x_1}\big[\mathbf{F}\big(\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot),\mathcal{Z}_{h_3}(x_1,\cdot)+\mathcal{Z}_{h_4}(x_3,\cdot)\big)\big]\big]\big] \\
&\quad = E_y\big[E_x\big[\mathbf{F}\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot),\mathcal{Z}_{\mathbf{s}(h_3,h_4)}(y,\cdot)\big)\big]\big].
\end{aligned}$$

Proof. Let the processes $\mathfrak{W}_{h_1,h_2,0},\mathfrak{W}_{h_3,0,h_4}: C_0^3[0,T]\times[0,T]\to\mathbb{R}$ be given by equation (4.7) and (4.8), respectively. Since h_i's are functions of bounded variation, for all $(x_1,x_2,x_3)\in C_0^3[0,T]$ the sample paths $\mathfrak{W}_{h_1,h_2,0}(x_1,x_2,x_3,\cdot)$ and $\mathfrak{W}_{h_3,0,h_4}(x_1,x_2,x_3,\cdot)$ of the processes are continuous functions on $[0,T]$. Let $Y_{h_1,h_2,0}$ and $Y_{h_3,0,h_4}$ be measurable transforms from $C_0^3[0,T]$ into $C_0^3[0,T]$ given by

$$Y_{h_1,h_2,0}(x_1,x_2,x_3)=\mathfrak{W}_{h_1,h_2,0}(x_1,x_2,x_3,\cdot)$$

and

$$Y_{h_3,0,h_4}(x_1,x_2,x_3)=\mathfrak{W}_{h_3,0,h_4}(x_1,x_2,x_3,\cdot),$$

respectively. Also let $M\equiv Y_{h_1,h_2,0}(C_0^3[0,T])$ and $N\equiv Y_{h_3,0,h_4}(C_0^3[0,T])$ be the image spaces of the measurable transforms $Y_{h_1,h_2,0}$ and $Y_{h_3,0,h_4}$, respectively. For simplicity, let $\mathfrak{m}^3$ denote the product Wiener measure $\mathfrak{m}\times\mathfrak{m}\times\mathfrak{m}$ on $C_0^3[0,T]$.

By Lemma 4.6, we see that $\mathfrak{W}_{h_1,h_2,0}$ and $\mathfrak{W}_{h_3,0,h_4}$ are independent processes on $C_0^3[0,T]$ and so $Y_{h_1,h_2,0}$ and $Y_{h_3,0,h_4}$ are independent measurable transforms. Thus, by the change of variables formula, the Fubini theorem, and (4.2), it follows that

$$\begin{aligned}
&E_{x_3}\big[E_{x_2}\big[E_{x_1}\big[\mathbf{F}\big(\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot),\mathcal{Z}_{h_3}(x_1,\cdot)+\mathcal{Z}_{h_4}(x_3,\cdot)\big)\big]\big]\big]\\
&\quad=\int_{C_0^3[0,T]}\mathbf{F}\big(Y_{h_1,h_2,0}(x_1,x_2,x_3),Y_{h_3,0,h_4}(x_1,x_2,x_3)\big)\,d\mathfrak{m}^3(x_1,x_2,x_3)\\
&\quad=\int_{N\times M}\mathbf{F}(w_1,w_2)\,d\big[(\mathfrak{m}^3\circ Y^{-1}_{h_1,h_2,0})\times(\mathfrak{m}^3\circ Y^{-1}_{h_3,0,h_4})\big](w_1,w_2)\\
&\quad=\int_N\Big[\int_M\mathbf{F}(w_1,w_2)\,d(\mathfrak{m}^3\circ Y^{-1}_{h_1,h_2,0})(w_1)\Big]\,d(\mathfrak{m}^3\circ Y^{-1}_{h_3,0,h_4})(w_2)\\
&\quad=\int_N\Big[\int_{C_0^3[0,T]}\mathbf{F}\big(Y_{h_1,h_2,0}(x_1,x_2,x_3),w_2\big)\,d\mathfrak{m}^3(x_1,x_2,x_3)\Big]\,d(\mathfrak{m}^3\circ Y^{-1}_{h_3,0,h_4})(w_2)\\
&\quad=\int_N\Big[\int_{C_0^2[0,T]}\mathbf{F}\big(\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot),w_2\big)\,d\mathfrak{m}^2(x_1,x_2)\Big]\,d(\mathfrak{m}^3\circ Y^{-1}_{h_3,0,h_4})(w_2)\\
&\quad=\int_N\Big[\int_{C_0[0,T]}\mathbf{F}\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot),w_2\big)\,d\mathfrak{m}(x)\Big]\,d(\mathfrak{m}^3\circ Y^{-1}_{h_3,0,h_4})(w_2)\\
&\quad=\int_{C_0[0,T]}\Big[\int_N\mathbf{F}\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot),w_2\big)\,d(\mathfrak{m}^3\circ Y^{-1}_{h_3,0,h_4})(w_2)\Big]\,d\mathfrak{m}(x)\\
&\quad=\int_{C_0[0,T]}\Big[\int_{C_0^3[0,T]}\mathbf{F}\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot),Y_{h_3,0,h_4}(x_1,x_2,x_3)\big)\,d\mathfrak{m}^3(x_1,x_2,x_3)\Big]\,d\mathfrak{m}(x)\\
&\quad=\int_{C_0[0,T]}\Big[\int_{C_0^2[0,T]}\mathbf{F}\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot),\mathcal{Z}_{h_3}(x_1,\cdot)+\mathcal{Z}_{h_4}(x_3,\cdot)\big)\,d\mathfrak{m}^2(x_1,x_3)\Big]\,d\mathfrak{m}(x)\\
&\quad=\int_{C_0[0,T]}\Big[\int_{C_0[0,T]}\mathbf{F}\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot),\mathcal{Z}_{\mathbf{s}(h_3,h_4)}(y,\cdot)\big)\,d\mathfrak{m}(y)\Big]\,d\mathfrak{m}(x).
\end{aligned}$$

Thus we obtain the desired result. □

The following corollary is a very simple consequence of Theorem 4.8.

Corollary 4.9. *Let h_1, h_2, h_3, and h_4 be as in Theorem 4.8. Let F and G be $\mathfrak{m}$-integrable functionals. Then*

$$\begin{aligned}
&E_{x_3}\big[E_{x_2}\big[E_{x_1}\big[F\big(\mathcal{Z}_{h_1}(x_1,\cdot)+\mathcal{Z}_{h_2}(x_2,\cdot)\big)G\big(\mathcal{Z}_{h_3}(x_1,\cdot)+\mathcal{Z}_{h_4}(x_3,\cdot)\big)\big]\big]\big]\\
&\quad=E_x\big[F\big(\mathcal{Z}_{\mathbf{s}(h_1,h_2)}(x,\cdot)\big)\big]E_x\big[G\big(\mathcal{Z}_{\mathbf{s}(h_3,h_4)}(x,\cdot)\big)\big].
\end{aligned}\tag{4.9}$$

5. Fourier–Feynman transforms and convolution products

In this section, we will establish that the GFFT of the GCP is a product of the GFFTs (see Theorem 5.3 below) and that the GCP of the GFFTs is also a product of the GFFTs (see Theorem 5.5 below).

5.1. **Transform of generalized convolution products.** It will be helpful to establish the following lemma before giving the first theorem in this section.

Lemma 5.1. *Let g_1, g_2, h_1, h_2, and k be nonzero functions in $BV[0,T]$. For each $j \in \{1,2\}$, let $\mathbf{s}(g_jh_j,h_j)$ be the function in $BV[0,T]$ satisfying equation (4.1) with h_1 and h_2 replaced with g_jhj and h_j. Also, let F and G be $\mathbb{C}$-valued scale-invariant measurable functionals on $C_0[0,T]$ such that the analytic transforms $T_{\lambda,\mathbf{s}(g_1k,h_1)}(F)(y)$, $T_{\lambda,\mathbf{s}(g_2k,h_2)}(G)(y)$, and the GCP $(F*G)_\lambda^{(g_1,g_2;h_1,h_2)}(y)$ exist for every $\lambda\in\mathbb{C}$ and s-a.e. $y\in C_0[0,T]$. Furthermore, assume that given nonzero function k in $BV[0,T]$, the analytic transform of $(F*G)_{\lambda_1}^{(g_1,g_2;h_1,h_2)}$, $T_{\lambda_2,k}((F*G)_{\lambda_1}^{(g_1,g_2;h_1,h_2)})(y)$ exists for every $(\lambda_1,\lambda_2)\in\mathbb{C}_+\times\mathbb{C}_+$ and s-a.e. $y\in C_0[0,T]$. Suppose that*

$$g_1g_2k^2+h_1h_2=0.$$

Then for each $\lambda\in\mathbb{C}_+$ and s-a.e. $y\in C_0[0,T]$,

$$\begin{aligned}
&T_{\lambda,k}\big((F*G)_\lambda^{(g_1,g_2;h_1,h_2)}\big)(y)\\
&\quad=T_{\lambda,\mathbf{s}(g_1k,h_1)}(F)\big(\mathcal{Z}_{g_1}(y,\cdot)\big)T_{\lambda,\mathbf{s}(g_2k,h_2)}(G)\big(\mathcal{Z}_{g_2}(y,\cdot)\big).
\end{aligned}\tag{5.1}$$

Remark 5.2 (Comments on the assumptions in Lemma 5.1). Let a function $y\in C_0[0,T]$ be given. For $(\lambda_1,\rho_2)\in\mathbb{C}_+\times(0,+\infty)$, let

$$\begin{aligned}
&J_{\lambda_1^*}(k;\rho_2)\\
&\quad\equiv T_{k,\rho_2}\big((F*G)_{\lambda_1}^{(g_1,g_2;h_1,h_2)}\big)(y)\\
&\quad=E_{x_2}\big[(F*G)_{\lambda_1}^{(g_1,g_2;h_1,h_2)}\big(y+\rho_2^{-1/2}\mathcal{Z}_k(x_2,\cdot)\big)\big]\\
&\quad=E_{x_2}\big[E_{x_1}^{\mathrm{an}\,w_{\lambda_1}}\big[F\big(\mathcal{Z}_{g_1}\big(y+\rho_2^{-1/2}\mathcal{Z}_k(x_2,\cdot)\big)+\mathcal{Z}_{h_1}(x_1,\cdot)\big)\\
&\qquad\times G\big(\mathcal{Z}_{g_2}\big(y+\rho_2^{-1/2}\mathcal{Z}_k(x_2,\cdot)\big)+\mathcal{Z}_{h_2}(x_1,\cdot)\big)\big]\big]\\
&\quad=E_{x_2}\big[E_{x_1}^{\mathrm{an}\,w_{\lambda_1}}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\rho_2^{-1/2}\mathcal{Z}_{g_1k}(x_2,\cdot)+\mathcal{Z}_{h_1}(x_1,\cdot)\big)\\
&\qquad\times G\big(\mathcal{Z}_{g_2}(y,\cdot)+\rho_2^{-1/2}\mathcal{Z}_{g_2k}(x_2,\cdot)+\mathcal{Z}_{h_2}(x_1,\cdot)\big)\big]\big].
\end{aligned}$$

For $(\rho_1,\lambda_2)\in(0,+\infty)\times\mathbb{C}_+$, let

$$\begin{aligned}
&J_{\lambda_2^*}(h_1,h_2;\rho_1)\\
&\quad\equiv T_{k,\lambda_2}\big((F*G)_{\rho_1}^{(g_1,g_2;h_1,h_2)}\big)(y)\\
&\quad= E_{x_2}^{\mathrm{an}\,w_{\lambda_2}}\big[(F*G)_{\rho_1}^{(g_1,g_2;h_1,h_2)}\big(y+\mathcal{Z}_k(x_2,\cdot)\big)\big]\\
&\quad= E_{x_2}^{\mathrm{an}\,w_{\lambda_2}}\big[E_{x_1}\big[F\big(\mathcal{Z}_{g_1}\big(y+\mathcal{Z}_k(x_2,\cdot)\big)+\rho_1^{-1/2}\mathcal{Z}_{h_1}(x_1,\cdot)\big)\\
&\qquad\times G\big(\mathcal{Z}_{g_2}\big(y+\mathcal{Z}_k(x_2,\cdot)\big)+\rho_1^{-1/2}\mathcal{Z}_{h_2}(x_1,\cdot)\big)\big]\big]\\
&\quad= E_{x_2}^{\mathrm{an}\,w_{\lambda_2}}\big[E_{x_1}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\mathcal{Z}_{g_1k}(x_2,\cdot)+\rho_1^{-1/2}\mathcal{Z}_{h_1}(x_1,\cdot)\big)\\
&\qquad\times G\big(\mathcal{Z}_{g_2}(y,\cdot)+\mathcal{Z}_{g_2k}(x_2,\cdot)+\rho_1^{-1/2}\mathcal{Z}_{h_2}(x_1,\cdot)\big)\big]\big].
\end{aligned}$$

Finally, for $(\rho_1,\rho_2)\in(0,+\infty)\times(0,+\infty)$, let

$$\begin{aligned}
&J_{(F,G)}(k,h_1,h_2;\rho_1,\rho_2)\\
&\quad\equiv T_{k,\rho_2}\big((F*G)_{\rho_1}^{(g_1,g_2;h_1,h_2)}\big)(y)\\
&\quad= E_{x_2}\big[(F*G)_{\rho_1}^{(g_1,g_2;h_1,h_2)}\big(y+\rho_2^{-1/2}\mathcal{Z}_k(x_2,\cdot)\big)\big]\\
&\quad= E_{x_2}\big[E_{x_1}\big[F\big(\mathcal{Z}_{g_1}\big(y+\rho_2^{-1/2}\mathcal{Z}_k(x_2,\cdot)\big)+\rho_1^{-1/2}\mathcal{Z}_{h_1}(x_1,\cdot)\big)\\
&\qquad\times G\big(\mathcal{Z}_{g_2}\big(y+\rho_2^{-1/2}\mathcal{Z}_k(x_2,\cdot)\big)+\rho_1^{-1/2}\mathcal{Z}_{h_2}(x_1,\cdot)\big)\big]\big]\\
&\quad= E_{x_2}\big[E_{x_1}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\rho_2^{-1/2}\mathcal{Z}_{g_1k}(x_2,\cdot)+\rho_1^{-1/2}\mathcal{Z}_{h_1}(x_1,\cdot)\big)\\
&\qquad\times G\big(\mathcal{Z}_{g_2}(y,\cdot)+\rho_2^{-1/2}\mathcal{Z}_{g_2k}(x_2,\cdot)+\rho_1^{-1/2}\mathcal{Z}_{h_2}(x_1,\cdot)\big)\big]\big].
\end{aligned}$$

Next, let $J_{\lambda_1^*}^*(k;\lambda_2)$, $\lambda_2\in\mathbb{C}_+$, denote the analytic continuation of $J_{\lambda_1^*}(k;\rho_2)$; let $J_{\lambda_2^*}^*(h_1,h_2;\lambda_1)$, $\lambda_1\in\mathbb{C}_+$, denote the analytic continuation of $J_{\lambda_2^*}(h_1,h_2;\rho_1)$; and let $J_{(F,G)}^{**}(k,h_1,h_2;\cdot,\cdot)$ denote the analytic continuation of $J_{(F,G)}(k,h_1,h_2;\rho_1,\rho_2)$ on $\mathbb{C}_+\times\mathbb{C}_+$. From the assumptions in Lemma 5.1, one can see that the three analytic Wiener integrals $J_{\lambda_1^*}^*(k;\lambda_2)$, $J_{\lambda_2^*}^*(h_1,h_2;\lambda_1)$, and $J_{(F,G)}^{**}(k,h_1,h_2;\lambda_1,\lambda_2)$ all exist, and that

$$J_{\lambda_1^*}^*(k;\lambda_2)=J_{\lambda_2^*}^*(h_1,h_2;\lambda_1)=J_{(F,G)}^{**}(k,h_1,h_2;\lambda_1,\lambda_2) \tag{5.2}$$

for all $(\lambda_1,\lambda_2)\in\mathbb{C}_+\times\mathbb{C}_+$.

Proof of Lemma 5.1. In view of equations (2.8) and (2.10), we first note that the existences of the analytic Wiener integrals

$$T_{\lambda,\mathbf{s}(g_1k,h_1)}(F)(y),T_{\lambda,\mathbf{s}(g_2k,h_2)}(G)(y),(F*G)_\lambda^{(g_1,g_2;h_1,h_2)}(y)$$

and

$$T_{\lambda_2,k}\big((F*G)_{\lambda_1}^{(g_1,g_2;h_1,h_2)}\big)(y)$$

guarantee that the five Wiener integrals

(i) $E_x\big[F\big(y+\lambda^{-1/2}\mathcal{Z}_{\mathbf{s}(g_1k,h_1)}(x,\cdot)\big)\big]$,

(ii) $E_x\big[F\big(y+\lambda^{-1/2}\mathcal{Z}_{\mathbf{s}(g_2k,h_2)}(x,\cdot)\big)\big]$,

(iii) $E_x\big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\lambda^{-1/2}\mathcal{Z}_{h_1}(x,\cdot)\big)G(\mathcal{Z}_{g_2}(y,\cdot)+\lambda^{-1/2}\mathcal{Z}_{h_2}(x,\cdot)\big]$,

$$\text{(iv)}\quad E_{x_2}\big[E_{x_1}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\lambda_2^{-1/2}\mathcal{Z}_{g_1k}(x_2,\cdot)+\lambda_1^{-1/2}\mathcal{Z}_{h_1}(x_1,\cdot)\big) \\ \times G\big(\mathcal{Z}_{g_2}(y,\cdot)+\lambda_2^{-1/2}\mathcal{Z}_{g_2k}(x_2,\cdot)+\lambda_1^{-1/2}\mathcal{Z}_{h_2}(x_1,\cdot)\big)\big]\big],$$

and

$$\text{(v)}\quad E_{x_2}\big[E_{x_1}^{\mathrm{an}\zeta_1}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\zeta_2^{-1/2}\mathcal{Z}_{g_1k}(x_2,\cdot)+\mathcal{Z}_{h_1}(x_1,\cdot)\big) \\ \times G\big(\mathcal{Z}_{g_2}(y,\cdot)+\zeta_2^{-1/2}\mathcal{Z}_{g_2k}(x_2,\cdot)+\mathcal{Z}_{h_2}(x_1,\cdot)\big)\big]\big]$$

all exist for any $\lambda>0, \lambda_1>0$, $\lambda_2>0$, $\zeta_1\in\mathbb{C}_+$, $\zeta_2>0$, and s-a.e. $y\in C_0[0,T]$.

Next, the existence of the analytic Wiener integral

$$\begin{aligned}\mathfrak{J}(\lambda_1,\lambda_2) &\equiv T_{\lambda_2,k}\big((F*G)_{\lambda_1}^{(g_1,g_2;h_1,h_2)}\big)\\ &= E_{x_2}^{\mathrm{an}\,w_{\lambda_2}}\big[E_{x_1}^{\mathrm{an}\,w_{\lambda_1}}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\mathcal{Z}_{g_1k}(x_2,\cdot)+\mathcal{Z}_{h_1}(x_1,\cdot)\big)\\ &\qquad\times G\big(\mathcal{Z}_{g_2}(y,\cdot)+\mathcal{Z}_{g_2k}(x_2,\cdot)+\mathcal{Z}_{h_2}(x_1,\cdot)\big)\big]\big]\end{aligned}\tag{5.3}$$

also ensures that the analytic Wiener integral

$$\mathfrak{J}(\lambda,\lambda)=E_{x_2}^{\mathrm{an}\,w_{\lambda}}\big[E_{x_1}^{\mathrm{an}\,w_{\lambda}}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\mathcal{Z}_{g_1k}(x_2,\cdot)+\mathcal{Z}_{h_1}(x_1,\cdot)\big) \\ \times G\big(\mathcal{Z}_{g_2}(y,\cdot)+\mathcal{Z}_{g_2k}(x_2,\cdot)+\mathcal{Z}_{h_2}(x_1,\cdot)\big)\big]\big]$$

is well-defined for all $\lambda\in\mathbb{C}_+$. In equation (5.3) above, by the observation in Remark 5.2, we see that $\mathfrak{J}(\lambda_1,\lambda_2)$ means the three analytic function space integrals in equation (5.2) above. On the other hand, using the Fubini theorem and (4.6), it follows that for all $\lambda>0$ and s-a.e. $y\in C_0[0,T]$,

$$\begin{aligned}&T_{\lambda,k}\big((F*G)_{\lambda}^{(g_1,g_2;h_1,h_2)}\big)(y)\\ &\quad=\mathfrak{J}(\lambda,\lambda)\\ &\quad=E_{x_2}\big[E_{x_1}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\lambda^{-1/2}\mathcal{Z}_{g_1k}(x_2,\cdot)+\lambda^{-1/2}\mathcal{Z}_{h_1}(x_1,\cdot)\big)\\ &\qquad\times G\big(\mathcal{Z}_{g_2}(y,\cdot)+\lambda^{-1/2}\mathcal{Z}_{g_2k}(x_2,\cdot)+\lambda^{-1/2}\mathcal{Z}_{h_2}(x_1,\cdot)\big)\big]\big]\\ &\quad=E_{x_2}\big[E_{x_1}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\lambda^{-1/2}\big[\mathcal{Z}_{g_1k}(x_2,\cdot)+\mathcal{Z}_{h_1}(x_1,\cdot)\big]\big)\\ &\qquad\times G\big(\mathcal{Z}_{g_2}(y,\cdot)+\lambda^{-1/2}\big[\mathcal{Z}_{g_2k}(x_2,\cdot)+\mathcal{Z}_{h_2}(x_1,\cdot)\big]\big)\big]\big]\\ &\quad=E_{x}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\lambda^{-1/2}\mathcal{Z}_{\mathbf{s}(g_1k,h_1)}(x,\cdot)\big)\big]\\ &\qquad\times E_{x}\big[G\big(\mathcal{Z}_{g_2}(y,\cdot)+\lambda^{-1/2}\mathcal{Z}_{\mathbf{s}(g_2k,h_2)}(x,\cdot)\big)\big]\\ &\quad=T_{\lambda,\mathcal{Z}_{\mathbf{s}(g_1k,h_1)}}(F)\big(\mathcal{Z}_{g_1}(y,\cdot)\big)T_{\lambda,\mathcal{Z}_{\mathbf{s}(g_2k,h_2)}}(G)\big(\mathcal{Z}_{g_2}(y,\cdot)\big).\end{aligned}$$

We now use the analytic continuation to obtain our desired conclusion. □

Theorem 5.3. *Let g_1, g_2, h_1, h_2, k, $\mathbf{s}(g_1h_1,h_1)$, and $\mathbf{s}(g_2h_2,h_2)$ be as in Lemma 5.1. Let q be a nonzero real number, and let F and G be $\mathbb{C}$-valued scale-invariant measurable functionals on $C_0[0,T]$ such that the L_1 analytic GFFTs $T^{(1)}_{q,\mathbf{s}(g_1k,h_1)}(F)(y)$, $T^{(1)}_{q,\mathbf{s}(g_2k,h_2)}(G)(y)$, and the GCP $(F*G)_q^{(g_1,g_2;h_1,h_2)}(y)$ exist for s-a.e. $y\in C_0[0,T]$. Furthermore, assume that given nonzero function k in*

$BV[0,T]$, the analytic GFFT of $(F*G)_q^{(g_1,g_2;h_1,h_2)}$, $T_{q,k}^{(1)}((F*G)_q^{(g_1,g_2;h_1,h_2)})(y)$ exists for s-a.e. $y \in C_0[0,T]$. Now suppose that

$$g_1 g_2 k^2 + h_1 h_2 = 0.$$

Then for s-a.e. $y \in C_0[0,T]$,

$$\begin{aligned} &T_{q,k}^{(1)}\big((F*G)_q^{(g_1,g_2;h_1,h_2)}\big)(y) \\ &\quad = T_{q,\mathbf{s}(g_1k,h_1)}^{(1)}(F)\big(\mathcal{Z}_{g_1}(y,\cdot)\big)T_{q,\mathbf{s}(g_2k,h_2)}^{(1)}(G)\big(\mathcal{Z}_{g_2}(y,\cdot)\big). \end{aligned} \tag{5.4}$$

Remark 5.4 (Comments on the assumptions in Theorem 5.3)*.* Before giving the proof of Theorem 5.3, we will emphasize the following assertions.

(i) The existence conditions for

$$T_{q,\mathbf{s}(g_1k,h_1)}^{(1)}(F), \qquad T_{q,\mathbf{s}(g_2k,h_2)}^{(1)}(G) \qquad \text{and} \qquad (F*G)_q^{(g_1,g_2;h_1,h_2)}$$

say that

$$T_{\lambda,\mathbf{s}(g_1k,h_1)}(F)(y), \qquad T_{\lambda,\mathbf{s}(g_2k,h_2)}(G)(y) \qquad \text{and} \qquad (F*G)_\lambda^{(g_1,g_2;h_1,h_2)}(y)$$

all exist for all $\lambda \in \mathbb{C}_+$ and s-a.e. $y \in C_0[0,T]$.

(ii) The existence conditions for $(F*G)_q^{(g_1,g_2;h_1,h_2)}$ and $T_{q,k}^{(1)}((F*G)_q^{(g_1,g_2;h_1,h_2)})$ say that

- $T_{\lambda,k}((F*G)_q^{(g_1,g_2;h_1,h_2)})(y)$ exists for every $\lambda \in \mathbb{C}_+$ and s-a.e. $y \in C_0[0,T]$; and
- $T_{\lambda_2,k}((F*G)_{\lambda_1}^{(g_1,g_2;h_1,h_2)})(y)$ exists for every $(\lambda_1,\lambda_2) \in \mathbb{C}_+ \times \mathbb{C}_+$ and s-a.e. $y \in C_0[0,T]$.

Thus the assumptions in Theorem 5.3 involve the assumptions in Lemma 5.1.

Proof of Theorem 5.3. To obtain equation (5.4), one may establish that

$$\begin{aligned} &T_{q,k}^{(1)}\big((F*G)_q^{(g_1,g_2;h_1,h_2)}\big)(y) \\ &\quad = \lim_{\substack{\lambda_2\to -iq \\ \lambda_2\in\mathbb{C}_+}} E_{x_2}^{\mathrm{an}\, w_{\lambda_2}}\big[(F*G)_q^{(g_1,g_2;h_1,h_2)}\big(y+\mathcal{Z}_k(x_2,\cdot)\big)\big] \\ &\quad = \lim_{\substack{\lambda_1,\lambda_2\to -iq \\ \lambda_1,\lambda_2\in\mathbb{C}_+}} E_{x_2}^{\mathrm{an}\, w_{\lambda_2}}\Big[E_{x_1}^{\mathrm{an}\, w_{\lambda_1}}\Big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\mathcal{Z}_{g_1k}(x_2,\cdot)+\mathcal{Z}_{h_1}(x_1,\cdot)\big) \\ &\qquad \times G\big(\mathcal{Z}_{g_2}(y,\cdot)+\mathcal{Z}_{g_2k}(x_2,\cdot)+\mathcal{Z}_{h_2}(x_1,\cdot)\big)\Big]\Big] \\ &\quad = \lim_{\substack{\lambda\to -iq \\ \lambda\in\mathbb{C}_+}} E_x^{\mathrm{an}\, w_\lambda}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\mathcal{Z}_{\mathbf{s}(g_1k,h_1)}(x,\cdot)\big)\big] \\ &\qquad \times \lim_{\substack{\lambda\to -iq \\ \lambda\in\mathbb{C}_+}} E_x^{\mathrm{an}\, w_\lambda}\big[G\big(\mathcal{Z}_{g_2}(y,\cdot)+\mathcal{Z}_{\mathbf{s}(g_2k,h_2)}(x,\cdot)\big)\big] \\ &\quad = T_{q,\mathbf{s}(g_1k,h_1)}^{(1)}(F)\big(\mathcal{Z}_g(y,\cdot)\big)T_{q,\mathbf{s}(g_2k,h_2)}^{(1)}(G)\big(\mathcal{Z}_g(y,\cdot)\big). \end{aligned}$$

But, as was shown in the proof of Lemma 5.1, the assertions in Remark 5.4 that the analytic Wiener integrals

$$T_{\lambda,\mathbf{s}(g_1k,h_1)}(F)(y) = E_x^{\mathrm{an}\,w_\lambda}\big[F\big(y+\mathcal{Z}_{\mathbf{s}(g_1k,h_1)}(x,\cdot)\big)\big],$$
$$T_{\lambda,\mathbf{s}(g_2k,h_2)}(G)(y) = E_x^{\mathrm{an}\,w_\lambda}\big[G\big(y+\mathcal{Z}_{\mathbf{s}(g_2k,h_2)}(x,\cdot)\big)\big]$$

and

$$(F*G)_\lambda^{(g_1,g_2;h_1,h_2)}(y) = E_x^{\mathrm{an}\,w_\lambda}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\mathcal{Z}_{h_1}(x,\cdot)\big)G\big(\mathcal{Z}_{g_2}(y,\cdot)+\mathcal{Z}_{h_2}(x,\cdot)\big)\big]$$

exist for every $\lambda\in\mathbb{C}_+$ and s-a.e. $y\in C_0[0,T]$ and the fact that the analytic Wiener integral

$$\begin{aligned}
&T_{\lambda_2,k}\big((F*G)_{\lambda_1}^{(g_1,g_2;h_1,h_2)}\big)(y)\\
&\quad= E_{x_2}^{\mathrm{an}\,w_{\lambda_2}}\big[E_{x_1}^{\mathrm{an}_{\lambda_1}}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\mathcal{Z}_{g_1k}(x_2,\cdot)+\mathcal{Z}_{h_1}(x_1,\cdot)\big)\\
&\qquad\times G\big(\mathcal{Z}_{g_2}(y,\cdot)+\mathcal{Z}_{g_2k}(x_2,\cdot)+\mathcal{Z}_{h_2}(x_1,\cdot)\big)\big]\big]
\end{aligned}$$

exists for every $(\lambda_1,\lambda_2)\in\mathbb{C}_+\times\mathbb{C}_+$ establish that $T_{\lambda,\mathbf{s}(g_1k,h_1)}(F)(y)$ and $T_{\lambda,\mathbf{s}(g_2k,h_2)}(G)(y)$ are analytic on $\mathbb{C}_+$, as functions of λ, and also establish that $T_{\lambda_2,k}((F*G)_{\lambda_1}^{(g_1,g_2;h_1,h_2)})(y)$ is analytic on $\mathbb{C}_+\times\mathbb{C}_+$, as a function of (λ_1,λ_2). Thus, to establish equation (5.4), it will suffice to show that

$$\begin{aligned}
&T_{q,k}^{(1)}\big((F*G)_q^{(g_1,g_2;h_1,h_2)}\big)(y)\\
&\quad=\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} E_{x_2}^{\mathrm{an}\,w_\lambda}\big[E_{x_1}^{\mathrm{an}\,w_\lambda}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\mathcal{Z}_{g_1k}(x_2,\cdot)+\mathcal{Z}_{h_1}(x_1,\cdot)\big)\\
&\qquad\times G\big(\mathcal{Z}_{g_2}(y,\cdot)+\mathcal{Z}_{g_2k}(x_2,\cdot)+\mathcal{Z}_{h_2}(x_1,\cdot)\big)\big]\big]\\
&\quad=\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} E_x^{\mathrm{an}\,w_\lambda}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot)+\mathcal{Z}_{\mathbf{s}(g_1k,h_1)}(x,\cdot)\big)\big]\\
&\qquad\times\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} E_x^{\mathrm{an}\,w_\lambda}\big[G\big(\mathcal{Z}_{g_2}(y,\cdot)+\mathcal{Z}_{\mathbf{s}(g_2k,h_2)}(x,\cdot)\big)\big]\\
&\quad=T_{q,\mathbf{s}(g_1k,h_1)}^{(1)}(F)\big(\mathcal{Z}_{g_1}(y,\cdot)\big)T_{q,\mathbf{s}(g_2k,h_2)}^{(1)}(G)\big(\mathcal{Z}_{g_2}(y,\cdot)\big).
\end{aligned}$$

But it follows from equation (5.1) and the analytic continuation. □

5.2. **Generalized convolution products of transforms.** In our second theorem, we establish that the GCP of the GFFTs is a product of the GFFTs.

Theorem 5.5. *Let g_1, g_2, k_1, k_2, h_3, and h_4 be nonzero functions in $BV[0,T]$ and let $\mathbf{s}(h_3,k_1)$ and $\mathbf{s}(h_4,k_2)$ be given as in equation (4.1). Also, let q be a nonzero real number, and let F and G be $\mathbb{C}$-valued scale-invariant measurable functionals on $C_0[0,T]$ such that the L_1 analytic GFFTs $T_{q,k_1}^{(1)}(F)$, $T_{q,k_2}^{(1)}(G)$, $T_{q,\mathbf{s}(g_1k,h_1)}^{(1)}(F)$, and $T_{q,\mathbf{s}(g_2k,h_2)}^{(1)}(G)$ exist for s-a.e. $y\in C_0[0,T]$. Furthermore, assume that the GCP $(T_{q,k_1}^{(1)}(F)*T_{q,k_2}^{(1)}(G))_q^{(g_1,g_2;h_3,h_4)}$ exists for s-a.e. $y\in C_0[0,T]$. Now suppose that*

$$m_L\big(\mathrm{supp}(h_3)\cap\mathrm{supp}(h_4)\big)=0. \tag{5.5}$$

Then s-a.e. $y \in C_0[0,T]$,

$$\begin{aligned}&\big(T^{(1)}_{q,k_1}(F) * T^{(1)}_{q,k_2}(G)\big)^{(g_1,g_2;h_3,h_4)}_q(y)\\ &\quad= T^{(1)}_{q,\mathbf{s}(h_3,k_1)}(F)\big(\mathcal{Z}_{g_1}(y,\cdot)\big)T^{(1)}_{q,\mathbf{s}(h_4,k_2)}(G)\big(\mathcal{Z}_{g_2}(y,\cdot)\big).\end{aligned} \tag{5.6}$$

Proof. By similar arguments in Remarks 5.2 and 5.4, the following analytic continuations of the following seven Wiener integrals

$$\begin{aligned}
J_1(\rho_1,\rho_2,\rho_3) &= \big(T_{\rho_1,k_1}(F) * T_{\rho_2,k_2}(G)\big)^{(g_1,g_2;h_3,h_4)}_{\rho_3}(y), \quad \rho_1,\rho_2,\rho_3 \in (0,+\infty),\\
J_2(\rho_2,\rho_3;\lambda_1) &= \big(T_{\lambda_1,k_1}(F) * T_{\rho_2,k_2}(G)\big)^{(g_1,g_2;h_3,h_4)}_{\rho_3}(y), \quad \rho_2,\rho_3 \in (0,+\infty), \lambda_1 \in \mathbb{C}_+,\\
J_3(\rho_1,\rho_3;\lambda_2) &= \big(T_{\rho_1,k_1}(F) * T_{\lambda_2,k_2}(G)\big)^{(g_1,g_2;h_3,h_4)}_{\rho_3}(y), \quad \rho_1,\rho_3 \in (0,+\infty), \lambda_2 \in \mathbb{C}_+,\\
J_4(\rho_1,\rho_2;\lambda_3) &= \big(T_{\rho_1,k_1}(F) * T_{\rho_2,k_2}(G)\big)^{(g_1,g_2;h_3,h_4)}_{\lambda_3}(y), \quad \rho_1,\rho_2 \in (0,+\infty), \lambda_3 \in \mathbb{C}_+,\\
J_5(\rho_3;\lambda_1,\lambda_2) &= \big(T_{\lambda_1,k_1}(F) * T_{\lambda_2,k_2}(G)\big)^{(g_1,g_2;h_3,h_4)}_{\rho_3}(y), \quad \rho_3 \in (0,+\infty), \lambda_1,\lambda_2 \in \mathbb{C}_+,\\
J_6(\rho_2;\lambda_1,\lambda_3) &= \big(T_{\lambda_1,k_1}(F) * T_{\rho_2,k_2}(G)\big)^{(g_1,g_2;h_3,h_4)}_{\lambda_3}(y), \quad \rho_2 \in (0,+\infty), \lambda_1,\lambda_3 \in \mathbb{C}_+
\end{aligned}$$

and

$$J_7(\rho_1;\lambda_2,\lambda_3) = \big(T_{\rho_1,k_1}(F) * T_{\lambda_2,k_2}(G)\big)^{(g_1,g_2;h_3,h_4)}_{\lambda_3}(y), \quad \rho_1 \in (0,+\infty), \lambda_2,\lambda_3 \in \mathbb{C}_+$$

all exist and have the same analytic continuation

$$J^*(\lambda_1,\lambda_2,\lambda_3) = \big(T_{\lambda_1,k_1}(F) * T_{\lambda_2,k_2}(G)\big)^{(g_1,g_2;h_3,h_4)}_{\lambda_3}, \quad \lambda_1,\lambda_2,\lambda_3 \in \mathbb{C}_+.$$

Thus, by similar arguments as in the proofs of Lemma 5.1 and Theorem 5.3, it will suffice to show that equation (5.6) holds for all $\lambda > 0$ and s-a.e. $y \in C_0[0,T]$.

Using the Fubini theorem and applying equation (4.9) with the condition (5.5), it follows that for all $\lambda > 0$ and s-a.e. $y \in C_0[0,T]$,

$$\begin{aligned}
&\big(T_{\lambda,k_1}(F) * T_{\lambda,k_2}(G)\big)^{(g_1,g_2;h_3,h_4)}_{\lambda}(y)\\
&\quad= E_{x_1}\big[T_{\lambda,k_1}(F)\big(\mathcal{Z}_{g_1}(y,\cdot) + \lambda^{-1/2}\mathcal{Z}_{h_3}(x_1,\cdot)\big)\\
&\qquad \times T_{\lambda,k_2}(G)\big(\mathcal{Z}_{g_2}(y,\cdot) + \lambda^{-1/2}\mathcal{Z}_{h_4}(x_1,\cdot)\big)\big]\\
&\quad= E_{x_1}\big[E_{x_2}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot) + \lambda^{-1/2}\mathcal{Z}_{h_3}(x_1,\cdot) + \lambda^{-1/2}\mathcal{Z}_{k_1}(x_2,\cdot)\big)\big]\\
&\qquad \times E_{x_3}\big[G\big(\mathcal{Z}_{g_2}(y,\cdot) + \lambda^{-1/2}\mathcal{Z}_{h_4}(x_1,\cdot) + \lambda^{-1/2}\mathcal{Z}_{k_2}(x_3,\cdot)\big)\big]\big]\\
&\quad= E_{x_1}\big[E_{x_2}\big[E_{x_3}\big[F\big(\mathcal{Z}_{g_1}(y,\cdot) + \lambda^{-1/2}\big(\mathcal{Z}_{h_3}(x_1,\cdot) + \mathcal{Z}_{k_1}(x_2,\cdot)\big)\big)\\
&\qquad \times G(\mathcal{Z}_{g_2}(y,\cdot) + \lambda^{-1/2}\big(\mathcal{Z}_{h_4}(x_1,\cdot) + \mathcal{Z}_{k_2}(x_3,\cdot)\big)\big]\big]\big]\\
&\quad= E_x\big[F\big(\mathcal{Z}_{g_1}(y,\cdot) + \lambda^{-1/2}\mathcal{Z}_{\mathbf{s}(h_3,k_1)}(x,\cdot)\big)\big]E_x\big[G\big(\mathcal{Z}_{g_2}(y,\cdot) + \lambda^{-1/2}\mathcal{Z}_{\mathbf{s}(h_4,k_2)}(x,\cdot)\big)\big]\\
&\quad= T_{\lambda,\mathbf{s}(h_3,k_1)}(F)\big(\mathcal{Z}_{g_1}(y,\cdot)\big)T_{\lambda,\mathbf{s}(h_4,k_2)}(G)\big(\mathcal{Z}_{g_2}(y,\cdot)\big)
\end{aligned}$$

as desired. □

6. Further results and examples

The assertions in Theorems 5.3 and 5.5 above can be applied to many large classes of functionals on $C_0[0,T]$. These classes of functionals are discussed in [3], [5], [10]–[14], [19]. In Theorem 5.3, we established that a GFFT of a GCP of functionals on $C_0[0,T]$ is a product of GFFTs, and in Theorem 5.5, we established that a GCP of GFFTs is also a product of GFFTs under appropriate conditions. This leads to the following question: how can we relate the two results in Theorems 5.3 and 5.5? In other words, how can we find the conditions on the GFFTs and the GCPs in the next equation?

$$T_{q,k}^{(1)}\big((F*G)_q^{(g_1,g_2;h_1,h_2)}\big)(y) = \big(T_{q,k_1}^{(1)}(F)*T_{q,k_2}^{(1)}(G)\big)_q^{(g_1,g_2;h_3,h_4)}(y). \tag{6.1}$$

In view of the assumptions in Theorems 5.3 and 5.5, we have to check that there exist solutions $\{g_1,g_2,k,k_1,k_2,h_1,h_2,h_3,h_4\}$ of the system

$$\begin{cases} \text{(i)} \quad g_1g_2k^2+h_1h_2=0, \\ \text{(ii)} \quad m_L(\operatorname{supp}(h_3)\cap\operatorname{supp}(h_4))=0, \\ \text{(iii)} \quad \mathbf{s}(g_1k,h_1)=\mathbf{s}(h_3,k_1), \\ \qquad \text{i.e., } g_1^2(t)k^2(t)+h_1^2(t)=h_3^2(t)+k_1^2(t) \quad m_L\text{-a.e. } t\in[0,T], \\ \text{(iv)} \quad \mathbf{s}(g_2k,h_2)=\mathbf{s}(h_4,k_2), \\ \qquad \text{i.e., } g_2^2(t)k^2(t)+h_2^2(t)=h_4^2(t)+k_2^2(t) \quad m_L\text{-a.e. } t\in[0,T]. \end{cases} \tag{6.2}$$

to establish equation (6.1) above.

In the remainder of this paper, we present three examples which shed light upon the conditions in Theorems 5.3 and 5.5 above, and which also illustrate that the conclusions of two theorems are valid.

Example 6.1. The set $\{g_1,g_2,k,k_1,k_2,h_1,h_2,h_3,h_4\}$ of functions in $BV[0,T]$ with

$$\begin{aligned} &g_1(t)=2\cos\Big(\frac{2\pi t}{T}\Big)\chi_{[0,T/2]}(t), \qquad g_2(t)=\Big[3-4\sin^2\Big(\frac{2\pi t}{T}\Big)\Big]\chi_{[T/2,T]}(t), \\ &k(t)=\sin\Big(\frac{2\pi t}{T}\Big) \\ &k_1(t)=\sin\Big(\frac{4\pi t}{T}\Big), \qquad k_2(t)=\sin\Big(\frac{6\pi t}{T}\Big), \\ &h_1(t)=\chi_{[T/2,T]}(t), \qquad h_2(t)=\chi_{[0,T/2]}(t), \\ &h_3(t)=\cos\Big(\frac{4\pi t}{T}\Big)\chi_{[T/2,T]}(t), \qquad h_4(t)=\cos\Big(\frac{6\pi t}{T}\Big)\chi_{[0,T/2]}(t), \end{aligned}$$

is a solution of the system (6.2).

Example 6.2. Given positive integers l, m, and n with $l < m < n$, let

$$\begin{aligned}
g_1(t) &= \sin\Big(\frac{l\pi t}{T}\Big), \qquad g_2(t) = \sin\Big(\frac{m\pi t}{T}\Big),\\
k(t) &= \cos\Big(\frac{n\pi t}{T}\Big),\\
k_1(t) &= \sqrt{2}\sin\Big(\frac{l\pi t}{T}\Big)\cos\Big(\frac{n\pi t}{T}\Big)\chi_B(t), \qquad k_2(t) = \sqrt{2}\sin\Big(\frac{m\pi t}{T}\Big)\cos\Big(\frac{n\pi t}{T}\Big)\chi_A(t),\\
h_1(t) &= \sin\Big(\frac{l\pi t}{T}\Big)\cos\Big(\frac{n\pi t}{T}\Big), \qquad h_2(t) = -\sin\Big(\frac{m\pi t}{T}\Big)\cos\Big(\frac{n\pi t}{T}\Big),\\
h_3(t) &= \sqrt{2}\sin\Big(\frac{l\pi t}{T}\Big)\cos\Big(\frac{n\pi t}{T}\Big)\chi_A(t), \qquad h_4(t) = \sqrt{2}\sin\Big(\frac{m\pi t}{T}\Big)\cos\Big(\frac{n\pi t}{T}\Big)\chi_B(t).
\end{aligned}$$

Then the set $\mathbf{S} = \{g_1, g_2, k, k_1, k_2, h_1, h_2, h_3, h_4\}$ is a solution of the system (6.2).

In fact, the solution sets $\mathbf{S}$ can be obtained by the following procedures. First, let $\{A, B\}$ be a measurable partition of $[0, T]$ with $m_L(A) > 0$ and $m_L(B) > 0$. Next, given any functions g_1, g_2, and k in $BV[0, T]$, let

$$\begin{aligned}
k_1(t) &= \sqrt{2}g_1(t)k(t)\chi_B(t), \qquad k_2(t) = \sqrt{2}g_2(t)k(t)\chi_A(t),\\
h_1(t) &= g_1(t)k(t), \qquad h_2(t) = -g_2(t)k(t),\\
h_3(t) &= \sqrt{2}g_1(t)k(t)\chi_A(t), \qquad h_4(t) = \sqrt{2}g_2(t)k(t)\chi_B(t).
\end{aligned}$$

Then one can see that the set $\{g_1, g_2, k, h_1, h_2, h_3, h_4, k_1, k_2\}$ is a solution of the system (6.2).

Example 6.3. Let $\mathcal{H} = \{\mathbf{h}_n\}_{n=1}^{\infty}$ be the sequence of Haar functions on $[0, T]$. (For more details, see [6].) It is well known that $\mathcal{H}$ is a complete orthonormal set on $L_2[0, T]$ which consists of nonsmooth functions.

Consider the intervals $A = [0, T/2]$ and $B = [T/2, T]$. Then, for each $n \in \mathbb{N}$ with $n > 2$, either $\operatorname{supp}(\mathbf{h}_n) \subset A$ or $\operatorname{supp}(\mathbf{h}_n) \subset B$.

Let $P_A = \{n \in \mathbb{N} : \operatorname{supp}(\mathbf{h}_n) \subset A\}$ and let $P_B = \{n \in \mathbb{N} : \operatorname{supp}(\mathbf{h}_n) \subset B\}$. Then, clearly,

$$\bigcup_{n\in P_A} \operatorname{supp}(\mathbf{h}_n) = A \qquad \text{and} \qquad \bigcup_{n\in P_B} \operatorname{supp}(\mathbf{h}_n) = B.$$

Let $\mathcal{P}^A = \{\mathbf{h}_1\chi_A\} \cup \{\mathbf{h}_n : n \in P_A\}$ and let $\mathcal{P}^B = \{\mathbf{h}_1\chi_B\} \cup \{\mathbf{h}_n : n \in P_B\}$. Next let $\mathcal{H}^A \equiv \{\mathbf{h}_n^A\}_{n=1}^{\infty}$ and $\mathcal{H}^B \equiv \{\mathbf{h}_n^B\}_{n=1}^{\infty}$ be the normalization of $\mathcal{P}^A$ and $\mathcal{P}^B$, respectively. Then it follows that

(i) $\mathcal{H}^A$ is a complete orthogonal set in $L_2(A) = L_2[0, T/2]$, and
(ii) $\mathcal{H}^B$ is a complete orthogonal set in $L_2(B) = L_2[T/2, T]$.

As discussed in Example 6.2 above, given g_1, g_2, and k in $BV[0, T]$, let

$$\begin{aligned}
k_1(t) &= \sqrt{2}g_1(t)k(t)\chi_B(t), \qquad k_2(t) = \sqrt{2}g_2(t)k(t)\chi_A(t),\\
h_1(t) &= g_1(t)k(t), \qquad h_2(t) = -g_2(t)k(t),\\
h_3(t) &= \sqrt{2}g_1(t)k(t)\chi_A(t), \qquad h_4(t) = \sqrt{2}g_2(t)k(t)\chi_B(t).
\end{aligned}$$

In these settings, for $j \in \{1,2\}$, let

$$\sum_{n=1}^{\infty} \alpha_n^{(j)} \mathbf{h}_n^A \equiv \sum_n \alpha_n^{(j)} \mathbf{h}_n^A$$

be the Fourier series of $\sqrt{2} g_j k$ with respect to $\mathcal{H}^A$ on $A \equiv [0, T/2]$, and let

$$\sum_{n=1}^{\infty} \beta_n^{(j)} \mathbf{h}_n^B \equiv \sum_n \beta_n^{(j)} \mathbf{h}_n^B$$

be the Fourier series of $\sqrt{2} g_j k$ with respect to $\mathcal{H}^B$ on $B \equiv [T/2, T]$. Then, one can see that

(i) $g_1 g_2(t) k^2(t) + h_1(t) h_2(t) = g_1 g_2(t) k^2(t) - g_1 g_2(t) k^2(t) = 0$,
(ii) $m_L(\mathrm{supp}(h_3) \cap \mathrm{supp}(h_4)) = m_L(A \cap B) = 0$,
(iii) for m_L-a.e. $t \in [0,T]$,

$$\begin{aligned} g_1^2(t) k^2(t) + h_1^2(t) = 2 g_1^2(t) k^2(t) &= \big[\sqrt{2} g_1(t) k(t)\big]^2 \\ &= \big[\sqrt{2} g_1(t) k(t) \chi_A(t) + \sqrt{2} g_1(t) k(t) \chi_B(t)\big]^2 \\ &= 2 g_1^2(t) k^2(t) \chi_A(t) + 2 g_1^2(t) k^2(t) \chi_B(t) \\ &= \Big(\sum_{n=1}^{\infty} \alpha_n^{(1)} \mathbf{h}_n^A\Big)^2(t) + \Big(\sum_{n=1}^{\infty} \beta_n^{(1)} \mathbf{h}_n^B\Big)^2(t), \end{aligned}$$

(iv) for m_L-a.e. $t \in [0,T]$,

$$\begin{aligned} g_2^2(t) k^2(t) + h_2^2(t) = 2 g_2^2(t) k^2(t) &= \big[\sqrt{2} g_2(t) k(t)\big]^2 \\ &= \big[\sqrt{2} g_2(t) k(t) \chi_A(t) + \sqrt{2} g_1(t) k(t) \chi_B(t)\big]^2 \\ &= 2 g_2^2(t) k^2(t) \chi_A(t) + 2 g_2^2(t) k^2(t) \chi_B(t) \\ &= \Big(\sum_{n=1}^{\infty} \alpha_n^{(2)} \mathbf{h}_n^A\Big)^2(t) + \Big(\sum_{n=1}^{\infty} \beta_n^{(2)} \mathbf{h}_n^B\Big)^2(t). \end{aligned}$$

Thus, given nonzero functions g_1, g_2, and k in $BV[0,T]$, it follows that

$$\begin{aligned} &T_{q,k}^{(1)}\big((F * G)_q^{(g_1, g_2; g_1 k, -g_2 k)}\big)(y) \\ &= \big(T^{(1)}_{q, \sum_n \beta_n^{(1)} \mathbf{h}_n^B}(F) * T^{(1)}_{q, \sum_n \alpha_n^{(2)} \mathbf{h}_n^A}(G)\big)_q^{(g_1, g_2; \sum_n \alpha_n^{(1)} \mathbf{h}_n^A, \sum_n \beta_n^{(2)} \mathbf{h}_n^B)}(y) \end{aligned}$$

for s-a.e. $y \in C_0[0,T]$.

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