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NEW FUNCTION SPACES RELATED TO MORREY SPACES AND THE FOURIER TRANSFORM

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Abstract. We introduce new function spaces to handle the Fourier transform on Morrey spaces and investigate fundamental properties of the spaces. As an application, we generalize the Stein–Tomas Strichartz estimate to our spaces. The geometric property of Morrey spaces and related function spaces will improve some well-known estimates.

1. Introduction

Although the Fourier transform is fundamental in mathematics, only a little is known about this transform. One of the traditional ways to understand its mapping property is to use the Lebesgue space L^p, but L^p, $p \neq 2$, is not enough to grasp the behavior of the Fourier transform. Among the properties of the Fourier transform, the relation between the Strichartz estimate and the Fourier transform is mysterious. In this article, we propose to use the Morrey space $\mathcal{M}^p_q$, which properly includes L^p whenever $1 \leq q \leq p < \infty$.

For some time, we have been developing the theory of the Fourier transform on L^p. There are so many results in this direction involving the application to PDEs. Meanwhile, our experience shows that the L^p result can be generalized to the Morrey space $\mathcal{M}^p_q$, $1 \leq q \leq p < \infty$ (see [1], [4], [9]). The Morrey space $\mathcal{M}^p_q$,

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$1 \le q \le p < \infty$ is the set of all measurable functions f for which the norm

$$\|f\|_{\mathcal{M}^p_q} \equiv \sup_Q |Q|^{\frac{1}{p}-\frac{1}{q}} \Big(\int_Q \big|f(x)\big|^q \, dx \Big)^{\frac{1}{q}}$$

is finite, where Q ranges over all cubes having their sides parallel to coordinate axes. By the use of the Hölder inequality,

$$L^p = \mathcal{M}^p_p \subset \mathcal{M}^p_{q_2} \subset \mathcal{M}^p_{q_1} \tag{1.1}$$

holds whenever $1 \le q_1 \le q_2 \le p$. Some mathematicians improved the L^p results by using Morrey or Morrey-type spaces. For example, in [2], the authors improved the Stein–Tomas Strichartz estimate by using the Morrey-type space $X_{p,q}$ (see Theorem 5.1 below). In [7], the authors improved the sharp maximal inequality by using Morrey spaces. We have many results together with applications to PDE on the estimate of the form $\|f \nabla g\|_{L^p}$ (see [8]). However, as far as we know, there are no results on the Fourier transform on Morrey spaces. The purpose of this paper is to introduce a new function space $\mathcal{M}\mathcal{F}^p_q$, to capture the behavior of the Fourier transform on Morrey spaces, and to investigate their properties.

We establish some basic notation. For a parameter $p \in [1, \infty]$, we denote its conjugate by p'; that is, $1/p + 1/p' = 1$. All cubes Q in $\mathbb{R}^n$ are assumed to have their sides parallel to the coordinate axes. We denote by $\mathcal{Q}$ the family of all cubes and by $\ell(Q)$ the sidelength of $Q \in \mathcal{Q}$. For $c > 0$ and $Q \in \mathcal{Q}$, we denote by cQ the cube with the same center as Q but with sidelength $c\ell(Q)$. We denote a cube centered at the origin with volume $(2r)^n$ by $Q(r)$:

$$Q(r) \equiv \big\{ x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \ \max\big(|x_1|, |x_2|, \ldots, |x_n|\big) \le r \big\}.$$

The set $Q(x_0, r)$ with $x_0 \in \mathbb{R}^n$ and $r > 0$ is the set of all points $x \in \mathbb{R}^n$ for which $x - x_0 \in Q(r)$. We denote the Lebesgue measure of the measurable set $E \subset \mathbb{R}^n$ by $|E|$. We define the Fourier transform and its inverse by

$$\mathcal{F}f(\xi) \equiv \int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi} f(x)\, dx \quad (\xi \in \mathbb{R}^n),$$

$$\mathcal{F}^{-1}f(\xi) \equiv \int_{\mathbb{R}^n} e^{2i\pi x\cdot\xi} f(x)\, dx \quad (\xi \in \mathbb{R}^n)$$

for $f \in \mathcal{S}$. Now we define a new space denoted by $\mathcal{M}\mathcal{F}^p_q$.

Definition 1.1. Let $1 \le p, q \le \infty$, and let $\psi \in C^\infty_{\rm c}$ satisfy

$$\chi_{Q(1)} \le \psi \le \chi_{Q(4)}. \tag{1.2}$$

For $f \in \mathcal{S}'$, the Fourier–Morrey space $\mathcal{M}\mathcal{F}^p_q = \mathcal{M}\mathcal{F}^p_q(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'$ for which the norm

$$\|f\|_{\mathcal{M}\mathcal{F}^p_q} \equiv \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{q}} \bigg(\int_{\mathbb{R}^n} \bigg| \mathcal{F}^{-1}\Big[\psi\Big(\frac{\cdot - c(Q)}{\ell(Q)}\Big)\Big] * f(x) \bigg|^{q'} dx \bigg)^{\frac{1}{q'}}$$

is finite. Here we denote the center of the cube Q by $c(Q)$.

Although the definition of $\mathcal{M}\mathcal{F}^p_q$ makes sense for all $1 \le p,\ q < \infty$, the case $1 \le p < q < \infty$ will be excluded (see Lemma 2.1). Since the function $\psi(\frac{\cdot - c(Q)}{\ell(Q)})$ is supported near Q and equals 1 on Q, we denote it by ψ_Q throughout this paper. It is easy to see that the space $\mathcal{M}\mathcal{F}^p_q$ does not depend on ψ satisfying (1.2). By using our space $\mathcal{M}\mathcal{F}^p_q$, we can describe the boundedness of the Fourier transform on Morrey spaces, which is our starting point for this research.

Theorem 1.2.

(1) *If $1 \le q \le p < \infty$ and $q < 2$, then the Fourier transform is bounded from $\mathcal{M}^p_q$ to $\mathcal{M}\mathcal{F}^p_q$; for some constant $C > 1$,*
$$\|\mathcal{F}f\|_{\mathcal{M}\mathcal{F}^p_q} \le C\|f\|_{\mathcal{M}^p_q} \tag{1.3}$$
for all $f \in \mathcal{M}^p_q$.

(2) *Let $p \ge 2$. Then the Fourier transform is an isomorphism from $\mathcal{M}^p_2$ to $\mathcal{M}\mathcal{F}^p_2$; for some constant $C > 1$,*
$$C^{-1}\|f\|_{\mathcal{M}^p_2} \le \|\mathcal{F}f\|_{\mathcal{M}\mathcal{F}^p_2} \le C\|f\|_{\mathcal{M}^p_2} \tag{1.4}$$
for all $f \in \mathcal{M}^p_2$.

Assertions (1) and (2) of Theorem 1.2 are not hard to prove. We can prove them directly by applying the Hausdorff–Young inequality to the definition of our norm. We also note that $\mathcal{F}^{-1}\psi = \mathcal{F}\psi(-\cdot)$. Meanwhile, using the space $\mathcal{M}\mathcal{F}^p_q$, we can refine some well-known estimates, as follows.

Theorem 1.3. *We have the following two estimates.*

(1) *The Fourier transform is bounded from $\mathcal{M}\mathcal{F}^\infty_\infty$ to L^∞; for some constant $C > 1$,*
$$\|\mathcal{F}f\|_{L^\infty} \le C\|f\|_{\mathcal{M}\mathcal{F}^\infty_\infty} \tag{1.5}$$
for all $f \in \mathcal{M}\mathcal{F}^\infty_\infty$.

(2) *For each $t \in \mathbb{R}$, let $\mathcal{T}(t)$ denote the Schrödinger propagator; see (1.7) for the precise definition. Then $\mathcal{T}(t)$ is bounded from $\mathcal{M}\mathcal{F}^\infty_\infty$ to L^∞; for some constant $C > 1$,*
$$\left\|\mathcal{T}(t)f(x)\right\|_{L^\infty_x} \le Ct^{-\frac{n}{2}}\|f\|_{\mathcal{M}\mathcal{F}^\infty_\infty} \tag{1.6}$$
for all $f \in \mathcal{M}\mathcal{F}^\infty_\infty$.

By Proposition 2.5, which will be proved below, we have $L^1 \hookrightarrow \mathcal{M}\mathcal{F}^\infty_\infty$. Hence Theorem 1.3 improves the well-known estimates $\mathcal{F} : L^1 \to L^\infty$ and $\mathcal{T}(t) : L^1 \to L^\infty$. These results motivate us to consider the space $\mathcal{M}\mathcal{F}^p_q$. We also introduce a new function space $\mathcal{H}\mathcal{F}^p_q$.

Definition 1.4. Let $1 \le p \le q \le \infty$. An $L^{q'}$ function b is said to be a (p,q)-Fourier block if the support of $\mathcal{F}b$ is contained in some cube Q and $\|b\|_{L^{q'}} \le |Q|^{\frac{1}{q}-\frac{1}{p}}$ holds. The space $\mathcal{H}\mathcal{F}^p_q$ is defined as the set of all $L^{p'}$ functions f for which f is expressed as
$$f = \sum_{j=1}^{\infty} \lambda_j b_j$$

in $L^{p'}$, where the b_k's are (p,q)-Fourier blocks, and $\{\lambda_j\}_{j=1}^\infty \in \ell^1$. One defines

$$\|f\|_{\mathcal{H}\mathcal{F}^p_q} \equiv \inf\left\{ \left\| \{\lambda_j\}_{j=1}^\infty \right\|_{\ell^1} : f = \sum_{j=1}^\infty \lambda_j b_j \right\},$$

where the infimum is taken over all possible decompositions of f above.

It is useful to see the fact that there exists a constant $C>0$ such that $\|b\|_{L^{p'}} \le C$ for all (p,q)-Fourier blocks. In fact, if we admit Corollary 2.4 for a moment, then it follows that $\|b\|_{L^{p'}} \le C|Q|^{\frac{1}{q'}-\frac{1}{p'}}\|b\|_{L^{q'}} \le C$. This definition dates back to 1986 (see [16, p. 589]).

Our main results in this paper are the following theorems. The first one is about a predual space of $\mathcal{M}\mathcal{F}^p_q$.

Theorem 1.5. *Let $1 \le p \le q < \infty$, and let $q>1$. Then $(\mathcal{H}\mathcal{F}^p_q)^* = \mathcal{M}\mathcal{F}^{p'}_{q'}$ in the following sense.*

(0) *The space $\mathcal{S}$ is dense in $\mathcal{H}\mathcal{F}^p_q$.*

(1) *Let $f \in \mathcal{M}\mathcal{F}^{p'}_{q'}$ be arbitrary. Then for any (p,q)-Fourier block $b \in \mathcal{S}$,*

$$\left|\langle f, b\rangle\right| \le C\|f\|_{\mathcal{H}\mathcal{F}^p_q};$$

hence, f can be extended to a bounded linear functional L_f on $\mathcal{H}\mathcal{F}^p_q$ with the estimate $\|L_f\|_{(\mathcal{H}\mathcal{F}^p_q)^*} \le C\|f\|_{\mathcal{M}\mathcal{F}^{p'}_{q'}}$.

(2) *For any $L \in (\mathcal{H}\mathcal{F}^p_q)^*$ there uniquely exists $f \in \mathcal{M}\mathcal{F}^{p'}_{q'}$ such that*

$$\|f\|_{\mathcal{M}\mathcal{F}^{p'}_{q'}} \le \|L\|_{(\mathcal{H}\mathcal{F}^p_q)^*}, \qquad L(g) = L_f(g) \quad (g \in \mathcal{H}\mathcal{F}^p_q),$$

holds.

The next result generalizes the Stein–Tomas Strichartz estimate (see [11], [12], [14]). To state our next result, we recall the Schrödinger propagator, and we define the mixed-type norm space denoted by $\mathcal{M}\mathcal{F}^p_q(\mathbb{R}^n_x, L^r(\mathbb{R}_t))$. The Schrödinger propagator $\mathcal{T}$ is defined by

$$\mathcal{T}(t)g(x) \equiv \mathcal{F}^{-1}[e^{-i4\pi^2|\xi|^2 t}\mathcal{F}g](x) = \int_{\mathbb{R}^n} e^{2i\pi(x\cdot\xi - 2\pi t|\xi|^2)}\mathcal{F}g(\xi)\,d\xi \tag{1.7}$$

for $g \in L^2(\mathbb{R}^n)$ and each $(x,t) \in \mathbb{R}^{n+1}$. Since the transformation

$$\mathcal{F}_{\mathbb{R}^n}F(\xi,t) = \int_{\mathbb{R}^n} F(x,t)e^{-2\pi i x\cdot\xi}\,dx$$

is an isomorphism in $\mathcal{S}(\mathbb{R}^{n+1})$, $\mathcal{T}(t)g(x)$ can be regarded as an element in $\mathcal{S}'(\mathbb{R}^{n+1})$.

We introduce a suitable mixed-type space to discuss the Schrödinger propagator on Morrey spaces.

Definition 1.6. Let $1 \le q \le p \le \infty$ and $1 < r < \infty$. Let $\psi^0 \in C^\infty_{\rm c}(\mathbb{R})$ satisfy $\chi_{[-1,1]} \le \psi^0 \le \chi_{[-2,2]}$, and set

$$\varphi^0_j \equiv \psi^0(2^{-j}\cdot) - \psi^0(2^{-j+1}\cdot).$$

Write

$$\psi_Q \otimes \varphi_j^0(x,t) \equiv \psi_Q(x)\varphi_j^0(t) \quad (x,t) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}.$$

Define $\mathcal{M}_{\mathcal{F}_q^p}(\mathbb{R}^n_x, L^r(\mathbb{R}_t))$ as the set of all distributions $F \in \mathcal{S}'(\mathbb{R}^{n+1})$ for which the norm

$$\begin{aligned}
&\|F\|_{\mathcal{M}_{\mathcal{F}_q^p}(\mathbb{R}^n_x,L^r(\mathbb{R}_t))} \\
&= \big\|F(x,t)\big\|_{\mathcal{M}_{\mathcal{F}_q^p}(\mathbb{R}^n_x,L^r(\mathbb{R}_t))} \\
&\equiv \sup_{Q\in\mathcal{D}(\mathbb{R}^n)} |Q|^{\frac{1}{p}-\frac{1}{q}} \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}} \big|\mathcal{F}^{-1}[\psi_Q \otimes \psi^0] * F(x,t)\big|^r \, dt\Big)^{\frac{q'}{r}} dx\Big)^{\frac{1}{q'}} \\
&\quad + \sup_{Q\in\mathcal{D}(\mathbb{R}^n)} |Q|^{\frac{1}{p}-\frac{1}{q}} \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}} \Big(\sum_{j=1}^{\infty} \big|\mathcal{F}^{-1}[\psi_Q \otimes \varphi_j^0] * F(x,t)\big|^2\Big)^{\frac{r}{2}} dt\Big)^{\frac{q'}{r}} dx\Big)^{\frac{1}{q'}}
\end{aligned}$$

is finite.

Although the above definition is quite complicated, we have the following formula (see Lemma 1.7(1) below). It is the advantage of the above definition that we can check the following embedding relation (see Lemma 1.7(2) below).

Lemma 1.7. *Let $1 \le q \le p \le \infty$, and let $1 < r < \infty$.*

(1) *If $F \in \mathcal{S}(\mathbb{R}^n_x \times \mathbb{R}_t)$, then*

$$\|F\|_{\mathcal{M}_{\mathcal{F}_q^p}(\mathbb{R}^n_x,L^r(\mathbb{R}_t))} \sim \sup_{Q\in\mathcal{D}(\mathbb{R}^n)} |Q|^{\frac{1}{p}-\frac{1}{q}} \Big(\int_{\mathbb{R}^n} \Big(\int_{\mathbb{R}} \big|\mathcal{F}^{-1}[\psi_Q] *_{\mathbb{R}^n} F(x,t)\big|^r \, dt\Big)^{\frac{q'}{r}} dx\Big)^{\frac{1}{q'}}.$$

Here the convolution $\mathcal{F}^{-1}\psi_Q *_{\mathbb{R}^n} F(x,t)$ is taken in the sense of $\mathbb{R}^n$; that is, $\mathcal{F}^{-1}\psi_Q *_{\mathbb{R}^n} F(x,t) \equiv \int_{\mathbb{R}^n} \mathcal{F}^{-1}\psi_Q(x-y)F(y,t)\,dy$.

(2) *The space $\mathcal{M}_{\mathcal{F}_q^p}(\mathbb{R}^n_x, L^r(\mathbb{R}_t))$ is continuously embedded into $\mathcal{S}'(\mathbb{R}^{n+1})$.*

We will prove this lemma in the beginning of Section 4. By using this mixed norm, we generalize the Stein–Tomas Strichartz estimate. Using the Stein–Tomas Strichartz estimate directly, we can show the following generalization easily.

Proposition 1.8. *Let*

$$q = \frac{2(n+2)}{n} (\ge 2), \tag{1.8}$$

and let

$$s \in [q', 2] = \Big[\frac{2(n+2)}{n+4}, 2\Big]. \tag{1.9}$$

Define $v = v_{s,q}$ by

$$\frac{1}{v_{s,q}} = \frac{1}{q} + \frac{1}{s} - \frac{1}{2}. \tag{1.10}$$

Then for all $g \in \mathcal{S}'$,

$$\big\|\mathcal{T}(t)g(x)\big\|_{\mathcal{M}_{\mathcal{F}^s_{q'}}(\mathbb{R}^n_x,L^q(\mathbb{R}_t))} \le C\|\mathcal{F}g\|_{\mathcal{M}^v_2(\mathbb{R}^n)} \le C\|g\|_{\mathcal{M}_{\mathcal{F}^v_2}(\mathbb{R}^n)}. \tag{1.11}$$

Note that $\mathcal{M}_{\mathcal{F}_{q'}}^{q'} = L^q$; hence $\mathcal{M}_{\mathcal{F}_{q'}}^{q'}(\mathbb{R}^n_x, L^q(\mathbb{R}_t)) = L^q(\mathbb{R}^{n+1}_{x,t})$. See Proposition 2.5 and the fact that the condition (1.10) turns into $v_{s,q} = 2$ when $s = q'$. Thus this proposition is a natural generalization of the Stein–Tomas Strichartz estimate since (1.11) turns into

$$\left\|\mathcal{T}(t)g(x)\right\|_{L^q(\mathbb{R}^{n+1})} \le C\|\mathcal{F}g\|_{L^2(\mathbb{R}^n)}, \tag{1.12}$$

which corresponds to the original Stein–Tomas Strichartz estimate (see [12], [14]) when $s = q'$.

It is difficult to improve the inequality (1.12) in the same framework. However, by using another function space, the inequality can be improved (see [2]). Meanwhile, we can improve the inequality (1.11) in the same framework as follows, which is our second main theorem.

Theorem 1.9. *Let q, s, and $v = v_{s,q}$ satisfy (1.8), (1.9), and (1.10), respectively.*

(1) *If ρ satisfies*

$$\max\left[\frac{2(n+1)(n+2)}{n^2+3n+4}, v - vq\left(-\frac{1}{s} + \frac{1}{q'}\right)\right] < \rho \le v, \tag{1.13}$$

then for all $g \in \mathcal{S}'$,

$$\left\|\mathcal{T}(t)g(x)\right\|_{\mathcal{M}_{\mathcal{F}_{q'}}^{s}(\mathbb{R}^n_x, L^q(\mathbb{R}_t))} \le C\|\mathcal{F}g\|_{\mathcal{M}^v_\rho(\mathbb{R}^n)}. \tag{1.14}$$

(2) *In the endpoint case, $s = 2$ if ρ satisfies*

$$\frac{2(n+1)(n+2)}{n^2+3n+4} < \rho \le q; \tag{1.15}$$

then for all $g \in \mathcal{S}'$, $\|\mathcal{T}(t)g(x)\|_{\mathcal{M}_{\mathcal{F}_{q'}}^{2}(\mathbb{R}^n_x, L^q(\mathbb{R}_t))} \le C\|\mathcal{F}g\|_{\mathcal{M}^q_\rho(\mathbb{R}^n)}$.

A couple of remarks may be in order.

Remark 1.10.

(1) Note that (1.13) and (1.15) are identical when $s = 2$. Meanwhile, when $s = q'$, the condition (1.13) turns into $2 < \rho \le 2$. In this case it will be understood that $\rho = 2$; hence the inequality corresponds to (1.12).
(2) When the parameter s moves from q' to 2, the parameter $v = v_{s,q}$ moves from 2 to q.
(3) In the condition (1.13), arithmetic shows $v - vq(-\frac{1}{s} + \frac{1}{q'}) < 2$ whenever $s \in (q', 2]$. Meanwhile, one knows that $\frac{2(n+1)(n+2)}{n^2+3n+4} < 2$. Hence the parameter ρ can be strictly less than 2. This together with (1.1) implies that Theorem 1.9 improves Proposition 1.8.

Throughout this article, we also use the following notation.

- For $f \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$, define $\varphi(D)f \equiv \mathcal{F}^{-1}[\varphi\mathcal{F}f]$.
- We will denote by $\mathcal{D}$ or $\mathcal{D}(\mathbb{R}^n)$ the family of all dyadic cubes $\{Q_{jm}\}_{j\in\mathbb{Z}, m\in\mathbb{Z}^n}$ in $\mathbb{R}^n$, where $Q_{jm} \equiv \prod_{i=1}^n [2^{-j}m_i, 2^{-j}(m_i+1))$.

This article is organized as follows. In Section 2, we investigate fundamental properties of the space $\mathcal{M}\mathcal{F}^p_q$. In Section 3, we investigate fundamental properties of the space $\mathcal{H}\mathcal{F}^p_q$ and prove Theorem 1.5. In Section 4, as an application of Theorem 1.5, we generalize the Fourier restriction theorem to our space $\mathcal{H}\mathcal{F}^p_q$. In Section 5, we will prove Theorem 1.9.

2. The fundamental properties of $\mathcal{M}\mathcal{F}^p_q$

We prove Theorem 1.2. For example, we prove (1.3) as follows. Fix a cube Q. Then we have

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{\mathbb{R}^n}\left|\mathcal{F}^{-1}\psi_Q*\mathcal{F}^{-1}f(x)\right|^{q'}dx\right)^{\frac{1}{q'}} \le C|Q|^{\frac{1}{p}-\frac{1}{q}}\left(\int_{\mathbb{R}^n}\left|\psi_Q(x)f(x)\right|^{q}dx\right)^{\frac{1}{q}} \le C\|f\|_{\mathcal{M}^p_q}$$

by Young's inequality. The proof of (1.4) is similar. We also prove Theorem 1.3 here. To obtain (1.5), we employ the estimate $\mathcal{F}: L^1\to L^\infty$ as follows:

$$\|\mathcal{F}f\|_{L^\infty}=\sup_{Q\in\mathcal{Q}}\|\psi_Q\mathcal{F}f\|_{L^\infty}\le C\sup_{Q\in\mathcal{Q}}\|\mathcal{F}^{-1}\psi_Q*f\|_{L^1}=C\|f\|_{\mathcal{M}\mathcal{F}^\infty_\infty}.$$

Meanwhile, to show (1.6), note that $\mathcal{M}\mathcal{F}^1_1=L^\infty$, which we will prove in Proposition 2.5 below. Further, we use the estimate $\|\mathcal{T}(t)g(x)\|_{L^\infty_x}\le Ct^{-\frac{n}{2}}\|g\|_{L^1}$ (see, e.g., [5]). With these in mind, we see that

$$\begin{aligned}\left\|\mathcal{T}(t)f(x)\right\|_{L^\infty_x} &\sim \left\|\mathcal{T}(t)f(x)\right\|_{\mathcal{M}\mathcal{F}^1_1(\mathbb{R}^n_x)}\\ &=\sup_{Q\in\mathcal{Q}}\left\|\mathcal{F}^{-1}\psi_Q*\left[\mathcal{T}(t)f(\cdot)\right]\right\|_{L^\infty}\\ &=\sup_{Q\in\mathcal{Q}}\left\|\mathcal{T}(t)[\mathcal{F}^{-1}\psi_Q*f]\right\|_{L^\infty}\\ &\le\sup_{Q\in\mathcal{Q}}Ct^{-\frac{n}{2}}\|\mathcal{F}^{-1}\psi_Q*f\|_{L^1}\\ &=Ct^{-\frac{n}{2}}\|f\|_{\mathcal{M}\mathcal{F}^\infty_\infty}.\end{aligned}$$

Having thus proved the fundamental theorem, we proceed to investigate the space $\mathcal{M}\mathcal{F}^p_q$.

According to the next lemma, it does not make sense to consider the case of $p<q$.

Lemma 2.1. *If $1\le p<q<\infty$ and if $f\in\mathcal{S}'$ satisfies $\|f\|_{\mathcal{M}\mathcal{F}^p_q}<\infty$, then $f=0$.*

Proof. For $N\in\mathbb{N}$, we define

$$f_N\equiv\mathcal{F}^{-1}\left[\psi\left(\frac{\cdot}{2N}\right)\right]*f.$$

Then each f_N satisfies $\|f_N\|_{L^{q'}}\le(2N)^{-\frac{n}{p}+\frac{n}{q}}\|f\|_{\mathcal{M}\mathcal{F}^p_q}$. Hence $f_N\to0$ in $L^{q'}$ as $N\to\infty$. Meanwhile, f_N tends to f in $\mathcal{S}'$ as $N\to\infty$ since $f\in\mathcal{S}'$. Thus $f=0$. $\square$

Before the next topic, we investigate the scaling property of $\mathcal{M}\mathcal{F}^p_q$.

Proposition 2.2. *Let $1 \le q \le p < \infty$, $R > 0$, and let $f \in \mathcal{S}'$. Write $f_R(x) \equiv f(Rx)$ for $x \in \mathbb{R}^n$. Then*

$$\|f_R\|_{\mathcal{M}\mathcal{F}^p_q} = R^{-\frac{n}{p'}} \|f\|_{\mathcal{M}\mathcal{F}^p_q}. \tag{2.1}$$

Proof. To calculate $\|f_R\|_{\mathcal{M}\mathcal{F}^p_q}$, we first observe that

$$\begin{aligned}\mathcal{F}^{-1}\psi_Q * f_R(x) &= \int_{\mathbb{R}^n} \mathcal{F}^{-1}\psi_Q(x-y) f(Ry)\, dy \\ &= \int_{\mathbb{R}^n} R^{-n}\mathcal{F}^{-1}\psi_Q\big(R^{-1}(Rx-w)\big) f(w)\, dw \\ &= \mathcal{F}^{-1}[\psi_Q]_R * f(Rx),\end{aligned}$$

where $[\psi_Q]_R(x) \equiv \psi_Q(Rx) = \psi_{Q_R}(x)$ and where $Q_R = \{x \in \mathbb{R}^n \colon Rx \in Q\}$. Then we see that

$$\begin{aligned}\|f_R\|_{\mathcal{M}\mathcal{F}^p_q} &= \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\mathbb{R}^n} \left|\mathcal{F}^{-1}\psi_{Q_R} * f(Rx)\right|^{q'} dx\right)^{\frac{1}{q'}} \\ &= R^{n(\frac{1}{p}-\frac{1}{q})-\frac{n}{q'}} \sup_{Q\in\mathcal{Q}} |Q_R|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\mathbb{R}^n} \left|\mathcal{F}^{-1}\psi_{Q_R} * f(y)\right|^{q'} dy\right)^{\frac{1}{q'}} \\ &= R^{-\frac{n}{p'}} \|f\|_{\mathcal{M}\mathcal{F}^p_q}.\end{aligned}$$

□

To investigate the fundamental embedding properties of $\mathcal{M}\mathcal{F}^p_q$, we recall the following result, which is a consequence of the Planchrel–Polya–Nikol'skii inequality.

Lemma 2.3 ([15, p. 18])**.** *Let $0 < p \le p_1 \le \infty$, and let $R > 0$. Then it holds that $\|f\|_{L^{p_1}} \le CR^{n(\frac{1}{p}-\frac{1}{p_1})}\|f\|_{L^p}$ for all $f \in \mathcal{S}'_{Q(R)} \equiv \{f \in \mathcal{S}' \colon \operatorname{supp}(\mathcal{F}f) \subset Q(R)\}$.*

As a direct corollary, we obtain the following.

Corollary 2.4. *Let $0 < p \le p_1 \le \infty$, let $R > 0$, and let $x_0 \in \mathbb{R}^n$ be arbitrary. Then it also holds that $\|f\|_{L^{p_1}} \le CR^{n(\frac{1}{p}-\frac{1}{p_1})}\|f\|_{L^p}$ for all $f \in \mathcal{S}'_{Q(x_0,R)} \equiv \{f \in \mathcal{S}' \colon \operatorname{supp}(\mathcal{F}f) \subset Q(x_0,R)\}$.*

Employing the above Planchrel–Polya–Nikol'skii-type inequality, we obtain the embedding relation similar to the case of Morrey spaces.

Proposition 2.5 (Fundamental embedding and special cases)**.**

(1) *Let $1 \le q \le q_1 \le p \le \infty$. Then $L^{p'} \hookrightarrow \mathcal{M}\mathcal{F}^p_{q_1} \hookrightarrow \mathcal{M}\mathcal{F}^p_q$ in the sense of continuous embedding.*
(2) *Let $1 \le p < \infty$. Then*

$$\mathcal{M}\mathcal{F}^p_p = L^{p'}. \tag{2.2}$$

(3) *Denote by δ_0 the Dirac delta massed at the origin. Then $\delta_0 \in \mathcal{M}\mathcal{F}^\infty_\infty$.*

Proof.

(1) Since $q \le q_1 \le p$ implies that $p' \le q_1' \le q'$, using Corollary 2.4 we obtain, for all $Q \in \mathcal{Q}$,

$$\begin{aligned}|Q|^{\frac{1}{p}-\frac{1}{q}}\Big(\int_{\mathbb{R}^n}\big|\mathcal{F}^{-1}\psi_Q * f(x)\big|^{q'}\,dx\Big)^{\frac{1}{q'}} &\le C|Q|^{(\frac{1}{p}-\frac{1}{q}+\frac{1}{q_1'}-\frac{1}{q'})}\Big(\int_{\mathbb{R}^n}\big|\mathcal{F}^{-1}\psi_Q * f(x)\big|^{q_1'}\,dx\Big)^{\frac{1}{q_1'}}\\ &= C|Q|^{\frac{1}{p}-\frac{1}{q_1}}\Big(\int_{\mathbb{R}^n}\big|\mathcal{F}^{-1}\psi_Q * f(x)\big|^{q_1'}\,dx\Big)^{\frac{1}{q_1'}},\end{aligned}$$

which implies that $\|f\|_{\mathcal{M}\mathcal{F}^p_q} \le C\|f\|_{\mathcal{M}\mathcal{F}^p_{q_1}}$. In particular, by taking $q_1 = p$, we see that

$$\|f\|_{\mathcal{M}\mathcal{F}^p_q} \le C\|f\|_{\mathcal{M}\mathcal{F}^p_p} = C\sup_{Q\in\mathcal{Q}}\Big(\int_{\mathbb{R}^n}\big|\mathcal{F}^{-1}\psi_Q * f(x)\big|^{p'}\,dx\Big)^{\frac{1}{p'}} \le C\|f\|_{L^{p'}}$$

by the Young inequality.

(2) Thanks to (1), we know that $L^{p'} \hookrightarrow \mathcal{M}\mathcal{F}^p_p$. To show the converse inclusion, we fix any $f \in \mathcal{M}\mathcal{F}^p_p$. Let us denote $\psi(2^{-j}\cdot) = \psi_j$. First, we consider the case of $1 < p < \infty$. If f is in $\mathcal{S}$, then it is easy to see that

$$\begin{aligned}\|f\|_{\mathcal{M}\mathcal{F}^p_p} &\ge \sup_{j\in\mathbb{Z}}\Big(\int_{\mathbb{R}^n}\big|\mathcal{F}^{-1}\psi_j * f(x)\big|^{p'}\,dx\Big)^{\frac{1}{p'}}\\ &\ge \lim_{j\to\infty}\Big(\int_{\mathbb{R}^n}\big|\mathcal{F}^{-1}\psi_j * f(x)\big|^{p'}\,dx\Big)^{\frac{1}{p'}} = \|f\|_{L^{p'}}\end{aligned}$$

by $p > 1$. However, since we have only $f \in \mathcal{M}\mathcal{F}^p_p \subset \mathcal{S}'$, we need to take care. We need to show $f \in L^1_{\rm loc}$. Notice that $f \in \mathcal{M}\mathcal{F}^p_p$ implies that $\{\mathcal{F}^{-1}[\psi_j] * f\}_{j\in\mathbb{Z}}$ forms a bounded set in $L^{p'}$:

$$\sup_{j\in\mathbb{Z}}\Big(\int_{\mathbb{R}^n}\big|\mathcal{F}^{-1}\psi_j * f(x)\big|^{p'}\,dx\Big)^{\frac{1}{p'}} \le \|f\|_{\mathcal{M}\mathcal{F}^p_p} < \infty.$$

We notice from $p < \infty$ that $L^{p'} = (L^p)^*$. Therefore, by Banach–Alaoglu's theorem, we may choose a subsequence $\{j_l\}_{l\in\mathbb{N}}$ such that $\mathcal{F}^{-1}[\psi_{j_l}] * f$ converges weakly to some $g \in L^{p'}$ as $l \to \infty$. In this case, we can show that $g = f$ in the sense of $\mathcal{S}'$ as follows: take any $\eta \in \mathcal{S}$, and calculate that

$$\langle g, \eta\rangle = \lim_{l\to\infty}\big\langle \mathcal{F}^{-1}[\psi_{j_l}] * f, \eta\big\rangle = \lim_{l\to\infty}\big\langle f, \mathcal{F}^{-1}[\psi_{j_l}] * \eta\big\rangle = \langle f, \eta\rangle.$$

Hence we see that $f = g \in L^{p'}$. Moreover, we have

$$\|f\|_{L^{p'}} = \|g\|_{L^{p'}} \le \sup_{j\in\mathbb{Z}}\Big(\int_{\mathbb{R}^n}\big|\mathcal{F}^{-1}\psi_j * f(x)\big|^{p'}\,dx\Big)^{\frac{1}{p'}} \le \|f\|_{\mathcal{M}\mathcal{F}^p_p},$$

which implies that $\mathcal{M}\mathcal{F}^p_p \hookrightarrow L^{p'}$. For the case of $p = 1$, we recall that

$$\|f\|_{L^\infty} \sim \|f\|_{H^\infty} \equiv \Big\|\sup_{j\in\mathbb{Z}}|\mathcal{F}^{-1}\psi_{Q(2^j)} * f|\Big\|_{L^\infty}.$$

Hence we see that

$$\|f\|_{L^\infty} \sim \Big\|\sup_{j\in\mathbb{Z}} |\mathcal{F}^{-1}\psi_{Q(2^j)} * f|\Big\|_{L^\infty} \le \sup_{Q\in\mathcal{Q}} \|\mathcal{F}^{-1}\psi_Q * f\|_{L^\infty} = \|f\|_{\mathcal{M}\mathcal{F}^1_1},$$

which implies that $L^\infty = \mathcal{M}\mathcal{F}^1_1$.

(3) Note that $\mathcal{F}^{-1}\psi_Q * \delta_0 = \mathcal{F}^{-1}\psi_Q$ and that $\|\mathcal{F}^{-1}\psi_Q\|_{L^1} = \|\mathcal{F}^{-1}\psi\|_{L^1}$. Hence $\|\mathcal{F}^{-1}\psi_Q * \delta_0\|_{L^1} = \|\mathcal{F}^{-1}\psi\|_{L^1}$ follows. □

Example 2.6. We give a couple of examples. Let $1 \le q \le p < \infty$, and let ψ satisfy (1.2). Write

$$C_{\psi,q} \equiv \left(\int_{\mathbb{R}^n} \left|\mathcal{F}^{-1}\psi_{Q_0}(x)\right|^{q'} dx\right)^{\frac{1}{q'}} > 0.$$

(1) Let $Q_0 \in \mathcal{D}$ with $\ell(Q_0) = 1$. Then we claim that

$$\|\mathcal{F}^{-1}\chi_{Q_0}\|_{\mathcal{M}\mathcal{F}^p_q} \sim C_{\psi,q} > 0.$$

In particular, by using the scaling law (2.1), we see that $\|\mathcal{F}^{-1}\chi_Q\|_{\mathcal{M}\mathcal{F}^p_q} \sim |Q|^{\frac{1}{p'}} C_{\psi,q}$.

(2) Let $1 \le q \le \min(p,2)$. Let $E_0 \equiv [0,1]^n$, and construct a set $E_m \subset E_0$ inductively. Supposing that we have defined $E_0, E_1, \dots, E_{m-1}$, we define E_m by

$$E_m \equiv \bigcup_{e\in\{0,1-\gamma\}^n} (e + \gamma E_{m-1}).$$

We can choose $\gamma \in (0,1/2)$, which depends on p, q so that

$$\|\chi_{E_m}\|_{\mathcal{M}^p_q} \sim \|\chi_{E_m}\|_{L^q} \sim \gamma^{\frac{mn}{p}}. \tag{2.3}$$

Then we have $\|\mathcal{F}^{-1}\chi_{E_m}\|_{\mathcal{M}\mathcal{F}^p_q} \sim \gamma^{\frac{mn}{p}}$. Hence we have an example of f satisfying $f \in \mathcal{M}\mathcal{F}^p_{q_0} \setminus \mathcal{M}\mathcal{F}^p_{q_1}$ with $1 \le q_0 < q_1 \le 2$.

One may observe that the norm of $\mathcal{M}\mathcal{F}^p_q$ is similar to the norm of the Besov space $B^{n(\frac{1}{p}-\frac{1}{q})}_{q'\infty}$. To see this, we recall the Besov norm. Let us denote the Littlewood–Paley decomposition by $\{\varphi_j\}_{j=0}^\infty$; that is, the sequence $\{\varphi_j\}_{j=0}^\infty \subset \mathcal{S}$ is defined as follows. Assume that $\varphi_0, \varphi \in C^\infty_{\rm c}(\mathbb{R}^n)$ satisfy

$$\chi_{Q(4)} \le \varphi_0 \le \chi_{Q(8)}, \qquad \chi_{Q(4)\setminus Q(2)} \le \varphi \le \chi_{Q(8)\setminus Q(1)},$$

and let $\varphi_j \equiv \varphi(2^{-j}\cdot)$ for $j \in \mathbb{N}$. Let $s \in \mathbb{R}$, and let $0 < p, q \le \infty$. The Besov norm of $f \in \mathcal{S}'$ is defined by

$$\|f\|_{B^s_{pq}} \equiv \left(\sum_{j\in\mathbb{N}_0} \left\|2^{js}\varphi_j(D)f\right\|_{L^p}^q\right)^{\frac{1}{q}},$$

where $\varphi_j(D)f \equiv \mathcal{F}^{-1}\varphi_j * f$.

Proposition 2.7. *Let $1 \le q \le p \le \infty$. The following embedding holds:*

$$L^{p'} \hookrightarrow \mathcal{M}\mathcal{F}^p_q \hookrightarrow B^{n(\frac{1}{p}-\frac{1}{q})}_{q'\infty} = B^{n(\frac{1}{q'}-\frac{1}{p'})}_{q'\infty}.$$

Proof. The left embedding is (2.2) itself; let us concentrate on the right embedding.

Let us abbreviate $Q(2^j)$ to Q_j, and let $\{\varphi_j\}_{j=0}^{\infty}$ denote a Littlewood–Paley decomposition as above. If we notice that $\varphi_j = \varphi_j \cdot \psi_{Q_{j+10}}$, then it follows that

$$\|f\|_{B^{n(\frac{1}{p}-\frac{1}{q})}_{q'\infty}} = \sup_{j\in\mathbb{N}_0} 2^{jn(\frac{1}{p}-\frac{1}{q})} \|\mathcal{F}^{-1}\varphi_j * \mathcal{F}^{-1}\psi_{Q_{j+10}} * f\|_{L^{q'}}$$
$$\leq C \sup_{j\in\mathbb{N}_0} |Q_j|^{\frac{1}{p}-\frac{1}{q}} \|\mathcal{F}^{-1}\psi_{Q_{j+10}} * f\|_{L^{q'}} \leq C\|f\|_{\mathcal{M}\mathcal{F}^p_q}. \qquad \square$$

From Proposition 2.7, we particularly have

$$L^1 \hookrightarrow \mathcal{M}\mathcal{F}^{\infty}_{\infty} \hookrightarrow B^0_{1\infty}.$$

In addition, we already know that $\mathcal{F} : B^0_{1\infty} \to L^\infty$; that is,

$$\|\mathcal{F}f\|_{L^\infty} \leq C\|f\|_{B^0_{1\infty}} \quad (f \in B^0_{1\infty}). \tag{2.4}$$

Hence our first observation (1.5) is weaker than the known estimate (2.4). However, it is possible to improve the estimate (2.4) further by defining the Besov-type space of $\mathcal{M}\mathcal{F}^p_q$.

Definition 2.8. Let us denote the Littlewood–Paley decomposition by $\{\varphi_j\}_{j\in\mathbb{N}_0}$. For $f \in \mathcal{S}'$, we define

$$\|f\|_{\mathcal{N}^0_{\mathcal{M}\mathcal{F}^\infty_\infty,\infty}} \equiv \sup_{j\in\mathbb{N}_0} \big\|\varphi_j(D)f\big\|_{\mathcal{M}\mathcal{F}^\infty_\infty}$$

and the space $\mathcal{N}^0_{\mathcal{M}\mathcal{F}^\infty_\infty,\infty}$ by all the functions $f \in \mathcal{S}'$ for which the norm $\|f\|_{\mathcal{N}^0_{\mathcal{M}\mathcal{F}^\infty_\infty,\infty}}$ is finite.

This space is closely related to the Besov–Morrey spaces (see [6, Definition 1.3]). Using the space $\mathcal{N}^0_{\mathcal{M}\mathcal{F}^\infty_\infty,\infty}$, we may improve the estimate (2.4), as follows.

Proposition 2.9.

(1) *We have the following embedding relation:*

$$B^0_{1\infty} \hookrightarrow \mathcal{N}^0_{\mathcal{M}\mathcal{F}^\infty_\infty,\infty}.$$

(2) *The Fourier transform $\mathcal{F}$ is bounded from $\mathcal{N}^0_{\mathcal{M}\mathcal{F}^\infty_\infty,\infty}$ to L^∞:*

$$\|\mathcal{F}f\|_{L^\infty} \leq C\|f\|_{\mathcal{N}^0_{\mathcal{M}\mathcal{F}^\infty_\infty,\infty}}.$$

Proof. The embedding (1) follows directly from the embedding $L^1 \hookrightarrow \mathcal{M}\mathcal{F}^\infty_\infty$. Indeed,

$$\|f\|_{\mathcal{N}^0_{\mathcal{M}\mathcal{F}^\infty_\infty,\infty}} = \sup_{j\in\mathbb{N}_0} \big\|\varphi_j(D)f\big\|_{\mathcal{M}\mathcal{F}^\infty_\infty} \leq C \sup_{j\in\mathbb{N}_0} \big\|\varphi_j(D)f\big\|_{L^1} = \|f\|_{B^0_{1\infty}}.$$

To show (2), we employ (1.5) to obtain

$$\|\mathcal{F}f\|_{L^\infty} = \sup_{j\in\mathbb{N}_0} \|\varphi_j\mathcal{F}f\|_{L^\infty} \leq C \sup_{j\in\mathbb{N}_0} \|\mathcal{F}^{-1}\varphi_j * f\|_{\mathcal{M}\mathcal{F}^\infty_\infty} = C\|f\|_{\mathcal{N}^0_{\mathcal{M}\mathcal{F}^\infty_\infty,\infty}}. \qquad \square$$

We next show the completeness of the space $\mathcal{M}\mathcal{F}^p_q$. We employ the argument in [10, Lemma 2.15] to show the completeness of $\mathcal{M}\mathcal{F}^p_q$.

Theorem 2.10. *Let $1 \le q \le p \le \infty$.*

(1) (Fatou property) *Let $\{f_j\}_{j=1}^{\infty} \subset \mathcal{S}'$ converge to f in $\mathcal{S}'$. Then*
$$\|f\|_{\mathcal{M}\mathcal{F}^p_q} \le \liminf_{j\to\infty} \|f_j\|_{\mathcal{M}\mathcal{F}^p_q}. \tag{2.5}$$

(2) (Completeness) *The space $\mathcal{M}\mathcal{F}^p_q$ is complete; that is, for any sequence $\{f_j\}_{j=1}^{\infty} \subset \mathcal{M}\mathcal{F}^p_q$ satisfying*
$$\lim_{j,k\to\infty} \|f_j - f_k\|_{\mathcal{M}\mathcal{F}^p_q} = 0, \tag{2.6}$$
there exists $f \in \mathcal{M}\mathcal{F}^p_q$ such that $f_j \to f$ in $\mathcal{M}\mathcal{F}^p_q$.

Proof. We may apply the argument as in the case of Besov spaces. Note that $f_j \to f$ in $\mathcal{S}'$ implies that $\mathcal{F}^{-1}\psi_Q * f_j(x) \to \mathcal{F}^{-1}\psi_Q * f(x)$ for each $x \in \mathbb{R}^n$ and each $Q \in \mathcal{Q}$. With this in mind, we calculate that
$$\begin{aligned}\|f\|_{\mathcal{M}\mathcal{F}^p_q} &= \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\mathbb{R}^n} \lim_{j\to\infty} \left|\mathcal{F}^{-1}\psi_Q * f_j(x)\right|^{q'} dx\right)^{\frac{1}{q'}} \\ &\le \liminf_{j\to\infty} \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_{\mathbb{R}^n} \left|\mathcal{F}^{-1}\psi_Q * f_j(x)\right|^{q'} dx\right)^{\frac{1}{q'}}.\end{aligned}$$
This proves (2.5).

Next we will show the completeness. Let $\{f_j\}_{j=1}^{\infty}$ satisfy (2.6). Since we have the embedding $\mathcal{M}\mathcal{F}^p_q \hookrightarrow B^{n(\frac{1}{p}-\frac{1}{q})}_{q'\infty}$ by Proposition 2.7, at least the sequence $\{f_j\}_{j=1}^{\infty}$ converges some $f \in B^{n(\frac{1}{p}-\frac{1}{q})}_{q'\infty}$ in the sense of $\mathcal{S}'$. Hence we employ the Fatou property of $\mathcal{M}\mathcal{F}^p_q$ to obtain
$$\|f - f_j\|_{\mathcal{M}\mathcal{F}^p_q} \le \liminf_{k\to\infty} \|f_k - f_j\|_{\mathcal{M}\mathcal{F}^p_q} \to 0$$
as $j \to \infty$. □

We have defined the norm $\|\cdot\|_{\mathcal{M}\mathcal{F}^p_q}$ without restricting the position of cubes. In the case of the central-type Fourier–Morrey norm, which we will define below, the converse inclusion holds in a certain sense.

Definition 2.11. Let $1 \le q \le p \le \infty$, and let $f \in \mathcal{S}'$. The central-type Fourier–Morrey norm $\|\cdot\|_{\mathcal{M}\mathcal{F}^p_{cq}}$ is defined by
$$\|f\|_{\mathcal{M}\mathcal{F}^p_{cq}} \equiv \sup_{j\in\mathbb{Z}} 2^{jn(\frac{1}{p}-\frac{1}{q})} \left(\int_{\mathbb{R}^n} \left|\mathcal{F}^{-1}\left[\psi(2^{-j}\cdot)\right] * f(x)\right|^{q'} dx\right)^{\frac{1}{q'}}.$$
The space $\mathcal{M}\mathcal{F}^p_{cq}$ collects all $f \in \mathcal{S}'$ for which the norm $\|f\|_{\mathcal{M}\mathcal{F}^p_{cq}}$ is finite.

Observe that $\mathcal{M}\mathcal{F}^p_q$ is continuously embedded into $\mathcal{M}\mathcal{F}^p_{cq}$.

Proposition 2.12. *Let $1 \le q \le p \le \infty$. Then*
$$B^{n(\frac{1}{p}-\frac{1}{q})}_{q'1} \hookrightarrow \mathcal{M}\mathcal{F}^p_{cq} \hookrightarrow B^{n(\frac{1}{p}-\frac{1}{q})}_{q'\infty}. \tag{2.7}$$
When $q = 2 \le p \le \infty$, the left embedding in (2.7) can be improved:
$$B^{n(\frac{1}{p}-\frac{1}{2})}_{22} \hookrightarrow \mathcal{M}\mathcal{F}^p_{c2} \hookrightarrow B^{n(\frac{1}{p}-\frac{1}{2})}_{2\infty}. \tag{2.8}$$

Proof. Let $\psi = \varphi_0$ be a function such that $\chi_{Q(1)} \le \psi \le \chi_{Q(2)}$. For each j, define $\varphi_j \equiv \psi(2^{-j}\cdot) - \psi(2^{-j+1}\cdot)$. The embedding $\mathcal{M}\mathcal{F}^p_{cq} \hookrightarrow B^{n(\frac{1}{p}-\frac{1}{q})}_{q'\infty}$ follows directly, similar to Proposition 2.7.

Let us show that $B^{n(\frac{1}{p}-\frac{1}{q})}_{q'1} \hookrightarrow \mathcal{M}\mathcal{F}^p_{cq}$. If we observe that $\psi_{Q_j} = \sum_{k=0}^{j+1} \varphi_k \cdot \psi_{Q_j}$, then

$$\begin{aligned}\|f\|_{\mathcal{M}\mathcal{F}^p_{cq}} &= \sup_{j\in\mathbb{N}_0} 2^{jn(\frac{1}{p}-\frac{1}{q})} \|\mathcal{F}^{-1}\psi_{Q_j} * f\|_{L^{q'}} \\ &\le \sup_{j\in\mathbb{N}_0} 2^{jn(\frac{1}{p}-\frac{1}{q})} \sum_{k=0}^{j+1} \|\mathcal{F}^{-1}\varphi_k * f\|_{L^{q'}} \cdot \|\mathcal{F}^{-1}\psi\|_{L^1}.\end{aligned}$$

In addition, since $\frac{1}{p} - \frac{1}{q} \le 0$, we have $2^{jn(\frac{1}{p}-\frac{1}{q})} \le 2^{kn(\frac{1}{p}-\frac{1}{q})}$ for all $k \le j$. As a result, it follows that

$$\|f\|_{\mathcal{M}\mathcal{F}^p_{cq}} \le \|\mathcal{F}^{-1}\psi\|_{L^1} \sum_{k\in\mathbb{N}_0} 2^{kn(\frac{1}{p}-\frac{1}{q})} \|\mathcal{F}^{-1}\varphi_k * f\|_{L^{q'}} = C_\psi \|f\|_{B^{n(\frac{1}{p}-\frac{1}{q})}_{q'1}},$$

which proves (2.7).

Let us prove (2.8). In the case of $q = 2$, since we may use Plancherel's theorem, it is possible to improve the above embedding. Note that

$$\|f\|_{\mathcal{M}\mathcal{F}^p_{cq}} = \sup_{j\in\mathbb{N}_0} 2^{jn(\frac{1}{p}-\frac{1}{2})} \|\psi_{Q_j}\mathcal{F}f\|_{L^2} \sim \sup_{j\in\mathbb{N}_0} 2^{jn(\frac{1}{p}-\frac{1}{2})} \left\| \left(\sum_{k=0}^{j+1} \varphi_k^2 \right)^{\frac{1}{2}} \psi_{Q_j}\mathcal{F}f \right\|_{L^2}.$$

A direct calculation gives us

$$\begin{aligned}\left\| \left(\sum_{k=0}^{j+1} \varphi_k^2 \right)^{\frac{1}{2}} \psi_{Q_j}\mathcal{F}f \right\|_{L^2} &= \left(\int_{\mathbb{R}^n} \sum_{k=0}^{j+1} \varphi_k^2(\xi)\psi_{Q_j}^2(\xi) \left|\mathcal{F}f(\xi)\right|^2 d\xi \right)^{\frac{1}{2}} \\ &= \left(\sum_{k=0}^{j+1} \|\varphi_k\psi_{Q_j}\mathcal{F}f\|_{L^2}^2 \right)^{\frac{1}{2}} \\ &= \left(\sum_{k=0}^{j+1} \|\mathcal{F}^{-1}\psi_{Q_j} * \mathcal{F}^{-1}\varphi_k * f\|_{L^2}^2 \right)^{\frac{1}{2}}.\end{aligned}$$

Thus, by recalling $\frac{1}{p} - \frac{1}{2} < 0$, we obtain

$$\|f\|_{\mathcal{M}\mathcal{F}^p_{cq}} \le \|\mathcal{F}^{-1}\psi\|_{L^1} \cdot \left(\sum_{k\in\mathbb{N}_0} 2^{2jn(\frac{1}{p}-\frac{1}{2})} \|\mathcal{F}^{-1}\varphi_k * f\|_{L^2} \right)^{\frac{1}{2}} = C_\psi \|f\|_{B^{n(\frac{1}{p}-\frac{1}{2})}_{22}}.$$

□

3. The fundamental properties of $\mathcal{H}\mathcal{F}^p_q$

Now we investigate the predual space $\mathcal{H}\mathcal{F}^p_q$ with $1 \le q \le p < \infty$ defined in Definition 1.4. The first one is a fundamental embedding.

Proposition 3.1. *Let $1 \le p \le q_1 \le q \le \infty$.*

(1) *The following embedding relation*

$$\mathcal{H}_{\mathcal{F}^p_q} \hookrightarrow \mathcal{H}_{\mathcal{F}^p_{q_1}} \hookrightarrow L^{p'} \tag{3.1}$$

holds.

(2) *If the b_k's are (p,q)-Fourier blocks and $\{\lambda_j\}_{j=1}^\infty \in \ell^1$, then*

$$\sum_{j=1}^\infty \lambda_j b_j$$

converges in $L^{p'}$.

Proof. We first show the left embedding in (3.1). If b is a (p,q)-Fourier block, by $q' \le q_1'$, then we have

$$\|b\|_{L^{q_1'}} \le C|Q|^{\frac{1}{q'}-\frac{1}{q_1'}}\|b\|_{L^{q'}} \le C|Q|^{\frac{1}{q'}-\frac{1}{q_1'}+\frac{1}{q}-\frac{1}{p}} = C|Q|^{\frac{1}{q_1}-\frac{1}{p}}.$$

This implies that b is a (p,q_1)-Fourier block; hence $\mathcal{H}_{\mathcal{F}^p_q} \hookrightarrow \mathcal{H}_{\mathcal{F}^p_{q_1}}$ holds.

Meanwhile, since for any $f \in \mathcal{H}_{\mathcal{F}^p_q}$ we can take $\{\lambda_j\}_{j=1}^\infty \in \ell^1$ and (p,q)-Fourier blocks $\{b_j\}_{j=1}^\infty$ such that $f = \sum \lambda_j b_j$ and $\|\{\lambda_j\}_{j=1}^\infty\|_{\ell^1} \le 2\|f\|_{\mathcal{H}_{\mathcal{F}^p_q}}$ hold, it follows that

$$\|f\|_{L^{p'}} \le \sum_{j=1}^\infty |\lambda_j|\|b_j\|_{L^{p'}} \le C\sum_{j=1}^\infty |\lambda_j|\|b_j\|_{L^{q'}}|Q_j|^{\frac{1}{q'}-\frac{1}{p'}} \le C\|f\|_{\mathcal{H}_{\mathcal{F}^p_q}}.$$

This implies that $\mathcal{H}_{\mathcal{F}^p_q} \hookrightarrow L^{p'}$, thereby implying assertion (2), as well. □

Let us consider further an example of (p,q)-Fourier blocks. We will employ the following lemma in the proof of Theorem 1.5.

Lemma 3.2. *Let $1 \le p \le q < \infty$, and let $Q \in \mathcal{Q}$. Let ψ and g satisfy (1.2) and $\|g\|_{L^{q'}} \le 1$, respectively. Then $|Q|^{\frac{1}{q}-\frac{1}{p}}(\|\mathcal{F}^{-1}\psi\|_{L^1})^{-1}\psi_Q(D)g$ is a (p,q)-Fourier block. In particular,*

$$\big\|\psi_Q(D)g\big\|_{\mathcal{H}_{\mathcal{F}^p_q}} \le C\|\mathcal{F}^{-1}\psi\|_{L^1}|Q|^{\frac{1}{p}-\frac{1}{q}}\|g\|_{L^{q'}}, \tag{3.2}$$

where the constant C depends only on the dimension.

Proof. By the Young inequality,

$$\begin{aligned}|Q|^{\frac{1}{q}-\frac{1}{p}}\big(\|\mathcal{F}^{-1}\psi\|_{L^1}\big)^{-1}\big\|\psi_Q(D)g\big\|_{L^{q'}} &\le \|\mathcal{F}^{-1}\psi_Q\|_{L^1}\big(\|\mathcal{F}^{-1}\psi\|_{L^1}\big)^{-1}|Q|^{\frac{1}{q}-\frac{1}{p}}\\ &= |Q|^{\frac{1}{q}-\frac{1}{p}}.\end{aligned}$$

As for the frequency support, we have $\mathrm{supp}(\mathcal{F}\psi_Q(D)g) \subset \mathrm{supp}(\psi_Q) \subset 4Q$. Thus $|Q|^{\frac{1}{q}-\frac{1}{p}}(\|\mathcal{F}^{-1}\psi\|_{L^1})^{-1}\psi_Q(D)g$ with some constant is a (p,q)-Fourier block. □

Before the next topic, we investigate the scaling property of $\mathcal{H}_{\mathcal{F}^p_q}$.

Proposition 3.3. *Let $1 \le p \le q \le \infty$. Let $f \in \mathcal{S}'$, and write $f_R(x) \equiv f(Rx)$ for $R > 0$. Then $\|f_R\|_{\mathcal{H}_{\mathcal{F}^p_q}} = R^{-\frac{n}{p'}}\|f\|_{\mathcal{H}_{\mathcal{F}^p_q}}$.*

Proof. If we let $b_R(x) \equiv b(Rx)$ for any (p,q)-Fourier block b with the associated cube Q, then we notice that the support of $\mathcal{F}b_R$ is contained in $Q^{(R)} = \{x \in \mathbb{R}^n \colon R^{-1}x \in Q\}$ and that

$$\|b_R\|_{L^{q'}} = R^{-\frac{n}{q'}}\|b\|_{L^{q'}} \le R^{-\frac{n}{q'}}|Q|^{\frac{1}{q}-\frac{1}{p}} = R^{-\frac{n}{p'}}|RQ|^{\frac{1}{q}-\frac{1}{p}},$$

which implies that $R^{\frac{n}{p'}}b_R$ is a (p,q)-Fourier block. Hence we see that $\|b_R\|_{\mathcal{H}\mathcal{F}^p_q} = R^{-\frac{n}{p'}}$ and that $\|f_R\|_{\mathcal{H}\mathcal{F}^p_q} = R^{-\frac{n}{p'}}\|f\|_{\mathcal{H}\mathcal{F}^p_q}$. □

We also observe the diagonal case: $p = q$.

Proposition 3.4.

(1) *If $1 < p \le \infty$, then $\mathcal{H}\mathcal{F}^p_p = L^{p'}$.*

(2) *The Sobolev space $W^{1,\infty} = \{f \in L^\infty \colon \nabla f \in L^\infty\}$ is dense in $\mathcal{H}\mathcal{F}^1_1$.*

Proof.
The inclusion $\mathcal{H}\mathcal{F}^p_p \subset L^{p'}$ is clear from the definition of (p,p)-Fourier blocks.

Let us check the reverse inclusion. Let $f \in L^{p'} \setminus \{0\}$ to this end. Choose $\psi \in \mathcal{S}$ so that it satisfies (1.2). Then $\lim_{j\to\infty}\psi(2^{-j}D)f = f$ in $L^{p'}$. This means that, for $f_j \equiv \varphi(2^{-j}D)f$, there exists a strictly increasing sequence $\{j_k\}$ of positive integers such that

$$\|f_{j_k} - f_{j_{k-1}}\|_{L^{p'}} \le 2^{-k}\|f\|_{L^{p'}}$$

for all $k = 1, 2, \dots$ and such that

$$\|f_{j_1}\|_{L^{p'}} \le 2\|f\|_{L^{p'}}.$$

In this case, $2^k(\|f\|_{L^{p'}})^{-1}(f_{j_k} - f_{j_{k-1}})$, $k \in \mathbb{N}$, and $(2\|f\|_{L^{p'}})^{-1}f_{j_1}$ are (p,p)-Fourier blocks. As a result, we see that $\mathcal{H}\mathcal{F}^p_p = L^{p'}$ holds. It is not so hard to see that $W^{1,\infty}$ is contained in $\mathcal{H}\mathcal{F}^1_1$. In fact, for any $f \in W^{1,\infty}$, we have

$$f = \psi(D)f + \lim_{j\to\infty}\sum_{l=1}^{j}\big(\psi(2^{-l}D) - \psi(2^{-l+1}D)\big)f$$

in L^∞ together with the estimate

$$\big\|\big(\psi(2^{-l}D) - \psi(2^{-l+1}D)\big)f\big\|_{L^\infty} = O(2^{-l})$$

as $l \to \infty$.

Let $f \in \mathcal{H}\mathcal{F}^1_1$. Then there exist $\{\lambda_j\}_{j=1}^\infty \in \ell^1$ and $\{b_j\}_{j=1}^\infty \in L^\infty$ such that $\mathrm{supp}(\mathcal{F}b_j)$ is compact, $\|b_j\|_{L^\infty} \le 1$, and

$$f = \sum_{j=1}^\infty \lambda_j b_j.$$

Since each b_j is in $W^{1,\infty}$, that is, $b_j, \nabla b_j \in L^\infty$, we see that f is in the closure of $W^{1,\infty}$. □

The following proposition is crucial when we discuss the Fourier restriction problem on our spaces.

Proposition 3.5. *Let $1 \le p \le q \le \infty$. If $b \in \mathcal{H}\mathcal{F}^p_q$ has compact frequency support, then there exists a finite decomposition*

$$b = \sum_{j=1}^{N} \lambda_j b_j,$$

where each b_j is a (p,q)-Fourier block, and

$$\sum_{j=1}^{N} |\lambda_j| \le \|b\|_{\mathcal{H}\mathcal{F}^p_q}.$$

Furthermore, if the frequency support of b is contained in $Q(2^J)$, then we can arrange that the frequency support be contained in $Q(2^{J+4})$.

Proof. Without loss of generality, we may assume that $\|b\|_{\mathcal{H}\mathcal{F}^p_q} = 1$. Since $b \in \mathcal{H}\mathcal{F}^p_q$, there exists an infinite decomposition

$$b = \sum_{j=1}^{\infty} \overline{\lambda}_j \overline{b}_j$$

in $L^{p'}$, where each $\overline{b}_j$ is a (p,q)-Fourier block with respect to a cube Q_j, and

$$\sum_{j=1}^{\infty} |\overline{\lambda}_j| \le 2\|b\|_{\mathcal{H}\mathcal{F}^p_q} = 2. \tag{3.3}$$

Let ψ satisfy (1.2). Then

$$b = \psi_{Q(2^J)}(D)b = \sum_{j=1}^{\infty} \overline{\lambda}_j \psi_{Q(2^J)}(D)\overline{b}_j \tag{3.4}$$

in $L^{p'}$. Note that if Q_j satisfies $|Q_j| \ge |Q(2^J)|$, then

$$\left\|\psi_{Q(2^J)}(D)\overline{b}_j\right\|_{L^{q'}} \lesssim \|\overline{b}_j\|_{L^{q'}} \le |Q_j|^{\frac{1}{q}-\frac{1}{p}} \le \left|Q(2^J)\right|^{\frac{1}{q}-\frac{1}{p}}.$$

Thus, by regarding $\psi_{Q(2^J)}(D)\overline{b}_j$ as a (p,q)-Fourier block with respect to $Q(2^J)$, we may assume that $|Q_j| \le 2^{nJ}$.

Next we let $\tau \in \mathcal{S}$ be a function such that the support of $\mathcal{F}\tau$ is contained in $Q(3)$ and such that $\tau(0) = 1$. Then we have

$$\lim_{t \downarrow 0} \left\|b - \tau(t\cdot)b\right\|_{L^{q'}} = 0. \tag{3.5}$$

With this in mind, we decompose

$$b = b - \tau(t\cdot)b + \sum_{j=1}^{\infty} \overline{\lambda}_j \tau(t\cdot)\psi_{Q(2^J)}(D)\overline{b}_j.$$

In view of (3.5), we may choose $t = t(J) \ll 2^J$ such that $\|b - \tau(t\cdot)b\|_{L^{q'}} \le |Q(2^J)|^{\frac{1}{q}-\frac{1}{p}}$, which implies that $b - \tau(t\cdot)b$ is a (p,q)-Fourier block with respect to $Q(2^J)$ since t is sufficiently small.

We also notice from $t \ll 2^J$ and $\operatorname{supp}(\mathcal{F}\tau(t^{-1}\cdot)) \subset Q(10t)$ that

$$\operatorname{supp}\big(\mathcal{F}\big[\tau(t\cdot)\psi_{Q(2^J)}(D)\bar{b}_j\big]\big) \subset Q(2^{J+4}). \tag{3.6}$$

Furthermore, we have $\|\bar{b}_j\|_{L^{p'}} \le 1$; hence

$$\begin{aligned}\big\|\tau(t\cdot)\psi_{Q(2^J)}(D)\bar{b}_j\big\|_{L^1} &\le \big\|\tau(t\cdot)\big\|_{L^p}\big\|\psi_{Q(2^J)}(D)\bar{b}_j\big\|_{L^{p'}}\\ &\le Ct^{-n/p}\|\bar{b}_j\|_{L^{p'}} \le Ct^{-n/p}.\end{aligned} \tag{3.7}$$

Additionally, we have a crude estimate:

$$\big\|\tau(t\cdot)\psi_{Q(2^J)}(D)\bar{b}_j\big\|_{L^{p'}} \le C\|\bar{b}_j\|_{L^{p'}} \le C. \tag{3.8}$$

Since q' is between 1 and p', by interpolating these two estimates, (3.7) and (3.8), it follows that $\|\tau(t\cdot)\psi_{Q(2^J)}(D)\bar{b}_j\|_{L^{q'}} \le C_t$. With this estimate in mind, we obtain from $\{\bar{\lambda}_j\}_{j=1}^{\infty} \in \ell^1$ that

$$\|B_N\|_{L^{q'}} \equiv \Big\|\sum_{j=N+1}^{\infty}\bar{\lambda}_j\tau(t\cdot)\psi_{Q(2^J)}(D)\bar{b}_j\Big\|_{L^{q'}} \le C_t\sum_{j=N+1}^{\infty}|\bar{\lambda}_j| \to 0 \tag{3.9}$$

as $N \to \infty$. Thus we take large $N = N_J \in \mathbb{N}$ so that $\|B_N\|_{L^{q'}} \le |Q(2^{J+4})|^{\frac{1}{q}-\frac{1}{p}}$. This together with (3.6) means that B_N is also a (p, q)-Fourier block.

Altogether, by letting

$$b = b - \tau(t\cdot)b + \sum_{j=1}^{N}\bar{\lambda}_j\tau(t\cdot)\psi_{Q(2^J)}(D)\bar{b}_j + B_N$$

and multiplying the suitable constant, we obtain the desired finite decomposition. □

As a dual of Proposition 2.7, the following holds.

Proposition 3.6. *Let $1 \le p \le q \le \infty$. Then the following inclusion holds:*

$$\mathcal{S} \hookrightarrow B^{n(\frac{1}{p}-\frac{1}{q})}_{q'1} \hookrightarrow \mathcal{H}\mathcal{F}^p_q \hookrightarrow L^{p'}.$$

Proof. The first inclusion is well known (see [15, p. 48]). The last inclusion is (3.1) itself, and so we concentrate on proving $B^{n(\frac{1}{p}-\frac{1}{q})}_{q'1} \hookrightarrow \mathcal{H}\mathcal{F}^p_q$. Let $\psi = \varphi_0 \in \mathcal{S}$ be chosen so that $\chi_{B(1)} \le \psi \le \chi_{B(2)}$. Define $\varphi_j \equiv \psi(2^{-j}\cdot) - \psi(2^{-j+1}\cdot)$ for $j \ge 1$. Take $f \in B^{n(\frac{1}{p}-\frac{1}{q})}_{q'1}$, and decompose

$$\begin{aligned} f &= \sum_{j\in\mathbb{N}_0}\mathcal{F}^{-1}\varphi_j * f\\ &= \sum_{j\in\mathbb{N}_0}|Q_j|^{\frac{1}{p}-\frac{1}{q}}\|\mathcal{F}^{-1}\varphi_j * f\|_{L^{q'}}\cdot|Q_j|^{\frac{1}{q}-\frac{1}{p}}\frac{\mathcal{F}^{-1}\varphi_j * f}{\|\mathcal{F}^{-1}\varphi_j * f\|_{L^{q'}}}\\ &=: \sum_{j\in\mathbb{N}_0}|Q_j|^{\frac{1}{p}-\frac{1}{q}}\|\mathcal{F}^{-1}\varphi_j * f\|_{L^{q'}}\cdot b_j,\end{aligned}$$

where $Q_j \equiv Q(0, 2^{j+2})$. Since b_j is a (p,q)-Fourier block associated to Q_j, we see that

$$\|f\|_{\mathcal{H}\mathcal{F}^p_q} \le \sum_{j\in\mathbb{N}_0} |Q_j|^{\frac{1}{p}-\frac{1}{q}} \|\mathcal{F}^{-1}\varphi_j * f\|_{L^{q'}} = \|f\|_{B^{n(\frac{1}{p}-\frac{1}{q})}_{q'1}}.$$

□

Proposition 3.7. *Let $1 \le p \le q \le \infty$, $q > 1$. Then $\mathcal{S}$ is dense in $\mathcal{H}\mathcal{F}^p_q$.*

Proof. First we note that if $\mathrm{supp}(\mathcal{F}f)$ is compact, then f is a C^∞ function; hence a Fourier block is also C^∞. Take any $f \in \mathcal{H}\mathcal{F}^p_q$, and take $\{\lambda_j\}_{j=1}^\infty \in \ell^1$ and a sequence $\{b_j\}_{j=1}^\infty$ of Fourier blocks such that $\|\lambda_j\|_{\ell^1} \le 2\|f\|_{\mathcal{H}\mathcal{F}^p_q}$ and $f = \sum_{j=1}^\infty \lambda_j b_j$ hold in $L^{p'}$.

First observe that $f_J \to f$ in $\mathcal{H}\mathcal{F}^p_q$, where $f_J \equiv \sum_{j=1}^J \lambda_j b_j$. In fact, since the relation $f - f_J = \sum_{j=J+1}^\infty \lambda_j b_j$ gives a Fourier block decomposition of $f - f_J$, we have

$$\|f - f_J\|_{\mathcal{H}\mathcal{F}^p_q} \le \sum_{j=J+1}^\infty |\lambda_j| \to 0 \quad (J \to \infty).$$

Next we will approximate f_J by a Schwartz function. Fix any $J \in \mathbb{N}$, and set $T_J \equiv \min\{\ell(Q_1), \dots, \ell(Q_J)\}$. Suppose that $\psi \in \mathcal{S}$ satisfies that $\mathrm{supp}(\mathcal{F}\psi) \subset Q(1)$ and $\psi(0) = 1$. For $t \in (0, T_J)$, we define

$$f_t^{(J)} \equiv \sum_{j=1}^J \lambda_j \psi(t\cdot) b_j.$$

Note that $\mathrm{supp}(\mathcal{F}(\psi(t\cdot)b_j))$ is included in $Q(T_J) + Q_j$. Thus $\mathrm{supp}(\mathcal{F}(\psi(t\cdot)b_j)) \subset 2Q_j$. Since we assume that $q > 1$, this implies that

$$\|f_J - f_t^{(J)}\|_{\mathcal{H}\mathcal{F}^p_q} \le \sum_{j=1}^J |\lambda_j| \cdot |2Q_j|^{\frac{1}{p}-\frac{1}{q}} \big\|\big(1 - \psi(t\cdot)\big)b_j\big\|_{L^{q'}} \to 0$$

as $t \to 0$ by the Lebesgue convergence theorem. Here we need the assumption $q > 1$ to justify $\|(1 - \psi(t\cdot))b_j\|_{L^{q'}} \to 0$. We also notice that $\psi(t\cdot)b_j \in \mathcal{S}$; hence $f_t^{(J)} \in \mathcal{S}$ for each fixed $t \in (0, T_J)$ since b_j is polynomially increasing. Thus $f^{(J)}$ can be approximated by a Schwartz function. □

We now prove Theorem 1.5 by using Proposition 3.7.

Proof of Theorem 1.5. Since (0) is included in Proposition 3.7, we prove (1) and (2).

(1) Take any $f \in \mathcal{M}\mathcal{F}^{p'}_{q'}$ and any (p,q)-Fourier block $b \in \mathcal{S}$ such that $\mathrm{supp}(\mathcal{F}b)$ is contained in some cube Q and such that $\|b\|_{L^{q'}} \le |Q|^{\frac{1}{q}-\frac{1}{p}}$. If we notice that $b = \mathcal{F}^{-1}\psi_Q * b$, then it follows that

$$\begin{aligned} \big|L_f(b)\big| &= \big|\langle f, \mathcal{F}^{-1}\psi_Q * b\rangle\big| \le \big\|\big[\mathcal{F}^{-1}\psi_Q(-\cdot)\big] * f\big\|_{L^q} \cdot \|b\|_{L^{q'}} \\ &\le |Q|^{\frac{1}{p'}-\frac{1}{q'}} \big\|\big[\mathcal{F}^{-1}\psi_Q(-\cdot)\big] * f\big\|_{L^q} \le \|f\|_{\mathcal{M}\mathcal{F}^{p'}_{q'}}. \end{aligned}$$

Since we may approximate any f by $f_t^{(J)} = \sum_{j=1}^J \lambda_j b_j^t$, where $\|\lambda_j\|_{\ell^1} \le 2\|f\|_{\mathcal{H}\mathcal{F}^p_q}$ and $b_j^t \in \mathcal{S}$, as in Proposition 3.7, the above L_f extends to the bounded operator on $\mathcal{H}\mathcal{F}^p_q$ with the estimate $\|L_f\|_{(\mathcal{H}\mathcal{F}^p_q)^*} \le C\|f\|_{\mathcal{M}\mathcal{F}^{p'}_{q'}}$.

(2) We start with a setup. First, let $L \in (\mathcal{H}\mathcal{F}^p_q)^*$ be arbitrary. Choose ψ satisfying (1.2). We fix $Q \in \mathcal{Q}$. We write

$$\Psi_Q \equiv \psi_Q \cdot \psi_Q + \sum_{j=1}^{\infty} (\psi_{2^j Q} - \psi_{2^{j-1}Q})^2 \tag{3.10}$$

and

$$\varphi_{0,Q} \equiv \frac{\psi_Q}{\Psi_Q}, \qquad \varphi_{j,Q} \equiv \frac{\psi_{2^j Q} - \psi_{2^{j-1}Q}}{\Psi_Q} \tag{3.11}$$

for $j = 1, 2, \ldots$. A direct consequence of (3.10) and (3.11) is that

$$\varphi_{0,Q} \cdot \psi_Q + \sum_{j=1}^{\infty} \varphi_{j,Q} \cdot (\psi_{2^j Q} - \psi_{2^{j-1}Q}) = 1. \tag{3.12}$$

Finally we define

$$L_j(g) \equiv L\big(\psi_{2^j Q}(D) g\big) \tag{3.13}$$

for $g \in L^{q'}$.

First, let us check that L_j is a bounded linear operator. Choose $g \in L^{q'}$ with norm 1 arbitrarily. Then

$$\big|L_j(g)\big| \le \|L\|_{(\mathcal{H}\mathcal{F}^p_q)^*} \big\|\psi_{2^j Q}(D) g\big\|_{\mathcal{H}\mathcal{F}^p_q} \le C 2^{jn(\frac{1}{p}-\frac{1}{q})} |Q|^{\frac{1}{p}-\frac{1}{q}} \|L\|_{(\mathcal{H}\mathcal{F}^p_q)^*}$$

from (3.2). Thus L_j is a bounded linear functional on $L^{q'}_{2^j Q}$ with the estimate

$$\|L_j\|_{(L^{q'}_{2^j Q})^*} \le C 2^{jn(\frac{1}{p}-\frac{1}{q})} |Q|^{\frac{1}{p}-\frac{1}{q}} \|L\|_{(\mathcal{H}\mathcal{F}^p_q)^*}. \tag{3.14}$$

Since we assumed that $1 < q \le \infty$ or, equivalently, that $1 \le q' < \infty$, we may employ the $L^{q'}$-L^q duality. Thus there exists a unique $f_j \in L^q$ such that $\|f_j\|_{L^q} = \|L_j\|_{(L^{q'}_{2^j Q})^*}$ and such that

$$L_j(g) = L(g) = \int_{\mathbb{R}^n} f_j(x) g(x)\, dx \quad (g \in L^{q'}_{2^j Q}). \tag{3.15}$$

Define

$$f_Q \equiv \varphi_{0,Q}(D) f_0 + \sum_{j=1}^{\infty} \varphi_{j,Q}(D)(f_j - f_{j-1}). \tag{3.16}$$

From (3.14) we see that $f_Q \in B^{-n(\frac{1}{p}-\frac{1}{q})}_{q\infty} \hookrightarrow \mathcal{S}'$. Note that the definition of f_Q is independent of the choice of Q; that is, for any $Q, Q' \in \mathcal{Q}$ we have $f_Q = f_{Q'}$ in the sense of $\mathcal{S}'$. Hence we denote it by f.

If b is a (p,q)-Fourier block which belongs to $\mathcal{S}$, then $\psi_{Q(2^J)}(D)b = b$ for any large $J \gg 1$. We decompose

$$L(b) = L\big[\varphi_{0,Q(1)}(D)\psi_{Q(1)}(D)b\big] + \sum_{j=1}^{J+10} L\big[\big(\psi_{Q(2^j)}(D) - \psi_{Q(2^{j-1})}(D)\big)\varphi_{j,Q(1)}(D)b\big]$$

in view of (3.12). According to (3.13), we have

$$L\big[\varphi_{0,Q(1)}(D)\psi_{Q(1)}(D)b\big] = L_0\big[\varphi_{0,Q(1)}(D)b\big] = \int_{\mathbb{R}^n} \varphi_{0,Q(1)}(D)b(x)f_0(x)\,dx$$

and

$$L\big[\big(\psi_{Q(2^j)}(D) - \psi_{Q(2^{j-1})}(D)\big)\varphi_{j,Q(1)}(D)b\big] = \int_{\mathbb{R}^n} \varphi_{j,Q(1)}(D)b(x)f_j(x)\,dx - \int_{\mathbb{R}^n} \varphi_{j,Q(1)}(D)b(x)f_{j-1}(x)\,dx.$$

Thus from (3.16), we have

$$L(b) = \int_{\mathbb{R}^n} b(x)f_{Q(2^J)}(x)\,dx = \langle f, b\rangle.$$

It thus remains to show that $f \in \mathcal{M}\mathcal{F}^{p'}_{q'}$. Once we prove $f \in \mathcal{M}\mathcal{F}^{p'}_{q'}$, then it follows from the assertions (0) and (1) that L_f can be extended to the bounded linear operator on $\mathcal{H}\mathcal{F}^p_q$ and can satisfy $L_f(g) = L(g)$ for any $g \in \mathcal{H}\mathcal{F}^p_q$. To this end, fix any $Q \in \mathcal{Q}$. Define

$$g(x) \equiv \overline{\operatorname{sgn}\big(\mathcal{F}^{-1}\psi_Q * f_Q(x)\big)}\big|\mathcal{F}^{-1}\psi_Q * f_Q(x)\big|^{q-1}.$$

Then we calculate that

$$|Q|^{\frac{1}{p'}-\frac{1}{q'}}\left(\int_{\mathbb{R}^n} \big|\mathcal{F}^{-1}\psi_Q * f(x)\big|^q\,dx\right)^{\frac{1}{q}}. \tag{3.17}$$

By recalling that $\mathcal{F}^{-1}\psi_Q * f = \mathcal{F}^{-1}\psi_Q * f_Q$, we learn that

$$\int_{\mathbb{R}^n} \big|\mathcal{F}^{-1}\psi_Q * f(x)\big|^q\,dx = \langle \mathcal{F}^{-1}\psi_Q * f_Q, g\rangle = \big\langle f_Q, \big[\mathcal{F}^{-1}\psi_Q(-\cdot)\big] * g\big\rangle.$$

Therefore, we obtain

$$\int_{\mathbb{R}^n} \big|\mathcal{F}^{-1}\psi_Q * f(x)\big|^q\,dx = L\big(\mathcal{F}^{-1}\psi_Q(-\cdot) * g\big) \le \|L\|_{(L^{q'}_Q)^*} \cdot \big\|\mathcal{F}^{-1}\psi_Q(-\cdot) * g\big\|_{L^{q'}}. \tag{3.18}$$

In addition, by recalling $\|f_Q\|_{L^q} = \|L\|_{(L^{q'}_Q)^*}$, we see that

$$\big\|\mathcal{F}^{-1}\psi_Q(-\cdot) * g\big\|_{L^{q'}} \le C\|g\|_{L^{q'}} = C\big\||\mathcal{F}^{-1}\psi_Q * f_Q|^{q'(q-1)}\big\|_{L^1}^{\frac{1}{q'}} = C\|\mathcal{F}^{-1}\psi_Q * f_Q\|_{L^q}^{\frac{q}{q'}} \le C\|f_Q\|_{L^q}^{\frac{q}{q'}} = C\|L\|_{(L^{q'}_Q)^*}^{\frac{q}{q'}}.$$

By inserting this inequality into (3.18), we obtain

$$\left(\int_{\mathbb{R}^n} \left|\mathcal{F}^{-1}\psi_Q * f(x)\right|^q dx\right)^{\frac{1}{q}} \leq C\|L\|_{(L^{q'}_Q)^*}^{(1+q/q')\cdot 1/q} = C\|L\|_{(L^{q'}_Q)^*}.$$

As a result, by using (3.14), we may estimate (3.17) as follows:

$$\begin{aligned}|Q|^{\frac{1}{p'}-\frac{1}{q'}}\left(\int_{\mathbb{R}^n} \left|\mathcal{F}^{-1}\psi_Q * f(x)\right|^q dx\right)^{\frac{1}{q}} &\leq C|Q|^{\frac{1}{p'}-\frac{1}{q'}}\|L\|_{(L^{q'}_Q)^*} \\ &\leq C\|L\|_{(\mathcal{H}\mathcal{F}^p_q)^*} < \infty,\end{aligned}$$

which implies $\|f\|_{\mathcal{M}\mathcal{F}^{p'}_{q'}} \leq C\|L\|_{(\mathcal{H}\mathcal{F}^p_q)^*}$. □

To investigate the role of $\mathcal{H}\mathcal{F}^p_q$, we now recall the predual space of $\mathcal{M}^p_q$, which is called the *block space* $\mathcal{H}^p_q$.

Definition 3.8 (see [3]). Let $1 \leq p \leq q < \infty$. A measurable function b is said to be a (p,q)-block if there is a cube Q supporting b and $\|b\|_{L^q} \leq |Q|^{\frac{1}{q}-\frac{1}{p}}$. The *block space* $\mathcal{H}^p_q$ is defined by the set of all L^p functions f with the norm

$$\|f\|_{\mathcal{H}^p_q} \equiv \inf\left\{\|\lambda_j\|_{\ell^1} : f = \sum_{j=1}^{\infty} \lambda_j b_j\right\},$$

where each b_j is a (p,q)-block and the infimum is taken over all possible decompositions of f.

It is well known that the block space $\mathcal{H}^p_q$ is a predual space of the Morrey space $\mathcal{M}^{p'}_{q'}$ for $1 < p \leq q < \infty$.

Theorem 3.9 ([3, Theorem 1])**.** *Let $1 < p \leq q < \infty$. Then $(\mathcal{H}^p_q)^* = \mathcal{M}^{p'}_{q'}$. Moreover, it holds that*

$$\|f\|_{\mathcal{H}^p_q} = \sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| : \|g\|_{\mathcal{M}^{p'}_{q'}} = 1\right\} \quad (f \in \mathcal{H}^p_q).$$

If we combine Theorems 1.2, 1.5, and 3.9, then we can describe the boundedness of the Fourier transform on $\mathcal{H}\mathcal{F}^p_q$.

Corollary 3.10. *Let $q \geq 2$, and let $1 \leq p < \infty$.*

(1) *If $q > 2$ and $q \geq p$, then the Fourier transform is bounded from $\mathcal{H}\mathcal{F}^p_q$ to $\mathcal{H}^p_q$:*

$$\|\mathcal{F}f\|_{\mathcal{H}^p_q} \leq C\|f\|_{\mathcal{H}\mathcal{F}^p_q} \quad (f \in \mathcal{H}\mathcal{F}^p_q). \tag{3.19}$$

(2) *If $p \leq 2 = q$, then the Fourier transform is isomorphic from $\mathcal{H}\mathcal{F}^p_2$ to $\mathcal{H}^p_2$:*

$$C^{-1}\|f\|_{\mathcal{H}\mathcal{F}^p_2} \leq \|\mathcal{F}f\|_{\mathcal{H}^p_2} \leq C\|f\|_{\mathcal{H}\mathcal{F}^p_2} \quad (f \in \mathcal{H}\mathcal{F}^p_2). \tag{3.20}$$

As in (1.5), we may improve the well-known inequality $\mathcal{F} : L^1 \to L^\infty$ by using $\mathcal{H}\mathcal{F}^1_1$ as follows.

Proposition 3.11. *The Fourier transform is bounded from L^1 to $\mathcal{H}\mathcal{F}^1_1$.*

Proof. Let $f \in L^1$. Let ψ satisfy (1.2). Write $\varphi_j \equiv \psi(2^{-j}\cdot) - \psi(2^{-j+1}\cdot)$ for $j = 1, 2, \ldots$. Then

$$\mathcal{F}f = \mathcal{F}[\psi \cdot f] + \sum_{j=1}^{\infty} \mathcal{F}[\varphi_j \cdot f].$$

Observe that $\frac{1}{\|\varphi_j \cdot f\|_{L^1}}\mathcal{F}[\varphi_j \cdot f]$ and $\frac{1}{\|\psi \cdot f\|_{L^1}}\mathcal{F}[\psi \cdot f]$ are (1.1)-Fourier blocks for all $j \in \mathbb{N}$. Thus,

$$\|\mathcal{F}f\|_{\mathcal{H}\mathcal{F}_1^1} \le \|\psi \cdot f\|_{L^1} + \sum_{j=1}^{\infty} \|\varphi_j \cdot f\|_{L^1} = \|f\|_{L^1},$$

as was to be shown. □

4. Some applications

We next generalize the Stein–Tomas Fourier restriction theorem to our spaces. As a preparatory step, we prove Lemma 1.7.

Proof of Lemma 1.7.

(1) This is a consequence of the Littlewood–Paley theory.

(2) Note that

$$\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}}\left|\mathcal{F}^{-1}[\psi_{Q(1)} \otimes \psi^0] * F(x,t)\right|^r dt\right)^{\frac{q'}{r}} dx\right)^{\frac{1}{q'}} \le \|F\|_{\mathcal{M}\mathcal{F}^p_q(\mathbb{R}^n_x, L^r(\mathbb{R}_t))} \tag{4.1}$$

and that for any $j, k \in \mathbb{N}$,

$$\begin{aligned} &2^{kn(\frac{1}{p}-\frac{1}{q})}\left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}}\left|\mathcal{F}^{-1}[\psi_{Q(2^k)} \otimes \varphi_j^0] * F(x,t)\right|^r dt\right)^{\frac{q'}{r}} dx\right)^{\frac{1}{q'}} \\ &\le C\|F\|_{\mathcal{M}\mathcal{F}^p_q(\mathbb{R}^n_x, L^r(\mathbb{R}_t))}. \end{aligned} \tag{4.2}$$

Thus $F \in B^s_{\infty\infty}(\mathbb{R}^{n+1}) \hookrightarrow \mathcal{S}'(\mathbb{R}^{n+1})$ holds for any $F \in \mathcal{M}\mathcal{F}^p_q(\mathbb{R}^n_x, L^r(\mathbb{R}_t))$ and for some $s \in \mathbb{R}$. □

We now move on to the application to PDEs. We first set up. Denote a C^∞ hypersurface by M and a surface-carried measure on M by $d\sigma$. In this paper, we always assume that M has nonzero Gauss curvature at each point of M.

For a cube Q in $\mathbb{R}^n$, we denote $\ell(Q \cap M) \equiv \sigma(Q \cap M)^{\frac{1}{n-1}}$, and always assume that

$$|Q| \sim \sigma(Q \cap M)^{\frac{n}{n-1}} \tag{4.3}$$

whenever Q is contained in a fixed compact set E and the center of Q lies in M. Further, let $\mathcal{R}$ and $\mathcal{R}^*$ denote the restriction operator and its dual operator, respectively; that is,

$$\begin{aligned} \mathcal{R}f(\xi) &\equiv \mathcal{F}f(\xi)|_M = \int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi} f(x)\,dx|_M \quad (\xi \in M), \\ \mathcal{R}^*F(x) &\equiv \int_M e^{2i\pi x\cdot\xi} F(\xi)\,d\sigma(\xi) \quad (x \in \mathbb{R}^n) \end{aligned}$$

for $f \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ and $F \in C^0(M)$.

We invoke the following result called the *Stein–Tomas restriction theorem.*

Theorem 4.1 ([13, Theorem 5.2])**.** *Suppose that M has nonzero Gauss curvature at each point of M. Let E be a compact set of M. Then for $p_n \equiv \frac{2(n+1)}{n+3}$, it holds that*

$$\|\chi_E \mathcal{R} f\|_{L^2(M)} \le C\|f\|_{L^{p_n}(\mathbb{R}^n)}. \tag{4.4}$$

By modifying the definition of (p,q)-blocks, we may define the space $\mathcal{H}^p_q$ on M. In fact, we can redefine the (p,q)-blocks on the manifold M as follows.

Definition 4.2. A function b is said to be a (p,q)-block if there exists a cube $Q \in \mathcal{Q}(\mathbb{R}^n)$ such that $\mathrm{supp}(b) \subset M \cap Q$ and $\|b\|_{L^q(M)} \le \sigma(Q\cap M)^{\frac{1}{q}-\frac{1}{p}}$.

The following theorem generalizes and strengthens Theorem 4.1.

Theorem 4.3. *Let $p_n = \frac{2(n+1)}{n+3}$, $p \le 2$, and take $r_p \ge p_n$ ($\Leftrightarrow r'_p \le p'_n$) so that*

$$\frac{1}{p'_n} \le \frac{1}{r'_p} \le \frac{1}{p'_n} + \left(\frac{1}{p}-\frac{1}{2}\right)\frac{n-1}{n}. \tag{4.5}$$

Then for any compact set E of M, there exists a constant C_E depending on E such that

$$\left\|\chi_E \mathcal{R}(f)\right\|_{\mathcal{H}^p_2(M)} \le C_E \|f\|_{\mathcal{H}\mathcal{F}^{r'_p}_{p'_n}(\mathbb{R}^n)} \tag{4.6}$$

for all $f \in \mathcal{H}\mathcal{F}^{r'_p}_{p'_n}(\mathbb{R}^n)$.

Before the proof, a couple of remarks may be in order.

Remark 4.4.

(1) In view of Proposition 3.4, the conclusion (4.4) is more general than (4.6). In fact, by letting $p = 2$ in Theorem 4.3, one deduces from (4.5) that $r_p = p_n$, which is (4.4).
(2) Arithmetic shows that
$$\frac{1}{p'_n} + \left(\frac{1}{p}-\frac{1}{2}\right)\frac{n-1}{n} < 1$$
holds for $n \ge 2$.

Proof of Theorem 4.3. Since E is compact, we may assume that the frequency support of f is compact. Let $Q \subset \mathbb{R}^n$ be a cube containing E and the frequency support of f. For any $f \in \mathcal{H}\mathcal{F}^{r'_p}_{p'_n}$, by Proposition 3.5, we can take $\{\lambda_j\}_{j=1}^\infty \in \ell^1$ and (r'_p, p'_n)-Fourier blocks $\{b_j\}_{j=1}^\infty$ such that

$$f = \sum_{j=1}^\infty \lambda_j b_j, \qquad \mathrm{supp}(\mathcal{F} b_j) \subset Q_j \in \mathcal{Q}(\mathbb{R}^n),$$

$$\|b_j\|_{L^{p_n}} \le |Q_j|^{\frac{1}{p'_n}-\frac{1}{r'_p}}, \qquad \|\lambda_j\|_{\ell^1} \le C\|f\|_{\mathcal{H}\mathcal{F}^{r'_p}_{p'_n}(\mathbb{R}^n)},$$

and such that

$$Q_j \subset 2Q, \quad \sharp\{j\in\mathbb{N}\colon \lambda_j \ne 0\} < \infty. \tag{4.7}$$

Then it is clear from (4.7) that $\mathcal{R}f = \sum_{j=1}^{\infty} \lambda_j \mathcal{R}b_j$ and $\mathrm{supp}(\mathcal{R}b_j) \subset M \cap Q_j$. In addition, from the Stein–Tomas theorem, it follows that

$$\|\chi_E \mathcal{R}b_j\|_{L^2(M)} \le C\|b_j\|_{L^{p_n}(\mathbb{R}^n)} \le C|Q_j|^{\frac{1}{p_n'}-\frac{1}{r_p'}} \sim \ell(Q_j \cap M)^{n(\frac{1}{p_n'}-\frac{1}{r_p'})}. \tag{4.8}$$

Here note that we may assume that $\ell(Q_j \cap M) \le 1$; hence $\ell(Q_j \cap M)^{-n} \ge 1$ since E is compact. In fact, if we have Q_j such that $\ell(Q_j \cap M) \ge 1$, then by dividing $Q_j = \bigcup_{k=1}^{K} Q_{jk}$, where $\ell(Q_{jk} \cap M) \le 1$, we get further decomposition $b_j = \sum_{k=1}^{K} \lambda_{jk} b_{jk}$. Here the coefficient λ_{jk} may depend on the size of E. This observation and our assumption (4.5) imply that

$$\ell(Q_j \cap M)^{n(\frac{1}{p_n'}-\frac{1}{r_p'})} \le \ell(Q_j \cap M)^{-n(\frac{1}{p}-\frac{1}{2})\frac{n-1}{n}} \sim \sigma(Q_j \cap M)^{\frac{1}{2}-\frac{1}{p}}$$

as long as Q_j intersects E. Altogether, we see that $\|\mathcal{R}b_j\|_{L^2(M)} \le C\sigma(Q_j \cap M)^{\frac{1}{2}-\frac{1}{p}}$; hence $\mathcal{R}b_j$ is a $(p,2)$-block on the hypersurface M modulo some unimportant multiplicative constant. This shows that $\|\mathcal{R}(f)\|_{\mathcal{H}_2^p(M)} \le C\|f\|_{\mathcal{H}\mathcal{F}^{r_p'}_{p_n'}(\mathbb{R}^n)}$ for $f \in \mathcal{H}\mathcal{F}^{r_p'}_{p_n'}$. □

One defines the Morrey space $\mathcal{M}_q^p(M)$ as the set of all σ-measurable functions f defined on M for which the norm

$$\|f\|_{\mathcal{M}_q^p(M)} \equiv \sup_{Q \in \mathcal{Q}(\mathbb{R}^n), Q\cap M \neq \emptyset} \sigma(Q \cap M)^{\frac{1}{p}-\frac{1}{q}} \left(\int_{Q\cap M} |f(y)|^q \, d\sigma(y)\right)^{\frac{1}{q}}$$

is finite.

Corollary 4.5. *Let the setting be as above. Then for all $F \in \mathcal{M}_2^{p'}(M)$,*

$$\left\|\mathcal{R}^*(\chi_E F)\right\|_{\mathcal{M}\mathcal{F}^{r_p}_{p_n}(\mathbb{R}^n)} \le C\|F\|_{\mathcal{M}_2^{p'}(M)}.$$

Proof. We will use the duality relation which is proved in Theorem 1.5 to get that

$$\begin{aligned}\left\|\mathcal{R}^*(\chi_E F)\right\|_{\mathcal{M}\mathcal{F}^{r_p}_{p_n}(\mathbb{R}^n)} &= \sup\left\{\left|\int_E F(\xi)\overline{\mathcal{R}(g)(\xi)}\, d\xi\right| : \|g\|_{\mathcal{H}\mathcal{F}^{r_p'}_{p_n'}(\mathbb{R}^n)} = 1\right\} \\ &\le \|F\|_{\mathcal{M}_2^{p'}(M)} \cdot \sup\left\{\left\|\chi_E \mathcal{R}(g)\right\|_{\mathcal{H}_2^p(M)} : \|g\|_{\mathcal{H}\mathcal{F}^{r_p'}_{p_n'}(\mathbb{R}^n)} = 1\right\} \\ &\le C\|F\|_{\mathcal{M}_2^{p'}(M)}.\end{aligned}$$

□

5. Proof of Theorem 1.9

Let us prove Proposition 1.8 and Theorem 1.9.

Proof of Proposition 1.8. Fix any $Q \in \mathcal{D}(\mathbb{R}^n)$, and let us calculate that

$$|Q|^{\frac{1}{s}-\frac{1}{q'}} \left(\int_{\mathbb{R}^{n+1}} \left|\mathcal{F}^{-1}\psi_Q * \mathcal{T}(t)g(x)\right|^q \, dx\, dt\right)^{\frac{1}{q}}.$$

Note that $\mathcal{F}^{-1}\psi_Q * \mathcal{T}(t)g(x) = \mathcal{T}(t)[\mathcal{F}^{-1}\psi_Q * g](x)$. In fact, by the definition, we see that

$$\mathcal{F}^{-1}\psi_Q * \mathcal{T}(t)g(x) = \mathcal{F}^{-1}\left[e^{-4\pi^2 it|\xi|^2}\mathcal{F}[\mathcal{F}^{-1}\psi_Q * g]\right](x).$$

With this in mind, by using the Stein–Tomas Strichartz estimate directly, we have

$$|Q|^{\frac{1}{s}-\frac{1}{q'}}\left(\int_{\mathbb{R}^{n+1}}\left|\mathcal{F}^{-1}\psi_Q*\mathcal{T}(t)g(x)\right|^q\,dx\,dt\right)^{\frac{1}{q}}$$
$$=|Q|^{\frac{1}{s}-\frac{1}{q'}}\left\|\mathcal{T}(t)[\mathcal{F}^{-1}\psi_Q*g](x)\right\|_{L^q(\mathbb{R}^{n+1})}\le C|Q|^{\frac{1}{s}-\frac{1}{q'}}\|\mathcal{F}^{-1}\psi_Q*g\|_{L^2(\mathbb{R}^n)}.$$

By recalling the condition for $v=v_{s,q}$: $\frac{1}{v}=\frac{1}{q}+\frac{1}{s}-\frac{1}{2}$, we conclude that

$$|Q|^{\frac{1}{s}-\frac{1}{q'}}\left(\int_{\mathbb{R}^{n+1}}\left|\mathcal{F}^{-1}\psi_Q*\mathcal{T}(t)g(x)\right|^q\,dx\,dt\right)^{\frac{1}{q}}\le C|Q|^{\frac{1}{v}-\frac{1}{2}}\|\mathcal{F}^{-1}\psi_Q*g\|_{L^2(\mathbb{R}^n)}$$
$$\le C\|g\|_{\mathcal{M}\mathcal{F}^v_2(\mathbb{R}^n)}.\qquad\square$$

Next we will show Theorem 1.9. Our proof is based on the refinement of the Stein–Tomas Strichartz estimate proved in [2, Theorem 2]. We recall the space $X_{p,q}$. Let $1\le p<\infty$, and let $1\le q<\infty$. The space $X_{p,q}$ is the set of all L^p local functions f for which the norm

$$\|f\|_{X_{p,q}}\equiv\left(\sum_{j\in\mathbb{Z}}2^{j\frac{n}{2}\frac{2-p}{p}q}\sum_{m\in\mathbb{Z}^n}\|f\cdot\chi_{Q_{jm}}\|_{L^p}^q\right)^{\frac{1}{q}}$$

is finite.

Theorem 5.1 ([2, Theorem 1.2]). *Let $q=\frac{2(n+2)}{n}$ and $p<2$ be such that $\frac{1}{p'}>\frac{n+3}{n+1}\frac{1}{q}$, or, equivalently,*

$$\frac{2(n+1)(n+2)}{n^2+3n+4}<p<2. \tag{5.1}$$

For every function g such that $\mathcal{F}g\in X_{p,q}$, we have

$$\left\|\mathcal{T}(t)g(x)\right\|_{L^q(\mathbb{R}^{n+1})}\le C\|\mathcal{F}g\|_{X_{p,q}(\mathbb{R}^n)},$$

where $C=C(n,p)$.

Furthermore, we will employ the boundedness of the fractional maximal operator M_α defined by

$$M_\alpha f(x)\equiv\sup_{Q\in\mathcal{D}}\frac{1}{|Q|^{1-\frac{\alpha}{n}}}\int_Q|f(y)|\,dy\cdot\chi_Q(x).$$

Let us recall the boundedness of M_α on Morrey spaces. To this end, we first recall the boundedness of I_α known as the *Adams inequality.*

Theorem 5.2 ([1, Theorem 3.1]). *Let $0\le\alpha<n$, $1<\sigma\le\lambda<\infty$, and $1<\gamma\le\beta<\infty$ satisfy*

$$\frac{1}{\lambda}=\frac{1}{\beta}-\frac{\alpha}{n},\qquad\frac{\lambda}{\sigma}=\frac{\beta}{\gamma}.$$

Then

$$\|I_\alpha f\|_{\mathcal{M}^\lambda_\sigma}\le C\|f\|_{\mathcal{M}^\beta_\gamma}$$

for some $C>0$ and all $f\in\mathcal{M}^\beta_\gamma$.

Since we know that $M_\alpha f \le I_\alpha(|f|)$, we obtain the following boundedness of M_α on Morrey spaces.

Theorem 5.3. *Keep the same assumption as in Theorem 5.2. Then*

$$\|M_\alpha f\|_{\mathcal{M}^\lambda_\sigma} \le C\|f\|_{\mathcal{M}^\beta_\gamma}$$

for some $C > 0$ and all $f \in \mathcal{M}^\beta_\gamma$.

Now let us show Theorem 1.9. Due to the original Stein–Tomas Strichartz estimate, we can assume that $s > q'$ (see Remark 1.10).

Lemma 5.4. *Let q, s, and $v_{s,q}$ satisfy (1.8), (1.9), and (1.10), respectively. Let $g \in \mathcal{S}'$. We set*

$$\mathrm{I} \equiv \Bigg(\sum_{l=-\infty}^{-\log_2(\ell(Q))} \sum_{\substack{m\in\mathbb{Z}^n:\\ Q\subset Q_{lm}}} 2^{l\times nr\frac{2-p}{p}} \|\psi_Q \chi_{Q_{lm}} \mathcal{F}g\|_{L^p(\mathbb{R}^n)}^q \Bigg)^{\frac{1}{q}}.$$

Then

$$\mathrm{I} \le C\Bigg(\sum_{\substack{l\in\mathbb{Z}:\\ 2^{-l}\le\ell(Q)}} \sum_{\substack{m\in\mathbb{Z}^n:\\ Q_{lm}\subset Q}} 2^{l\times nr\frac{2-p}{p}} \|\chi_{Q_{lm}} \mathcal{F}g\|_{L^p(\mathbb{R}^n)}^q \Bigg)^{\frac{1}{q}}; \tag{5.2}$$

hence

$$\big\|\mathcal{T}(t)[\mathcal{F}^{-1}\psi_Q * g](x)\big\|_{L^q(\mathbb{R}^{n+1})}{}^q \le C \sum_{\substack{l\in\mathbb{Z}:\\ 2^{-l}\le\ell(Q)}} \sum_{\substack{m\in\mathbb{Z}^n:\\ Q_{lm}\subset Q}} 2^{l\times nr\frac{2-p}{p}} \|\chi_{Q_{lm}} \mathcal{F}g\|_{L^p(\mathbb{R}^n)}^q. \tag{5.3}$$

Proof. Since for each $l \le -\log_2(\ell(Q))$ there exists a unique $m(l) \in \mathbb{Z}^n$ such that $Q \subset Q_{lm(l)}$, we obtain

$$\mathrm{I} = \Bigg(\sum_{l=-\infty}^{-\log_2(\ell(Q))} 2^{l\times nr\frac{2-p}{p}} \|\psi_Q \chi_{Q_{lm(l)}} \mathcal{F}g\|_{L^p(\mathbb{R}^n)}^q \Bigg)^{\frac{1}{q}} \sim \Big(|Q|^{-r\frac{2-p}{p}} \|\psi_Q \mathcal{F}g\|_{L^p(\mathbb{R}^n)}^q \Big)^{\frac{1}{q}}$$

since we have the power $r\frac{2-p}{p} > 0$ by our assumption $p < 2$. By observing that

$$\Big(|Q|^{-r\frac{2-p}{p}} \|\psi_Q \mathcal{F}g\|_{L^p(\mathbb{R}^n)}^q \Big)^{\frac{1}{q}} \le C\Bigg(\sum_{\substack{l\in\mathbb{Z}:\\ 2^{-l}\le\ell(Q)}} \sum_{\substack{m\in\mathbb{Z}^n:\\ Q_{lm}\subset Q}} 2^{l\times nr\frac{2-p}{p}} \|\chi_{Q_{lm}} \mathcal{F}g\|_{L^p(\mathbb{R}^n)}^q \Bigg)^{\frac{1}{q}},$$

we obtain (5.3). □

Proof of Theorem 1.9. We may assume that $\rho < v$. As in the proof of Proposition 1.8, fix any $Q \in \mathcal{D}(\mathbb{R}^n)$, and calculate

$$|Q|^{\frac{1}{s}-\frac{1}{q'}} \Big(\int_{\mathbb{R}^{n+1}} \big|\mathcal{F}^{-1}\psi_Q * \mathcal{T}(t)g(x)\big|^q \,dx\,dt \Big)^{\frac{1}{q}}.$$

Again we note that $\mathcal{F}^{-1}\psi_Q * \mathcal{T}(t)g(x) = \mathcal{T}(t)[\mathcal{F}^{-1}\psi_Q * g](x)$. Hence we focus on the quantity $\|\mathcal{T}(t)[\mathcal{F}^{-1}\psi_Q * g](x)\|_{L^q(\mathbb{R}^{n+1})}$. In view of assumptions (1.13) and (5.1), we choose the parameters p and r so that

$$\frac{2(n+1)(n+2)}{n^2+3n+4} < p < \rho, \quad r \equiv \frac{q}{2}. \tag{5.4}$$

Let $g^l_m \equiv \mathcal{F}^{-1}[\chi_{Q_{lm}}\mathcal{F}g]$. Recall that I is defined in Lemma 5.4. By applying Theorem 5.1, we obtain

$$\begin{aligned}
&\big\|\mathcal{T}(t)[\mathcal{F}^{-1}\psi_Q * g](x)\big\|_{L^q(\mathbb{R}^{n+1})} \\
&\le C\Big(\sum_{l\in\mathbb{Z}}\sum_{m\in\mathbb{Z}^n} 2^{l\times nr\frac{2-p}{p}}\big\|\mathcal{F}[\mathcal{F}^{-1}\psi_Q * g^l_m]\big\|^{2r}_{L^p(\mathbb{R}^n)}\Big)^{\frac{1}{2r}} \\
&\le C\Big(\sum_{\substack{l\in\mathbb{Z}:\\ 2^{-l}\le\ell(Q)}}\sum_{\substack{m\in\mathbb{Z}^n:\\ Q_{lm}\subset Q}} 2^{l\times nr\frac{2-p}{p}}\|\chi_{Q_{lm}}\mathcal{F}g\|^q_{L^p(\mathbb{R}^n)}\Big)^{\frac{1}{q}}.
\end{aligned} \tag{5.5}$$

Here for given parameters s and ρ, we take $\varepsilon > 0$ so that

$$\varepsilon \equiv \frac{1}{q}\Big(1-\frac{\rho}{v}\Big). \tag{5.6}$$

By inserting the volume of Q_{lm} with certain power to (5.5), the right-hand side of (5.5) equals

$$\Big(\sum_{\substack{l\in\mathbb{Z}:\\ 2^{-l}\le\ell(Q)}}\sum_{\substack{m\in\mathbb{Z}^n:\\ Q_{lm}\subset Q}} |Q_{lm}|^{(-\frac{1}{s}+\frac{1}{q'}-\varepsilon)q+1}\cdot|Q_{lm}|^{(\frac{1}{s}-\frac{1}{q'}+\varepsilon)q-1}\cdot|Q_{lm}|^{-r\frac{2-p}{p}}\|\chi_{Q_{lm}}\mathcal{F}g\|^q_{L^p}\Big)^{\frac{1}{q}}.$$

Now we notice that the fractional maximal operator will appear. More precisely, we observe that

$$\begin{aligned}
&|Q_{lm}|^{(\frac{1}{s}-\frac{1}{q'}+\varepsilon)q-1}\cdot|Q_{lm}|^{-r\frac{2-p}{p}}\|\chi_{Q_{lm}}\mathcal{F}g\|^q_{L^p(\mathbb{R}^n)} \\
&= \Big(\frac{|Q_{lm}|^{p(\frac{1}{s}+\varepsilon-\frac{1}{2})}}{|Q_{lm}|}\int_{Q_{lm}}\big|\mathcal{F}g(\xi)\big|^p\,d\xi\Big)^{\frac{q}{p}} \\
&\le \inf_{\eta\in Q_{lm}} M^{(p)}_\alpha[\mathcal{F}g](\eta)^q,
\end{aligned}$$

where

$$\alpha \equiv np\Big(\frac{1}{s}-\frac{1}{2}+\varepsilon\Big) \ge 0 \tag{5.7}$$

(since we assumed that $s \le 2$), and the powered fractional maximal operator $M^{(p)}_\alpha$ is defined by

$$M^{(p)}_\alpha f(x) \equiv \sup_{Q\in\mathcal{D}}\Big(\frac{|Q|^{\frac{\alpha}{n}}}{|Q|}\int_Q\big|f(y)\big|^p\,dy\Big)^{\frac{1}{p}}\cdot\chi_Q(x).$$

Therefore, it follows that

$$\begin{aligned}&\left\|\mathcal{T}(t)[\mathcal{F}^{-1}\psi_Q * g](x)\right\|_{L^q(\mathbb{R}^{n+1})}^q\\&\le C\sum_{\substack{l\in\mathbb{Z}:\\2^{-l}\le\ell(Q)}}\sum_{\substack{m\in\mathbb{Z}^n:\\Q_{lm}\subset Q}}|Q_{lm}|^{(-\frac{1}{s}+\frac{1}{q'}-\varepsilon)q}\int_{Q_{lm}}M_\alpha^{(p)}[\mathcal{F}g](\eta)^q\,d\eta\\&=C\sum_{\substack{l\in\mathbb{Z}:\\2^{-l}\le\ell(Q)}}2^{-l\times n(-\frac{1}{s}+\frac{1}{q'}-\varepsilon)q}\int_Q M_\alpha^{(p)}[\mathcal{F}g](\eta)^q\,d\eta.\end{aligned}$$

Now we recall the assumption of ρ in (1.13): $\rho > v - vq(-\frac{1}{s}+\frac{1}{q'})$. With this and the definition of ε, and keeping (5.6) in mind, arithmetic shows that $-\frac{1}{s}+\frac{1}{q'}-\varepsilon>0$; hence,

$$\sum_{\substack{l\in\mathbb{Z}:\\2^{-l}\le\ell(Q)}}2^{-l\times n(-\frac{1}{s}+\frac{1}{q'}-\varepsilon)q}\sim|Q|^{(-\frac{1}{s}+\frac{1}{q'}-\varepsilon)q}.$$

As a result, we obtain

$$\begin{aligned}&\left\|\mathcal{T}(t)[\mathcal{F}^{-1}\psi_Q * g](x)\right\|_{L^q(\mathbb{R}^{n+1})}\\&\le C|Q|^{-\frac{1}{s}+\frac{1}{q'}-\varepsilon}\left(\int_Q M_\alpha^{(p)}[\mathcal{F}g](\eta)^q d\eta\right)^{\frac{1}{q}}.\end{aligned}$$

Hence, by choosing $\lambda > q$ so that

$$\frac{1}{\lambda}\equiv\frac{1}{q}-\varepsilon, \tag{5.8}$$

we obtain

$$\begin{aligned}|Q|^{\frac{1}{s}-\frac{1}{q'}}\left\|\mathcal{T}(t)[\mathcal{F}^{-1}\psi_Q * g](x)\right\|_{L^q(\mathbb{R}^{n+1})}&\le C|Q|^{-\varepsilon}\left(\int_Q M_\alpha^{(p)}[\mathcal{F}g](\eta)^q d\eta\right)^{\frac{1}{q}}\\&\le\left\|M_\alpha^{(p)}[\mathcal{F}g]\right\|_{\mathcal{M}_q^\lambda(\mathbb{R}^n)}\\&=\left\|M_\alpha\left[|\mathcal{F}g|^p\right]\right\|_{\mathcal{M}_{q/p}^{\lambda/p}(\mathbb{R}^n)}^{\frac{1}{p}}.\end{aligned}$$

Here let us invoke the Adams inequality, Theorem 5.3; that is, for the parameters $1<\gamma\le\beta<\infty$ satisfying that

$$\frac{p}{\lambda}=\frac{1}{\beta}-\frac{\alpha}{n},\qquad\frac{\lambda}{q}=\frac{\beta}{\gamma}, \tag{5.9}$$

it follows that

$$\left\|M_\alpha\left[|\mathcal{F}g|^p\right]\right\|_{\mathcal{M}_{q/p}^{\lambda/p}(\mathbb{R}^n)}^{\frac{1}{p}}\le C\left\||\mathcal{F}g|^p\right\|_{\mathcal{M}_\gamma^\beta(\mathbb{R}^n)}^{\frac{1}{p}}=\|\mathcal{F}g\|_{\mathcal{M}_{\gamma p}^{\beta p}(\mathbb{R}^n)}.$$

By inserting the definition of several parameters, (5.6), (5.7), (5.8), and (5.9), we may calculate β and γ as follows:

$$\frac{p}{\lambda}=\frac{1}{\beta}-\frac{\alpha}{n}\Leftrightarrow\beta p=v,\qquad\frac{\lambda}{q}=\frac{\beta}{\gamma}\Leftrightarrow\gamma p=\rho.$$

Note that $\gamma = \frac{\rho}{p} > 1$ by the choice of p (5.4), which ensures the application of the Adams inequality in the above. In summary, we obtain

$$\big\|\mathcal{T}(t)g(x)\big\|_{\mathcal{M}\mathcal{F}^{s}_{q'}(\mathbb{R}^n_x, L^q(\mathbb{R}_t))} \le C\|\mathcal{F}g\|_{\mathcal{M}^{v}_{\rho}(\mathbb{R}^n)}. \qquad \square$$

Remark 5.5. The restriction of v is natural. For given s, q, the inequality (1.14) holds if and only if the parameter v satisfies the condition $\frac{1}{v} = \frac{1}{q} + \frac{1}{s} - \frac{1}{2}$. To see this, we have only to check the scaling exponent.

Finally we note another estimate for the operator $\mathcal{T}$.

Proposition 5.6. *Let $q \ge 2$, and take $s \in [2, \infty]$ so that*

$$\frac{1}{q} + \frac{1}{s} \le \frac{1}{2}. \tag{5.10}$$

Let $p \in [s, \infty]$. Then

$$\sup_{t\ge 0}\big\|\mathcal{T}(t)f(x)\big\|_{\mathcal{M}\mathcal{F}^{p}_{q'}(\mathbb{R}^n_x)} \le C\|f\|_{\mathcal{M}\mathcal{F}^{p}_{s}} \quad (f \in \mathcal{M}\mathcal{F}^{p}_{s}).$$

Proof. We first take r such that $\frac{1}{q} = \frac{1}{r'} + \frac{1}{s'} - 1$. By our assumption (5.10), we notice that $\frac{1}{r'} = \frac{1}{q} + \frac{1}{s} \le \frac{1}{2}$, namely, $r' \ge 2$. With this in mind, we employ the Young inequality and the boundedness of the Fourier transform on L^r to get

$$\begin{aligned}
\big\|\mathcal{T}(t)f(x)\big\|_{\mathcal{M}\mathcal{F}^{p}_{q'}(\mathbb{R}^n_x)} &= \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{q'}} \big\|\mathcal{F}^{-1}[e^{-i4\pi^2|\xi|^2 t}\psi_Q] * \mathcal{F}^{-1}\psi_Q * f\big\|_{L^q} \\
&\le \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{q'}} \big\|\mathcal{F}^{-1}[e^{-i4\pi^2|\xi|^2 t}\psi_Q]\big\|_{L^{r'}} \cdot \|\mathcal{F}^{-1}\psi_Q * f\|_{L^{s'}} \\
&\le C\sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{q'}} \|\psi_Q\|_{L^r} \cdot \|\mathcal{F}^{-1}\psi_Q * f\|_{L^{s'}}.
\end{aligned}$$

Note that $\|\psi_Q\|_{L^r} \sim |Q|^{\frac{1}{r}}$. Arithmetic gives us $\frac{1}{p} - \frac{1}{q'} + \frac{1}{r} = \frac{1}{p} - \frac{1}{s}$; hence we obtain

$$\big\|\mathcal{T}(t)f\big\|_{\mathcal{M}\mathcal{F}^{p}_{q'}} \le C\sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{s}} \|\mathcal{F}^{-1}\psi_Q * f\|_{L^{s'}} = C\|f\|_{\mathcal{M}\mathcal{F}^{p}_{s}(\mathbb{R}^n)}. \qquad \square$$

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