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A GENERALIZATION OF KANTOROVICH OPERATORS FOR CONVEX COMPACT SUBSETS

FRANCESCO ALTOMARE,[1*] MIRELLA CAPPELLETTI MONTANO,[1]
VITA LEONESSA,[2] and IOAN RAŞA[3]

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Abstract. In this article, we introduce and study a new sequence of positive linear operators acting on function spaces defined on a convex compact subset. Their construction depends on a given Markov operator, a positive real number, and a sequence of Borel probability measures. By considering special cases of these parameters for particular convex compact subsets, we obtain the classical Kantorovich operators defined in the 1-dimensional and multidimensional setting together with several of their wide-ranging generalizations scattered in the literature. We investigate the approximation properties of these operators by also providing several estimates of the rate of convergence. Finally, we discuss the preservation of Lipschitz-continuity and of convexity.

1. Introduction

The last twenty years have seen a growing interest in, and relevance of, the study of positive approximation processes on convex compact subsets as more and more evidence emerges. This has been mainly due to their useful connections with approximation problems both for functions defined on these domains and for the solutions of special classes of initial-boundary value differential problems. In such a setting, a prominent role is played by Bernstein–Schnabl operators which are generated by Markov operators. (In our monograph [4] (see also [2]), we

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*Corresponding author.
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provide a rather complete overview of the main results in these fields of research, together with their main applications.)

In this article, we introduce and study a new sequence of positive linear operators acting on function spaces defined on a convex compact subset K of some locally convex Hausdorff space. Their construction depends on a given Markov operator $T : C(K) \to C(K)$, a real number $a \geq 0$, and a sequence $(\mu_n)_{n\geq 1}$ of Borel probability measures on K. By considering special cases of these parameters for particular convex compact subsets such as the unit interval or the multidimensional hypercube and simplex, we obtain all the Kantorovich operators defined on these settings, together with several other wide-ranging generalizations (see [3], [6], [8], [10]–[12], [17], [18]).

Moreover, for $a = 0$, the new operators turn into Bernstein–Schnabl operators and so, by means of the real continuous parameter $a \geq 0$, our sequence of operators represents, indeed, a link between the Bernstein operators ($a = 0$) and the Kantorovich operators ($a = 1$) on the classical 1-dimensional and multidimensional domains where they are defined.

This article is mainly devoted to investigating the approximation properties of the above-mentioned operators in spaces of continuous functions and, for special settings, in L^p-spaces as well. Several estimates of the rate of convergence are also provided. In the final section, we discuss some conditions under which these operators preserve Lipschitz-continuity or convexity. In a future paper, we intend to investigate whether (and for which class of initial-boundary value differential problems) our operators, like Bernstein–Schnabl operators, can be useful in approximating the relevant solutions.

2. Notation and preliminaries

Throughout the article, we will fix a locally convex Hausdorff space X and a convex compact subset K of X. The symbol X' will denote the dual space of X, and the symbol $L(K)$ will stand for the space

$$L(K) := \{\varphi_{|K} \mid \varphi \in X'\}. \tag{2.1}$$

As usual, we will denote by $C(K)$ the space of all real-valued continuous functions on K; $C(K)$ is a Banach lattice if endowed with the natural (pointwise) ordering and the sup-norm $\|\cdot\|_\infty$. Furthermore, we will denote by $A(K)$ the space of all continuous affine functions on K.

Whenever X is the real Euclidean space $\mathbf{R}^d$ of dimension d ($d \geq 1$), we will denote by $\|\cdot\|_2$ the Euclidean norm on $\mathbf{R}^d$. Additionally, we will denote by λ_d the Borel–Lebesgue measure on $K \subset \mathbf{R}^d$, and we will denote by $|K|$ the measure of K with respect to λ_d. Finally, for every $i = 1, \dots, d$, pr_i will stand for the ith coordinate function on K; that is, $pr_i(x) := x_i$ for every $x = (x_1, \dots, x_d) \in K$.

Coming back to an arbitrary convex compact subset K, let B_K be the σ-algebra of all Borel subsets of K, and let $M^+(K)$ (resp., $M_1^+(K)$) be the cone of all regular Borel measures on K (resp., the cone of all regular Borel probability measures on K). For every $x \in K$, the symbol ϵ_x stands for the Dirac measure concentrated at x. If $\mu \in M^+(K)$ and $1 \leq p < +\infty$, we will denote by $L^p(K, \mu)$ the space of all μ-integrable in the pth power functions on K; moreover, we will denote by $L^\infty(K,\mu)$ the space of all μ-essentially bounded measurable functions on K. In particular, if $\mu=\lambda_d$, then we will use the symbols $L^p(K)$ and $L^\infty(K)$.

From now on, let $T : C(K) \to C(K)$ be a Markov operator, that is, a positive linear operator on $C(K)$ such that $T(\mathbf{1}) = \mathbf{1}$, where the symbol $\mathbf{1}$ stands for the function of constant value 1 on K. Furthermore, let $(\tilde{\mu}_x^T)_{x\in K}$ be the continuous selection of Borel probability measures on K corresponding to T via the Riesz representation theorem, that is,

$$\int_K f\, d\tilde{\mu}_x^T = T(f)(x) \quad \big(f \in C(K), x \in K\big). \tag{2.2}$$

In [4, Chapter 3] (see also [2, Chapter 6]), the authors introduced and studied the so-called *Bernstein–Schnabl operators* associated with the Markov operator T and defined, for every $f \in C(K)$ and $x \in K$, as follows:

$$B_n(f)(x) = \int_K \cdots \int_K f\Big(\frac{x_1+\cdots+x_n}{n}\Big)\, d\tilde{\mu}_x^T(x_1)\cdots d\tilde{\mu}_x^T(x_n). \tag{2.3}$$

Note that, for every $n \geq 1$, B_n is a linear positive operator from $C(K)$ into $C(K)$, and that $B_n(\mathbf{1}) = \mathbf{1}$ and hence $\|B_n\| = 1$. Moreover, $B_1 = T$.

The operators B_n generalize the classical Bernstein operators on the unit interval, on multidimensional simplices and hypercubes, and they share with them several preservation properties also investigated in [4] and [2]. If, in addition, we suppose that the Markov operator T satisfies the following condition

$$T(h) = h \quad \text{for every } h \in A(K), \tag{2.4}$$

or, equivalently (see (2.2)),

$$\int_K h\, d\tilde{\mu}_x^T = h(x) \quad \text{for every } h \in A(K) \text{ and } x \in K, \tag{2.5}$$

then the sequence $(B_n)_{n\geq 1}$ is an approximation process on $C(K)$. Namely, for every $f \in C(K)$,

$$\lim_{n\to\infty} B_n(f) = f \tag{2.6}$$

uniformly on K.

Finally, for every $h, k \in A(K)$ and $n \geq 1$, the following useful formulas hold:

$$B_n(h) = h \tag{2.7}$$

and

$$B_n(hk) = \frac{1}{n}T(hk) + \frac{n-1}{n}hk. \tag{2.8}$$

In particular,

$$B_n(h^2) = \frac{1}{n}T(h^2) + \frac{n-1}{n}h^2. \tag{2.9}$$

(For a proof of (2.6)–(2.9), see [4, Theorem 3.2.1] or [5, Theorem 3.2].)

3. Generalized Kantorovich operators

In this section, we introduce the main object of interest of the article and we show some examples. Let $T : C(K) \to C(K)$ be a Markov operator satisfying condition (2.4) (or, equivalently, (2.5)). Moreover, fix $a \geq 0$ and a sequence $(\mu_n)_{n\geq 1}$ of Borel probability measures on K. Then, for every $n \geq 1$, we consider the positive linear operator C_n defined by setting

$$C_n(f)(x) = \int_K \cdots \int_K f\Big(\frac{x_1 + \cdots + x_n + a x_{n+1}}{n+a}\Big)\, d\tilde{\mu}_x^T(x_1) \cdots d\tilde{\mu}_x^T(x_n)\, d\mu_n(x_{n+1}) \quad (3.1)$$

for every $x \in K$ and for every $f \in C(K)$.

The germ of the idea of the construction of (3.1) goes back to [6], where the first and third authors considered the particular case of the unit interval. Subsequently, in [3] a natural generalization of C_n's to the multidimensional setting—that is, to hypercubes and simplices—was presented, which encompassed, as a particular case, the multidimensional Kantorovich operators on these frameworks.

Here we develop this idea in full generality, obtaining a new class of positive linear operators which encompasses not only several well-known approximation processes in both univariate and multivariate settings, but also new ones in finite- and infinite-dimensional frameworks as well. Clearly, in the special case $a = 0$, the operators C_n correspond to the B_n ones (see (2.3)). Moreover, by introducing the auxiliary continuous function

$$I_n(f)(x) := \int_K f\Big(\frac{n}{n+a}x + \frac{a}{n+a}t\Big)\, d\mu_n(t) \quad \big(f \in C(K), x \in K\big), \quad (3.2)$$

for every $n \geq 1$, we then have

$$C_n(f) = B_n\big(I_n(f)\big). \quad (3.3)$$

Therefore $C_n(f) \in C(K)$, and the operator $C_n : C(K) \to C(K)$, being linear and positive, is continuous with norm equal to 1, because $C_n(\mathbf{1}) = \mathbf{1}$.

We point out that the operators C_n are well defined on the larger linear space of all Borel measurable functions $f : K \to \mathbf{R}$ for which the multiple integral in (3.1) is absolutely convergent. This space contains, among other things, all the bounded Borel measurable functions on K as well as a suitable subspace of $\bigcap_{n\geq 1} L^1(K, \mu_n)$.

Here, we prefer not to provide more information in this regard and have decided to postpone a more thorough analysis until a subsequent future article. However, in Section 4 we will discuss the approximation properties of these operators also in the setting of $L^p(K)$-spaces in the particular cases where K is a simplex or a hypercube of $\mathbf{R}^d$.

Note that assumption (2.4) is not essential in defining the operators C_n, but as we will see in the next sections, it will be needed in order to prove that $(C_n)_{n\geq 1}$ is an approximation process on $C(K)$ and on $L^p(K)$. By specifying the Markov operator T (i.e., the family of representing measures $(\tilde{\mu}_x^T)_{x\in K}$ and the parameter $a \geq 0$, as well as the sequence of measures $(\mu_n)_{n\geq 1}$), we obtain several classes of approximating operators which can be tracked down in different articles. In particular, when $a = 1$, for a special class of T and of the sequence $(\mu_n)_{n\geq 1}$, we get the Kantorovich operators on the unit interval, on simplices, and on hypercubes (see the next examples).

Another interesting case is covered when a is a positive integer. Indeed, given $\mu \in M_1^+(K)$, we may consider the measure $\mu_a \in M_1^+(K)$ defined by

$$\int_K f\,d\mu_a := \int_K \cdots \int_K f\Big(\frac{y_1 + \cdots + y_a}{a}\Big)\,d\mu(y_1)\cdots d\mu(y_a)$$

($f \in C(K)$) and hence, for $\mu_n := \mu_a$ for every $n \geq 1$, the corresponding operators in (3.1) reduce to

$$C_n(f)(x) \\ = \int_K \cdots \int_K f\Big(\frac{x_1 + \cdots + x_n + y_1 + \cdots + y_a}{n+a}\Big)\,d\tilde{\mu}_x^T(x_1)\cdots d\tilde{\mu}_x^T(x_n)\,d\mu(y_1)\cdots d\mu(y_a) \tag{3.4}$$

($n \geq 1$, $f \in C(K)$, $x \in K$).

When $K = [0, 1]$, for particular T's and μ's we get the so-called *Kantorovich operators of order* a (see [8, Examples (A)] and [12] for $a = 2$).

We also mention another particular case which, while seemingly simple, is not devoid of interest. Assume that $a > 0$, and consider a sequence $(b_n)_{n\geq 1}$ in X such that $b_n/a \in K$ for every $n \geq 1$. Then, setting $\mu_n := \epsilon_{b_n/a}$ ($n \geq 1$), from (3.1) we get

$$C_n(f)(x) = \int_K \cdots \int_K f\Big(\frac{x_1 + \cdots + x_n + b_n}{n+a}\Big)\,d\tilde{\mu}_x^T(x_1)\cdots d\tilde{\mu}_x^T(x_n) \tag{3.5}$$

($n \geq 1$, $f \in C(K)$, $x \in K$).

We proceed to show more specific examples.

Examples 3.1.

1. Assume that $K = [0, 1]$, and consider the Markov operator $T_1 : C([0, 1]) \to C([0, 1])$ defined, for every $f \in C([0, 1])$ and $0 \leq x \leq 1$, by

$$T_1(f)(x) := (1 - x)f(0) + xf(1). \tag{3.6}$$

Then, the Bernstein–Schnabl operators associated with T_1 are the classical Bernstein operators

$$B_n(f)(x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} f\Big(\frac{k}{n}\Big)$$

($n \geq 1$, $f \in C([0, 1])$, $x \in [0, 1]$), and, considering $a \geq 0$ and $(\mu_n)_{n\geq 1}$ in $M_1^+([0, 1])$, from (3.1) and (3.3) we get

$$C_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1 - x)^{n-k} \int_0^1 f\Big(\frac{k + as}{n+a}\Big)\,d\mu_n(s) \tag{3.7}$$

($n \geq 1$, $f \in C([0, 1])$, $x \in [0, 1]$). In particular, if all the μ_n's are equal to the Borel–Lebesgue measure λ_1 on $[0, 1]$ and $a = 1$, then formula (3.7) gives the classical Kantorovich operators (see [2, Section 5.3.7]). Moreover, as already remarked, for $a = 0$ we obtain the Bernstein operators; thus, by means of (3.7), we obtain a link between these fundamental sequences of approximating operators in terms of a continuous parameter $a \in [0, 1]$. Special cases of operators (3.7) have been also considered in [6] and [10] and we omit the details for the sake of brevity.

When a is a positive integer, from (3.4) we obtain

$$C_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} \times \int_0^1 \cdots \int_0^1 f\Big(\frac{k + y_1 + \cdots + y_a}{n+a}\Big)\, d\mu(y_1) \cdots d\mu(y_a) \tag{3.8}$$

($n \geq 1$, $f \in C([0,1])$, $0 \leq x \leq 1$), $\mu \in M_1^+([0,1])$ being fixed. When μ is the Borel–Lebesgue measure on $[0, 1]$, we obtain the previously mentioned Kantorovich operators of order a (see [8, Examples (A)] and [12] for $a = 2$). Finally, from (3.5) and with $b_n \leq a$ ($n \geq 1$), we get the operators

$$C_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\Big(\frac{k + b_n}{n+a}\Big) \tag{3.9}$$

($n \geq 1$, $f \in C([0,1])$, $x \in [0,1]$), which were first considered in [17] for a constant sequence $(b_n)_{n\geq 1}$.

2. Let $Q_d := [0,1]^d$, $d \geq 1$, and consider the Markov operator $S_d : C(Q_d) \to C(Q_d)$ defined by

$$S_d(f)(x) := \sum_{h_1,\ldots,h_d=0}^{1} f(\delta_{h_1 1}, \ldots, \delta_{h_d 1}) x_1^{h_1} (1-x_1)^{1-h_1} \cdots x_d^{h_d} (1-x_d)^{1-h_d} \tag{3.10}$$

($f \in C(Q_d)$, $x = (x_1, \ldots, x_d) \in Q_d$), where δ_{ij} stands for the Kronecker symbol.

In this case, the Bernstein–Schnabl operators associated with S_d are the classical Bernstein operators on Q_d defined by

$$B_n(f)(x) = \sum_{h_1,\ldots,h_d=0}^{n} \prod_{i=1}^{d} \binom{n}{h_i} x_i^{h_i} (1-x_i)^{n-h_i} f\Big(\frac{h_1}{n}, \ldots, \frac{h_d}{n}\Big)$$

($n \geq 1$, $f \in C(Q_d)$, $x = (x_1, \ldots, x_d) \in Q_d$). Then, taking (3.3) into account, the operators C_n given by (3.1) become

$$C_n(f)(x) = \sum_{h_1,\ldots,h_d=0}^{n} \prod_{i=1}^{d} \binom{n}{h_i} x_i^{h_i} (1-x_i)^{n-h_i} \times \int_{Q_d} f\Big(\frac{h_1 + a s_1}{n+a}, \ldots, \frac{h_d + a s_d}{n+a}\Big)\, d\mu_n(s_1, \ldots, s_d) \tag{3.11}$$

($n \geq 1$, $f \in C(Q_d)$, $x = (x_1, \ldots, x_d) \in Q_d$).

When all the μ_n's coincide with the Borel–Lebesgue measure λ_d on Q_d and $a = 1$, the operators C_n turn into a generalization of Kantorovich operators introduced in [18]. (Another special case of (3.11) has been studied in [3].)

3. Denote by K_d the canonical simplex in $\mathbf{R}^d$, $d \geq 1$, that is,

$$K_d := \Big\{(x_1, \dots, x_d) \in \mathbf{R}^d \;\Big|\; x_i \geq 0 \ (i = 1, \dots, d) \text{ and } \sum_{i=1}^{d} x_i \leq 1\Big\},$$

and consider the canonical Markov operator $T_d : C(K_d) \to C(K_d)$ defined by

$$T_d(f)(x) := \Big(1 - \sum_{i=1}^{d} x_i\Big) f(0) + \sum_{i=1}^{d} x_i f(e_i) \tag{3.12}$$

($f \in C(K_d)$, $x = (x_1, \dots, x_d) \in K_d$), where, for every $i = 1, \dots, d$, $e_i := (\delta_{ij})_{1 \leq j \leq d}$, with δ_{ij} being the Kronecker symbol (see, e.g., [2, Section 6.3.3]).

The Bernstein–Schnabl operators associated with T_d are the classical Bernstein operators on K_d defined by

$$B_n(f)(x) = \sum_{\substack{h_1, \dots, h_d = 0, \dots, n \\ h_1 + \cdots + h_d \leq n}} \frac{n!}{h_1! \cdots h_d!(n - h_1 - \cdots - h_d)!} x_1^{h_1} \cdots x_d^{h_d} \\ \times \Big(1 - \sum_{i=1}^{d} x_i\Big)^{n - \sum_{i=1}^{d} h_i} f\Big(\frac{h_1}{n}, \dots, \frac{h_d}{n}\Big)$$

($n \geq 1$, $f \in C(K_d)$, $x = (x_1, \dots, x_d) \in K_d$). By once again using (3.3), we obtain

$$C_n(f)(x) \\ = \sum_{\substack{h_1, \dots, h_d = 0, \dots, n \\ h_1 + \cdots + h_d \leq n}} \frac{n!}{h_1! \cdots h_d!(n - h_1 - \cdots - h_d)!} x_1^{h_1} \cdots x_d^{h_d} \Big(1 - \sum_{i=1}^{d} x_i\Big)^{n - \sum_{i=1}^{d} h_i} \\ \times \int_{K_d} f\Big(\frac{h_1 + as_1}{n + a}, \frac{h_2 + as_2}{n + a}, \dots, \frac{h_d + as_d}{n + a}\Big) \, d\mu_n(s_1, \dots, s_d) \tag{3.13}$$

($n \geq 1$, $f \in C(K_d)$, $x = (x_1, \dots, x_d) \in K_d$).

When all the μ_n's are equal to the Borel–Lebesgue measure λ_d on K_d and $a = 1$, these operators are referred to as the *Kantorovich operators on* $C(K_d)$ and were introduced in [18]. Another particular case of (3.13) has been investigated in [3, Section 3].

For the sake of brevity, we omit details of the operators corresponding to (3.4) and (3.5) in the setting of $C(Q_d)$ and $C(K_d)$.

4. Approximation properties in $C(K)$

In this section, we present some approximation properties of the sequence $(C_n)_{n \geq 1}$ on $C(K)$, showing several estimates of the rate of convergence. In order to prove that the sequence $(C_n)_{n \geq 1}$ is a (positive) approximation process on $C(K)$, we need the following preliminary result.

Lemma 4.1. *Let $(C_n)_{n\geq 1}$ be the sequence of operators defined by (3.1) and associated with a Markov operator satisfying (2.4) (or, equivalently, (2.5)). Then, for every $h, k \in A(K)$, we have*

$$C_n(h) = \frac{a}{n+a}\int_K h\,d\mu_n \cdot \mathbf{1} + \frac{n}{n+a}h \tag{4.1}$$

and

$$\begin{aligned} C_n(hk) = {} & \frac{a^2}{(n+a)^2}\int_K hk\,d\mu_n \cdot \mathbf{1} + \frac{na}{(n+a)^2}\int_K h\,d\mu_n \cdot k \\ & + \frac{na}{(n+a)^2}\int_K k\,d\mu_n \cdot h + \frac{n^2}{(n+a)^2}B_n(hk). \end{aligned} \tag{4.2}$$

In particular,

$$C_n(h^2) = \frac{a^2}{(n+a)^2}\int_K h^2\,d\mu_n \cdot \mathbf{1} + \frac{2na}{(n+a)^2}\int_K h\,d\mu_n \cdot h + \frac{n^2}{(n+a)^2}B_n(h^2). \tag{4.3}$$

Proof. By (3.2), if $h \in A(K)$, then $I_n(h) = \frac{n}{n+a}h + \frac{a}{n+a}\int_K h\,d\mu_n \cdot \mathbf{1}$, and hence (4.1) follows by taking (3.3) and (2.7) into account. Analogously, (4.2) is a consequence of the identity

$$\begin{aligned} I_n(hk) = {} & \frac{n^2}{(n+a)^2}hk + \frac{a^2}{(n+a)^2}\int_K hk\,d\mu_n \cdot \mathbf{1} \\ & + \frac{na}{(n+a)^2}\Big(\int_K h\,d\mu_n\Big)k + \frac{na}{(n+a)^2}\Big(\int_K k\,d\mu_n\Big)h \end{aligned}$$

and (2.8). Formula (4.3) is a direct consequence of (4.2). □

The following approximation result holds.

Theorem 4.2. *Under assumption (2.4), for every $f \in C(K)$ we have*

$$\lim_{n\to\infty} C_n(f) = f \quad \text{uniformly on } K. \tag{4.4}$$

Proof. First, observe that the space $A(K)$ contains the constant functions and separates the points of K by the Hahn–Banach theorem. Then, according to [2, Theorem 4.4.6, Example 3] (see also [4, Theorem 1.2.8]), $A(K) \cup A(K)^2$ is a Korovkin subset for $C(K)$. Hence the claim will be proved if we show that, for every $h \in A(K)$,

$$\lim_{n\to\infty} C_n(h) = h \qquad \text{and} \qquad \lim_{n\to\infty} C_n(h^2) = h^2$$

uniformly on K. All these assertions follow from Lemma 4.1, observing that the sequences $(\int_K h\,d\mu_n)_{n\geq 1}$ and $(\int_K h^2\,d\mu_n)_{n\geq 1}$ are bounded for every $h \in A(K)$. □

Now we present some quantitative estimates of the rate of convergence in (4.4) by means of suitable moduli of continuity in both the finite-dimensional and infinite-dimensional settings. To this end, we need to recall some useful definitions. We begin with the finite-dimensional case, that is, K is a convex compact subset of $\mathbf{R}^d$, $d \geq 1$. Then we can estimate the rate of uniform convergence of the sequence $(C_n(f))_{n\geq 1}$ to f by means of the first and second moduli of continuity, respectively, defined as

$$\omega(f,\delta) := \sup\{|f(x)-f(y)| \mid x,y\in K, \|x-y\|_2 \leq \delta\} \tag{4.5}$$

and

$$\omega_2(f,\delta) := \sup\Big\{\Big|f(x)-2f\Big(\frac{x+y}{2}\Big)+f(y)\Big| \;\Big|\; x,y\in K, \|x-y\|_2 \leq 2\delta\Big\},$$

for any $f\in C(K)$ and $\delta>0$.

In the general case of a locally convex Hausdorff space X (not necessarily of finite dimension), we will use the total modulus of continuity which we are going to define. First, if $m\geq 1$, $h_1,\dots,h_m\in L(K)$ (see (2.1)) and $\delta>0$, we set

$$H(h_1,\dots,h_m,\delta) := \Big\{(x,y)\in K\times K \;\Big|\; \sum_{j=1}^m \big(h_j(x)-h_j(y)\big)^2 \leq \delta^2\Big\}.$$

Fix a bounded function $f: K\to \mathbf{R}$; the modulus of continuity of f with respect to $h_1,\dots,h_m$ is defined as

$$\omega(f;h_1,\dots,h_m,\delta) := \sup\{|f(x)-f(y)| \mid (x,y)\in H(h_1,\dots,h_m,\delta)\}.$$

Furthermore, we define the total modulus of continuity of f as

$$\begin{aligned}\Omega(f,\delta) &:= \inf\Big\{\omega(f;h_1,\dots,h_m,\delta) \;\Big|\; m\geq 1, h_1,\dots,h_m\in L(K), \Big\|\sum_{j=1}^m h_j^2\Big\|_\infty = 1\Big\}\\ &= \inf\Big\{\omega(f;h_1,\dots,h_m,1) \;\Big|\; m\geq 1, h_1,\dots,h_m\in L(K), \Big\|\sum_{j=1}^m h_j^2\Big\|_\infty = \frac{1}{\delta^2}\Big\}.\end{aligned} \tag{4.6}$$

If $X=\mathbf{R}^d$, then there is a simple relationship between $\Omega(f,\delta)$ and the (first) modulus of continuity $\omega(f,\delta)$ defined by (4.5); indeed,

$$\omega(f;pr_1,\dots,pr_d,\delta) = \omega(f,\delta), \tag{4.7}$$

so that, setting $r(K) := \max\{\|x\|_2 \mid x\in K\}$,

$$\Omega(f,\delta) \leq \omega\big(f,\delta r(K)\big),$$

the last inequality being an equality if $d=1$.

By using Proposition 1.6.5 in [4] (see also [2, Proposition 5.1.4]), we get the following result.

Proposition 4.3. *For every $n\geq 1$ and $f\in C(K)$,*

$$\big\|C_n(f)-f\big\|_\infty \leq 2\Omega\Big(f,\sqrt{\frac{4a^2+1}{n+a}}\Big).$$

Proof. Since $C_n(\mathbf{1}) = \mathbf{1}$ $(n \geq 1)$, we can apply estimate (1.6.14) in [4, Proposition 1.6.5] obtaining, for every $n \geq 1$, $f \in C(K)$, $x \in K$ and $\delta > 0$, and for every $h_1, \dots, h_m \in L(K)$, $m \geq 1$,

$$\big|C_n(f)(x) - f(x)\big| \leq \Big(1 + \frac{1}{\delta^2}\sum_{j=1}^{m} \mu(x, C_n, h_j)\Big)\omega(f; h_1, \dots, h_m, \delta), \tag{1}$$

where $\mu(x, C_n, h_j) = C_n((h_j - h_j(x)\mathbf{1})^2)(x)$, $j = 1, \dots, m$. Therefore,

$$\big|C_n(f)(x) - f(x)\big| \leq \big(1 + \tau_n(\delta, x)\big)\Omega(f, \delta), \tag{2}$$

where $\Omega(f, \delta)$ is the total modulus of continuity (see (4.6)) and

$$\tau_n(\delta, x) := \sup\Big\{\sum_{j=1}^{m} \mu(x, C_n, h_j)(x) \;\Big|\; m \geq 1, h_1, \dots, h_m \in L(K) \text{ and } \Big\|\sum_{j=1}^{m} h_j^2\Big\|_\infty = \frac{1}{\delta^2}\Big\}.$$

Fix $h \in L(K)$. Keeping (4.1), (4.3), and (2.9) in mind, we have

$$\begin{aligned}
&C_n\big(\big(h - h(x)\mathbf{1}\big)^2\big)(x)\\
&\quad = C_n(h^2)(x) - 2h(x)C_n(h)(x) + h^2(x)\\
&\quad = \frac{a^2}{(n+a)^2}\int_K h^2\, d\mu_n + \frac{2na}{(n+a)^2}h(x)\int_K h\, d\mu_n + \frac{n^2}{(n+a)^2}B_n(h^2)(x)\\
&\qquad - \frac{2a}{n+a}h(x)\int_K h\, d\mu_n - \frac{2n}{n+a}h^2(x) + h^2(x)\\
&\quad = \frac{a^2}{(n+a)^2}\int_K h^2\, d\mu_n - \frac{2a^2}{(n+a)^2}h(x)\int_K h\, d\mu_n + \frac{n}{(n+a)^2}T(h^2)(x)\\
&\qquad + \frac{a^2 - n}{(n+a)^2}h^2(x)\\
&\quad \leq \frac{a^2}{(n+a)^2}\int_K h^2\, d\mu_n + \frac{2a^2}{(n+a)^2}\Big|h(x)\int_K h\, d\mu_n\Big| + \frac{n}{(n+a)^2}T(h^2)(x)\\
&\qquad + \frac{a^2}{(n+a)^2}h^2(x).
\end{aligned}$$

Then, using the Cauchy–Schwarz and the Jensen inequalities, we get

$$\begin{aligned}
&\sum_{j=1}^{m} C_n\big(\big(h_j - h_j(x)\mathbf{1}\big)^2\big)(x)\\
&\quad \leq \frac{a^2}{(n+a)^2}\int_K \sum_{j=1}^{m} h_j^2\, d\mu_n + \frac{2a^2}{(n+a)^2}\sum_{j=1}^{m}\Big|h_j(x)\int_K h_j\, d\mu_n\Big|\\
&\qquad + \frac{n}{(n+a)^2}T\Big(\sum_{j=1}^{m} h_j^2\Big)(x) + \frac{a^2}{(n+a)^2}\sum_{j=1}^{m} h_j^2(x)
\end{aligned}$$

$$\leq \frac{a^2}{(n+a)^2}\Big\|\sum_{j=1}^m h_j^2\Big\|_\infty + \frac{2a^2}{(n+a)^2}\Big(\sum_{j=1}^m h_j^2(x)\Big)^{1/2}\Big(\sum_{j=1}^m\Big|\int_K h_j\,d\mu_n\Big|^2\Big)^{1/2}$$
$$+\frac{n}{(n+a)^2}\Big\|\sum_{j=1}^m h_j^2\Big\|_\infty + \frac{a^2}{(n+a)^2}\Big\|\sum_{j=1}^m h_j^2\Big\|_\infty.$$

Then, for every $\delta > 0$ and for every $h_1, \dots h_m \in L(K)$, $m \geq 1$, such that $\|\sum_{j=1}^m h_j^2\|_\infty = 1/\delta^2$, we have

$$\begin{aligned}
&\sum_{j=1}^m C_n\big(\big(h_j - h_j(x)\mathbf{1}\big)^2\big)(x)\\
&\quad\leq \frac{2a^2}{(n+a)^2}\frac{1}{\delta^2} + \frac{2a^2}{(n+a)^2}\Big(\Big\|\sum_{j=1}^m h_j^2\Big\|_\infty\Big)^{1/2}\Big(\sum_{j=1}^m\int_K h_j^2\,d\mu_n\Big)^{1/2} + \frac{n}{(n+a)^2}\frac{1}{\delta^2}\\
&\quad\leq \frac{2a^2}{(n+a)^2}\frac{1}{\delta^2} + \frac{2a^2}{(n+a)^2}\Big\|\sum_{j=1}^m h_j^2\Big\|_\infty + \frac{n}{(n+a)^2}\frac{1}{\delta^2}\\
&\quad= \frac{4a^2+n}{\delta^2(n+a)^2} \leq \frac{4a^2+1}{(n+a)\delta^2}.
\end{aligned}$$

From (2), we infer that

$$\big|C_n(f)(x) - f(x)\big| \leq \Big(1 + \frac{4a^2+1}{(n+a)\delta^2}\Big)\Omega(f,\delta)$$

for every $\delta > 0$. Then setting $\delta = \sqrt{(4a^2+1)/(n+a)}$, we get the claim. □

We proceed to establish some estimates in the finite-dimensional case. First, we state the following result involving $\omega(f,\delta)$. In the rest of this article, $e_2 : K \longrightarrow \mathbf{R}$ will denote the function

$$e_2(x) := \|x\|_2^2 = \sum_{i=1}^d pr_i^2(x) \quad (x \in K)$$

and, for a given $x \in K$, $d_x : K \longrightarrow \mathbf{R}$ will stand for the function

$$d_x(y) := \|y - x\|_2 \quad (y \in K).$$

Proposition 4.4. *For every $f \in C(K)$, $n \geq 1$, and $x \in K$,*

$$\big|C_n(f)(x) - f(x)\big| \leq 2\omega\Big(f, \frac{1}{n+a}\sqrt{a^2\int_K d_x^2\,d\mu_n + n\big(T(e_2)(x) - e_2(x)\big)}\Big). \quad (4.8)$$

Moreover,

$$\big\|C_n(f) - f\big\|_\infty \leq 2\omega\Big(f, \frac{\max(a\delta(K)^2, \|T(e_2) - e_2\|_\infty)}{\sqrt{n+a}}\Big), \quad (4.9)$$

where $\delta(K) := \sup\{\|x-y\|_2 \mid x, y \in K\}$.

Proof. Considering the coordinate functions pr_i, $i = 1, \dots, d$, clearly, for every $x \in K$,

$$d_x^2 = \sum_{i=1}^{d} \big(pr_i - pr_i(x)\big)^2.$$

Therefore, from formula (1) of the proof of Proposition 4.3 and from (4.7) (see also [2, Proposition 5.1.4 and (5.1.13)]), it follows that, for any $n \geq 1$ and $\delta > 0$,

$$\big|C_n(f)(x) - f(x)\big| \leq \Big(1 + \frac{1}{\delta^2} C_n(d_x^2)(x)\Big)\omega(f, \delta).$$

If $C_n(d_x^2)(x) = 0$, then, letting $\delta \to 0^+$, we get $C_n(f)(x) = f(x)$ and, in this case, (4.8) is obviously satisfied. If $C_n(d_x^2)(x) > 0$, then for $\delta := \sqrt{C_n(d_x^2)(x)}$, we obtain

$$\big|C_n(f)(x) - f(x)\big| \leq 2\omega\big(f, \sqrt{C_n(d_x^2)(x)}\big).$$

On the other hand, by applying Lemma 4.1 to each pr_i, $i = 1, \dots, d$, we have

$$C_n(d_x^2)(x) = \frac{a^2}{(n+a)^2} \int_K d_x^2 \, d\mu_n + \frac{n}{(n+a)^2} \big(T(e_2)(x) - e_2(x)\big),$$

and hence (4.8) holds. Clearly, (4.9) follows from (4.8). □

In the next result, we show a further estimate of the rate of convergence in (4.4) by means of $\omega_2(f, \delta)$. To that end, according to [7], we set

$$\lambda_{n,\infty} := \max_{0 \leq i \leq d+1} \big\|C_n(\varphi_i) - \varphi_i\big\|_\infty, \tag{4.10}$$

where the functions φ_i are defined by

$$\varphi_0 := \mathbf{1}, \qquad \varphi_i := pr_i \quad (i = 1, \dots, d) \qquad \text{and} \qquad \varphi_{d+1} := \sum_{i=1}^{d} pr_i^2. \tag{4.11}$$

Then we have the following result.

Proposition 4.5. *For every $f \in C(K)$ and $n \geq 1$,*

$$\big\|C_n(f) - f\big\|_\infty \leq C\Big(\frac{M}{n+a}\|f\|_\infty + \omega_2\Big(f, \sqrt{\frac{M}{n+a}}\Big)\Big),$$

where the constants C and M do not depend on f.

Proof. Since every convex bounded set has the cone property (see [1, p. 66]), from [7, Theorem 2′] we infer that, for every $f \in C(K)$ and $n \geq 1$,

$$\big\|C_n(f) - f\big\|_\infty \leq C\big(\lambda_{n,\infty}\|f\|_\infty + \omega_2(f, \lambda_{n,\infty}^{1/2})\big),$$

where $\lambda_{n,\infty}$ is defined by (4.10) and the constant C does not depend on f. In order to estimate $\lambda_{n,\infty}$, note that, for every $n \geq 1$, $C_n(\mathbf{1}) = \mathbf{1}$ and, for $i = 1, \dots, d$ and $x = (x_1, \dots, x_d) \in K$, taking (4.1) into account,

$$\big|C_n(pr_i)(x) - pr_i(x)\big| = \frac{a}{n+a}\Big|\int_K pr_i \, d\mu_n - x_i\Big| \leq \frac{2a\|x\|_2}{n+a};$$

hence

$$\left\|C_n(\varphi_i)-\varphi_i\right\|_\infty \le \frac{2ar(K)}{n+a},$$

where $r(K) := \max\{\|x\|_2 \mid x \in K\}$.

On the other hand, by virtue of (4.3) and (2.9), for every $n \ge 1$, $i = 1,\dots,d$ and $x = (x_1,\dots,x_d) \in K$ we get

$$\begin{aligned}C_n(pr_i^2)(x) - pr_i^2(x) = \frac{1}{(n+a)^2}\Big(a^2\int_K pr_i^2\,d\mu_n + 2na\Big(\int_K pr_i\,d\mu_n\Big)x_i \\ + n\big(T(pr_i^2)(x) - x_i^2\big) - a(2n+a)x_i^2\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}&\left|C_n(\varphi_{d+1})(x) - \varphi_{d+1}(x)\right| \\ &\quad \le \sum_{i=1}^d \left|C_n(pr_i^2)(x) - pr_i^2(x)\right| \\ &\quad \le \frac{1}{(n+a)^2}\Big\{a^2 r(K)^2 + 2adnr(K)^2 + n\sum_{i=1}^d\left|T(pr_i^2)(x) - x_i^2\right| \\ &\qquad + a(2n+a)r(K)^2\Big\} \\ &\quad \le \frac{1}{(n+a)^2}\Big\{\big(2a(n+a)+2adn+n\big)r(K)^2 + n\sum_{i=1}^d T(pr_i^2)(x)\Big\} \\ &\quad = \frac{1}{(n+a)^2}\Big\{\big(2a(n+a)+2adn+n\big)r(K)^2 + nT\Big(\sum_{i=1}^d pr_i^2\Big)(x)\Big\} \\ &\quad \le \frac{r(K)^2}{(n+a)^2}\big\{2a(n+a)+2adn+2n\big\}.\end{aligned}$$

Consequently,

$$\left\|C_n(\varphi_{d+1}) - \varphi_{d+1}\right\|_\infty \le \frac{2a+2ad+2}{n+a}r(K)^2$$

and, if we set $M := \max\{2ar(K), (2a+2ad+2)r(K)^2\}$, we get

$$\lambda_{n,\infty} \le \frac{M}{n+a};$$

this completes the proof. □

5. Approximation properties in L^p-spaces

In this section, we investigate particular subclasses of the operators C_n, $n \ge 1$, which are well defined in $L^p(K)$-spaces, $1 \le p < +\infty$. This analysis will be carried out in the special cases where K is the d-dimensional unit hypercube (in particular, the unit interval) and the d-dimensional simplex.

In order to estimate the rate of convergence, we recall here the definition of some moduli of smoothness. Let K be a convex compact subset of $\mathbf{R}^d$, $d \ge 1$, having nonempty interior. If $f : K \to \mathbf{R}$ is a Borel measurable bounded function and if $\delta > 0$, then we define the (multivariate) averaged modulus of smoothness of the first order for f and step δ in L^p-norm, $1 \le p < +\infty$, as

$$\tau(f,\delta)_p := \big\|\omega(f,\cdot;\delta)\big\|_p, \tag{5.1}$$

where, for every $x \in K$,

$$\begin{aligned}\omega(f,x;\delta) := \sup\big\{\big|f(t+h)-f(t)\big| \;\big|\; & t, t+h \in K, \|t-x\|_2 \le \delta/2,\\ & \|t+h-x\|_2 \le \delta/2\big\}.\end{aligned}$$

Furthermore, if $f \in L^p(K)$, $1 \le p < +\infty$, and $\delta > 0$, then the (multivariate) modulus of smoothness for f of order k and step δ in L^p-norm is defined by

$$\omega_{k,p}(f,\delta) := \sup_{0<|h|\le\delta}\Big(\int_K \big|\Delta_h^k f(x)\big|^p\,dx\Big)^{1/p},$$

where, for $x \in K$,

$$\Delta_h^k f(x) := \begin{cases}\sum_{l=0}^k (-1)^{k-l}\binom{k}{l} f(x+lh) & \text{if } x+hk \in K,\\ 0 & \text{otherwise}\end{cases}$$

(see [16, Section 1.3], [13], [14]).

We now restrict ourselves to considering the d-dimensional unit hypercube $Q_d := [0,1]^d$, $d \ge 1$. For every $n \ge 1$, $h = (h_1,\dots,h_d) \in \{0,\dots,n\}^d$, and $x = (x_1,\dots,x_d) \in Q_d$, we set

$$P_{n,h}(x) := \prod_{i=1}^d \binom{n}{h_i} x_i^{h_i}(1-x_i)^{n-h_i};$$

then

$$P_{n,h} \ge 0 \qquad \text{and} \qquad \sum_{h\in\{0,\dots,n\}^d} P_{n,h} = 1 \quad \text{on } Q_d. \tag{5.2}$$

Further,

$$\int_{Q_d} P_{n,h}(x)\,dx = \prod_{i=1}^d \binom{n}{h_i}\int_0^1 x_i^{h_i}(1-x_i)^{n-h_i}\,dx_i = \frac{1}{(n+1)^d}. \tag{5.3}$$

Moreover, for any given $a \ge 0$, we set

$$Q_{n,h}(a) := \prod_{i=1}^d \Big[\frac{h_i}{n+a}, \frac{h_i+a}{n+a}\Big] \subset Q_d;$$

in particular,

$$\bigcup_{h\in\{0,\dots,n\}^d} Q_{n,h}(a) = Q_d.$$

Consider the operators C_n, $n \ge 1$, defined by (3.11) with all the measures μ_n equal to the Borel–Lebesgue measure on Q_d. In this case, the operators C_n are well defined on $L^1(Q_d)$ as well and the operator S_d satisfies (2.4). Furthermore, if $f \in L^1(Q_d)$ and $x \in Q_d$, then

$$C_n(f)(x) = \sum_{h\in\{0,\dots,n\}^d} P_{n,h}(x) \int_{Q_d} f\Big(\frac{h+au}{n+a}\Big)\,du; \tag{5.4}$$

in particular, if $a > 0$, then

$$C_n(f)(x) = \sum_{h\in\{0,\dots,n\}^d} P_{n,h}(x) \Big(\frac{n+a}{a}\Big)^d \int_{Q_{n,h}(a)} f(v)\,dv. \tag{5.5}$$

If $d = 1$, formulas (5.4) and (5.5) turn into

$$C_n(f)(x) = \sum_{h=0}^{n} \binom{n}{h} x^h (1-x)^{n-h} \int_0^1 f\Big(\frac{h+as}{n+a}\Big)\,ds$$

and, if $a > 0$, then

$$C_n(f)(x) = \sum_{h=0}^{n} \binom{n}{h} x^h (1-x)^{n-h} \Big(\frac{n+a}{a}\Big) \int_{\frac{h}{n+a}}^{\frac{h+a}{n+a}} f(t)\,dt.$$

Theorem 5.1. *Assume that $a > 0$. If $f \in L^p(Q_d)$, $1 \le p < +\infty$, then $\lim_{n\to\infty} C_n(f) = f$ in $L^p(Q_d)$.*

Proof. Since $C(Q_d)$ is dense in $L^p(Q_d)$ with respect to the L^p-norm $\|\cdot\|_p$, and since, on account of Theorem 4.2, $\lim_{n\to\infty} C_n(f) = f$ in $L^p(Q_d)$ for every $f \in C(Q_d)$, it is enough to show that the sequence $(C_n)_{n\ge 1}$ is equibounded from $L^p(Q_d)$ into $L^p(Q_d)$.

Fix, indeed, $f \in L^p(Q_d)$, $n \ge 1$, and $x \in Q_d$. By recalling that the function $|t|^p$ ($t \in \mathbf{R}$) is convex and that

$$\Big|\int_{Q_d} g(u)\,du\Big|^p \le \int_{Q_d} \big|g(u)\big|^p\,du$$

for every $g \in L^p(Q_d)$, setting $M := \sup_{n\ge1}(\frac{n+a}{a(n+1)})^d$, on account of (5.2) we get

$$\begin{aligned}\big|C_n(f)(x)\big|^p &\le \sum_{h\in\{0,\dots,n\}^d} P_{n,h}(x) \int_{Q_d} \Big|f\Big(\frac{h+au}{n+a}\Big)\Big|^p\,du \\ &= \sum_{h\in\{0,\dots,n\}^d} P_{n,h}(x) \Big(\frac{n+a}{a}\Big)^d \int_{Q_{n,h}(a)} \big|f(v)\big|^p\,dv.\end{aligned}$$

Therefore, by using (5.3), we obtain

$$\begin{aligned}\int_{Q_d} \big|C_n(f)(x)\big|^p\,dx &\le \sum_{h\in\{0,\dots,n\}^d} \Big(\frac{n+a}{a(n+1)}\Big)^d \int_{Q_{n,h}(a)} \big|f(v)\big|^p\,dv \\ &\le M \int_{Q_d} \big|f(v)\big|^p\,dv;\end{aligned}$$

that is, $\|C_n(f)\|_p \le M^{1/p}\|f\|_p$, and so the result follows. $\square$

We now present some estimates of the rate of convergence in Theorem 5.1.

Proposition 5.2. *For every Borel measurable bounded function f on Q_d, $1 \leq p < +\infty$ and $n \geq 1$,*

$$\left\|C_n(f) - f\right\|_p \leq C\tau\Big(f;\ \sqrt[2d]{\frac{3n+a^2}{12(n+a)^2}}\Big)$$

(see (5.1)), where the positive constant C does not depend on f.

Proof. Let f be a Borel measurable bounded function on Q_d, and let $p \geq 1$. For a fixed $x \in Q_d$, set $\Psi_x(y) := y - x$ for every $y \in Q_d$. By virtue of [14, Remark, p. 285], defining $M := \sup\{C_n((pr_i \circ \Psi_x)^2)(x) \mid i = 1, \dots, d, x \in Q_d\}$, we have that there exists a constant C such that

$$\left\|C_n(f) - f\right\|_p \leq C\tau\big(f;\ \sqrt[2d]{M}\big)_p,$$

provided $M \leq 1$.

Therefore, in order to obtain the desired result it is enough to estimate M. Since for every $n \geq 1$, $i = 1, \dots, d$ and $x \in Q_d$, we have $(pr_i \circ \Psi_x)^2 = pr_i^2 - 2x_i pr_i + x_i\mathbf{1}$, then

$$C_n\big((pr_i \circ \Psi_x)^2\big)(x) = C_n(pr_i^2)(x) - 2x_iC_n(pr_i)(x) + x_i^2C_n(\mathbf{1})(x).$$

One has $C_n(\mathbf{1})(x) = 1$ and, from Lemma 4.1,

$$C_n(pr_i)(x) = \frac{a}{2(n+a)} + \frac{n}{n+a}x_i.$$

Moreover, since $B_n(pr_i^2) = \frac{1}{n}x_i + \frac{n-1}{n}x_i^2$ (see (2.9) and (3.10)), we have

$$C_n(pr_i^2)(x) = \frac{a^2}{3(n+a)^2} + \frac{n(a+1)}{(n+a)^2}x_i + \frac{n(n-1)}{(n+a)^2}x_i^2.$$

Then

$$\begin{aligned} C_n\big((pr_i \circ \Psi_x)^2\big)(x) &= \frac{a^2-n}{(n+a)^2}x_i^2 + \frac{n-a^2}{(n+a)^2}x_i + \frac{a^2}{3(n+a)^2} \\ &= \frac{n-a^2}{(n+a)^2}x_i(1-x_i) + \frac{a^2}{3(n+a)^2} \\ &\leq \frac{n-a^2}{4(n+a)^2} + \frac{a^2}{3(n+a)^2} = \frac{3n+a^2}{12(n+a)^2}. \end{aligned}$$

Therefore, $M \leq \frac{3n+a^2}{12(n+a)^2} \leq 1$ and the result follows. $\square$

Now we present some estimates of the approximation error $\|C_n(f) - f\|_p$ by applying the results contained in [7]. To this end, for every $n \geq 1$ and $p \in [1, +\infty[$, we have to estimate the quantity $\lambda_{n,p}$ defined by

$$\lambda_{n,p} := \max_{0 \leq i \leq d+1} \left\|C_n(\varphi_i) - \varphi_i\right\|_p, \tag{5.6}$$

where the functions φ_i are defined by (4.11). Note that $\|C_n(\mathbf{1})-\mathbf{1}\|_p=0$. For every $i=1,\dots,d$, by virtue of Lemma 4.1 we get

$$C_n(pr_i)(x)-pr_i(x)=\frac{a}{n+a}\Big(\frac{1}{2}-x_i\Big);$$

therefore,

$$\begin{aligned}\big\|C_n(pr_i)-pr_i\big\|_p&=\frac{a}{n+a}\Big(\int_{Q_d}\Big|\frac{1}{2}-x_i\Big|^p\,dx\Big)^{1/p}\\&=\frac{a}{n+a}\Big(\int_0^1\Big|\frac{1}{2}-x_i\Big|^p\,dx_i\Big)^{1/p}\\&=\frac{a}{2(n+a)(p+1)^{1/p}}\le\frac{a}{4(n+a)}.\end{aligned}$$

Moreover,

$$\begin{aligned}&\big\|C_n(\varphi_{d+1})-\varphi_{d+1}\big\|_p\\&\quad=\frac{1}{(n+a)^2}\Big(\int_{Q_d}\Big|\frac{a^2d}{3}+n(a+1)\sum_{i=1}^d x_i-(n+2na+a^2)\sum_{i=1}^d x_i^2\Big|^p\,dx\Big)^{1/p}\\&\quad\le\frac{1}{(n+a)^2}\Big(\int_{Q_d}\Big(\frac{a^2d}{3}+n(a+1)d+(n+2na+a^2)d\Big)^p\,dx\Big)^{1/p}\\&\quad\le\frac{d}{(n+a)^2}\Big(\frac{a^2}{3}+n(a+1)+(n+2na+a^2)\Big)\\&\quad\le\frac{d}{n+a}\Big(\frac{a^2}{3}+(a+1)+(a+1)^2\Big)\le\frac{3d(a+1)^2}{n+a}.\end{aligned}$$

Hence,

$$\lambda_{n,p}\le\frac{3d(a+1)^2}{n+a}. \tag{5.7}$$

Consider the Sobolev space $W^2_\infty(K)$ of all functions $f\in L^\infty(K)$ such that, for every $|k|\le 2$, D^kf exists (in the Sobolev sense) and $D^kf\in L^\infty(K)$, endowed with the norm $\|f\|_{2,\infty}:=\max_{|k|\le 2}\|D^kf\|_{L^\infty(K)}$.

Proposition 5.3. *If $f\in W^2_\infty(Q_d)$, then, for every $1\le p<+\infty$ and $n\ge 1$,*

$$\big\|C_n(f)-f(x)\big\|_p\le C\|f\|_{2,\infty}\lambda_{n,p}\le\tilde{C}\|f\|_{2,\infty}\frac{1}{n+a}, \tag{5.8}$$

where the constants C and $\tilde{C}$ do not depend on f. Furthermore, if $f\in L^1(Q_d)$, then, for every $n\ge 1$,

$$\begin{aligned}&\big\|C_n(f)-f(x)\big\|_1\\&\quad\le C\big(\lambda_{n,1}\|f\|_1+\omega_{d+2,1}(f,\lambda_{n,1}^{1/(d+2)})\big)\\&\quad\le C\Big(\frac{3d(a+1)^2}{n+a}\|f\|_1+\omega_{d+2,1}\Big(f,\Big(\frac{3d(a+1)^2}{n+a}\Big)^{1/(d+2)}\Big)\Big),\end{aligned} \tag{5.9}$$

with C not depending on f.

Proof. Keeping (5.6) and (5.7) in mind, estimate (5.8) follows from Theorem 1 in [7]; moreover, formula (5.9) is a consequence of [7, Theorem 2] and (5.7). □

To obtain a result similar to Theorem 5.1 for the d-dimensional simplex K_d, $d \geq 1$, we will adapt the proof for Q_d by making the necessary modifications. As usual, for every $n \geq 1$ and $h = (h_1, \ldots, h_d) \in \{0, \ldots, n\}^d$, we set $|h| = h_1 + \cdots + h_d$. For every $n \geq 1$, $h = (h_1, \ldots, h_d) \in \{0, \ldots, n\}^d$, $|h| \leq n$ and $x = (x_1, \ldots, x_d) \in K_d$, we set

$$P^*_{n,h}(x) := \frac{n!}{h_1! \cdots h_d!(n - h_1 - \cdots - h_d)!} x_1^{h_1} \cdots x_d^{h_d} \Big(1 - \sum_{i=1}^{d} x_i\Big)^{n - \sum_{i=1}^{d} h_i}.$$

Then

$$P^*_{n,h} \geq 0 \qquad \text{and} \qquad \sum_{h \in \{0,\ldots,n\}^d, |h| \leq n} P^*_{n,h} = 1 \quad \text{on } K_d,$$

and, on account of [9, Section 976],

$$\int_{K_d} P^*_{n,h}(x)\, dx = \frac{n!}{(n+d)!} = \frac{1}{(n+1)(n+2)\cdots(n+d)} \leq \frac{1}{(n+1)^d}.$$

Moreover, for any $a \geq 0$, we set

$$K_{n,h}(a) := \Big\{ (x_1, \ldots, x_d) \in \mathbf{R}^d \;\Big|\; \frac{h_i}{n+a} \leq x_i \text{ for each } i = 1, \ldots, d \text{ and } \sum_{i=1}^{d} x_i \leq \frac{1}{n+a}\Big(a + \sum_{i=1}^{d} h_i\Big) \Big\};$$

in particular,

$$K_{n,h}(a) \subset K_d \qquad \text{and} \qquad K_d = \bigcup_{\substack{h \in \{0,\ldots,n\}^d \\ |h| \leq n}} K_{n,h}(a).$$

Now consider the operators C_n, $n \geq 1$, defined by (3.13), where each μ_n is the normalized Borel–Lebesgue measure $d!\lambda_d$ on K_d. Also, in this case the operators C_n are well defined on $L^1(K_d)$ and the operator (3.12) satisfies (2.4). Moreover, for every $f \in L^1(K_d)$ and $x \in K_d$,

$$C_n(f)(x) = \sum_{\substack{h \in \{0,\ldots,n\}^d \\ |h| \leq n}} d! P^*_{n,h}(x) \int_{K_d} f\Big(\frac{h + au}{n+a}\Big)\, du$$

and, if $a > 0$, then

$$C_n(f)(x) = \sum_{\substack{h \in \{0,\ldots,n\}^d \\ |h| \leq n}} P^*_{n,h}(x) d! \Big(\frac{n+a}{a}\Big)^d \int_{K_{n,h}(a)} f(v)\, dv.$$

Therefore, if $1 \leq p < +\infty$ and $a > 0$, then

$$\int_{K_d} \big|C_n(f)(x)\big|^p \, dx \leq \sum_{\substack{h\in\{0,\dots,n\}^d \\ |h|\leq n}} \int_{K_d} P^*_{n,h}(x)\, dx\, d!\Big(\frac{n+a}{a}\Big)^d \int_{K_{n,h}(a)} \big|f(v)\big|^p \, dv$$
$$\leq d!\Big(\frac{n+a}{a(n+1)}\Big)^d \int_{K_d} \big|f(v)\big|^p \, dv,$$

and hence, setting $\overline{M} := \sup_{n\geq 1} d!(\frac{n+a}{a(n+1)})^d$, we get

$$\big\|C_n(f)\big\|_p \leq \overline{M}^{1/p} \|f\|_p.$$

By the same reasoning as in the proof of Theorem 5.1, we infer the following.

Theorem 5.4. *If $a > 0$, then, for every $f \in L^p(K_d)$, $1 \leq p < +\infty$, we have $\lim_{n\to\infty} C_n(f) = f$ in $L^p(K_d)$.*

For Theorem 5.4, we will also furnish some estimates of the rate of convergence by applying the results contained in [7]. In order to do this, we first evaluate $\lambda_{n,p}$ for each $n \geq 1$ and $1 \leq p < +\infty$ (see (5.6)). First of all (see [9, Section 976]), we recall that, for every $k \in \mathbf{N}$,

$$\int_{K_d} x_i^k \, dx = \frac{\Gamma(k+1)}{\Gamma(k+d)} \frac{1}{k+d} = \frac{1}{(k+d)(k+d-1)\cdots(k+1)}. \tag{5.10}$$

As in the case of a hypercube, $\|C_n(\mathbf{1}) - \mathbf{1}\|_p = 0$. For every $i = 1, \dots, d$, by virtue of Lemma 4.1 and (5.10) for $k = 0, 1$, we get that, for every $x = (x_1, \dots, x_d) \in K_d$,

$$\big|C_n(pr_i)(x) - pr_i(x)\big| = \frac{a}{n+a}\Big|\frac{1}{d+1} - x_i\Big| \leq \frac{a}{n+a},$$

and therefore,

$$\big\|C_n(pr_i) - pr_i\big\|_p \leq \frac{a}{(n+a)(d!)^{1/p}}.$$

Moreover, from (2.9) and (3.12) it follows that $B_n(pr_i^2) = \frac{n-1}{n} pr_i^2 + \frac{1}{n} pr_i$.

Thus, from Lemma 4.1 and (5.10), for $k = 2$, for every $x = (x_1, \dots, x_d) \in K_d$, we get

$$C_n(pr_i^2)(x) = \frac{a^2 d!}{(n+a)^2} \frac{2}{(d+2)!} + \frac{2nad!}{(n+a)^2} \frac{1}{(d+1)!} x_i$$
$$+ \frac{n^2}{(n+a)^2}\Big[\frac{1}{n} x_i + \frac{n-1}{n} x_i^2\Big]$$
$$= \frac{2a^2}{(n+a)^2(d+2)(d+1)}$$
$$+ \frac{n(2a+d+1)}{(n+a)^2(d+1)} x_i + \frac{n(n-1)}{(n+a)^2} x_i^2$$

and hence (see (4.11))

$$
\begin{aligned}
&\big|C_n(\varphi_{d+1})(x) - \varphi_{d+1}(x)\big| \\
&\quad = \frac{1}{(n+a)^2} \times \Big|\frac{2a^2 d}{(d+2)(d+1)} + \frac{n(2a+d+1)}{(d+1)} \sum_{i=1}^{d} x_i - (n + 2na + a^2) \sum_{i=1}^{d} x_i^2\Big| \\
&\quad \le \frac{1}{(n+a)^2}\Big(\frac{2a^2 d}{(d+2)(d+1)} + \frac{n(2a+d+1)}{(d+1)} + n + 2na + a^2\Big).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\big\|C_n(\varphi_{d+1}) - \varphi_{d+1}\big\|_p \\
&\quad \le \frac{1}{(n+a)^2 (d!)^{1/p}}\Big(\frac{2a^2 d}{(d+2)(d+1)} + \frac{n(2a+d+1)}{(d+1)} + n + 2na + a^2\Big) \\
&\quad \le \frac{2a^2 + 2d(a+1) + (d+1)(a+1)^2}{(n+a)(d!)^{1/p}(d+1)} \le \frac{3d+3}{(d!)^{1/p}(d+1)} \frac{(a+1)^2}{n+a}.
\end{aligned}
$$

Accordingly,

$$
\lambda_{n,p} \le \frac{3(a+1)^2}{(d!)^{1/p}(n+a)}.
$$

As a consequence, we get the following result.

Proposition 5.5. *If $f \in W^2_\infty(K_d)$, then, for every $n \ge 1$,*

$$
\big\|C_n(f) - f\big\|_p \le C\|f\|_{2,\infty}\lambda_{n,p} \le \tilde{C}\|f\|_{2,\infty}\frac{1}{n+a},
$$

where the constants C and $\tilde{C}$ do not depend on f. Moreover, if $f \in L^1(K_d)$, then, for every $1 \le p < +\infty$ and $n \ge 1$,

$$
\begin{aligned}
\big\|C_n(f) - f(x)\big\|_1 &\le C\big(\lambda_{n,1}\|f\|_1 + \omega_{d+2,1}(f, \lambda_{n,1}^{1/(d+2)})\big) \\
&\le C\Big(\frac{3(a+1)^2}{d!(n+a)}\|f\|_1 + \omega_{d+2,1}\Big(f, \Big(\frac{3(a+1)^2}{d!(n+a)}\Big)^{1/(d+2)}\Big)\Big)
\end{aligned}
$$

with C not depending on f.

6. Preservation properties

In this section, we will investigate some shape- and regularity-preserving properties of the C_n's by proving that, under suitable assumptions, they preserve convexity and Lipschitz-continuity. (Relation (3.3) yields that some of the preservation properties of the B_n's (see (2.3)) are naturally shared by the C_n's.)

First of all, we investigate the behavior of the sequence $(C_n)_{n\ge 1}$ on Lipschitz-continuous functions and, to this end, we recall some basic definitions. Here we assume that K is metrizable, and we denote by ρ the metric on K which induces its topology. The ρ-modulus of continuity of a given $f \in C(K)$ with respect to $\delta > 0$ is then defined by

$$
\omega_\rho(f,\delta) := \sup\big\{\big|f(x) - f(y)\big| \;\big|\; x, y \in K, \rho(x,y) \le \delta\big\}.
$$

Furthermore, for any $M \geq 0$ and $0 < \alpha \leq 1$, we denote by

$$\mathrm{Lip}(M,\alpha) := \big\{ f \in C(K) \;\big|\; \big|f(x) - f(y)\big| \leq M\rho(x,y)^\alpha \text{ for every } x, y \in K \big\}$$

the space of all Hölder continuous functions with exponent α and constant M. In particular, $\mathrm{Lip}(M,1)$ is the space of all Lipschitz continuous functions with constant M. Assume that

$$\omega_\rho(f, t\delta) \leq (1+t)\omega_\rho(f,\delta)$$

for every $f \in C(K)$, $\delta, t > 0$. From now on, we suppose that there exists $c \geq 1$ such that

$$T\big(\mathrm{Lip}(1,1)\big) \subset \mathrm{Lip}(c,1), \tag{6.1}$$

or, equivalently,

$$T\big(\mathrm{Lip}(M,1)\big) \subset \mathrm{Lip}(cM,1),$$

for every $M \geq 0$.

For instance, the Markov operators T_1, S_d, and T_d of parts 1, 2, and 3 of Examples 3.1 satisfy condition (6.1) with $c = 1$, by considering on $[0,1]$ the usual metric and on Q_d and K_d the l_1-metric, that is, the metric generated by the l_1-norm (see [4, p. 124]). Under condition (6.1) it has been shown that (see [4, Theorem 3.3.1])

$$B_n(f) \in \mathrm{Lip}(cM,1) \tag{6.2}$$

for every $f \in \mathrm{Lip}(M,1)$ and $n \geq 1$. Furthermore (see [4, Corollary 3.3.2]), for every $f \in C(K)$, $\delta > 0$ and $n \geq 1$,

$$\omega_\rho\big(B_n(f),\delta\big) \leq (1+c)\omega_\rho(f,\delta) \tag{6.3}$$

and

$$B_n\big(\mathrm{Lip}(M,\alpha)\big) \subset \mathrm{Lip}(c^\alpha M,\alpha) \tag{6.4}$$

for every $M > 0$ and $\alpha \in]0,1]$. Then, taking (6.2) into account, after noting that $I_n(f) \in \mathrm{Lip}(M,\alpha)$ whenever $f \in \mathrm{Lip}(M,\alpha)$ (see (3.2)), we easily deduce the following proposition.

Proposition 6.1. *Assume that condition (6.1) is satisfied. Then, for every $f \in \mathrm{Lip}(M,1)$ and $n \geq 1$, $C_n(f) \in \mathrm{Lip}(cM,1)$.*

Moreover, we see how (6.3) and (6.4) yield the following result, since $\omega_\rho(I_n(f),\delta) \leq \omega_\rho(f,\delta)$.

Proposition 6.2. *Assume that condition (6.1) is satisfied. Then, for every $f \in C(K)$, $\delta > 0$ and $n \geq 1$,*

$$\omega_\rho\big(C_n(f),\delta\big) \leq (1+c)\omega_\rho(f,\delta)$$

and

$$C_n\big(\mathrm{Lip}(M,\alpha)\big) \subset \mathrm{Lip}(c^\alpha M,\alpha)$$

for every $M > 0$ and $0 < \alpha \leq 1$. In particular, if $T(\mathrm{Lip}(1,1)) \subset \mathrm{Lip}(1,1)$, then

$$\omega_\rho\big(C_n(f),\delta\big) \leq 2\omega_\rho(f,\delta)$$

and

$$C_n\big(\mathrm{Lip}(M,\alpha)\big) \subset \mathrm{Lip}(M,\alpha)$$

for every $M > 0$ *and* $0 < \alpha \le 1$.

Below we state some further properties of the operators C_n for special functions $f \in C(K)$. To this end, we need some additional concepts. We first recall that, if T is an arbitrary Markov operator on $C(K)$, then a function $f \in C(K)$ is said to be T-*convex* if

$$f_{z,\alpha} \le T(f_{z,\alpha}) \quad \text{for every } z \in K \text{ and } \alpha \in [0,1],$$

where $f_{z,\alpha}$ is defined by $f_{z,\alpha}(x) := f(\alpha x + (1-\alpha)z)$ $(x \in K)$. If $K = [0,1]$ and T_1 denotes the operator (3.6), then a function $f \in C([0,1])$ is T_1-convex if and only if it is convex. For $K = Q_d$ and $T = S_d$ (see (3.10)), a function $f \in C(Q_d)$ is S_d-convex if and only if f is convex with respect to each variable. Finally, for $K = K_d$ and $T = T_d$ (see (3.12)), a function $f \in C(K_d)$ is T_d-convex if and only if it is axially convex, that is, it is convex on each segment parallel to a segment joining two extreme points of K_d (see [4, Section 3.5] for more details).

In general, each convex function $f \in C(K)$ is T-convex. Moreover, T-axially convex functions are T-convex as well (see [4, Definition 3.5.1 and remarks on p. 148]). In [4, Theorem 3.5.2] it has been shown that, if $(B_n)_{n\ge1}$ is the sequence of Bernstein–Schnabl operators associated with T and if T satisfies hypothesis (2.4) (or (2.5)), then

$$f \le B_n(f) \le T(f), \quad n \ge 1$$

whenever $f \in C(K)$ is T-convex. As a consequence, we have the following result.

Proposition 6.3. *Under hypothesis (2.4) (or (2.5)), if* $f \in C(K)$ *is* T*-convex, then, for any* $n \ge 1$,

$$C_n(f) \le C_n\big(T(f)\big).$$

In particular, if f *is* T*-convex and each* $I_n(f)$ *is* T*-convex, then, for every* $n \ge 1$,

$$I_n(f) \le C_n(f) \le T\big(I_n(f)\big).$$

Apart from the case of the interval $[0,1]$ and the classical Bernstein operators, in general, Bernstein–Schnabl operators do not preserve convexity. A simple counterexample is given by the function $f := |pr_1 - pr_2|$ defined on the 2-dimensional simplex K_2 (see [15, p. 468]). Hence, in general, the C_n's do not preserve convexity either. But it is possible to determine sufficient conditions in order that the B_n's (and, hence, the C_n's) preserve convexity.

For a given $f \in C(K)$, we set

$$\tilde{f}(s,t) := f\Big(\frac{s+t}{2}\Big) \quad (s,t \in K)$$

and

$$\begin{aligned}\Delta(\tilde{f};x,y) := &\iint_{K^2} \tilde{f}(s,t)\, d\tilde{\mu}_x^T(s)\, d\tilde{\mu}_x^T(t)\\ &+ \iint_{K^2} \tilde{f}(s,t)\, d\tilde{\mu}_y^T(s)\, d\tilde{\mu}_y^T(t)\end{aligned}$$

$$
\begin{aligned}
&- 2 \iint_{K^2} \tilde{f}(s,t)\, d\tilde{\mu}_x^T(s)\, d\tilde{\mu}_y^T(t) \\
&= B_2(f)(x) + B_2(f)(y) - 2 \iint_{K^2} f\Big(\frac{s+t}{2}\Big)\, d\tilde{\mu}_x^T(s)\, d\tilde{\mu}_y^T(t)
\end{aligned}
$$

for every $x, y \in K$.

Theorem 6.4. *Suppose that T satisfies the following assumptions:*

(c_1) *T maps continuous convex functions into (continuous) convex functions,*

(c_2) *$\Delta(\tilde{f}; x, y) \geq 0$ for every convex function $f \in C(K)$ and for every $x, y \in K$.*

Then each C_n maps continuous convex functions into (continuous) convex functions.

Proof. According to [4, Theorem 3.4.3], under assumptions (c_1) and (c_2), each Bernstein–Schnabl operator B_n maps continuous convex functions into (continuous) convex functions. Therefore, the result follows from (3.3) taking into account that each $I_n(f)$ is convex, provided that $f \in C(K)$ is convex. □

Remark 6.5. In [4, Remark 3.4.4 and Examples 3.4.5–3.4.11] there are several examples of settings where conditions (c_1) and (c_2) are satisfied. This is the case, in particular, when $K = [0, 1]$ and $T = T_1$ (see (3.6)). Therefore, all the operators defined by (3.7), (3.8), and (3.9) preserve the convexity.

Finally, we point out that if $K = K_d$, $d \geq 1$, then the Bernstein operators on $C(K_d)$, that is, the Bernstein–Schnabl operators associated with the Markov operator (3.12), preserve the axial convexity (see [2, Theorem 6.3.2], [4, Theorem 3.5.9]). On the other hand, if $f \in C(K_d)$ is axially convex, then $I_n(f)$ is axially convex too for every $n \geq 1$. Therefore, on account of (3.3), we conclude with the following.

Corollary 6.6. *Considering the canonical simplex K_d of $\mathbf{R}^d$, $d \geq 1$, then the operators C_n defined by (3.13) map continuous axially convex functions on K_d into (continuous) axially convex functions.*

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[1]Dipartimento di Matematica, Università degli Studi di Bari Aldo Moro, Campus Universitario, Via E. Orabona n. 4, 70125-Bari, Italy.

E-mail address: francesco.altomare@uniba.it; mirella.cappellettimontano@uniba.it

[2]Dipartimento di Matematica, Informatica ed Economia, Università degli Studi della Basilicata, Viale Dell' Ateneo Lucano n. 10, Campus di Macchia Romana 85100-Potenza, Italy.

E-mail address: vita.leonessa@unibas.it

[3]Department of Mathematics, Technical University of Cluj-Napoca, Str. Memorandumului 28 RO-400114 Cluj-Napoca, Romania.

E-mail address: Ioan.Rasa@math.utcluj.ro