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TWO-SIDED AND ONE-SIDED INVERTIBILITY OF WIENER-TYPE FUNCTIONAL OPERATORS WITH A SHIFT AND SLOWLY OSCILLATING DATA

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Abstract. Let α be an orientation-preserving homeomorphism of $[0,\infty]$ onto itself with only two fixed points at 0 and ∞, whose restriction to $\mathbb{R}_+ = (0,\infty)$ is a diffeomorphism, and let U_α be the isometric shift operator acting on the Lebesgue space $L^p(\mathbb{R}_+)$ with $p \in [1,\infty]$ by the rule $U_\alpha f = (\alpha')^{1/p}(f \circ \alpha)$. We establish criteria of the two-sided and one-sided invertibility of functional operators of the form

$$A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \quad \text{where } \|A\|_W = \sum_{k\in\mathbb{Z}} \|a_k\|_{L^\infty(\mathbb{R}_+)} < \infty,$$

on the spaces $L^p(\mathbb{R}_+)$ under the assumptions that the functions $\log \alpha'$ and a_k for all $k \in \mathbb{Z}$ are bounded and continuous on $\mathbb{R}_+$ and may have slowly oscillating discontinuities at 0 and ∞. The unital Banach algebra $\mathfrak{A}_W$ of such operators is inverse-closed: if $A \in \mathfrak{A}_W$ is invertible on $L^p(\mathbb{R}_+)$ for $p \in [1,\infty]$, then $A^{-1} \in \mathfrak{A}_W$. Obtained criteria are of two types: in terms of the two-sided or one-sided invertibility of so-called *discrete operators* on the spaces l^p and in terms of conditions related to the fixed points of α and the orbits $\{\alpha^n(t) : n \in \mathbb{Z}\}$ of points $t \in \mathbb{R}_+$.

1. Introduction

Let $\mathcal{B}(X,Y)$ be the Banach space of all bounded linear operators acting from a Banach space X to a Banach space Y. We abbreviate $\mathcal{B}(X,X)$ to $\mathcal{B}(X)$. An

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operator $A \in \mathcal{B}(X, Y)$ is called *left invertible* (resp., right invertible) if there exists an operator $B \in \mathcal{B}(Y, X)$ such that $BA = I_X$ (resp., $AB = I_Y$) where $I_X \in \mathcal{B}(X)$ and $I_Y \in \mathcal{B}(Y)$ are the identity operators on X and Y, respectively. The operator B is called a *left* (resp., right) *inverse* of A. An operator $A \in \mathcal{B}(X, Y)$ is considered to be (two-sided) invertible if it is left invertible and right invertible simultaneously. We say that A is strictly *left* (resp., right) *invertible* if it is left (resp., right) invertible, but not invertible. If the operator A is invertible only from one side, then the corresponding inverse is not uniquely defined (see [9, Section 2.5] for further properties of one-sided invertible operators).

Let $C_b(\mathbb{R}_+)$ denote the C^*-algebra of all bounded continuous functions on $\mathbb{R}_+ := (0, +\infty)$. Following [36, p. 820] and [29], a function $f \in C_b(\mathbb{R}_+)$ is called *slowly oscillating* (at 0 and ∞) if for each (equivalently, for some) $\lambda \in (0, 1)$,

$$\lim_{r \to s} \operatorname{osc}\big(f, [\lambda r, r]\big) = 0, \quad s \in \{0, \infty\},$$

where

$$\operatorname{osc}\big(f, [\lambda r, r]\big) := \sup\big\{\big|f(t) - f(\tau)\big| : t, \tau \in [\lambda r, r]\big\}$$

is the oscillation of f on the segment $[\lambda r, r] \subset \mathbb{R}_+$. Obviously, the set $\mathrm{SO}(\mathbb{R}_+)$ of all slowly oscillating (at 0 and ∞) functions in $C_b(\mathbb{R}_+)$ is a unital commutative C^*-algebra. This algebra properly contains $C(\overline{\mathbb{R}}_+)$, the C^*-algebra of all continuous functions on $\overline{\mathbb{R}}_+ := [0, +\infty]$.

Let α be an orientation-preserving homeomorphism of $\overline{\mathbb{R}}_+$ onto itself, which has only two fixed points at 0 and ∞. Thus, $\alpha(0) = 0$, $\alpha(\infty) = \infty$, but $\alpha(t) \neq t$ for all $t \in \mathbb{R}_+$. The function α is referred to as a *shift*. Since the function α strictly monotonically increases on $\overline{\mathbb{R}}_+$, its derivative exists and is positive almost everywhere on $\mathbb{R}_+$. If $\log \alpha' \in L^\infty(\mathbb{R}_+)$, then the weighted shift operator U_α defined by

$$U_\alpha \varphi = (\alpha')^{1/p}(\varphi \circ \alpha)$$

is an isometry on the Lebesgue space $L^p(\mathbb{R}_+)$ for every $p \in [1, \infty]$, and therefore the operator U_α is invertible on this space.

We say that the considered homeomorphism $\alpha : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ is a *slowly oscillating shift* if its restriction to $\mathbb{R}_+$ is a diffeomorphism and $\log \alpha' \in \mathrm{SO}(\mathbb{R}_+)$. The set of all slowly oscillating shifts is denoted by $\mathrm{SOS}(\mathbb{R}_+)$ (see [13]).

Given $p \in [1, \infty]$, let $\mathfrak{A}_{p,\mathrm{SO}}$ stand for the unital Banach subalgebra of $\mathcal{B}(L^p(\mathbb{R}_+))$ which is generated by all operators of multiplication by functions in $\mathrm{SO}(\mathbb{R}_+)$ and by the shift operators U_α and U_α^{-1}, where $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$. Operators $A \in \mathfrak{A}_{p,\mathrm{SO}}$ are called *functional operators*.

Let $\mathfrak{A}_W := W_{p,\mathrm{SO}}$ be the unital Banach algebra of all operators of the form

$$A = \sum_{k \in \mathbb{Z}} a_k U_\alpha^k \in \mathcal{B}\big(L^p(\mathbb{R}_+)\big), \tag{1.1}$$

where $a_k \in \mathrm{SO}(\mathbb{R}_+)$ for all $k \in \mathbb{Z}$, $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$, and the norm is given by

$$\|A\|_W := \sum_{k \in \mathbb{Z}} \|a_k\|_{C_b(\mathbb{R}_+)} < \infty. \tag{1.2}$$

By analogy with the Wiener algebra of absolutely convergent Fourier series, we call $\mathfrak{A}_W$ the *Wiener algebra*. Obviously, $\mathfrak{A}_W \subset \mathfrak{A}_{p,\mathrm{SO}}$ for all $p \in [1,\infty]$.

Functional operators of the form $A = \sum_{g\in G} a_g U_g$ with different classes of coefficients a_g and shift operators $U_g : f \mapsto f \circ g$ associated with groups $G = \{g\}$ of diffeomorphisms play an important role in studying the solvability of functional and integro-functional equations (see [1], [10], [19]), functional-differential and pseudodifference equations (see [2], [20], [33]), and nonlocal boundary value problems (see [22]–[24] and the references therein). The theory of functional operators is closely related to the theory of dynamical systems and the theory of representations of C^*-algebras extended by automorphisms, endomorphisms, and partial isometries (see, e.g., [3], [6], [21]).

The invertibility criterion for the binomial functional operators $A = aI - bU_\alpha$ with data $a, b, \log \alpha' \in C(\overline{\mathbb{R}}_+)$ on the Lebesgue spaces $L^p(\mathbb{R}_+)$, where $p \in (1,\infty)$, easily follows from that established by Kravchenko in [18] (also see [19]). Criteria of the one-sided invertibility of such operators on the spaces $L^p(\mathbb{R}_+)$ are similar to those obtained in [25] (also see [11] for generalizations of these results to the case of reflexive rearrangement-invariant spaces). On the other hand, replacing $(0,1)$ by $\mathbb{R}_+$ in [12], we get a criterion for the two-sided invertibility of binomial functional operators with coefficients in $L^\infty(\mathbb{R}_+)$ and $\log \alpha' \in L^\infty(\mathbb{R}_+)$ on every space $L^p(\mathbb{R}_+)$ with $p \in [1,\infty]$.

The study of the two-sided and one-sided invertibility of binomial functional operators with slowly oscillating coefficients and a slowly oscillating shift was culminated in [15], where the criteria of the two-sided and one-sided invertibility of the operators $A = aI - bU_\alpha$ with data $a, b, \log \alpha' \in \mathrm{SO}(\mathbb{R}_+)$ were established on the Lebesgue spaces $L^p(\mathbb{R}_+)$ with $p \in (1,\infty)$. The slow oscillation of data functions in comparison with data functions in $L^\infty(\mathbb{R}_+)$ allows one to obtain sufficiently effective criteria of the two-sided and one-sided invertibility for considered operators.

For $p \in (1,\infty)$, two-sided invertibility criteria for any operator in the Banach algebra $\mathfrak{A}_{p,C(\overline{\mathbb{R}}_+)} \subset \mathcal{B}(L^p(\mathbb{R}_+))$ generated by the operators $A_n = \sum_{|k|\le n} a_k U_\alpha^k$ $(n \in \mathbb{N})$ with coefficients $a_k \in C(\overline{\mathbb{R}}_+)$ and $\log \alpha' \in C(\overline{\mathbb{R}}_+)$ easily follow from the criteria obtained in [17] (see also [19], [26], [27]).

We also mention the article [12], where, for $p \in [1,\infty]$, criteria for the two-sided invertibility of operators in the Banach subalgebra $\mathfrak{A}_{p,L^\infty(0,1)}$ of $\mathcal{B}(L^p(0,1))$ generated by the operators $A_n = \sum_{|k|\le n} a_k U_\alpha^k$ $(n \in \mathbb{N})$ with coefficients $a_k \in L^\infty(0,1)$ and $\log \alpha' \in L^\infty(0,1)$ were established in terms of discrete operators associated with the orbits $\mathcal{O}_t$ of points $t \in (0,1)$ under the action of the cyclic group G generated by the shift α.

The present article is devoted to studying the two-sided invertibility of operators $A \in \mathfrak{A}_{p,\mathrm{SO}}$ and the one-sided invertibility of operators $A \in \mathfrak{A}_W$ on the Lebesgue spaces $L^p(\mathbb{R}_+)$. Thus, we generalize results of [15] to the Wiener-type series of functional operators, as well as uniform limits of functional polynomials. The main difficulty in studying such operators is related to oscillations of coefficients and the shift derivative α' in neighborhoods of 0 and ∞.

This article has the following organization. In Section 2, we collect the most important properties of slowly oscillating functions and slowly oscillating shifts, including the description of the maximal ideal space of $\mathrm{SO}(\mathbb{R}_+)$. In Section 3, for every $p \in [1,\infty]$, we study relations of functional operators A acting on the space $L^p(\mathbb{R}_+)$ and discrete operators $\mathcal{A}(t)$ acting on the space l^p and associated with the orbits $\mathcal{O}_t$ of the points $t \in \mathbb{R}_+$ under the actions of the cyclic group G generated by the shift α, and we obtain two-sided invertibility criteria for functional operators belonging to the Banach subalgebra $\mathfrak{A}_{p,\mathrm{SO}}$ of $\mathcal{B}(L^p(\mathbb{R}_+))$ in terms of invertibility of discrete operators $\mathcal{A}(t)$ on l^p. We prove that in the case of slowly oscillating behavior of coefficients and the derivatives of shifts the $\mathcal{B}(l^p)$-valued function $t \mapsto \mathcal{A}(t)$ is bounded and continuous on $\mathbb{R}_+$, which allowed us to avoid the condition of uniform boundedness of the norms of inverse discrete operators (which arose, e.g., in [4, Theorem 8.4]) in the invertibility criterion for considered functional operators (see Theorem 3.4). Thus, by this theorem, the invertibility of any functional operator $A \in \mathfrak{A}_{p,\mathrm{SO}}$ for $p \in [1,\infty]$ is equivalent to the invertibility of all discrete operators $\mathcal{A}(t)$ for $t \in \mathbb{R}_+$. These results were generalized here to a bigger Banach subalgebra $\mathfrak{A}_{p,\mathfrak{S}}$ of $\mathcal{B}(L^p(\mathbb{R}_+))$ as well.

In Section 4, we study limit operators for the one-sided invertible functional operators $A \in \mathfrak{A}_W \subset \mathcal{B}(L^p(\mathbb{R}_+))$ for $p \in (1,\infty)$ and for their discrete analogues $\mathcal{A}(t) \in \mathcal{B}(l^p)$ $(t \in \mathbb{R}_+)$ (see [32] and [33] for the definition, properties, and applications of limit operators), and we prove according to [15, Theorem 3.1] that the limit operators for any left invertible operator $A \in \mathfrak{A}_W$ have trivial kernels and closed images. The Fredholmness and invertibility of discrete (band-dominated) operators with slowly oscillating or bounded coefficients on the spaces l^p with entries in a Banach space X in terms of limit operators were studied in [30], [31], [33, Chapter 2], and [37].

Section 5 is devoted to necessary conditions for the one-sided invertibility of operators $A \in \mathfrak{A}_W$ and $\mathcal{A}(t)$ for $t \in \mathbb{R}_+$, which are obtained on the basis of Section 4 and are related to the fixed points of the shift α. For any left or right invertible functional operator $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ with coefficients $a_k \in \mathrm{SO}(\mathbb{R}_+)$, we prove the two-sided invertibility of its limit operators $A_\xi = \sum_{k\in\mathbb{Z}} a_k(\xi) U_\alpha^k \in \mathfrak{A}_W$ associated with the points $\xi \in \Delta$, where

$$\Delta := M_0\big(\mathrm{SO}(\mathbb{R}_+)\big) \cup M_\infty\big(\mathrm{SO}(\mathbb{R}_+)\big) \tag{1.3}$$

and $M_s(\mathrm{SO}(\mathbb{R}_+))$ are fibers of the maximal ideal space of $\mathrm{SO}(\mathbb{R}_+)$ over points $s \in \overline{\mathbb{R}}_+$. This implies for operators A_ξ the invertibility of their Gelfand transforms given by

$$A_\xi(z) = \sum_{k\in\mathbb{Z}} a_k(\xi) z^k \quad \text{for all } z \in \mathbb{T} := \big\{\zeta \in \mathbb{C} : |\zeta| = 1\big\}. \tag{1.4}$$

In Section 6, applying the representation of functional operators $A \in \mathfrak{A}_W$ as a 3×3 operator matrix, we first prove the two-sided invertibility of the outermost blocks and then show the equivalence of the one-sided invertibility for the operator A and a modified central block of the mentioned operator matrix. The proof of the two-sided invertibility of the outermost blocks is based on the simultaneous factorization of absolutely convergent Fourier series (1.4) parameterized by the points $\xi \in M_s(\mathrm{SO}(\mathbb{R}_+))$ for each $s \in \{0, \infty\}$ and a subsequent operator factorization of the mentioned blocks (see key Theorem 6.1). This important result gives necessary conditions for the two-sided and one-sided invertibility of functional operators, which are related to points $t \in \mathbb{R}_+$. A discrete version of these results is also presented in this section.

Section 7 deals with studying relations of the one-sided invertibility of functional operators $A \in \mathfrak{A}_W$ and corresponding discrete operators. For the Wiener-type functional operators $A \in \mathfrak{A}_W = W_{p,\mathrm{SO}}$ and $p \in (1, \infty)$, we obtain here the important analogues of Theorem 3.4, which says that the one-sided invertibility of functional operators $A \in \mathfrak{A}_W$ on the space $L^p(\mathbb{R}_+)$ is equivalent to the corresponding one-sided invertibility of all discrete operators $\mathcal{A}(t)$ for $t \in \mathbb{R}_+$ on the space l^p (see Theorem 7.8). We also prove here the inverse closedness of the Banach algebra $\mathfrak{A}_W$ in the Banach algebra $\mathcal{B}(L^p(\mathbb{R}_+))$ for every $p \in [1, \infty]$ (see Theorem 7.4).

Finally, in Section 8 we establish a criterion for the two-sided invertibility of operators $A \in \mathfrak{A}_W$ on the spaces $L^p(\mathbb{R}_+)$ for $p \in [1, \infty]$ (see Theorem 8.1), which is more effective than that obtained in Section 3, and sufficiently effective criteria for the one-sided invertibility of operators $A \in \mathfrak{A}_W$ on the spaces $L^p(\mathbb{R}_+)$ for $p \in (1, \infty)$ (see Theorems 8.2 and 8.3). Theorems 8.1–8.3, along with Theorems 3.4, 7.4, and 7.8, are our main results here.

2. Slowly oscillating functions and shifts

2.1. **The maximal ideal space of** $\mathrm{SO}(\mathbb{R}_+)$**.** Let $M(\mathfrak{A})$ denote the maximal ideal space of a unital commutative Banach algebra $\mathfrak{A}$. Identifying the points $t \in \overline{\mathbb{R}}_+$ with the evaluation functionals $t(f) = f(t)$ for $f \in C(\overline{\mathbb{R}}_+)$, we get $M(C(\overline{\mathbb{R}}_+)) = \overline{\mathbb{R}}_+$. Consider the fibers

$$M_s\big(\mathrm{SO}(\mathbb{R}_+)\big) := \big\{\xi \in M\big(\mathrm{SO}(\mathbb{R}_+)\big) : \xi|_{C(\overline{\mathbb{R}}_+)} = s\big\}$$

of the maximal ideal space $M(\mathrm{SO}(\mathbb{R}_+))$ over points $s \in \{0, \infty\}$. As $M_t(\mathrm{SO}(\mathbb{R}_+)) = \{t\}$ for all $t \in \mathbb{R}_+$, we get $M(\mathrm{SO}(\mathbb{R}_+)) = \Delta \cup \mathbb{R}_+$, where Δ is given by (1.3). In what follows, we write $a(\xi) := \xi(a)$ for every $a \in \mathrm{SO}(\mathbb{R}_+)$ and every $\xi \in \Delta$.

By analogy with [5, Propositions 4.1, 4.2, and Corollary 4.3], we have the following two assertions.

Lemma 2.1 ([16, Proposition 2.1])**.** *The set* $\Delta = M_0(\mathrm{SO}(\mathbb{R}_+)) \cup M_\infty(\mathrm{SO}(\mathbb{R}_+))$ *coincides with the set* $\mathrm{clos}_{\mathrm{SO}^*} \mathbb{R}_+ \setminus \mathbb{R}_+$*, where* $\mathrm{clos}_{\mathrm{SO}^*} \mathbb{R}_+$ *is the weak-star closure of* $\mathbb{R}_+$ *in the dual space of* $\mathrm{SO}(\mathbb{R}_+)$*.*

Lemma 2.2 ([16, Proposition 2.2])**.** *Suppose that* $\{a_k\}_{k\in\mathbb{N}}$ *is a countable subset of the space* $\mathrm{SO}(\mathbb{R}_+)$ *and* $s \in \{0, \infty\}$*. For each* $\xi \in M_s(\mathrm{SO}(\mathbb{R}_+))$ *there exists a sequence* $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ *such that* $t_n \to s$ *as* $n \to \infty$ *and*

$$\xi(a_k) = a_k(\xi) = \lim_{n\to\infty} a_k(t_n) \quad \text{for all } k \in \mathbb{N}. \tag{2.1}$$

Conversely, if $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ *is a sequence such that* $t_n \to s$ *as* $n \to \infty$ *and the limits* $\lim_{n\to\infty} a_k(t_n)$ *exist for all* $k \in \mathbb{N}$*, then there exists a functional* $\xi \in M_s(\mathrm{SO}(\mathbb{R}_+))$ *such that (2.1) holds.*

We also need below the following important fact.

Lemma 2.3 ([14, Lemma 2.2])**.** *The fibers* $M_0(\mathrm{SO}(\mathbb{R}_+))$ *and* $M_\infty(\mathrm{SO}(\mathbb{R}_+))$ *are connected compact Hausdorff spaces.*

2.2. **Properties of slowly oscillating functions and shifts.** Let α be an orientation-preserving homeomorphism of $\overline{\mathbb{R}}_+$ onto itself, which has only two fixed points at 0 and ∞. Then $\alpha(0) = 0$, $\alpha(\infty) = \infty$, but $\alpha(t) \neq t$ for all $t \in \mathbb{R}_+$. Suppose that $\alpha_0(t) := t$ and $\alpha_n(t) := \alpha[\alpha_{n-1}(t)]$ for all $n \in \mathbb{Z}$ and $t \in \mathbb{R}_+$. Let $G := \{\alpha_n\}_{n\in\mathbb{Z}}$ be the cyclic group generated by the shift α. Then G is isomorphic to the group $\mathbb{Z}$. Given $\tau \in \mathbb{R}_+$, we put

$$\tau_- := \lim_{n\to-\infty} \alpha_n(\tau), \qquad \tau_+ := \lim_{n\to+\infty} \alpha_n(\tau).$$

The points τ_+ and τ_- are called *attracting* and *repelling* points of α, respectively. Then either $\tau_- = 0$ and $\tau_+ = \infty$, or $\tau_- = \infty$ and $\tau_+ = 0$.

Fix $\tau \in \mathbb{R}_+$, and let γ be a semisegment of $\mathbb{R}_+$ with endpoints τ and $\alpha(\tau)$, where $\tau \in \gamma$ and $\alpha(\tau) \notin \gamma$. Then we obtain the following orbital decomposition

$$\mathbb{R}_+ = \bigcup_{n\in\mathbb{Z}} \alpha_n(\gamma), \qquad \alpha_i(\gamma) \cap \alpha_j(\gamma) = \emptyset \quad \text{for } i \neq j. \tag{2.2}$$

Below we also assume that $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$, that is, the restriction of α to $\mathbb{R}_+$ is a diffeomorphism and $\log \alpha' \in \mathrm{SO}(\mathbb{R}_+)$.

Lemma 2.4 ([13, Lemmas 2.3–2.4])**.** *If* $c \in \mathrm{SO}(\mathbb{R}_+)$ *and* $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$*, then* $\alpha_{-1} \in \mathrm{SOS}(\mathbb{R}_+)$*,* $c \circ \alpha$ *belongs to* $\mathrm{SO}(\mathbb{R}_+)$ *and*

$$\lim_{t\to s} \big[c(t) - c\big(\alpha(t)\big)\big] = 0 \quad \text{for } s \in \{0, \infty\}.$$

Lemma 2.4 immediately implies the following assertion.

Corollary 2.5. *If* $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$*, then* $\alpha_j \in \mathrm{SOS}(\mathbb{R}_+)$ *for every* $j \in \mathbb{Z}$*.*

3. Two-sided invertibility of functional operators in terms of discrete operators

3.1. **Functional operators with slowly oscillating data.** Let $p \in [1, \infty]$. Consider the unital Banach algebra $\mathfrak{A}_W = W_{p,\mathrm{SO}} \subset \mathcal{B}(L^p(\mathbb{R}_+))$ consisting of all operators of the form (1.1) with norm (1.2), where $a_k \in \mathrm{SO}(\mathbb{R}_+)$ for all $k \in \mathbb{Z}$ and $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$. Obviously, $\mathfrak{A}_W \subset \mathfrak{A}_{p,\mathrm{SO}} \subset \mathcal{B}(L^p(\mathbb{R}_+))$ for all $p \in [1, \infty]$, where $\mathfrak{A}_{p,\mathrm{SO}}$ is the closure of $\mathfrak{A}_W$ in $\mathcal{B}(L^p(\mathbb{R}_+))$.

Given $p \in [1, \infty]$, we also consider the Banach space l^p consisting of the vectors $f = \{f_j\}_{j\in\mathbb{Z}}$ with entries in $\mathbb{C}$ and the norm

$$\|f\|_{l^p} = \begin{cases} \left(\sum_{j\in\mathbb{Z}} |f_j|^p\right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \sup_{j\in\mathbb{Z}} |f_j| & \text{if } p = \infty, \end{cases}$$

and define the unital commutative Banach algebra $\widehat{\mathfrak{D}} \subset \mathcal{B}(l^p)$ given by

$$\widehat{\mathfrak{D}} := \big\{d = \operatorname{diag}\{d_j\}_{j\in\mathbb{Z}} I : d_j \in \mathbb{C}, \lim_{n\to\pm\infty}(d_{n+j} - d_n) = 0 \text{ for all } j \in \mathbb{Z}, \\ \|d\|_{\mathcal{B}(l^p)} = \sup_{j\in\mathbb{Z}} |d_j| < \infty\big\}. \tag{3.1}$$

It is easily seen that, for every $a \in SO(\mathbb{R}_+)$ and all $t \in \mathbb{R}_+$, the associated operators $d = \operatorname{diag}\{a[\alpha_j(t)]\}_{j\in\mathbb{Z}} I$ belong to the Banach algebra $\widehat{\mathfrak{D}}$.

We now define the unital Banach subalgebra $\mathfrak{D}_p$ of $\mathcal{B}(l^p)$ generated by the isometric operators $\mathcal{V}, \mathcal{V}^{-1} \in \mathcal{B}(l^p)$ and by all operators $d \in \widehat{\mathfrak{D}}$, where $\mathcal{V} = (\delta_{i,j-1})_{i,j\in\mathbb{Z}} I$ and $\delta_{i,j}$ stands for the Kronecker delta. The operators $B \in \mathfrak{D}_p$ are called *discrete (or band-dominated) operators*.

We also consider the unital Banach algebra $\mathcal{W}_p \subset \mathfrak{D}_p$ given by

$$\mathcal{W}_p := \Big\{B = \sum_{k\in\mathbb{Z}} b_k \mathcal{V}^k : \text{ all } b_k \in \widehat{\mathfrak{D}} \text{ and } \|B\|_W := \sum_{k\in\mathbb{Z}} \|b_k\|_{\mathcal{B}(l^p)} < \infty\Big\}. \tag{3.2}$$

Furthermore, we introduce the isometric isomorphism

$$\begin{aligned} &\sigma : L^p(\mathbb{R}_+) \to L^p(\gamma, l^p), \qquad f \mapsto \psi; \\ &\psi : \gamma \to l^p, \qquad t \mapsto \big\{(U_\alpha^n f)(t)\big\}_{n\in\mathbb{Z}}. \end{aligned} \tag{3.3}$$

Let us show that for every functional operator $A \in \mathfrak{A}_{p,\mathrm{SO}}$ the discrete operators $\mathcal{A}(t) \in \mathcal{B}(l^p)$ can be defined for all $t \in \mathbb{R}_+$, and the function $t \mapsto \mathcal{A}(t)$ is bounded and continuous on $\mathbb{R}_+$.

Theorem 3.1. *If $p \in [1, \infty]$ and $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W \subset \mathcal{B}(L^p(\mathbb{R}_+))$, then the function $\mathcal{A}$, defined by*

$$\mathcal{A}(t) = \big(a_{j-i}\big[\alpha_i(t)\big]\big)_{i,j\in\mathbb{Z}} I \quad \text{for all } t \in \mathbb{R}_+, \tag{3.4}$$

is a bounded continuous $\mathcal{B}(l^p)$-valued operator function on $\mathbb{R}_+$, and

$$\max_{t\in\mathbb{R}_+} \big\|\mathcal{A}(t)\big\|_{\mathcal{B}(l^p)} \le \|A\|_W. \tag{3.5}$$

Proof. If $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$, then for every $t \in \mathbb{R}_+$ and every $p \in [1, \infty]$ the operator $\mathcal{A}(t)$ given by (3.4) is bounded on the space l^p. Indeed, take the operators

$$d_k(t) := \operatorname{diag}\big\{a_k\big[\alpha_j(t)\big]\big\}_{j\in\mathbb{Z}} I \in \widehat{\mathfrak{D}}, \qquad \mathcal{V} = (\delta_{i,j-1})_{i,j\in\mathbb{Z}} I, \tag{3.6}$$

where $\widehat{\mathfrak{D}}$ is given by (3.1). By (3.2), (3.4), and (3.6), we infer that the operator $\mathcal{A}(t) = \sum_{k\in\mathbb{Z}} d_k(t)\mathcal{V}^k$ belongs to the algebra $\mathcal{W}_p$ for every $t \in \mathbb{R}_+$ and

$$\big\|\mathcal{A}(t)\big\|_{\mathcal{B}(l^p)} \le \sum_{k\in\mathbb{Z}} \big\|d_k(t)\big\|_{\mathcal{B}(l^p)} = \sum_{k\in\mathbb{Z}} \sup_{j\in\mathbb{Z}} \big|a_k\big[\alpha_j(t)\big]\big| \le \|A\|_W. \tag{3.7}$$

Moreover, by (3.4) (see also [12, Lemma 8]),

$$\mathcal{A}\big[\alpha_n(t)\big] = \mathcal{V}^n \mathcal{A}(t) \mathcal{V}^{-n} \quad \text{for all } t \in \gamma \text{ and all } n \in \mathbb{Z}. \tag{3.8}$$

Fix $\varepsilon > 0$, and consider the operators $A_n = \sum_{|k|\le n} a_k U_\alpha^k$ ($n \in \mathbb{N}$) associated with the operator $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$. Obviously, there is an $n \in \mathbb{N}$ such that $\|A - A_n\|_W < \varepsilon/3$. Taking $t_1, t_2 \in \overline{\gamma}$, where $\overline{\gamma} \subset \mathbb{R}_+$ is the closure of γ, we infer from (3.4) and (3.7) that

$$\begin{aligned}\big\|\mathcal{A}(t_1) - \mathcal{A}(t_2)\big\|_{\mathcal{B}(l^p)} &< \big\|\mathcal{A}_n(t_1) - \mathcal{A}_n(t_2)\big\|_{\mathcal{B}(l^p)} + 2\varepsilon/3\\ &\le \sum_{|k|\le n} \sup_{i\in\mathbb{Z}} \big|a_k\big[\alpha_i(t_1)\big] - a_k\big[\alpha_i(t_2)\big]\big| + 2\varepsilon/3.\end{aligned}$$

Thus, $\|\mathcal{A}(t_1) - \mathcal{A}(t_2)\|_{\mathcal{B}(l^p)} < \varepsilon$ if there exists a $\delta > 0$ such that

$$\sup_{|k|\le n} \sup_{i\in\mathbb{Z}} \big|a_k\big[\alpha_i(t_1)\big] - a_k\big[\alpha_i(t_2)\big]\big| < \widetilde{\varepsilon} := \frac{\varepsilon}{3(2n+1)} \tag{3.9}$$

if $t_1, t_2 \in \overline{\gamma}$ and $|t_1 - t_2| < \delta$.

Let, for example, $\overline{\gamma} = [\tau, \alpha(\tau)]$ (for $\overline{\gamma} = [\alpha(\tau), \tau]$ the proof is analogous). Then $\alpha_j(t_1), \alpha_j(t_2) \in [\alpha_j(\tau), \alpha_{j+1}(\tau)]$ for every $j \in \mathbb{Z}$. Since $M := \sup_{x\in\mathbb{R}_+} \alpha'(x) > 1$ because

$$1 < \frac{\alpha_{j+1}(\tau)}{\alpha_j(\tau)} = \frac{1}{\alpha_j(\tau)} \int_0^{\alpha_j(\tau)} \alpha'(x)\, dx \le M,$$

we conclude that $[\alpha_j(\tau), \alpha_{j+1}(\tau)] \subset [r_j, M r_j]$, where $r_j = \alpha_j(\tau)$. Hence, for every function $a_k \in SO(\mathbb{R}_+)$ ($|k| \le n$) and arbitrary points $t_1, t_2 \in \overline{\gamma}$,

$$\lim_{j\to\pm\infty} \big|a_k\big[\alpha_j(t_1)\big] - a_k\big[\alpha_j(t_2)\big]\big| \le \lim_{j\to\pm\infty} \sup_{x,y\in[r_j, Mr_j]} \big|a_k(x) - a_k(y)\big| = 0.$$

Thus, there is a number $N \in \mathbb{N}$ such that, for all $|k| \le n$ and all $|j| > N$,

$$\big|a_k\big[\alpha_j(t_1)\big] - a_k\big[\alpha_j(t_2)\big]\big| < \widetilde{\varepsilon} \quad \text{if } t_1, t_2 \in \overline{\gamma}. \tag{3.10}$$

Let $\Gamma = \bigcup_{|j|\le N} \alpha_j(\overline{\gamma})$. Since the functions a_k are uniformly continuous on the segment $\Gamma \subset \mathbb{R}_+$, we conclude that there is a $\widetilde{\delta} > 0$ such that

$$\big|a_k(x) - a_k(y)\big| < \widetilde{\varepsilon} \quad \text{for all } |k| \le n \text{ if } x, y \in \Gamma \text{ and } |x - y| < \widetilde{\delta}. \tag{3.11}$$

Taking $K = \inf_{x\in\mathbb{R}_+} \alpha'(x) > 0$ and $M = \sup_{x\in\mathbb{R}_+} \alpha'(x) < \infty$, it remains to put $\delta := \widetilde{\delta} \min\{M^{-j}, K^j : j = 0, 1, \ldots, N\}$. Then $|\alpha_j(t_1) - \alpha_j(t_2)| < \widetilde{\delta}$ for all $|j| \le N$ and all $t_1, t_2 \in \overline{\gamma}$ with $|t_1 - t_2| < \delta$. Hence, taking into account (3.11), we infer that, for all $|k| \le n$ and all $|j| \le N$,

$$\big|a_k\big[\alpha_j(t_1)\big] - a_k\big[\alpha_j(t_2)\big]\big| < \widetilde{\varepsilon} \quad \text{if } t_1, t_2 \in \overline{\gamma} \text{ and } |t_1 - t_2| < \delta. \tag{3.12}$$

Combining (3.10) and (3.12), we obtain (3.9) for all $t_1, t_2 \in \overline{\gamma}$ such that $|t_1 - t_2| < \delta$. Thus, the function $t \mapsto \mathcal{A}(t)$ with values in $\mathcal{B}(l^p)$ is continuous on the segment $\overline{\gamma}$. Applying (3.8), we obtain the continuity of the function $t \mapsto \mathcal{A}(t)$ on the whole $\mathbb{R}_+$ and the property

$$\max_{t\in\alpha_n(\overline{\gamma})} \big\|\mathcal{A}(t)\big\|_{\mathcal{B}(l^p)} = \max_{t\in\overline{\gamma}} \big\|\mathcal{A}(t)\big\|_{\mathcal{B}(l^p)} \quad \text{for all } n \in \mathbb{Z}, \tag{3.13}$$

which implies (3.5) in view of (2.2) and (3.7). $\square$

Applying Theorem 3.1 and [12, Proposition 6], we obtain the following.

Theorem 3.2. *If $p \in [1, \infty]$ and $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W \subset \mathcal{B}(L^p(\mathbb{R}_+))$, then the operator $\widehat{A} := \sigma A \sigma^{-1} \in \mathcal{B}(L^p(\gamma, l^p))$ is given by $\widehat{A} = \mathcal{A}|_\gamma I$, where the operator-valued function $\mathcal{A} \in C_b(\mathbb{R}_+, \mathcal{B}(l^p))$ is defined by (3.4) and*

$$\|A\|_{\mathcal{B}(L^p(\mathbb{R}_+))} = \max_{t\in\overline{\gamma}} \big\|\mathcal{A}(t)\big\|_{\mathcal{B}(l^p)} = \max_{t\in\mathbb{R}_+} \big\|\mathcal{A}(t)\big\|_{\mathcal{B}(l^p)}. \tag{3.14}$$

Proof. If $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$, then for every $p \in [1, \infty]$ and every $\psi \in L^p(\gamma, l^p)$ the direct computation shows that

$$(\widehat{A}\psi)(t) = (\sigma A \sigma^{-1}\psi)(t) = \mathcal{A}(t)\psi(t) \quad \text{for every } t \in \gamma,$$

where $\mathcal{A}(t)$ is given by (3.4). Then

$$\|A\|_{\mathcal{B}(L^p(\mathbb{R}_+))} = \|\widehat{A}\|_{\mathcal{B}(L^p(\gamma,l^p))}. \tag{3.15}$$

Since by Theorem 3.1 the operator function $\mathcal{A} : \mathbb{R}_+ \to \mathcal{B}(l^p)$ is continuous on $\overline{\gamma}$, we infer from [12, Proposition 6] that

$$\|\widehat{A}\|_{\mathcal{B}(L^p(\gamma,l^p))} = \Big\|\big\|\mathcal{A}(\cdot)\big\|_{\mathcal{B}(l^p)}\Big\|_{L^\infty(\gamma)} = \max_{t\in\overline{\gamma}} \big\|\mathcal{A}(t)\big\|_{\mathcal{B}(l^p)},$$

which gives (3.14) in view of (3.15) and (3.13). ☐

If now $A \in \mathfrak{A}_{p,\mathrm{SO}} \setminus \mathfrak{A}_W$, then there is a sequence of operators $A_n \in \mathfrak{A}_W$ such that $\lim_{n\to\infty} \|A - A_n\|_{\mathcal{B}(L^p(\mathbb{R}_+))} = 0$. Then, by Theorem 3.1, $\mathcal{A}_n \in C_b(\mathbb{R}_+, \mathcal{B}(l^p))$ for all $n \in \mathbb{N}$. The operator-valued function $\mathcal{A} : \mathbb{R}_+ \to \mathcal{B}(l^p)$ is defined as the uniform limit of the sequence of the operator-valued functions $\mathcal{A}_n \in C_b(\mathbb{R}_+, \mathcal{B}(l^p))$ in the norm of $L^\infty(\mathbb{R}_+, \mathcal{B}(l^p))$, where $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in view of the equalities

$$\max_{t\in\mathbb{R}_+} \big\|\mathcal{A}_n(t) - \mathcal{A}_m(t)\big\|_{\mathcal{B}(l^p)} = \|A_n - A_m\|_{\mathcal{B}(L^p(\mathbb{R}_+))} \quad (n, m \in \mathbb{N})$$

followed from Theorem 3.2. Then the limit $\mathcal{A} = \lim_{n\to\infty} \mathcal{A}_n$ belongs to the Banach algebra $C_b(\mathbb{R}_+, \mathcal{B}(l^p))$, is independent of a choice of the sequence $\{A_n\} \subset \mathfrak{A}_W$, and satisfies (3.8) and (3.14) along with all $\mathcal{A}_n$. Moreover, the matrix function $\mathcal{A}(\cdot)$ has the form (3.4) with entries being uniform limits of the corresponding entries of the matrix functions $\mathcal{A}_n(\cdot)$, and

$$\sigma A \sigma^{-1} = \mathcal{A}|_\gamma I \quad \text{for all } A \in \mathfrak{A}_{p,\mathrm{SO}}. \tag{3.16}$$

Thus, Theorems 3.1 and 3.2, by the density of $\mathfrak{A}_W$ in $\mathfrak{A}_{p,\mathrm{SO}}$, imply the following.

Corollary 3.3. *For every $p \in [1, \infty]$ and every $A \in \mathfrak{A}_{p,\mathrm{SO}}$, the operator-valued function $\mathcal{A} : \mathbb{R}_+ \to \mathcal{B}(l^p)$ is bounded and continuous on $\mathbb{R}_+$, and satisfies (3.8), (3.14), and (3.16).*

Applying Corollary 3.3 and (3.16), we immediately establish the following invertibility criterion for the operators $A \in \mathfrak{A}_{p,\mathrm{SO}}$, which strengthens [12, Theorem 9].

Theorem 3.4. *For $p \in [1, \infty]$, a functional operator $A \in \mathfrak{A}_{p,\mathrm{SO}}$ is invertible on the space $L^p(\mathbb{R}_+)$ if and only if for all $t \in \overline{\gamma}$ (equivalently, for all $t \in \mathbb{R}_+$) the discrete operators $\mathcal{A}(t)$, given by (3.4) and Corollary 3.3, are invertible on the space l^p.*

Remark 3.5. Since the discrete operators $\mathcal{A}(\tau)$ and $\mathcal{A}[\alpha(\tau)]$ defined at the endpoints τ and $\alpha(\tau)$ of γ are invertible on the space l^p only simultaneously in view of (3.8), Theorem 3.4 remains valid with $\overline{\gamma}$ replaced by γ.

3.2. **Functional operators with coefficients in a subalgebra $\mathfrak{S}$ of $L^\infty(\mathbb{R}_+)$.** A function $f \in L^\infty(\mathbb{R}_+)$ is called *slowly oscillating* (at 0 and ∞) if for each (equivalently, for some) $\lambda \in (0,1)$,

$$\lim_{r\to s} \operatorname{ess\,sup}\big\{\big|f(t)-f(\tau)\big| : t,\tau \in [\lambda r, r]\big\} = 0, \quad s \in \{0,\infty\}.$$

Let $\mathfrak{S}$ denote the C^*-subalgebra of $L^\infty(\mathbb{R}_+)$ consisting of all functions on $\mathbb{R}_+$ that are continuous on every semisegment $\alpha_n(\gamma)$ $(n \in \mathbb{Z})$, have finite one-sided limits at the points $\alpha_n(\tau)$ $(n \in \mathbb{Z})$, where τ is the endpoint of γ that belongs to γ, and are slowly oscillating at 0 and ∞.

We now consider the unital Banach algebra $W_{p,\mathfrak{S}}$ consisting of all functional operators of the form $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k$ with coefficients $a_k \in \mathfrak{S}$ and the norm $\|A\|_W := \sum_{k\in\mathbb{Z}} \|a_k\|_{L^\infty(\mathbb{R}_+)} < \infty$. Let the unital Banach algebra $\mathfrak{A}_{p,\mathfrak{S}}$ be the closure of $W_{p,\mathfrak{S}}$ in the norm of $\mathcal{B}(L^p(\mathbb{R}_+))$.

Slightly modifying the proof of Theorem 3.1, we obtain the following.

Lemma 3.6. *If $A \in W_{p,\mathfrak{S}}$ for $p \in [1,\infty]$, then the operator-valued function $\mathcal{A} : \gamma \to \mathcal{B}(l^p)$ given by (3.4) for all $t \in \overline{\gamma}$, where its values at the endpoints τ and $\alpha(\tau)$ of γ are defined by*

$$\begin{aligned} \mathcal{A}(\tau) &= \Big(\lim_{t\in\gamma, t\to\tau} a_{j-i}\big[\alpha_i(t)\big]\Big)_{i,j\in\mathbb{Z}}, \\ \mathcal{A}\big[\alpha(\tau)\big] &= \Big(\lim_{t\in\gamma, t\to\alpha(\tau)} a_{j-i}\big[\alpha_i(t)\big]\Big)_{i,j\in\mathbb{Z}}, \end{aligned} \tag{3.17}$$

is continuous on the segment $\overline{\gamma}$.

Since the algebra $W_{p,\mathfrak{S}}$ is dense in the Banach algebra $\mathfrak{A}_{p,\mathfrak{S}}$, we conclude from Lemma 3.6 that for every $A \in \mathfrak{A}_{p,\mathfrak{S}}$ the operator-valued function $\mathcal{A} : \gamma \to \mathcal{B}(l^p)$ is also continuous on the segment $\overline{\gamma}$. Applying now [12, Theorem 9], and the continuity of the function $\mathcal{A} : \overline{\gamma} \to \mathcal{B}(l^p)$, we immediately obtain the following generalization of Theorem 3.4.

Theorem 3.7. *Given $p \in [1,\infty]$, a functional operator $A \in \mathfrak{A}_{p,\mathfrak{S}}$ is invertible on the space $L^p(\mathbb{R}_+)$ if and only if for all $t \in \overline{\gamma}$ the discrete operators $\mathcal{A}(t)$ are invertible on the space l^p.*

4. Limit operators and their application to left invertible functional operators

4.1. **Abstract approach.** Let X be a Banach space, let X^* be its dual space, let $A \in \mathcal{B}(X)$, and let $\mathcal{U} = \{U_n\}_{n=1}^\infty$ be a sequence of isometries. If the strong limit

$$A_{\mathcal{U}} := \operatorname*{s-lim}_{n\to\infty} (U_n^{-1} A U_n)$$

exists in $\mathcal{B}(X)$, then it is referred to as the limit operator for A with respect to the sequence $\mathcal{U}$. Note that sometimes the existence of the strong limit

$$A_{\mathcal{U}^*} := \operatorname*{s-lim}_{n\to\infty}(U_n^{-1}AU_n)^*$$

in $\mathcal{B}(X^*)$ is also required in the definition of limit operator, and then $(A_{\mathcal{U}})^* = A_{\mathcal{U}^*}$.

The following statement is a variation of [8, Chapter 3, Lemma 1.1].

Theorem 4.1 ([15, Theorem 3.1])**.** *Let X be a Banach space, let $A \in \mathcal{B}(X)$, and let $\mathcal{U} = \{U_n\}_{n=1}^{\infty} \subset \mathcal{B}(X)$ be a sequence of isometries such that the limit operator $A_{\mathcal{U}}$ exists. If the operator A is left invertible in $\mathcal{B}(X)$, then $\operatorname{Ker} A_{\mathcal{U}} = \{0\}$, $\operatorname{Im} A_{\mathcal{U}}$ is a closed subspace of X, and the limit operator $A_{\mathcal{U}}$ is invertible as the operator acting from X to $\operatorname{Im} A_{\mathcal{U}}$.*

Theorem 4.1 implies the following corollary (see [8, Chapter 3, Lemma 1.1] and [33, Proposition 1.2.9]).

Corollary 4.2. *If an operator $A \in \mathcal{B}(X)$ is invertible in $\mathcal{B}(X)$ and the strong limits $A_{\mathcal{U}} \in \mathcal{B}(X)$ and $A_{\mathcal{U}^*} \in \mathcal{B}(X^*)$ exist with respect to a sequence $\mathcal{U}$ of isometries in $\mathcal{B}(X)$, then the limit operator $A_{\mathcal{U}}$ is also invertible in $\mathcal{B}(X)$.*

4.2. **An application of limit operators to functional operators $A \in \mathfrak{A}_W$.** We now apply the technique of limit operators [5] to studying the two-sided and one-sided invertibility of operators $A \in \mathfrak{A}_W$ given by (1.1)–(1.2) on the Lebesgue spaces $L^p(\mathbb{R}_+)$, $p \in (1,\infty)$. Limit operators are related to points ξ of the set Δ given by (1.3).

By analogy with [15, Lemma 3.2], we obtain the following.

Lemma 4.3. *Let $p \in [1,\infty)$, and let $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W \subset \mathcal{B}(L^p(\mathbb{R}_+))$, where all $a_k \in SO(\mathbb{R}_+)$ and $\alpha \in SOS(\mathbb{R}_+)$. Then for every $\xi \in \Delta$ there is a sequence of isometries $\mathcal{U}$ on $L^p(\mathbb{R}_+)$ such that*

$$A_{\mathcal{U}} = A_\xi := \sum_{k\in\mathbb{Z}} a_k(\xi) U_\alpha^k \in \mathfrak{A}_W. \tag{4.1}$$

Proof. Let us show that for every $s \in \{0,\infty\}$ and every $\xi \in M_s(SO(\mathbb{R}_+))$ there is a strictly increasing sequence $\{k_n\}_{n\in\mathbb{N}}$ of numbers $k_n \in \mathbb{N}$ such that (4.1) holds for the sequence of isometries $\mathcal{U} \subset \mathcal{B}(L^p(\mathbb{R}_+))$ given by

$$\mathcal{U} := \begin{cases} \{U_\alpha^{-k_n}\}_{n\in\mathbb{N}} & \text{if } s \text{ is the attracting point of } \alpha, \\ \{U_\alpha^{k_n}\}_{n\in\mathbb{N}} & \text{if } s \text{ is the repelling point of } \alpha. \end{cases}$$

Let, for example, $s = \infty$ be the attracting point of α, that is,

$$\lim_{n\to+\infty} \alpha_n(t) = \tau_+ = s = \infty \quad \text{for all } t \in \mathbb{R}_+.$$

Then $0 < t < \alpha_m(t)$ for all $m \in \mathbb{N}$ and all $t \in \mathbb{R}_+$, and hence

$$\frac{\alpha_m(t)}{t} = \frac{1}{t}\int_0^t \alpha_m'(x)\,dx \le \sup_{t\in\mathbb{R}_+}\big[\alpha'(t)\big]^m := M^m,$$

where $1 < M := \sup_{t\in\mathbb{R}_+} \alpha'(t) < \infty$. Fix $\xi \in M_s(\mathrm{SO}(\mathbb{R}_+))$. By Lemma 2.2, for the countable set $\{a_k \in \mathrm{SO}(\mathbb{R}_+) : k \in \mathbb{Z}\}$ there is a monotonically increasing sequence $\{t_n\} \subset \mathbb{R}_+$ such that $t_n \to \tau_+$ as $n \to \infty$ and

$$\lim_{n\to\infty} a_k(t_n) = a_k(\xi) \quad \text{for all } k \in \mathbb{Z}. \tag{4.2}$$

Let $k_n \in \mathbb{N}$ be such that $t_n \in [\alpha_{k_n}(1), \alpha_{k_n+1}(1)]$, where $t_1 \geq \alpha(1)$. Since

$$U_\alpha^{k_n}\Big(\sum_{k\in\mathbb{Z}} a_k U_\alpha^k\Big)U_\alpha^{-k_n} = \sum_{k\in\mathbb{Z}}(a_k\circ\alpha_{k_n})U_\alpha^k \quad \text{for all } n \in \mathbb{N},$$

where $a_k\circ\alpha_{k_n} \in \mathrm{SO}(\mathbb{R}_+)$ by Lemma 2.4 and Corollary 2.5, it is sufficient to prove that

$$\operatorname*{s-lim}_{n\to\infty}(a_k\circ\alpha_{k_n})I = a_k(\xi)I \quad \text{for every } k \in \mathbb{Z}. \tag{4.3}$$

Then we will conclude that

$$\operatorname*{s-lim}_{n\to\infty}\sum_{k\in\mathbb{Z}}(a_k\circ\alpha_{k_n})U_\alpha^k = \sum_{k\in\mathbb{Z}} a_k(\xi)U_\alpha^k = A_\xi,$$

where the operator A_ξ belongs to the algebra $\mathfrak{A}_W$ along with A because

$$\|A_\xi\|_W = \sum_{k\in\mathbb{Z}}\big|a_k(\xi)\big| \leq \sum_{k\in\mathbb{Z}}\|a_k\circ\alpha_{k_n}\|_{C_b(\mathbb{R}_+)} = \sum_{k\in\mathbb{Z}}\|a_k\|_{C_b(\mathbb{R}_+)} < \infty.$$

Thus, to get (4.3), it remains to prove that, for any $c \in \{a_k \in \mathrm{SO}(\mathbb{R}_+) : k \in \mathbb{Z}\}$, any $m \in \mathbb{N}$, and any function $f \in L^p(\mathbb{R}_+)$ with support in $[\alpha_{-m}(1), \alpha_m(1)]$,

$$\lim_{n\to\infty}\big\|(c\circ\alpha_{k_n})f - c(t_n)f\big\|_{L^p(\mathbb{R}_+)} = 0.$$

This will follow if

$$\lim_{n\to\infty}\max_{t\in[\alpha_{-m}(1),\alpha_m(1)]}\big|c\big[\alpha_{k_n}(t)\big] - c(t_n)\big| = 0. \tag{4.4}$$

Clearly, for every $t \in [\alpha_{-m}(1), \alpha_m(1)]$ we get

$$\alpha_{k_n}(t), \quad t_n \in \big[\alpha_{k_n-m}(1), \alpha_{k_n+m}(1)\big] \subset [r_n, M^{2m}r_n], \tag{4.5}$$

where $r_n := \alpha_{k_n-m}(1)$. Since $c \in \mathrm{SO}(\mathbb{R}_+)$, we infer from (4.5) that

$$\lim_{n\to\infty}\max_{t\in[\alpha_{-m}(1),\alpha_m(1)]}\big|c\big[\alpha_{k_n}(t)\big] - c(t_n)\big| \leq \lim_{n\to\infty}\max_{t,\tau\in[r_n,M^{2m}r_n]}\big|c(t) - c(\tau)\big| = 0,$$

which implies (4.4). Hence

$$\operatorname*{s-lim}_{n\to\infty}\big[(a_k\circ\alpha_{k_n})I\big] = \operatorname*{s-lim}_{n\to\infty}\big[a_k(t_n)I\big] = \lim_{n\to\infty}\big[a_k(t_n)I\big] = a_k(\xi)I \quad (k \in \mathbb{Z}),$$

which completes the proof in the case $s = \tau_+ = \infty$.

The cases $s = \tau_- = \infty$, $s = \tau_+ = 0$, and $s = \tau_- = 0$ are treated analogously. □

Theorem 4.1 and Lemma 4.3 imply the following.

Corollary 4.4. *If $p \in [1,\infty)$ and the operator $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ with coefficients $a_k \in SO(\mathbb{R}_+)$ and a shift $\alpha \in SOS(\mathbb{R}_+)$ is left invertible on the space $L^p(\mathbb{R}_+)$, then for all $\xi \in \Delta = M_0(SO(\mathbb{R}_+)) \cup M_\infty(SO(\mathbb{R}_+))$ the limit operators*

$$A_\xi = \sum_{k\in\mathbb{Z}} a_k(\xi) U_\alpha^k \in \mathfrak{A}_W \tag{4.6}$$

possess the properties: $\operatorname{Ker} A_\xi = \{0\}$, $\operatorname{Im} A_\xi$ *is a closed subspace of $L^p(\mathbb{R}_+)$, and the A_ξ's are invertible operators from $L^p(\mathbb{R}_+)$ onto* $\operatorname{Im} A_\xi$.

Applying Corollary 4.4 to the invertible operator $A \in \mathfrak{A}_W \subset \mathcal{B}(L^p(\mathbb{R}_+))$ and its adjoint operator $A^* \in \mathfrak{A}_W \subset \mathcal{B}(L^q(\mathbb{R}_+))$, where $p \in (1,\infty)$ and $1/p + 1/q = 1$, we immediately obtain the following.

Corollary 4.5. *If $p \in (1,\infty)$ and the operator $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ with coefficients $a_k \in SO(\mathbb{R}_+)$ and a shift $\alpha \in SOS(\mathbb{R}_+)$ is invertible on the space $L^p(\mathbb{R}_+)$, then for all $\xi \in \Delta$ the limit operators (4.6) also are invertible on this space.*

Lemma 4.6 ([15, Theorem 2.5])**.** *If $\alpha \in SOS(\mathbb{R}_+)$, then the spectrum of the isometric operator U_α coincides with the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.*

Consider the unital commutative Banach algebra of all absolutely convergent series

$$W = \Big\{ f = \sum_{k\in\mathbb{Z}} c_k z^k : c_k \in \mathbb{C}, z \in \mathbb{T}, \|f\|_W := \sum_{k\in\mathbb{Z}} |c_k| < \infty \Big\} \subset C(\mathbb{T}). \tag{4.7}$$

As is well known (see, e.g., [7, pp. 22–23]), the maximal ideal space $M(W)$ of the algebra W can be identified with the unit circle $\mathbb{T}$.

Along with W we also consider the unital commutative Banach subalgebra $W_{p,\mathbb{C}}$ of $\mathfrak{A}_W \subset \mathcal{B}(L^p(\mathbb{R}_+))$ that consists of all functional operators $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ with constant coefficients $a_k \in \mathbb{C}$. Since the Banach algebras W and $W_{p,\mathbb{C}}$ are isometrically isomorphic, the maximal ideal space of the algebra $W_{p,\mathbb{C}}$ can also be identified with $\mathbb{T}$, and the Gelfand transform of the operators $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in W_{p,\mathbb{C}}$ is given by $A(z) := \sum_{k\in\mathbb{Z}} a_k z^k$ for all $z \in \mathbb{T}$, where the function $A(\cdot)$ belongs to the algebra W.

Hence, for each $\xi \in \Delta$ the limit operator $A_\xi = \sum_{k\in\mathbb{Z}} a_k(\xi) U_\alpha^k \in \mathfrak{A}_W$ is invertible on the Lebesgue space $L^p(\mathbb{R}_+)$ $(1 \le p < \infty)$ if and only if its Gelfand transform $A_\xi(\cdot)$ is invertible in $C(\mathbb{T})$, that is,

$$A_\xi(z) := \sum_{k\in\mathbb{Z}} a_k(\xi) z^k \neq 0 \quad \text{for all } z \in \mathbb{T}. \tag{4.8}$$

While all limit operators A_ξ are invertible for each invertible operator $A \in \mathfrak{A}_W$ by Corollary 4.5, and hence (4.8) holds, this fact for strictly one-sided invertible operators $A \in \mathfrak{A}_W$ we still need to prove.

4.3. **Limit operators for discrete operators associated with operators $A \in \mathfrak{A}_W$.** Consider the isometry $\mathcal{V}$ of the space l^p onto itself, which is given by $\mathcal{V}f = \{f_{k+1}\}_{k\in\mathbb{Z}} \in l^p$ for every vector $f = \{f_k\}_{k\in\mathbb{Z}} \in l^p$. Taking the orbit $\mathcal{O}(t) := \{\alpha_n(t) : n \in \mathbb{Z}\}$ of any point $t \in \mathbb{R}_+$, one can easily deduce from the proof of Lemma 4.3 the following discrete analogue of Lemma 4.3.

Lemma 4.7. *Let $p \in [1,\infty)$, and let $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ with all $a_k \in SO(\mathbb{R}_+)$ and $\alpha \in SOS(\mathbb{R}_+)$. Then for every $s \in \{0,\infty\}$ and every $\xi \in M_s(SO(\mathbb{R}_+))$ there is a sequence $\{k_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ of numbers $k_n \in \mathbb{N}$ such that $k_n \to \infty$ as $n \to \infty$ and, for every $t \in \mathbb{R}_+$ and every discrete operator $\mathcal{A}(t) \in \mathcal{W}_p$ given by (3.4) on the space l^p,*

$$\operatorname*{s-lim}_{n\to\infty} \mathcal{V}^{\pm k_n}\mathcal{A}(t)\mathcal{V}^{\mp k_n} = \widehat{A}_\xi := \sum_{k\in\mathbb{Z}} a_k(\xi)\mathcal{V}^k \quad \textit{for } s = \tau_\pm. \tag{4.9}$$

Proof. Let $A \in \mathfrak{A}_W$, and let $\lim_{n\to+\infty} \alpha_n(t) = \tau_+ = s = \infty$ for all $t \in \mathbb{R}_+$. Applying (4.2) and (4.4) for all $c \in \{a_k : k \in \mathbb{Z}\}$, we infer that for any $\xi \in M_{\tau_+}(SO(\mathbb{R}_+))$ there is a sequence $\{k_n\}_{n\in\mathbb{N}}$ of numbers $k_n \in \mathbb{N}$ such that $\lim_{n\to\infty} k_n = \infty$ and

$$\lim_{n\to\infty} \max_{t\in[\alpha_{-m}(1),\alpha_m(1)]} \big|a_k\big[\alpha_{k_n}(t)\big] - a_k(\xi)\big| = 0 \quad \text{for all } k \in \mathbb{Z}. \tag{4.10}$$

Hence, from (4.10) and the definition of $SO(\mathbb{R}_+)$ it follows that

$$\lim_{n\to\infty} a_k\big[\alpha_{k_n}(t)\big] = a_k(\xi) \quad \text{for all } t \in \mathbb{R}_+ \text{ and all } k \in \mathbb{Z}. \tag{4.11}$$

Analogously, for every $\xi \in M_{\tau_-}(SO(\mathbb{R}_+))$ there exists a sequence $\{k_n\}_{n\in\mathbb{N}}$ of numbers $k_n \in \mathbb{N}$ such that $\lim_{n\to\infty} k_n = \infty$ and

$$\lim_{n\to\infty} a_k\big[\alpha_{-k_n}(t)\big] = a_k(\xi) \quad \text{for all } t \in \mathbb{R}_+ \text{ and all } k \in \mathbb{Z}. \tag{4.12}$$

Taking the discrete operator $\mathcal{A}(t)$ with entries in the orbit $\mathcal{O}(t) = \{\alpha_n(t) : n \in \mathbb{Z}\}$ of any point $t \in \mathbb{R}_+$, we infer from (4.11), (4.12), and (3.8) by analogy with the proof of Lemma 4.3 that for every $\xi \in \Delta = M_0(SO(\mathbb{R}_+)) \cup M_\infty(SO(\mathbb{R}_+))$ and every $t \in \mathbb{R}_+$ there exists the limit operator $\widehat{A}_\xi = \sum_{k\in\mathbb{Z}} a_k(\xi)\mathcal{V}^k \in \mathcal{B}(l^p)$ satisfying (4.9). Moreover, the set of limit operators $\{\widehat{A}_\xi : \xi \in \Delta\}$ is the same for every discrete operator $\mathcal{A}(t) \in \mathcal{W}_p$ given for $t \in \mathbb{R}_+$. ☐

Theorem 4.1 and Lemma 4.7 imply the following.

Corollary 4.8. *If $p \in [1,\infty)$, $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$, and for some $t \in \mathbb{R}_+$ the discrete operator $\mathcal{A}(t) \in \mathcal{W}_p$ given by (3.4) is left invertible on the space l^p, then for all $\xi \in \Delta$ the limit operators*

$$\widehat{A}_\xi := \sum_{k\in\mathbb{Z}} a_k(\xi)\mathcal{V}^k \in \mathcal{W}_p \tag{4.13}$$

possess the properties: $\operatorname{Ker} \widehat{A}_\xi = \{0\}$, $\operatorname{Im} \widehat{A}_\xi$ *is a closed subspace of l^p, and $\widehat{A}_\xi$ are invertible operators from l^p onto* $\operatorname{Im} \widehat{A}_\xi$.

Applying Corollary 4.8 to invertible operators $\mathcal{A}(t) \in \mathcal{W}_p \subset \mathcal{B}(l^p)$ and $(\mathcal{A}(t))^* \in \mathcal{W}_q \subset \mathcal{B}(l^q)$ associated with $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ and $t \in \mathbb{R}_+$, where $p \in (1,\infty)$ and $1/p + 1/q = 1$, we immediately obtain the following.

Corollary 4.9. *If $p \in (1,\infty)$, $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$, and for some $t \in \mathbb{R}_+$ the discrete operator $\mathcal{A}(t) \in \mathcal{W}_p$ given by (3.4) is invertible on the space l^p, then for all $\xi \in \Delta$ the limit operators (4.13) also are invertible on the space l^p.*

Given $p \in (1,\infty)$, it is easily seen that for every $\xi \in \Delta$ the limit operators $A_\xi \in \mathfrak{A}_W$ related to the operators $A \in \mathfrak{A}_W$ are isometrically isomorphic to the limit operators $\widehat{A}_\xi \in \mathcal{W}_p$ related to the discrete operators $\mathcal{A}(t) \in \mathcal{W}_p$ for any point $t \in \mathbb{R}_+$. Since the Banach algebras of such limit operators are commutative and the Gelfand transforms for the limit operators $A_\xi \in \mathfrak{A}_W$ and $\widehat{A}_\xi \in \mathcal{W}_p$ can be identified, we conclude that for every $\xi \in \Delta$ the limit operators $A_\xi \in \mathfrak{A}_W$ related to the operators $A \in \mathfrak{A}_W$ are invertible on the space $L^p(\mathbb{R}_+)$ if and only if the limit operators $\widehat{A}_\xi \in \mathcal{W}_p$ related to discrete operators $\mathcal{A}(t) \in \mathcal{W}_p$ for any point $t \in \mathbb{R}_+$ are invertible on the space l^p. Hence, the two-sided invertibility of any discrete operator $\mathcal{A}(t) \in \mathcal{W}_p$ related to an operator $A \in \mathfrak{A}_W$ implies that $A_\xi(z) \neq 0$ for all $\xi \in \Delta$ and all $z \in \mathbb{T}$.

5. Necessary one-sided invertibility conditions at fixed points of the shift

In what follows, we assume that $1 < p < \infty$.

5.1. Necessary conditions for functional operators at fixed points of the shift.

Lemma 5.1. *If $p \in (1,\infty)$, $M \in \mathbb{N}$, and the functional operator $A = \sum_{|k|\le M} a_k U_\alpha^k$ with coefficients $a_k \in SO(\mathbb{R}_+)$ and a shift $\alpha \in SOS(\mathbb{R}_+)$ is left or right invertible on the Lebesgue space $L^p(\mathbb{R}_+)$, then*

$$A_\xi(z) = \sum_{|k|\le M} a_k(\xi) z^k \neq 0 \quad \text{for all } \xi \in \Delta \text{ and all } z \in \mathbb{T}. \tag{5.1}$$

Proof. Let the operator $A = \sum_{|k|\le M} a_k U_\alpha^k$ be left invertible on the space $L^p(\mathbb{R}_+)$, and, on the contrary, suppose that $A_\xi(z_0) = 0$ for some $\xi \in \Delta$ and some $z_0 \in \mathbb{T}$. Take the limit operator

$$A_\xi = \sum_{|k|\le M} a_k(\xi) U_\alpha^k \in \mathcal{B}\big(L^p(\mathbb{R}_+)\big).$$

Since the operator A is left invertible on the space $L^p(\mathbb{R}_+)$, we infer from Corollary 4.4 that $\operatorname{Ker} A_\xi = \{0\}$ and $\operatorname{Im} A_\xi$ is closed in $L^p(\mathbb{R}_+)$.

Since $\operatorname{Ker} A_\xi = \{0\}$, we conclude that $A_\xi(z) \not\equiv 0$. Indeed, if $A_\xi(z) \equiv 0$, then $a_k(\xi) = 0$ for all $|k| \le M$, and therefore $\operatorname{Ker} A_\xi = L^p(\mathbb{R}_+)$, which is impossible. Hence, there exist numbers $m_1, m_2 \in \{-M, \dots, M\}$ such that $m_1 < m_2$, $a_k(\xi) \neq 0$ for $k \in \{m_1, m_2\}$ and $a_k(\xi) = 0$ for all $k < m_1$ and all $k > m_2$. Representing the function $A_\xi(\cdot)$ in the form

$$A_\xi(z) = \sum_{k=m_1}^{m_2} a_k(\xi) z^k = a_{m_2}(\xi) z^{m_1} \prod_{k=1}^{m_2-m_1} (z - z_k) \quad \text{for } z \in \mathbb{T}, \tag{5.2}$$

where all $z_k \in \mathbb{C}$ and where at least one z_k coincides with $z_0 \in \mathbb{T}$ because $A_\xi(z_0) = 0$, we infer that

$$A_\xi = a_{m_2}(\xi) U_\alpha^{m_1} \prod_{k=1}^{m_2-m_1} (U_\alpha - z_k I).$$

Passing to the adjoint operator, we obtain

$$A_\xi^* = \overline{a_{m_2}(\xi)} U_{\alpha_{-1}}^{m_1} \prod_{k=1}^{m_2-m_1} (U_{\alpha_{-1}} - \overline{z_k} I) \in \mathcal{B}\big(L^q(\mathbb{R}_+)\big), \tag{5.3}$$

where $1/p + 1/q = 1$ and for every $k = 1, 2, \dots, m_2 - m_1$ either $|z_k| \neq 1$ and then the operator $U_{\alpha_{-1}} - \overline{z_k} I$ is invertible on the space $L^q(\mathbb{R}_+)$, or $|z_0| = 1$ and then it follows that $\operatorname{Ker}(U_{\alpha_{-1}} - \overline{z_0} I) = \{0\}$. Indeed, in the latter case, according to the proof of [15, Lemma 4.1], we consider the equation

$$(U_{\alpha_{-1}} f)(x) - \overline{z_0} f(x) = 0 \quad \text{for } f \in L^q(\mathbb{R}_+) \text{ and } x \in \mathbb{R}_+. \tag{5.4}$$

Let $\gamma_n = \alpha_n(\gamma)$ $(n \in \mathbb{Z})$, where γ is the segment of $\mathbb{R}_+$ with endpoints 1 and $\alpha(1)$. Because $|z_0| = 1$, we deduce from (5.4) that

$$\|f|_{\gamma_n}\|_{L^q(\gamma_n)}^q = \big\|(U_{\alpha_{-1}}^n f)|_{\gamma_n}\big\|_{L^q(\gamma_n)}^q = \|f|_{\gamma_0}\|_{L^q(\gamma_0)}^q \quad \text{for all } n \in \mathbb{Z},$$

which implies for $q \in (1, \infty)$ that $\operatorname{Ker}(U_{\alpha_{-1}} - \overline{z_0} I) = \{0\}$. Hence, by (5.3), $\operatorname{Ker} A_\xi^* = \{0\}$. Since $\operatorname{Im} A_\xi$ is closed in $L^p(\mathbb{R}_+)$, $\operatorname{Ker} A_\xi = \{0\}$, and $\operatorname{Ker} A_\xi^* = \{0\}$, we conclude that the operator A_ξ is invertible on the space $L^p(\mathbb{R}_+)$. But then the Gelfand transform $A_\xi(\cdot)$ of the operator A_ξ should be separated from zero for all $z \in \mathbb{T}$, which contradicts the assumption that $A_\xi(z_0) = 0$.

If the operator $A = \sum_{|k| \le M} a_k U_\alpha^k$ is right invertible on the space $L^p(\mathbb{R}_+)$, then passing to the left invertible adjoint operator

$$A^* = \sum_{|k| \le M} (\overline{a_k} \circ \alpha_{-k}) U_{\alpha_{-1}}^k \in \mathcal{B}\big(L^q(\mathbb{R}_+)\big),$$

where $1/p + 1/q = 1$ and, by Lemma 2.4 and Corollary 2.5,

$$\overline{a_k} \circ \alpha_{-k} \in SO(\mathbb{R}_+) \qquad \text{and} \qquad (\overline{a_k} \circ \alpha_{-k})(\xi) = \overline{a_k(\xi)} \quad \text{for all } \xi \in \Delta,$$

we obtain (5.1) from the part already proved. □

We now pass to functional operators A in the Banach algebra $\mathfrak{A}_W$.

Theorem 5.2. *Let $p \in (1, \infty)$. If the functional operator $A = \sum_{k \in \mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ with coefficients $a_k \in SO(\mathbb{R}_+)$ and a shift $\alpha \in SOS(\mathbb{R}_+)$ is left or right invertible on the Lebesgue space $L^p(\mathbb{R}_+)$, then*

$$A_\xi(z) = \sum_{k \in \mathbb{Z}} a_k(\xi) z^k \neq 0 \quad \textit{for all } \xi \in \Delta \textit{ and all } z \in \mathbb{T}. \tag{5.5}$$

Proof. Let the operator $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k$ be left invertible on the space $L^p(\mathbb{R}_+)$, and, on the contrary, suppose that $A_\xi(z_0) = 0$ for some $\xi \in \Delta$ and some $z_0 \in \mathbb{T}$. Take the limit operator

$$A_\xi = \sum_{k\in\mathbb{Z}} a_k(\xi) U_\alpha^k \in \mathfrak{A}_W \subset \mathcal{B}\big(L^p(\mathbb{R}_+)\big).$$

By the stability of the left invertibility, there is an $\varepsilon > 0$ such that every operator $\widetilde{A} \in \mathcal{B}(L^p(\mathbb{R}_+))$ with $\|A - \widetilde{A}\|_{\mathcal{B}(L^p(\mathbb{R}_+))} < \varepsilon$ also is left invertible on the space $L^p(\mathbb{R}_+)$. We now choose $M \in \mathbb{N}$ such that the polynomial functional operator $\widetilde{A} := \sum_{|k|\le M} a_k U_\alpha^k$ satisfies the inequalities

$$\|A - \widetilde{A}\|_{\mathcal{B}(L^p(\mathbb{R}_+))} \le \|A - \widetilde{A}\|_W < \varepsilon/2. \tag{5.6}$$

Since $A_\xi(z_0) = 0$ and since

$$\max_{z\in\mathbb{T}} \big|A_\xi(z) - \widetilde{A}_\xi(z)\big| \le \|A_\xi - \widetilde{A}_\xi\|_{\mathcal{B}(L^p(\mathbb{R}_+))} \le \|A - \widetilde{A}\|_{\mathcal{B}(L^p(\mathbb{R}_+))} < \varepsilon/2$$

for all $\xi \in \Delta$, we deduce that

$$\big|\widetilde{A}_\xi(z_0)\big| < \varepsilon/2. \tag{5.7}$$

We now consider the polynomial functional operator $B := \widetilde{A} - \widetilde{A}_\xi(z_0)I$. Then we infer from (5.6) and (5.7) that

$$\|A - B\|_{\mathcal{B}(L^p(\mathbb{R}_+))} \le \|A - \widetilde{A}\|_{\mathcal{B}(L^p(\mathbb{R}_+))} + \big|\widetilde{A}_\xi(z_0)\big| < \varepsilon,$$

and therefore the operator B is left invertible on the space $L^p(\mathbb{R}_+)$ along with A. Since $B_\xi = \widetilde{A}_\xi - \widetilde{A}_\xi(z_0)I$ and hence $B_\xi(z) = \widetilde{A}_\xi(z) - \widetilde{A}_\xi(z_0)$, it follows that $B_\xi(z_0) = 0$, which is impossible in view of Lemma 5.1.

If the operator $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ is right invertible on the space $L^p(\mathbb{R}_+)$, then passing to the left invertible adjoint operator

$$A^* = \sum_{k\in\mathbb{Z}} (\overline{a_k} \circ \alpha_{-k}) U_{\alpha_{-1}}^k \in \mathfrak{A}_W \subset \mathcal{B}\big(L^q(\mathbb{R}_+)\big),$$

where $1/p + 1/q = 1$, we obtain (5.5) from the part already proved. □

Corollary 5.3. *If $p \in (1,\infty)$ and a functional operator $A \in \mathfrak{A}_W$ is left or right invertible on the Lebesgue space $L^p(\mathbb{R}_+)$, then for all $\xi \in \Delta$ the limit operators $A_\xi \in \mathfrak{A}_W$ are two-sided invertible on the space $L^p(\mathbb{R}_+)$.*

Proof. Since the operator $A \in \mathfrak{A}_W$ is one-sided invertible on the space $L^p(\mathbb{R}_+)$ for $p \in (1,\infty)$, we infer from Theorem 5.2 that $A_\xi(z) \ne 0$ for all $\xi \in \Delta$ and all $z \in \mathbb{T}$. Hence, for every $\xi \in \Delta$ the Gelfand transform $A_\xi(\cdot)$ of the limit operator A_ξ is invertible in the algebra $C(\mathbb{T})$, which implies the invertibility of the operator A_ξ on the space $L^p(\mathbb{R}_+)$. □

Under conditions of Theorem 5.2, for every $\xi \in \Delta$, we can define the Cauchy indices for the invertible functions $z \mapsto A_\xi(z)$ by letting

$$\operatorname{ind} A_\xi(\cdot) := \frac{1}{2\pi} \big\{\arg A_\xi(z)\big\}_{z\in\mathbb{T}}, \tag{5.8}$$

where $\{\arg A_\xi(z)\}_{z\in\mathbb{T}}$ denotes the increment of any continuous branch of $\arg A_\xi(z)$ as z traces $\mathbb{T}$ counterclockwise.

Corollary 5.4. *If the functional operator $A=\sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ with coefficients $a_k\in \mathrm{SO}(\mathbb{R}_+)$ and a shift $\alpha\in \mathrm{SOS}(\mathbb{R}_+)$ is left or right invertible on the Lebesgue space $L^p(\mathbb{R}_+)$ with $p\in(1,\infty)$, then the Cauchy indices $\operatorname{ind} A_\xi(\cdot)$ given by (5.8) coincide for all $\xi\in M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))$ and, respectively, for all $\xi\in M_{\tau_-}(\mathrm{SO}(\mathbb{R}_+))$.*

Proof. By Theorem 5.2, the one-sided invertibility of the operator $A\in\mathfrak{A}_W$ implies the fulfillment of (5.5). Hence the Cauchy indices $\operatorname{ind} A_\xi(\cdot)$ are defined for all $\xi\in M_{\tau_\pm}(\mathrm{SO}(\mathbb{R}_+))$. Since the fibers $M_{\tau_\pm}(\mathrm{SO}(\mathbb{R}_+))$ for $\tau_\pm\in\{0,\infty\}$ are connected compact Hausdorff spaces by Lemma 2.3, and since the function $\xi\mapsto A_\xi(\cdot)$ with values in the unital Banach algebra $W\subset C(\mathbb{T})$ given by (4.7) is continuous on the compacts $M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))$ and $M_{\tau_-}(\mathrm{SO}(\mathbb{R}_+))$ for every left or right invertible operator $A\in\mathfrak{A}_W$, we infer that the function $\xi\mapsto \operatorname{ind} A_\xi(\cdot)$ is a constant for all $\xi\in M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))$ and, respectively, for all $\xi\in M_{\tau_-}(\mathrm{SO}(\mathbb{R}_+))$. □

By Corollary 5.4, the numbers $\operatorname{ind} A_\xi(\cdot)$ do not depend on $\xi\in M_s(\mathrm{SO}(\mathbb{R}_+))$ and can only depend on $s\in\{0,\infty\}$. Hence, we uniquely define the numbers

$$N_\pm := \operatorname{ind} A_\xi(\cdot) \quad \text{for all } \xi\in M_{\tau_\pm}\big(\mathrm{SO}(\mathbb{R}_+)\big), \tag{5.9}$$

where $\tau_\pm\in\{0,\infty\}$.

5.2. **Necessary conditions for discrete operators at fixed points of the shift.** Since the Gelfand transforms for the limit operators $A_\xi\in\mathfrak{A}_W$ and $\widehat{A}_\xi\in\mathcal{W}_p$ are the same, and since for every $p\in(1,\infty)$ and every $z_0\in\mathbb{C}$ the kernels of the operators $\mathcal{V}^{\pm1}-z_0I$ on the space l^p are trivial, we obtain the following result by analogy with Lemma 5.1.

Lemma 5.5. *If $p\in(1,\infty)$, $M\in\mathbb{N}$, $A=\sum_{|k|\le M} a_k U_\alpha^k\in\mathfrak{A}_W$, and for some $t\in\mathbb{R}_+$ the discrete operator*

$$\mathcal{A}(t)=\sum_{|k|\le M} \operatorname{diag}\big\{a_k\big[\alpha_j(t)\big]\big\}_{j\in\mathbb{Z}}\mathcal{V}^k\in\mathcal{W}_p \tag{5.10}$$

is left or right invertible on the space l^p, then (5.1) holds.

Proof. Let $p\in(1,\infty)$, and let for some $t\in\mathbb{R}_+$ the discrete operator (5.10) be left invertible on the space l^p. On the contrary, suppose that $A_\xi(z_0)=0$ for some $\xi\in\Delta$ and some $z_0\in\mathbb{T}$. Then we deduce from Corollary 4.8 that for this $\xi\in\Delta$ the limit operator $\widehat{A}_\xi:=\sum_{|k|\le M} a_k(\xi)\mathcal{V}^k\in\mathcal{W}_p$ possesses the properties: $\operatorname{Ker}\widehat{A}_\xi=\{0\}$ and $\operatorname{Im}\widehat{A}_\xi$ is a closed subspace of l^p. Since $\operatorname{Ker}\widehat{A}_\xi=\{0\}$, we conclude that the Gelfand transform $z\mapsto\widehat{A}_\xi(z)$ of the operator $\widehat{A}_\xi$ is not equal to zero identically, because otherwise $\operatorname{Ker}\widehat{A}_\xi=l^p$. As the Gelfand transforms for the limit operators $\widehat{A}_\xi\in\mathcal{W}_p$ and $A_\xi\in\mathfrak{A}_W$ are the same, we conclude that there exist numbers $m_1,m_2\in\{-M,\dots,M\}$ such that $m_1<m_2$, $a_k(\xi)\ne0$ for $k\in\{m_1,m_2\}$ and, according to (5.2),

$$\widehat{A}_\xi(z)=A_\xi(z)=a_{m_2}(\xi)z^{m_1}\prod_{k=1}^{m_2-m_1}(z-z_k) \quad \text{for } z\in\mathbb{T}, \tag{5.11}$$

where all $z_k \in \mathbb{C}$. Since $\widehat{A}_\xi(z_0) = A_\xi(z_0) = 0$, we conclude that at least one z_k in (5.11) coincides with $z_0 \in \mathbb{T}$. Further, from (5.11) it follows that

$$\widehat{A}_\xi = a_{m_2}(\xi)\mathcal{V}^{m_1} \prod_{k=1}^{m_2-m_1} (\mathcal{V} - z_k I) \in \mathcal{B}(l^p).$$

Passing to the adjoint operator, we obtain

$$\widehat{A}_\xi^* = \overline{a_{m_2}(\xi)}\mathcal{V}^{-m_1} \prod_{k=1}^{m_2-m_1} (\mathcal{V}^{-1} - \overline{z_k} I) \in \mathcal{B}(l^q), \tag{5.12}$$

where $1/p+1/q = 1$ and for every $k = 1, 2, \dots, m_2-m_1$ either $|z_k| \neq 1$ or $|z_0| = 1$. If $|z_k| \neq 1$, then the operator $\mathcal{V}^{-1} - \overline{z_k} I$ is invertible on the space l^q because $\mathcal{V}$ is an isometric operator on l^q. If $|z_0| = 1$, then any solution $f = \{f_j\}_{j\in\mathbb{Z}} \in l^q$ of the equation $\mathcal{V}^{-1} f - \overline{z_0} f = 0$ possesses the property: $|f_j| = |f_0|$ for all $j \in \mathbb{Z}$, which implies that $\operatorname{Ker}(\mathcal{V}^{-1} - \overline{z_0} I) = \{0\}$. Hence, by (5.12), $\operatorname{Ker} \widehat{A}_\xi^* = \{0\}$. Since $\operatorname{Im} A_\xi$ is closed in l^p, $\operatorname{Ker} \widehat{A}_\xi = \{0\}$, and $\operatorname{Ker} \widehat{A}_\xi^* = \{0\}$, we conclude that the operator $\widehat{A}_\xi$ is invertible on the space l^p. But then the Gelfand transform $\widehat{A}_\xi(\cdot)$ of the operator $\widehat{A}_\xi$ should be separated from zero for all $z \in \mathbb{T}$, which contradicts the assumption that $A_\xi(z_0) = 0$.

If the operator $\mathcal{A}(t) = \sum_{|k|\le M} \operatorname{diag}\{a_k[\alpha_j(t)]\}_{j\in\mathbb{Z}} \mathcal{V}^k \in \mathcal{W}_p$ is right invertible on the space l^p, then passing to the left invertible adjoint operator

$$\big(\mathcal{A}(t)\big)^* = \sum_{|k|\le M} \operatorname{diag}\big\{\overline{a_k\big[\alpha_{j-k}(t)\big]}\big\}_{j\in\mathbb{Z}} \mathcal{V}^{-k} \in \mathcal{W}_q,$$

where $1/p + 1/q = 1$ and taking into account the fact that $\overline{a_k} \circ \alpha_{j-k} \in SO(\mathbb{R}_+)$ and that $(\overline{a_k} \circ \alpha_{-k})(\xi) = \overline{a_k(\xi)}$ for all $\xi \in \Delta$ (see Lemma 2.4 and Corollary 2.5), we conclude from the part already proved that

$$\widehat{A}_\xi^*(z) = \sum_{|k|\le M} \overline{a_k(\xi)} z^{-k} \neq 0 \quad \text{for all } \xi \in \Delta \text{ and all } z \in \mathbb{T},$$

which implies (5.1). □

Lemma 5.5, similarly to Theorem 5.2, leads to the following assertion.

Theorem 5.6. *If $p \in (1, \infty)$ and for some $t \in \mathbb{R}_+$ the discrete operator*

$$\mathcal{A}(t) = \sum_{k\in\mathbb{Z}} \operatorname{diag}\big\{a_k\big[\alpha_j(t)\big]\big\}_{j\in\mathbb{Z}} \mathcal{V}^k \in \mathcal{W}_p, \tag{5.13}$$

associated with an operator $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$, is left or right invertible on the space l^p, then (5.5) holds and the numbers $N_\pm$ are uniquely defined by (5.9).

Proof. Let for some $t \in \mathbb{R}_+$ the discrete operator (5.13) be left invertible on the space l^p, and, contrary to (5.5), we suppose that $A_\xi(z_0) = 0$ for some $\xi \in \Delta$ and some $z_0 \in \mathbb{T}$. For $\mathcal{A}(t) \in \mathcal{W}_p$ and given $\xi \in \Delta$, we take the limit operator

$$\widehat{A}_\xi = \sum_{k\in\mathbb{Z}} a_k(\xi)\mathcal{V}^k \in \mathcal{W}_p.$$

By the stability of the left invertibility, there is an $\varepsilon > 0$ such that every discrete operator $\mathcal{F}(t) \in \mathcal{W}_p$ with $\|\mathcal{A}(t) - \mathcal{F}(t)\|_{\mathcal{B}(l^p)} < \varepsilon$ is also left invertible on the space l^p. We now choose $M \in \mathbb{N}$ such that the discrete operator

$$\mathcal{F}(t) := \sum_{|k| \le M} \mathrm{diag}\big\{a_k\big[\alpha_j(t)\big]\big\}_{j\in\mathbb{Z}} \mathcal{V}^k \in \mathcal{W}_p$$

satisfies the inequalities

$$\big\|\mathcal{A}(t) - \mathcal{F}(t)\big\|_{\mathcal{B}(l^p)} \le \big\|\mathcal{A}(t) - \mathcal{F}(t)\big\|_W < \varepsilon/2. \tag{5.14}$$

Since the functions $A_\xi(z)$ and $F_\xi(z) := \sum_{|k|\le M} a_k(\xi) z^k$ on $\mathbb{T}$ are the Gelfand transforms for the limit operators $\widehat{A}_\xi \in \mathcal{W}_p$ and $\widehat{F}_\xi := \sum_{|k|\le M} a_k(\xi)\mathcal{V}^k \in \mathcal{W}_p$, and since

$$\max_{z\in\mathbb{T}} \big|A_\xi(z) - F_\xi(z)\big| \le \|\widehat{A}_\xi - \widehat{F}_\xi\|_{\mathcal{B}(l^p)} \le \big\|\mathcal{A}(t) - \mathcal{F}(t)\big\|_{\mathcal{B}(l^p)} < \varepsilon/2$$

for all $\xi \in \Delta$, we deduce from the equality $A_\xi(z_0) = 0$ that

$$\big|F_\xi(z_0)\big| < \varepsilon/2. \tag{5.15}$$

We now consider the discrete operator $\mathcal{H}(t) := \mathcal{F}(t) - F_\xi(z_0)I \in \mathcal{W}_p$. Then we infer from (5.14) and (5.15) that

$$\big\|\mathcal{A}(t) - \mathcal{H}(t)\big\|_{\mathcal{B}(l^p)} \le \big\|\mathcal{A}(t) - \mathcal{F}(t)\big\|_{\mathcal{B}(l^p)} + \big|F_\xi(z_0)\big| < \varepsilon,$$

and therefore the discrete operator $\mathcal{H}(t) \in \mathcal{W}_p$ is left invertible on the space l^p along with $\mathcal{A}(t)$. Finally, since $\widehat{H}_\xi = \widehat{F}_\xi - F_\xi(z_0)I$ and hence $H_\xi(z) = F_\xi(z) - F_\xi(z_0)$, we conclude that $H_\xi(z_0) = 0$, which is impossible in view of Lemma 5.5.

If the discrete operator (5.13) is right invertible on the space l^p, then passing to the left invertible adjoint operator

$$\big(\mathcal{A}(t)\big)^* = \sum_{k\in\mathbb{Z}} \mathrm{diag}\big\{a_k\big[\alpha_{j-k}(t)\big]\big\}_{j\in\mathbb{Z}} \mathcal{V}^{-k} \in \mathcal{W}_q,$$

where $1/p + 1/q = 1$, we conclude from the part already proved that

$$A_\xi^*(z) = \sum_{k\in\mathbb{Z}} \overline{a_k(\xi)} z^{-k} \ne 0 \quad \text{for all } \xi \in \Delta \text{ and all } z \in \mathbb{T},$$

which implies (5.5).

Finally, by the proof of Corollary 5.4, the numbers $N_\pm$ are uniquely defined by (5.9) as soon as (5.5) holds. □

In view of Theorem 5.6, for any one-sided invertible discrete operator $\mathcal{A}(t) \in \mathcal{W}_p$, we also can uniquely define the numbers $N_\pm$.

6. Necessary one-sided invertibility conditions related to points $t \in \mathbb{R}_+$

6.1. **Invertibility of outermost blocks.** Fix a point $\tau \in \mathbb{R}_+$, and let γ be a semisegment of $\mathbb{R}_+$ with endpoints τ and $\alpha(\tau)$, where $\tau \in \gamma$ and $\alpha(\tau) \notin \gamma$. Consider the intervals

$$\gamma_n^+ := \bigcup_{k=n}^{\infty} \alpha_k(\gamma), \qquad \gamma_n^- := \bigcup_{k=n}^{\infty} \alpha_{-k}(\gamma), \qquad \gamma_n^0 := \bigcup_{k=-n+1}^{n-1} \alpha_k(\gamma). \tag{6.1}$$

Let $\chi_n^\pm$ and χ_n^0 denote the operators of multiplication by the characteristic functions of $\gamma_n^\pm$ and γ_n^0, respectively.

Let $\mathfrak{A}_W^\pm$ be the unital Banach subalgebras of the unital Banach algebra $\mathfrak{A}_W$, which are given, respectively, by

$$\mathfrak{A}_W^\pm := \Big\{ A = \sum_{k\in\mathbb{Z}_+} a_k^\pm U_\alpha^{\pm k} : a_k^\pm \in SO(\mathbb{R}_+), \|A\|_W = \sum_{k\in\mathbb{Z}_+} \|a_k^\pm\|_{C_b(\mathbb{R}_+)} < \infty \Big\},$$

where $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$. Let $W \subset C(\mathbb{T})$ be the unital Banach algebra (4.7), and let $W^\pm$ be the unital Banach subalgebras of W which are given by

$$W^\pm = \Big\{ f = \sum_{k\in\mathbb{Z}_+} a_k^\pm z^{\pm k} : a_k^\pm \in \mathbb{C}, z \in \mathbb{T}, \|f\|_W := \sum_{k\in\mathbb{Z}_+} |a_k^\pm| < \infty \Big\}. \tag{6.2}$$

Theorem 6.1. *If $p \in [1,\infty]$, $A \in \mathfrak{A}_W \subset \mathcal{B}(L^p(\mathbb{R}_+))$, $A_\xi(z) \neq 0$ for all $(\xi, z) \in \Delta \times \mathbb{T}$, and the numbers $N_\pm$ are given by (5.9), then there exists a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the operators*

$$\chi_n^+ A \chi_{n+N_+}^+ : \chi_{n+N_+}^+ L^p(\mathbb{R}_+) \to \chi_n^+ L^p(\mathbb{R}_+), \tag{6.3}$$

$$\chi_n^- A \chi_{n-N_-}^- : \chi_{n-N_-}^- L^p(\mathbb{R}_+) \to \chi_n^- L^p(\mathbb{R}_+) \tag{6.4}$$

are invertible.

Proof. It is sufficient to prove the invertibility of operator (6.3) (the proof for the operator (6.4) is analogous). Moreover, we may assume without loss of generality that $N_+ = 0$. Indeed, taking into account the equalities

$$\begin{cases} \chi_n^+ U_\alpha^k \chi_n^+ = \chi_n^+ U_\alpha^k = U_\alpha^k \chi_{n+k}^+ \\ \chi_n^+ U_\alpha^{-k} \chi_n^+ = U_\alpha^{-k} \chi_n^+ = \chi_{n+k}^+ U_\alpha^{-k} \end{cases} \quad \text{for } k \in \mathbb{Z}_+, \tag{6.5}$$

we infer for $N_+ \in \mathbb{Z}$ that

$$(\chi_n^+ A \chi_{n+N_+}^+)(\chi_{n+N_+}^+ U_\alpha^{-N_+} \chi_n^+) = \chi_n^+ (A U_\alpha^{-N_+}) \chi_n^+,$$

where $\operatorname{ind}(A U_\alpha^{-N_+})_\xi(\cdot) = 0$ for all $\xi \in M_{\tau_+}(SO(\mathbb{R}_+))$ and

$$(\chi_{n+N_+}^+ U_\alpha^{-N_+} \chi_n^+)^{-1} = \chi_n^+ U_\alpha^{N_+} \chi_{n+N_+}^+.$$

Thus, it remains to prove that for sufficiently large $n \in \mathbb{N}$ the operator $\chi_n^+ A \chi_n^+$ is invertible on the space $\chi_n^+ L^p(\mathbb{R}_+)$ if $\operatorname{ind} A_\xi(\cdot) = 0$ for all $\xi \in M_{\tau_+}(SO(\mathbb{R}_+))$.

Since $A_\xi(z) \neq 0$ for all $\xi \in M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))$ and all $z \in \mathbb{T}$, and since $\operatorname{ind} A_\xi(\cdot) = 0$ for these ξ, we infer from [9, Chapter 3, Corollary 3.2] that for all $\xi \in M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))$ the functions $z \mapsto A_\xi(z)$ admit canonical factorizations

$$A_\xi(z) = A_\xi^+(z) A_\xi^-(z) \quad (z \in \mathbb{T}), \tag{6.6}$$

where the functions $A_\xi^\pm(\cdot), (A_\xi^\pm(\cdot))^{-1}$ belong to the unital Banach algebras $W^\pm \subset C(\mathbb{T})$ given by (6.2). Then $\frac{1}{2\pi}\int_{\mathbb{T}} A_\xi^\pm(z)|dz| \neq 0$, and therefore factorization (6.6) is unique if, for example, $\frac{1}{2\pi}\int_{\mathbb{T}} A_\xi^+(z)|dz| = 1$. Note that $\xi \mapsto A_\xi^\pm(\cdot)$ and $\xi \mapsto (A_\xi^\pm(\cdot))^{-1}$ are continuous $W^\pm$-valued functions with respect to $\xi \in M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))$.

The functions $A_\xi^\pm(\cdot)$ and their inverses are represented in the form

$$A_\xi^\pm(z) = \sum_{k \in \mathbb{Z}_+} b_{k,\xi}^\pm z^{\pm k}, \qquad \big(A_\xi^\pm(z)\big)^{-1} = \sum_{k \in \mathbb{Z}_+} c_{k,\xi}^\pm z^{\pm k}, \tag{6.7}$$

where $b_{0,\xi}^+ = c_{0,\xi}^+ = 1$, $b_{0,\xi}^- c_{0,\xi}^- = 1$, $b_{k,\xi}^\pm, c_{k,\xi}^\pm \in \mathbb{C}$ for all $k \in \mathbb{N}$ and

$$\big\|A_\xi^\pm(\cdot)\big\|_W = \sum_{k \in \mathbb{Z}_+} |b_{k,\xi}^\pm| < \infty, \qquad \big\|\big(A_\xi^\pm(\cdot)\big)^{-1}\big\|_W = \sum_{k \in \mathbb{Z}_+} |c_{k,\xi}^\pm| < \infty. \tag{6.8}$$

The functions $A_\xi^\pm(\cdot), (A_\xi^\pm(\cdot))^{-1}$ are analytic on the domains $D_\pm := \{z \in \mathbb{C} : |z|^{\pm 1} < 1\}$.

The functions $\xi \mapsto b_{k,\xi}^\pm$ and $\xi \mapsto c_{k,\xi}^\pm$ for $k \in \mathbb{Z}_+$ are continuous on the compact $M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))$ of the Hausdorff space $M(\mathrm{SO}(\mathbb{R}_+))$. With functions (6.7) given for every $\xi \in M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))$ we associate the functional operators

$$B^\pm = \sum_{k \in \mathbb{Z}_+} b_k^\pm U_\alpha^{\pm k} \in \mathfrak{A}_W^\pm, \qquad C^\pm = \sum_{k \in \mathbb{Z}_+} c_k^\pm U_\alpha^{\pm k} \in \mathfrak{A}_W^\pm, \tag{6.9}$$

where $b_k^\pm$ and $c_k^\pm$ are functions in $\mathrm{SO}(\mathbb{R}_+)$ such that $b_k^\pm(\xi) = b_{k,\xi}^\pm$ and $c_k^\pm(\xi) = c_{k,\xi}^\pm$ for all $\xi \in M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))$ and all $k \in \mathbb{Z}_+$. Representing the functions $b_k^\pm$ and $c_k^\pm$ as continuous functions on the normal topological space $M(\mathrm{SO}(\mathbb{R}_+))$ that are extensions of continuous functions $b_k^\pm|_{M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))}$ and $c_k^\pm|_{M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))}$ (we save here the notation $b_k^\pm$ and $c_k^\pm$ for the Gelfand transforms of these functions) and applying the Urysohn–Tietze extension theorem to their real and imaginary parts, we can attain the properties:

$$\begin{aligned}
\|\operatorname{Re} b_k^\pm\|_{C_b(\mathbb{R}_+)} &= \|\operatorname{Re} b_k^\pm|_{M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))}\|_{C(M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+)))},\\
\|\operatorname{Im} b_k^\pm\|_{C_b(\mathbb{R}_+)} &= \|\operatorname{Im} b_k^\pm|_{M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))}\|_{C(M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+)))},\\
\|\operatorname{Re} c_k^\pm\|_{C_b(\mathbb{R}_+)} &= \|\operatorname{Re} c_k^\pm|_{M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))}\|_{C(M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+)))},\\
\|\operatorname{Im} c_k^\pm\|_{C_b(\mathbb{R}_+)} &= \|\operatorname{Im} c_k^\pm|_{M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))}\|_{C(M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+)))}.
\end{aligned}$$

These equalities imply that

$$\begin{aligned}
\|b_k^\pm\|_{C_b(\mathbb{R}_+)} &\le 2\|b_k^\pm|_{M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))}\|_{C(M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+)))} = 2 \max_{\xi \in M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))} |b_{k,\xi}^\pm|,\\
\|c_k^\pm\|_{C_b(\mathbb{R}_+)} &\le 2\|c_k^\pm|_{M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))}\|_{C(M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+)))} = 2 \max_{\xi \in M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))} |c_{k,\xi}^\pm|.
\end{aligned}$$

Hence the operators $B^\pm, C^\pm$ belong to the Banach subalgebras $\mathfrak{A}_W^\pm$ of $\mathfrak{A}_W$, and

$$\begin{aligned} D^\pm &:= B^\pm C^\pm = \sum_{k\in\mathbb{Z}_+} d_k^\pm U_\alpha^{\pm k} \in \mathfrak{A}_W^\pm, \qquad d_k^\pm = \sum_{j=0}^{k} b_j^\pm (c_{k-j}^\pm \circ \alpha_{\pm j}), \\ \widetilde{D}^\pm &:= C^\pm B^\pm = \sum_{k\in\mathbb{Z}_+} \widetilde{d}_k^\pm U_\alpha^{\pm k} \in \mathfrak{A}_W^\pm, \qquad \widetilde{d}_k^\pm = \sum_{j=0}^{k} c_j^\pm (b_{k-j}^\pm \circ \alpha_{\pm j}), \end{aligned} \tag{6.10}$$

where $d_k^\pm, \widetilde{d}_k^\pm \in \mathrm{SO}(\mathbb{R}_+)$ according to Lemma 2.4 and Corollary 2.5.

Applying the equalities (6.5), we obtain the following relations:

$$\begin{aligned} (\chi_n^+ B^+ \chi_n^+)(\chi_n^+ C^+ \chi_n^+) &= \chi_n^+ D^+ = \chi_n^+ \sum_{k\in\mathbb{Z}_+} d_k^+ U_\alpha^k, \\ (\chi_n^+ C^+ \chi_n^+)(\chi_n^+ B^+ \chi_n^+) &= \chi_n^+ \widetilde{D}^+ = \chi_n^+ \sum_{k\in\mathbb{Z}_+} \widetilde{d}_k^+ U_\alpha^k, \\ (\chi_n^+ B^- \chi_n^+)(\chi_n^+ C^- \chi_n^+) &= D^- \chi_n^+ = \sum_{k\in\mathbb{Z}_+} d_k^- U_\alpha^{-k} \chi_n^+, \\ (\chi_n^+ C^- \chi_n^+)(\chi_n^+ B^- \chi_n^+) &= \widetilde{D}^- \chi_n^+ = \sum_{k\in\mathbb{Z}_+} \widetilde{d}_k^- U_\alpha^{-k} \chi_n^+, \end{aligned} \tag{6.11}$$

where for every $\xi \in M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))$ we infer in view of (6.9) and the equalities $b_k^\pm(\xi) = b_{k,\xi}^\pm$, $c_k^\pm(\xi) = c_{k,\xi}^\pm$, $b_{0,\xi}^+ = c_{0,\xi}^+ = 1$, and $b_{0,\xi}^- c_{0,\xi}^- = 1$ that $d_0^\pm(\xi) = \widetilde{d}_0^\pm(\xi) = 1$ and $d_k^\pm(\xi) = \widetilde{d}_k^\pm(\xi) = 0$ for all $k \in \mathbb{N}$. Hence, the functions $d_k^\pm$ and $\widetilde{d}_k^\pm$ for all $k \in \mathbb{Z}_+$ are continuous at the point τ_+, and therefore for every $\varepsilon \in (0,1)$ there exists an $N \in \mathbb{N}$ such that for every $n \geq N$ the following inequalities hold:

$$\begin{aligned} \Big\| \chi_n^+ \Big(\sum_{k\in\mathbb{Z}_+} d_k^\pm U_\alpha^{\pm k} - I \Big) \chi_n^+ \Big\|_{\mathcal{B}(\chi_n^+ L^p(\mathbb{R}_+))} &< \varepsilon, \\ \Big\| \chi_n^+ \Big(\sum_{k\in\mathbb{Z}_+} \widetilde{d}_k^\pm U_\alpha^{\pm k} - I \Big) \chi_n^+ \Big\|_{\mathcal{B}(\chi_n^+ L^p(\mathbb{R}_+))} &< \varepsilon. \end{aligned}$$

To this end, we first approximate the corresponding operator series by functional polynomials $\sum_{k=0}^K d_k^\pm U_\alpha^{\pm k}$ and $\sum_{k=0}^K \widetilde{d}_k^\pm U_\alpha^{\pm k}$ with sufficiently large $K \in \mathbb{N}$ in the Banach algebras $\mathfrak{A}_W^\pm$, respectively, and then choose $n \in \mathbb{N}$ in view of the continuity of the coefficients of these operators at the point τ_+. Consequently, the operators

$$\chi_n^+ D^\pm \chi_n^+ = \chi_n^+ \Big(\sum_{k\in\mathbb{Z}_+} d_k^\pm U_\alpha^{\pm k} \Big) \chi_n^+ \qquad \text{and} \qquad \chi_n^+ \widetilde{D}^\pm \chi_n^+ = \chi_n^+ \Big(\sum_{k\in\mathbb{Z}_+} \widetilde{d}_k^\pm U_\alpha^{\pm k} \Big) \chi_n^+$$

are invertible on the subspace $\chi_n^+ L^p(\mathbb{R}_+)$, which implies (due to (6.11)) the invertibility of the operators $\chi_n^+ B^\pm \chi_n^+$ and $\chi_n^+ C^\pm \chi_n^+$ on this subspace. Moreover, the operators $\chi_n^+ D^\pm \chi_n^+$, $\chi_n^+ \widetilde{D}^\pm \chi_n^+$, and hence the operators $\chi_n^+ B^\pm \chi_n^+$, $\chi_n^+ C^\pm \chi_n^+$ are invertible in the Banach subalgebras $\chi_n^+ \mathfrak{A}_W^\pm \chi_n^+$ of $W_{p,\mathfrak{G}}$, respectively.

The operator C^+AC^- belongs to the Banach algebra $\mathfrak{A}_W$ along with A, and from (6.7) and (6.9) it follows that, for all $\xi \in M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))$ and all $z \in \mathbb{T}$,

$$(C^+AC^-)_\xi(z) = \big(A_\xi^+(z)\big)^{-1} A_\xi(z)\big(A_\xi^-(z)\big)^{-1} = 1,$$

whence all coefficients of the operator C^+AC^- are continuous at the point τ_+, and therefore for every sufficiently large $n \in \mathbb{N}$, the operator $\chi_n^+(C^+AC^-)\chi_n^+$ is close to the identity operator on the subspace $\chi_n^+L^p(\mathbb{R}_+)$. Consequently, for such $n \in \mathbb{N}$ the operator $\chi_n^+(C^+AC^-)\chi_n^+$ is invertible on the subspace $\chi_n^+L^p(\mathbb{R}_+)$. Since

$$(\chi_n^+C^+\chi_n^+)(\chi_n^+A\chi_n^+)(\chi_n^+C^-\chi_n^+) = \chi_n^+(C^+AC^-)\chi_n^+,$$

we conclude that for every sufficiently large $n \in \mathbb{N}$, the operator $\chi_n^+A\chi_n^+$ is invertible on the subspace $\chi_n^+L^p(\mathbb{R}_+)$ along with $\chi_n^+(C^+AC^-)\chi_n^+$ and $\chi_n^+C^\pm\chi_n^+$. Moreover, the operator $\chi_n^+(C^+AC^-)\chi_n^+$ is close to the identity operator in the Banach subalgebra $\chi_n^+W_{p,\mathfrak{G}}\chi_n^+$ of the Wiener-type algebra $W_{p,\mathfrak{G}}$. Hence, the operators $\chi_n^+(C^+AC^-)\chi_n^+$ and $\chi_n^+A\chi_n^+$ are invertible in the algebra $\chi_n^+W_{p,\mathfrak{G}}\chi_n^+$.

Similarly, since $A_\xi(z) \neq 0$ for all $\xi \in M_{\tau_-}(\mathrm{SO}(\mathbb{R}_+))$ and all $z \in \mathbb{T}$, and therefore $N_- = \operatorname{ind} A_\xi(\cdot)$ for all these ξ's, we can reduce the study to the case $N_- = 0$. Hence, we conclude that for every $\xi \in M_{\tau_-}(\mathrm{SO}(\mathbb{R}_+))$ the function $z \mapsto A_\xi(z)$ admits the canonical factorization (6.6) with functions $A_\xi^\pm(\cdot), (A_\xi^\pm(\cdot))^{-1} \in W^\pm$ satisfying (6.7) and (6.8), where $b_{0,\xi}^+ = c_{0,\xi}^+ = 1$, $b_{0,\xi}^- c_{0,\xi}^- = 1$ and $b_{k,\xi}^\pm, c_{k,\xi}^\pm \in \mathbb{C}$ for all $k \in \mathbb{N}$ and all $\xi \in M_{\tau_-}(\mathrm{SO}(\mathbb{R}_+))$.

With functions (6.7) given for every $\xi \in M_{\tau_-}(\mathrm{SO}(\mathbb{R}_+))$, we associate the functional operators (6.9) where now $b_k^\pm$ and $c_k^\pm$ are functions in $\mathrm{SO}(\mathbb{R}_+)$ such that $b_k^\pm(\xi) = b_{k,\xi}^\pm$ and $c_k^\pm(\xi) = c_{k,\xi}^\pm$ for all $k \in \mathbb{Z}_+$ and all $\xi \in M_{\tau_-}(\mathrm{SO}(\mathbb{R}_+))$.

Applying the equalities

$$\begin{cases} \chi_n^- U_\alpha^k \chi_n^- = U_\alpha^k \chi_n^- = \chi_{n+k}^- U_\alpha^k \\ \chi_n^- U_\alpha^{-k} \chi_n^- = \chi_n^- U_\alpha^{-k} = U_\alpha^{-k} \chi_{n+k}^- \end{cases} \quad \text{for } k \in \mathbb{Z}_+,$$

we deduce in contrast to (6.11) that

$$\begin{aligned}
(\chi_n^- B^+ \chi_n^-)(\chi_n^- C^+ \chi_n^-) &= D^+ \chi_n^- = \sum_{k \in \mathbb{Z}_+} d_k^+ U_\alpha^k \chi_n^-, \\
(\chi_n^- C^+ \chi_n^-)(\chi_n^- B^+ \chi_n^-) &= \widetilde{D}^+ \chi_n^- = \sum_{k \in \mathbb{Z}_+} \widetilde{d}_k^+ U_\alpha^k \chi_n^-, \\
(\chi_n^- B^- \chi_n^-)(\chi_n^- C^- \chi_n^-) &= \chi_n^- D^- = \chi_n^- \sum_{k \in \mathbb{Z}_+} d_k^- U_\alpha^{-k}, \\
(\chi_n^- C^- \chi_n^-)(\chi_n^- B^- \chi_n^-) &= \chi_n^- \widetilde{D}^- = \chi_n^- \sum_{k \in \mathbb{Z}_+} \widetilde{d}_k^- U_\alpha^{-k},
\end{aligned}$$

where the operators $D^\pm, \widetilde{D}^\pm$ and coefficients $d_k^\pm, \widetilde{d}_k^\pm \in \mathrm{SO}(\mathbb{R}_+)$ are given by (6.10). We can then infer by analogy with the part already proved that for all sufficiently large $n \in \mathbb{N}$ the operators $\chi_n^- B^\pm \chi_n^-$ and $\chi_n^- C^\pm \chi_n^-$ are invertible on the subspace $\chi_n^- L^p(\mathbb{R}_+)$.

The operator C^-AC^+ belongs to the Banach algebra $\mathfrak{A}_W$ along with A, and from (6.7) and (6.9) it follows that, for all $\xi \in M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))$ and all $z \in \mathbb{T}$,

$$(C^-AC^+)_\xi(z) = \big(A_\xi^-(z)\big)^{-1} A_\xi(z) \big(A_\xi^+(z)\big)^{-1} = 1,$$

whence all coefficients of the operator C^-AC^+ are continuous at the point τ_-, and therefore for every sufficiently large $n \in \mathbb{N}$ the operator $\chi_n^-(C^-AC^+)\chi_n^-$ is close to the identity operator on the subspace $\chi_n^- L^p(\mathbb{R}_+)$. Consequently, for such $n \in \mathbb{N}$ the operator $\chi_n^-(C^-AC^+)\chi_n^-$ is invertible on the subspace $\chi_n^- L^p(\mathbb{R}_+)$. Since

$$(\chi_n^- C^- \chi_n^-)(\chi_n^- A \chi_n^-)(\chi_n^- C^- \chi_n^-) = \chi_n^-(C^-AC^+)\chi_n^-,$$

it follows that for every sufficiently large $n \in \mathbb{N}$ the operator $\chi_n^- A \chi_n^-$ is invertible on the subspace $\chi_n^- L^p(\mathbb{R}_+)$ along with $\chi_n^-(C^-AC^+)\chi_n^-$ and $\chi_n^- C^\pm \chi_n^-$. Moreover, by analogy with the case of $M_{\tau_+}(\mathrm{SO}(\mathbb{R}_+))$, the operator $\chi_n^- A \chi_n^-$ is invertible in the Banach subalgebra $\chi_n^- W_{p,\mathfrak{S}} \chi_n^-$ of $W_{p,\mathfrak{S}}$, which completes the proof. □

Remark 6.2. It follows from the proof of Theorem 6.1 that under the conditions of this theorem the inverses to the operators (6.3) and (6.4) belong to the subsets $\chi_{n+N_+}^+ W_{p,\mathfrak{S}} \chi_n^+$ and $\chi_{n-N_-}^- W_{p,\mathfrak{S}} \chi_n^-$ of $W_{p,\mathfrak{S}}$, respectively.

Theorem 5.2, Corollary 5.4, Theorem 6.1, and Remark 6.2 imply the following.

Theorem 6.3. *If $p \in (1,\infty)$ and the functional operator $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ with coefficients $a_k \in \mathrm{SO}(\mathbb{R}_+)$ and a shift $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$ is left or right invertible on the space $L^p(\mathbb{R}_+)$, then there exists a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the operators (6.3) and (6.4) are invertible, and their inverses belong, respectively, to the subsets $\chi_{n+N_+}^+ W_{p,\mathfrak{S}} \chi_n^+$ and $\chi_{n-N_-}^- W_{p,\mathfrak{S}} \chi_n^-$ of $W_{p,\mathfrak{S}}$.*

6.2. **One-sided invertibility of a modified central block.** Representing the operator $A \in \mathfrak{A}_W$ acting from the direct sum of spaces

$$\chi_{n-N_-}^- L^p(\mathbb{R}_+) \dot{+} \chi_{n-N_-,n+N_+}^0 L^p(\mathbb{R}_+) \dot{+} \chi_{n+N_+}^+ L^p(\mathbb{R}_+)$$

to the direct sum of spaces $\chi_n^- L^p(\mathbb{R}_+) \dot{+} \chi_n^0 L^p(\mathbb{R}_+) \dot{+} \chi_n^+ L^p(\mathbb{R}_+)$ as the operator matrix

$$D := \begin{bmatrix} \chi_n^- A \chi_{n-N_-}^- & \chi_n^- A \chi_{n-N_-,n+N_+}^0 & \chi_n^- A \chi_{n+N_+}^+ \\ \chi_n^0 A \chi_{n-N_-}^- & \chi_n^0 A \chi_{n-N_-,n+N_+}^0 & \chi_n^0 A \chi_{n+N_+}^+ \\ \chi_n^+ A \chi_{n-N_-}^- & \chi_n^+ A \chi_{n-N_-,n+N_+}^0 & \chi_n^+ A \chi_{n+N_+}^+ \end{bmatrix}, \tag{6.12}$$

where $\chi_n^\pm$, χ_n^0 and $\chi_{n-N_-,n+N_+}^0$ are the operators of multiplication by the characteristic functions of the sets $\gamma_n^\pm$, γ_n^0 given in (6.1) and the set $\gamma_{n-N_-,n+N_+}^0 := \bigcup_{k=-n+N_-+1}^{n+N_+-1} \alpha_k(\gamma)$, respectively. Since

$$\lim_{n\to\infty} \|\chi_n^- A \chi_{n+N_+}^+\|_{\mathcal{B}(\chi_{n+N_+}^+ L^p(\mathbb{R}_+), \chi_n^- L^p(\mathbb{R}_+))} = 0,$$
$$\lim_{n\to\infty} \|\chi_n^+ A \chi_{n-N_-}^+\|_{\mathcal{B}(\chi_{n-N_-}^- L^p(\mathbb{R}_+), \chi_n^+ L^p(\mathbb{R}_+))} = 0,$$

we conclude that the invertibility of the operators

$$\chi_n^- A \chi_{n-N_-}^- \in \mathcal{B}\big(\chi_{n-N_-}^- L^p(\mathbb{R}_+), \chi_n^- L^p(\mathbb{R}_+)\big),$$
$$\chi_n^+ A \chi_{n+N_+}^+ \in \mathcal{B}\big(\chi_{n+N_+}^+ L^p(\mathbb{R}_+), \chi_n^+ L^p(\mathbb{R}_+)\big)$$

for all sufficiently large $n \in \mathbb{N}$ implies the invertibility of the operator

$$D_{n,\infty} := \begin{bmatrix} \chi_n^- A \chi_{n-N_-}^- & \chi_n^- A \chi_{n+N_+}^+ \\ \chi_n^+ A \chi_{n-N_-}^- & \chi_n^+ A \chi_{n+N_+}^+ \end{bmatrix}, \tag{6.13}$$

which acts from the Banach space $\chi_{n-N_-}^- L^p(\mathbb{R}_+) \dotplus \chi_{n+N_+}^+ L^p(\mathbb{R}_+)$ onto the Banach space $\chi_n^- L^p(\mathbb{R}_+) \dotplus \chi_n^+ L^p(\mathbb{R}_+)$. Moreover, from Theorem 6.3 it follows that the operator $D_{n,\infty}$, which belongs to the set $(\chi_n^- + \chi_n^+) W_{p,\mathfrak{S}} (\chi_{n-N_-}^- + \chi_{n+N_+}^+)$ has the inverse operator $D_{n,\infty}^{-1} \in (\chi_{n-N_-}^- + \chi_{n+N_+}^+) W_{p,\mathfrak{S}} (\chi_n^- + \chi_n^+)$.

Setting $\widetilde{D}_{n,0} := \chi_n^0 A \chi_{n-N_-,n+N_+}^0$,

$$D_{n,1} := \begin{bmatrix} \chi_n^0 A \chi_{n-N_-}^- & \chi_n^0 A \chi_{n+N_+}^+ \end{bmatrix}, \qquad D_{n,2} := \begin{bmatrix} \chi_n^- A \chi_{n-N_-,n+N_+}^0 \\ \chi_n^+ A \chi_{n-N_-,n+N_+}^0 \end{bmatrix},$$

we can write the operator matrix D in the form

$$D = \begin{bmatrix} \widetilde{D}_{n,0} & D_{n,1} \\ D_{n,2} & D_{n,\infty} \end{bmatrix}$$

with the entries $\widetilde{D}_{n,0} \in \chi_n^0 W_{p,\mathfrak{S}} \chi_{n-N_-,n+N_+}^0$, $D_{n,1} \in \chi_n^0 W_{p,\mathfrak{S}} (\chi_{n-N_-}^- + \chi_{n+N_+}^+)$, $D_{n,2} \in (\chi_n^- + \chi_n^+) W_{p,\mathfrak{S}} \chi_{n-N_-,n+N_+}^0$, and $D_{n,\infty} \in (\chi_n^- + \chi_n^+) W_{p,\mathfrak{S}} (\chi_{n-N_-}^- + \chi_{n+N_+}^+)$.

Since the block $D_{n,\infty}$ given by (6.13) is invertible, we infer that

$$D = \begin{bmatrix} \widetilde{D}_{n,0} & D_{n,1} \\ D_{n,2} & D_{n,\infty} \end{bmatrix} = \begin{bmatrix} I & D_{n,1} \\ 0 & D_{n,\infty} \end{bmatrix} \begin{bmatrix} \widetilde{D}_{n,0} - D_{n,1} D_{n,\infty}^{-1} D_{n,2} & 0 \\ D_{n,\infty}^{-1} D_{n,2} & I \end{bmatrix}. \tag{6.14}$$

Hence, the left (resp., right) invertibility of the operator matrix D is equivalent to the left (resp., right) invertibility of the operator

$$D_{n,0} := \widetilde{D}_{n,0} - D_{n,1} D_{n,\infty}^{-1} D_{n,2} \in \chi_n^0 W_{p,\mathfrak{S}} \chi_{n-N_-,n+N_+}^0, \tag{6.15}$$

where $N_+ \le N_-$ in the case of left invertibility, and $N_+ \ge N_-$ in the case of right invertibility.

Given $n \in \mathbb{N}$, let $L_n^p(\gamma)$ be the Banach space of n-dimensional vector functions $\psi = \{\psi_k\}_{k=1}^n$ with entries $\psi_k \in L^p(\gamma)$ and the norm $\|\psi\|_{L_n^p(\gamma)} = (\sum_{k=1}^n \|\psi_k\|_{L^p(\gamma)}^p)^{1/p}$. Consider the isometric isomorphisms

$$\begin{aligned} \sigma_0 : \chi_n^0 L^p(\mathbb{R}_+) \to L_{2n-1}^p(\gamma), \qquad & \chi_n^0 f \mapsto \big\{ (U_\alpha^k f)|_\gamma \big\}_{k=-n+1}^{n-1}, \\ \widetilde{\sigma}_0 : \chi_{n-N_-,n+N_+}^0 L^p(\mathbb{R}_+) \to L_{2n-1+N_+-N_-}^p(\gamma), & \\ \chi_{n-N_-,n+N_+}^0 f \mapsto \big\{ (U_\alpha^k f)|_\gamma \big\}_{k=-n+N_-+1}^{n+N_+-1}. & \end{aligned}$$

Then the operator

$$\sigma_0 D_{n,0} \widetilde{\sigma}_0^{-1} : L_{2n-1+N_+-N_-}^p(\gamma) \to L_{2n-1}^p(\gamma)$$

is the operator of multiplication by a $(2n-1) \times (2n-1+N_+-N_-)$ matrix function $\mathcal{D}_{n,0}(\cdot)$, which according to the definition of the algebra $W_{p,\mathfrak{S}}$ becomes continuous on $\overline{\gamma}$ if we put $\mathcal{D}_{n,0}(\alpha(\tau)) := \lim_{t \to \alpha(\tau), t \in \gamma} \mathcal{D}_{n,0}(t)$.

Let the operator $D_{n,0}$ be left invertible (the case of right invertibility is treated analogously). The operator $\mathcal{D}_{n,0}(\cdot)I \in \mathcal{B}(L^p_{2n-1+N_+-N_-}(\gamma), L^p_{2n-1}(\gamma))$ is left invertible along with $D_{n,0}$, and hence $N_+ \le N_-$. Moreover, then $\operatorname{rank} \mathcal{D}_{n,0}(t) = 2n - 1 + N_+ - N_-$ for all $t \in \overline{\gamma}$. Indeed, if

$$\operatorname{rank} \mathcal{D}_{n,0}(t_0) < 2n - 1 + N_+ - N_- \quad \text{for some } t_0 \in \overline{\gamma}, \tag{6.16}$$

then by a small perturbation of the operator $\mathcal{D}_{n,0}(\cdot)I$ we can attain the equality $\mathcal{D}_{n,0}(t) = \mathcal{D}_{n,0}(t_0)$ for all t's in a small neighborhood of t_0, which in view of (6.16) implies the existence of a nontrivial kernel for the perturbed operator. This contradicts the stability of the left invertibility for the perturbed operator.

Conversely, let $N_+ \le N_-$ and $\operatorname{rank} \mathcal{D}_{n,0}(t) = 2n-1+N_+-N_-$. Then for every point $t \in \overline{\gamma}$ the $(2n-1) \times (2n-1+N_+-N_-)$ matrix function $\mathcal{D}_{n,0}(\cdot)$ contains its $(2n-1+N_+-N_-) \times (2n-1+N_+-N_-)$ block $M_t(\cdot)$ with $\det M_t(t) \ne 0$. In view of the continuity on $\overline{\gamma}$ of all entries of the matrix function $\mathcal{D}_{n,0}(\cdot)$, for every $t \in \overline{\gamma}$ there exists its open neighborhood $u_t \subset \mathbb{R}_+$ such that

$$\inf_{x \in u_t \cap \overline{\gamma}} \big|\det M_t(x)\big| > 0. \tag{6.17}$$

Consider the open covering of the segment $\overline{\gamma}$ by the sets u_t $(t \in \overline{\gamma})$, and choose a finite subcovering u_{t_i} $(i = 1, 2, \dots, k)$ of $\overline{\gamma}$. According to [35, Theorem 2.13], there exists a partition $\{\eta_1, \dots, \eta_k\}$ of unity on $\overline{\gamma}$, subordinate to the covering $\{u_{t_1}, \dots, u_{t_k}\}$, that is, $\eta_i \in C(\mathbb{R}_+)$, $0 \le \eta_i \le 1$, $\operatorname{supp} \eta_i \subset u_{t_i}$ for all $i = 1, 2, \dots, k$ and

$$\sum_{i=1}^{k} \eta_i(x) = 1 \quad \text{for all } x \in \overline{\gamma}. \tag{6.18}$$

By (6.17), for each $i = 1, 2, \dots, k$, there exists a $(2n-1+N_+-N_-) \times (2n-1)$ matrix function $F_{t_i}(\cdot)$ defined on $u_{t_i} \cap \overline{\gamma}$, which consists of the block $M^{-1}_{t_i}(\cdot)$ and the zero supplementary block, and such that

$$F_{t_i}(x)\mathcal{D}_{n,0}(x) = M^{-1}_{t_i}(x) M_{t_i}(x) = I_{2n-1+N_+-N_-} \quad \text{for all } x \in u_{t_i} \cap \overline{\gamma}.$$

Then, by (6.18), the continuous on $\overline{\gamma}$ matrix function $\mathcal{D}^L_{n,0}(\cdot) := \sum_{i=1}^k \eta_i F_{t_i}$ is a left inverse to the matrix function $\mathcal{D}_{n,0}(\cdot)$. Hence, the operator $D^L_{n,0} = \widetilde{\sigma}_0^{-1}(\mathcal{D}^L_{n,0}(\cdot)I)\sigma_0$ is a left inverse of the operator $D_{n,0}$. In view of the continuity on $\overline{\gamma}$ of all entries of the matrix function $\mathcal{D}^L_{n,0}(\cdot)$, we infer from the definitions of the isomorphisms $\widetilde{\sigma}_0^{-1}$ and σ_0 that $D^L_{n,0} \in \chi^0_{n-N_-,n+N_+} W_{p,\mathfrak{G}} \chi^0_n$.

Thus, if the operator $D_{n,0}$ is one-sided invertible or, equivalently, if the operator $\mathcal{D}_{n,0}(\cdot)I$ is one-sided invertible from $L^p_{2n-1+N_+-N_-}(\gamma)$ to $L^p_{2n-1}(\gamma)$, then there is its one-sided inverse $\mathcal{D}^{(-1)}_{n,0}(\cdot)I : L^p_{2n-1}(\gamma) \to L^p_{2n-1+N_+-N_-}(\gamma)$ such that the operator

$$D^{(-1)}_{n,0} = \widetilde{\sigma}_0^{-1}\big(\mathcal{D}^{(-1)}_{n,0}(\cdot)I\big)\sigma_0 \in \chi^0_{n-N_-,n+N_+} W_{p,\mathfrak{G}} \chi^0_n$$

is the corresponding one-sided inverse to the operator $D_{n,0}$.

Clearly, the operator $D^{(-1)}_{n,0} \in \chi^0_{n-N_-,n+N_+} W_{p,\mathfrak{G}} \chi^0_n$ is a right inverse to the operator $D_{n,0}$ if $N_+ \ge N_-$, and a left inverse to the operator $D_{n,0}$ if $N_+ \le N_-$. By

(6.14) and (6.15), the operator

$$D^{(-1)} = \begin{bmatrix} D_{n,0}^{(-1)} & -D_{n,0}^{(-1)} D_{n,1} D_{n,\infty}^{-1} \\ -D_{n,\infty}^{-1} D_{n,2} D_{n,0}^{(-1)} & D_{n,\infty}^{-1} D_{n,2} D_{n,0}^{(-1)} D_{n,1} D_{n,\infty}^{-1} + D_{n,\infty}^{-1} \end{bmatrix}$$

is a right (resp., left) inverse to the operator D if $D_{n,0}^{(-1)}$ is a right (resp., left) inverse to the operator $D_{n,0}$. Since $D_{n,0}^{(-1)} \in \chi_{n-N_-,n+N_+}^0 W_{p,\mathfrak{G}} \chi_n^0$ and

$$\begin{aligned} -D_{n,\infty}^{-1} D_{n,2} D_{n,0}^{(-1)} &\in (\chi_{n-N_-}^- + \chi_{n+N_+}^+) W_{p,\mathfrak{G}} \chi_n^0, \\ -D_{n,0}^{(-1)} D_{n,1} D_{n,\infty}^{-1} &\in \chi_{n-N_-,n+N_+}^0 W_{p,\mathfrak{G}} (\chi_n^- + \chi_n^+), \\ D_{n,\infty}^{-1} D_{n,2} D_{n,0}^{(-1)} D_{n,1} D_{n,\infty}^{-1} + D_{n,\infty}^{-1} &\in (\chi_{n-N_-}^- + \chi_{n+N_+}^+) W_{p,\mathfrak{G}} (\chi_n^- + \chi_n^+), \end{aligned}$$

we conclude that the corresponding one-sided inverse $D^{(-1)}$ to the operator D belongs to the Wiener-type algebra $W_{p,\mathfrak{G}}$.

Thus, we have proved the following result.

Theorem 6.4. *If $1 < p < \infty$, $A \in \mathfrak{A}_W$, and the operators*

$$\begin{aligned} \chi_n^- A \chi_{n-N_-}^- &\in \mathcal{B}\big(\chi_{n-N_-}^- L^p(\mathbb{R}_+), \chi_n^- L^p(\mathbb{R}_+)\big), \\ \chi_n^+ A \chi_{n+N_+}^+ &\in \mathcal{B}\big(\chi_{n+N_+}^+ L^p(\mathbb{R}_+), \chi_n^+ L^p(\mathbb{R}_+)\big) \end{aligned}$$

are invertible for all sufficiently large $n \in \mathbb{N}$, then there exists a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the left (resp., right) invertibility of the operator $A \in \mathfrak{A}_W$ is equivalent to the left (resp., right) invertibility of the operator

$$\begin{aligned} D_{n,0} = \chi_n^0 A \chi_{n-N_-,n+N_+}^0 &- \begin{bmatrix} \chi_n^0 A \chi_{n-N_-}^- & \chi_n^0 A \chi_{n+N_+}^+ \end{bmatrix} \\ &\times \begin{bmatrix} \chi_n^- A \chi_{n-N_-}^- & \chi_n^- A \chi_{n+N_+}^+ \\ \chi_n^+ A \chi_{n-N_-}^- & \chi_n^+ A \chi_{n+N_+}^+ \end{bmatrix}^{-1} \begin{bmatrix} \chi_n^- A \chi_{n-N_-,n+N_+}^0 \\ \chi_n^+ A \chi_{n-N_-,n+N_+}^0 \end{bmatrix}, \end{aligned} \tag{6.19}$$

which acts from the space $\chi_{n-N_-,n+N_+}^0 L^p(\mathbb{R}_+)$ to the space $\chi_n^0 L^p(\mathbb{R}_+)$. Moreover, then there exists a left (resp., right) one-sided inverse operator $A^{(-1)}$ to A which belongs to the Wiener-type algebra $W_{p,\mathfrak{G}}$.

Thus, if $N_+ \neq N_-$, then the operator $D_{n,0}$ cannot be invertible. Moreover, if the operator $D_{n,0}$ is invertible (resp., strictly left invertible, strictly right invertible), then $N_- = N_+$ (resp., $N_- > N_+$, $N_- < N_+$).

Hence, Theorems 6.3 and 6.4 imply the following corollary.

Corollary 6.5. *If the functional operator $A \in \mathfrak{A}_W$ is invertible (resp., strictly left invertible, strictly right invertible) on the Lebesgue space $L^p(\mathbb{R}_+)$ with $p \in (1, \infty)$, then $N_- = N_+$ (resp., $N_- > N_+$, $N_- < N_+$).*

6.3. **Discrete version.** Let $A \in \mathfrak{A}_W$. Given $n \in \mathbb{Z}$, we consider the projections

$$P_n^{\pm} = \operatorname{diag}\{P_{s,n}^{\pm}\}_{s \in \mathbb{Z}} I \in \mathcal{B}(l^p),$$

where

$$P_{s,n}^+ = \begin{cases} 0 & \text{if } s < n, \\ 1 & \text{if } s \geq n, \end{cases} \qquad P_{s,n}^- = \begin{cases} 1 & \text{if } s \leq -n, \\ 0 & \text{if } s > -n. \end{cases}$$

Theorem 6.6. *If $p \in [1, \infty]$, $A \in \mathfrak{A}_W \subset \mathcal{B}(L^p(\mathbb{R}_+))$, $A_\xi(z) \neq 0$ for all $(\xi, z) \in \Delta \times \mathbb{T}$, and the numbers $N_\pm$ are given by (5.9), then there exists a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the operators*

$$\begin{aligned} \mathcal{A}_n^+(t) &= P_n^+ \mathcal{A}(t) P_{n+N_+}^+ : P_{n+N_+}^+ l^p \to P_n^+ l^p, \\ \mathcal{A}_n^-(t) &= P_n^- \mathcal{A}(t) P_{n-N_-}^- : P_{n-N_-}^- l^p \to P_n^- l^p \end{aligned} \tag{6.20}$$

are invertible for all $t \in \gamma$.

Proof. Since the operators

$$\begin{aligned} \chi_n^+ A \chi_{n+N_+}^+ &: \chi_{n+N_+}^+ L^p(\mathbb{R}_+) \to \chi_n^+ L^p(\mathbb{R}_+), \\ \chi_n^- A \chi_{n-N_-}^- &: \chi_{n-N_-}^- L^p(\mathbb{R}_+) \to \chi_n^- L^p(\mathbb{R}_+) \end{aligned}$$

are invertible by Theorem 6.1, we conclude that the operators

$$\begin{aligned} &\chi_n^+ A \chi_{n+N_+}^+ + (I - \chi_n^+) U_\alpha^{N_+} (I - \chi_{n+N_+}^+), \\ &\chi_n^- A \chi_{n-N_-}^- + (I - \chi_n^-) U_\alpha^{N_-} (I - \chi_{n-N_-}^+) \end{aligned} \tag{6.21}$$

are invertible on the space $L^p(\mathbb{R}_+)$. As $\sigma \chi_n^\pm \sigma^{-1} = P_n^\pm$, where σ is given by (3.3) and I is the identity operator on the space $L^p(\gamma, l^p)$, we infer that

$$\begin{aligned} \mathcal{A}_n^+(\cdot) I &= P_n^+ \mathcal{A}(\cdot) P_{n+N_+}^+ I = \sigma(\chi_n^+ A \chi_{n+N_+}^+) \sigma^{-1}, \\ \mathcal{A}_n^-(\cdot) I &= P_n^- \mathcal{A}(\cdot) P_{n-N_-}^- I = \sigma(\chi_n^- A \chi_{n-N_-}^-) \sigma^{-1}. \end{aligned}$$

Then, by (6.21) and Theorem 3.7, the discrete operators

$$\begin{aligned} &\mathcal{A}_n^+(t) + (I - P_n^+) \mathcal{V}^{N_+} (I - P_{n+N_+}^+), \\ &\mathcal{A}_n^-(t) + (I - P_n^-) \mathcal{V}^{N_-} (I - P_{n-N_-}^-) \end{aligned}$$

are invertible on the space l^p, which in view of the invertibility of the operators

$$\begin{aligned} (I - P_n^+) \mathcal{V}^{N_+} (I - P_{n+N_+}^+) &: P_{-n-N_++1}^- l^p \to P_{-n+1}^- l^p, \\ (I - P_n^-) \mathcal{V}^{N_-} (I - P_{n-N_-}^-) &: P_{-n+N_-+1}^+ l^p \to P_{-n+1}^+ l^p \end{aligned}$$

implies the assertion of the theorem. ☐

Combining Theorem 5.2, Corollary 5.4, and Theorem 6.6, we obtain the following.

Theorem 6.7. *If the functional operator $A = \sum_{k \in \mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ with coefficients $a_k \in \mathrm{SO}(\mathbb{R}_+)$ and a shift $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$ is left or right invertible on the Lebesgue space $L^p(\mathbb{R}_+)$ with $p \in (1, \infty)$, then there exists a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the operators (6.20) are invertible for all $t \in \gamma$.*

We immediately infer the following result from Theorems 5.6 and 6.6.

Theorem 6.8. *If $p \in (1, \infty)$, $A = \sum_{k \in \mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$, and for some $t \in \gamma$ the discrete operator $\mathcal{A}(t) = \sum_{k \in \mathbb{Z}} \mathrm{diag}\{a_k[\alpha_j(t)]\}_{j \in \mathbb{Z}} \mathcal{V}^k \in \mathcal{W}_p$ is left or right invertible on the space l^p, then for this $t \in \gamma$ there exists a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the operators (6.20) are invertible.*

6.4. **Necessary one-sided invertibility conditions for discrete operators.** Furthermore, by applying the transform $A \mapsto \sigma A \sigma^{-1}$, we deduce from (6.12) the following representation of every discrete operator $\mathcal{A}(t) \in \mathcal{B}(l^p)$ $(t \in \gamma)$ as the operator matrix

$$\mathcal{A}(t) = \begin{bmatrix} P_n^- \mathcal{A}(t) P_{n-N_-}^- & P_n^- \mathcal{A}(t) P_{n-N_-,n+N_+}^0 & P_n^- \mathcal{A}(t) P_{n+N_+}^+ \\ P_n^0 \mathcal{A}(t) P_{n-N_-}^- & P_n^0 \mathcal{A}(t) P_{n-N_-,n+N_+}^0 & P_n^0 \mathcal{A}(t) P_{n+N_+}^+ \\ P_n^+ \mathcal{A}(t) P_{n-N_-}^- & P_n^+ \mathcal{A}(t) P_{n-N_-,n+N_+}^0 & P_n^+ \mathcal{A}(t) P_{n+N_+}^+ \end{bmatrix}$$

acting from the space $P_{n-N_-}^- l^p \dotplus P_{n-N_-,n+N_+}^0 l^p \dotplus P_{n+N_+}^+ l^p$ to the space $P_n^- l^p \dotplus P_n^0 l^p \dotplus P_n^+ l^p$, where $P_{n-N_-,n+N_+}^0 := I - P_{n-N_-}^- - P_{n+N_+}^+$ and $P_n^0 := I - P_n^- - P_n^+$. If the discrete operator $\mathcal{A}(t)$ is one-sided invertible for some $t \in \gamma$, then by Theorem 6.8 the operators (6.20) are invertible for this $t \in \gamma$ and all sufficiently large $n \in \mathbb{N}$. Hence, the operator

$$\mathcal{D}_{n,\infty}(t) := \begin{bmatrix} P_n^- \mathcal{A}(t) P_{n-N_-}^- & P_n^- \mathcal{A}(t) P_{n+N_+}^+ \\ P_n^+ \mathcal{A}(t) P_{n-N_-}^- & P_n^+ \mathcal{A}(t) P_{n+N_+}^+ \end{bmatrix}, \tag{6.22}$$

which acts from the space $P_{n-N_-}^- l^p \dotplus P_{n+N_+}^+ l^p$ onto the space $P_n^- l^p \dotplus P_n^+ l^p$, is also invertible for all sufficiently large $n \in \mathbb{N}$ along with the operators (6.20). Then we infer by analogy with Theorem 6.4 and (6.19) that the left (resp., right) invertibility of the operator $\mathcal{A}(t)$ is equivalent to the left (resp., right) invertibility of the operator

$$\begin{aligned} \mathcal{D}_{n,0}(t) := P_n^0 \mathcal{A}(t) P_{n-N_-,n+N_+}^0 - \begin{bmatrix} P_n^0 \mathcal{A}(t) P_{n-N_-}^- & P_n^0 \mathcal{A}(t) P_{n+N_+}^+ \end{bmatrix} \\ \times \begin{bmatrix} P_n^- \mathcal{A}(t) P_{n-N_-}^- & P_n^- \mathcal{A}(t) P_{n+N_+}^+ \\ P_n^+ \mathcal{A}(t) P_{n-N_-}^- & P_n^+ \mathcal{A}(t) P_{n+N_+}^+ \end{bmatrix}^{-1} \begin{bmatrix} P_n^- \mathcal{A}(t) P_{n-N_-,n+N_+}^0 \\ P_n^+ \mathcal{A}(t) P_{n-N_-,n+N_+}^0 \end{bmatrix}, \end{aligned} \tag{6.23}$$

which acts from the space $P_{n-N_-,n+N_+}^0 l^p$ to the space $P_n^0 l^p$ (below we identify the operator $\mathcal{D}_{n,0}(t)$ and the matrix $\mathcal{D}_{n,0}(t)$). Clearly, if $N_+ \neq N_-$, then the operator $\mathcal{D}_{n,0}(t)$ cannot be invertible. Thus, the operator $\mathcal{D}_{n,0}(t)$ is two-sided invertible if and only if $N_- = N_+$ and $\det \mathcal{D}_{n,0}(t) \neq 0$, the operator $\mathcal{D}_{n,0}(t)$ is strictly left invertible if and only if $N_- > N_+$ and $\operatorname{rank} \mathcal{D}_{n,0}(t) = 2n - 1 + N_+ - N_-$, and the operator $\mathcal{D}_{n,0}(t)$ is strictly right invertible if $N_- < N_+$ and $\operatorname{rank} \mathcal{D}_{n,0}(t) = 2n - 1$. This in view of Theorems 5.6 and 6.8 immediately implies the following result.

Theorem 6.9. *Given $p \in (1, \infty)$ and $t \in \gamma$, an operator $\mathcal{A}(t) \in \mathcal{W}_p$ is two-sided (resp., strictly left, strictly right) invertible on the space l^p if and only if (5.5) holds, $N_- = N_+$, and there exists a number $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the operator $\mathcal{D}_{n,\infty}(t)$ is invertible and $\det \mathcal{D}_{n,0}(t) \neq 0$ (resp., (5.5) holds, $N_- > N_+$, and there exists a number $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the operator $\mathcal{D}_{n,\infty}(t)$ is invertible and $\operatorname{rank} \mathcal{D}_{n,0}(t) = 2n - 1 + N_+ - N_-$; (5.5) holds, $N_- < N_+$, and there exists a number $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ the operator $\mathcal{D}_{n,\infty}(t)$ is invertible and $\operatorname{rank} \mathcal{D}_{n,0}(t) = 2n - 1$), where $\mathcal{D}_{n,\infty}(t)$ and $\mathcal{D}_{n,0}(t)$ are given by (6.22) and (6.23), respectively.*

7. One-sided invertibility of functional operators in terms of discrete operators

According to [28, Section 23, Corollary 2], the left (resp., right) invertibility of an element b in a C^*-algebra $\mathcal{B}$ is equivalent to the two-sided invertibility of the element b^*b (resp., bb^*) in the C^*-algebra $\mathcal{B}$. Applying this fact to the C^*-algebra $\mathfrak{A}_{2,\mathrm{SO}} \subset \mathcal{B}(L^2(\mathbb{R}_+))$, we immediately infer from Theorem 3.4 the following criterion for the one-sided invertibility of functional operators $A \in \mathfrak{A}_{2,\mathrm{SO}}$ on the space $L^2(\mathbb{R}_+)$.

Theorem 7.1. *A functional operator $A \in \mathfrak{A}_{2,\mathrm{SO}}$ is invertible (left invertible, right invertible) on the space $L^2(\mathbb{R}_+)$ if and only if for all $t \in \gamma$, the discrete operators $\mathcal{A}(t)$ are invertible (left invertible, right invertible) on the space l^2.*

Given $p \in [1,\infty]$ and following [12, Section 2.3], we denote by W_{p,L^∞} the unital Banach algebra of all functional operators of the form $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathcal{B}(L^p(\mathbb{R}_+))$, where $a_k \in L^\infty(\mathbb{R}_+)$ for all $k \in \mathbb{Z}$, $\log \alpha' \in L^\infty(\mathbb{R}_+)$ and

$$\|A\|_{W_{p,L^\infty}} := \sum_{k\in\mathbb{Z}} \|a_k\|_{L^\infty(\mathbb{R}_+)} < +\infty.$$

For a unital algebra $\mathcal{B}$, let $\mathcal{GB}$ denote the group of all invertible elements in $\mathcal{B}$. Let $\mathcal{C}$ be a subalgebra of $\mathcal{B}$ with the same identity element. The algebra $\mathcal{C}$ is said to be *inverse-closed* in $\mathcal{B}$ (see [34, Section 1.2.5]), if for every $c \in \mathcal{C}$ such that $c \in \mathcal{GB}$, we have $c \in \mathcal{GC}$.

Slightly modifying the proof of [12, Theorem 3], we obtain the following.

Theorem 7.2. *For every $p \in [1,\infty]$, the Wiener algebra W_{p,L^∞} is inverse-closed in the unital Banach algebra $\mathcal{B}(L^p(\mathbb{R}_+))$.*

Theorem 7.2 immediately implies the following.

Corollary 7.3. *Every functional operator $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in W_{p,L^\infty}$ is invertible on all spaces $L^p(\mathbb{R}_+)$ with $p \in [1,\infty]$ if it is invertible on some space $L^{p_0}(\mathbb{R}_+)$, where $p_0 \in [1,\infty]$.*

Theorem 6.4 and Corollary 7.3 allow us to get the following analogue of Theorem 7.2.

Theorem 7.4. *The Wiener algebra $\mathfrak{A}_W$ is inverse-closed in the Banach algebra $\mathcal{B}(L^p(\mathbb{R}_+))$ for every $p \in [1,\infty]$.*

Proof. Fix $p \in (1,\infty)$, and take an operator $A \in \mathfrak{A}_W$ invertible on the space $L^p(\mathbb{R}_+)$. Then from Theorem 3.4 it follows that for all $t \in \overline{\gamma}$ the discrete operators $\mathcal{A}(t)$ given by Theorem 3.1 are invertible on the space l^p, and hence, in view of Corollary 3.3, the operator-valued function $\mathcal{A}^{-1} : \overline{\gamma} \to \mathcal{B}(l^p)$ is continuous. Moreover, by (3.8),

$$\mathcal{A}^{-1}\big[\alpha(\tau)\big] = \mathcal{V}\mathcal{A}^{-1}(\tau)\mathcal{V}^{-1}. \tag{7.1}$$

Further, by Theorem 6.4, the inverse operator A^{-1} for $A \in \mathfrak{A}_W$ belongs to the Wiener-type algebra $W_{p,\mathfrak{S}}$, and therefore the coefficients of the operator A^{-1} being functions in $\mathfrak{S}$ can admit discontinuities on $\mathbb{R}_+$ in view of (3.17) only on the set $\mathcal{O}_\tau = \{\alpha_n(\tau) : n \in \mathbb{Z}\}$. By Lemma 3.6, the operator-valued function $\mathcal{A}^{-1} : \overline{\gamma} \to \mathcal{B}(l^p)$ is continuous. On the other hand, since $\mathrm{SO} \subset \mathfrak{S}$ and therefore (7.1) holds, we infer from that equality that the coefficients of the operator A^{-1} are continuous at the points $t \in \mathcal{O}_\tau$ as well. Hence, the coefficients of the operator A^{-1} are continuous on $\mathbb{R}_+$. But every continuous function in $\mathfrak{S}$ belongs to $\mathrm{SO}(\mathbb{R}_+)$, which implies that $A^{-1} \in \mathfrak{A}_W$.

Let now an operator $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ be invertible on the space $L^p(\mathbb{R}_+)$ with $p \in \{1, \infty\}$. Since $\mathfrak{A}_W \subset W_{p,L^\infty}$, we infer from Corollary 7.3 that the operator A is invertible on the space $L^2(\mathbb{R}_+)$ and therefore, by the part already proved, $A^{-1} \in \mathfrak{A}_W = W_{2,\mathrm{SO}}$. But this means that $A^{-1} \in \mathfrak{A}_W = W_{p,\mathrm{SO}}$ for all $p \in [1, \infty]$. □

For every operator $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ we define its formally adjoint operator $A^\diamond := \sum_{k\in\mathbb{Z}} (\overline{a_k} \circ \alpha_{-k}) U_\alpha^{-k} \in \mathfrak{A}_W$.

Theorem 7.5. *If a functional operator $A \in \mathfrak{A}_W$ is left invertible (resp., right invertible) on the space $L^p(\mathbb{R}_+)$ for some $p \in (1, \infty)$, then for all $t \in \gamma$ the discrete operators $\mathcal{A}(t)$ are left invertible (resp., right invertible) on the space l^p.*

Proof. Consider the case of left invertibility (the case of right invertibility is reduced to the previous one by passing to adjoint operators).

Let an operator $A \in \mathfrak{A}_W$ be left invertible on some space $L^p(\mathbb{R}_+)$. Then from Theorems 6.3 and 6.4, it follows that A is left invertible in the unital Banach algebra $W_{p,\mathfrak{S}}$. Then A is left invertible in the Banach algebra $W_{2,\mathfrak{S}}$ and hence in the C^*-algebra $\mathcal{B}(L^2(\mathbb{R}_+))$. Therefore, by [28, Section 23, Corollary 2], the operator A^*A is two-sided invertible in the C^*-algebra $\mathcal{B}(L^2(\mathbb{R}_+))$. But the operator $A^\diamond A$ belongs to the Banach algebra $\mathfrak{A}_W = W_{p,\mathrm{SO}}$ along with A. Consequently, by Corollary 7.3, the operator $A^\diamond A$ is two-sided invertible on the space $L^p(\mathbb{R}_+)$. Hence, by Theorem 3.4, for all $t \in \gamma$ the discrete operators $\mathcal{A}^\diamond(t)\mathcal{A}(t)$ are two-sided invertible on the space l^p. Then for every $t \in \gamma$ the operator $(\mathcal{A}^\diamond(t)\mathcal{A}(t))^{-1}\mathcal{A}^\diamond(t)$ is a left inverse of the operator $\mathcal{A}(t)$ in the Banach algebra $\mathcal{B}(l^p)$. □

For every operator $B = \sum_{k\in\mathbb{Z}} b_k \mathcal{V}^k \in \mathcal{W}_p$, where $b_k \in \widehat{\mathfrak{D}}$ for all $k \in \mathbb{Z}$, $\widehat{\mathfrak{D}}$ is given by (3.1), $\mathcal{V}f = \{f_{j+1}\}_{j\in\mathbb{Z}}$ for $f = \{f_j\}_{j\in\mathbb{Z}} \in l^p$, and $\|B\|_W := \sum_{k\in\mathbb{Z}} \|b_k\|_{\mathcal{B}(l^p)} < \infty$, we define the formally adjoint operator by $B^\diamond := \sum_{k\in\mathbb{Z}} \mathrm{diag}\{\overline{b_{k,j-k}}\}_{j\in\mathbb{Z}} \mathcal{V}^{-k} \in \mathcal{W}_p$.

Theorem 7.6. *For every $p \in (1, \infty)$, an operator $A \in \mathfrak{A}_W$ is left invertible (resp., right invertible) on the Lebesgue space $L^p(\mathbb{R}_+)$ if for all $t \in \gamma$ the discrete operators $\mathcal{A}(t) \in \mathcal{W}_p$ are left invertible (resp., right invertible) in the Banach algebra $\mathcal{W}_p$.*

Proof. For every $t \in \gamma$, let the discrete operator $\mathcal{A}(t) \in \mathcal{W}_p$ be left invertible in the Banach algebra $\mathcal{W}_p$. Then $\mathcal{A}(t)$ is left invertible in the Banach algebra $\mathcal{W}_2$ and hence in the C^*-algebra $\mathcal{B}(l^2)$. Therefore, by [28, Section 23, Corollary 2], for every $t \in \gamma$ the discrete operator $(\mathcal{A}(t))^*\mathcal{A}(t) \in \mathcal{W}_2$ is two-sided invertible in the C^*-algebra $\mathcal{B}(l^2)$. Since $A \in \mathfrak{A}_W$, we conclude that $A^*A \in \mathfrak{A}_W$ as well. Consequently, we infer from Theorem 3.4 that the operator $A^*A \in \mathfrak{A}_W$ is two-sided invertible on the space $L^2(\mathbb{R}_+)$. Then from Corollary 7.3, it follows that the operator $A^\diamond A \in \mathfrak{A}_W$ is two-sided invertible on the space $L^p(\mathbb{R}_+)$. Finally, the operator $(A^\diamond A)^{-1}A^\diamond$ is a left inverse of the operator A on the space $L^p(\mathbb{R}_+)$.

The case of right invertibility is treated by passing to adjoint operators on dual spaces. □

Theorem 7.7. *If $A \in \mathfrak{A}_W$ and for some $t \in \gamma$ the discrete operator $\mathcal{A}(t)$ is left (resp., right) invertible on the space l^p for some $p \in (1,\infty)$, then there exists its left (resp., right) inverse $(\mathcal{A}(t))^{(-1)}$ which belongs to the Wiener-type algebra $\mathcal{W}_p$.*

Proof. Fix $p \in (1,\infty)$, and for $A \in \mathfrak{A}_W$ suppose that the discrete operator $\mathcal{A}(t) \in \mathcal{W}_p$ is one-sided invertible on the space l^p for some $t \in \gamma$. We then infer from Theorem 5.6 that $A_\xi(z) \neq 0$ for all $(\xi, z) \in \Delta \times \mathbb{T}$ and therefore, by the proof of Corollary 5.4, the numbers $N_\pm$ are given by (5.9). Hence, from Theorem 6.1 it follows that the operators

$$\begin{aligned}\chi_n^+ A\chi_{n+N_+}^+ &: \chi_{n+N_+}^+ L^p(\mathbb{R}_+) \to \chi_n^+ L^p(\mathbb{R}_+),\\ \chi_n^- A\chi_{n-N_-}^- &: \chi_{n-N_-}^- L^p(\mathbb{R}_+) \to \chi_n^- L^p(\mathbb{R}_+)\end{aligned}$$

are invertible. Since by Theorem 6.3 the inverse operators

$$\begin{aligned}(\chi_n^+ A\chi_{n+N_+}^+)^{-1} &: \chi_n^+ L^p(\mathbb{R}_+) \to \chi_{n+N_+}^+ L^p(\mathbb{R}_+),\\ (\chi_n^- A\chi_{n-N_-}^-)^{-1} &: \chi_n^- L^p(\mathbb{R}_+) \to \chi_{n-N_-}^- L^p(\mathbb{R}_+)\end{aligned}$$

belong, respectively, to the subsets $\chi_{n+N_+}^+ W_{p,\mathfrak{S}}\chi_n^+$ and $\chi_{n-N_-}^- W_{p,\mathfrak{S}}\chi_n^-$ of $W_{p,\mathfrak{S}}$, we deduce by applying the mapping

$$(\chi_n^\pm A\chi_{n\pm N_\pm}^\pm)^{-1} \mapsto \sigma(\chi_n^\pm A\chi_{n\pm N_\pm}^\pm)^{-1}\sigma^{-1}$$

that for every $t \in \gamma$ the operators

$$\begin{aligned}\big(\mathcal{A}_n^+(t)\big)^{-1} &= \big(P_n^+\mathcal{A}(t)P_{n+N_+}^+\big)^{-1} : P_n^+ l^p \to P_{n+N_+}^+ l^p,\\ \big(\mathcal{A}_n^-(t)\big)^{-1} &= \big(P_n^-\mathcal{A}(t)P_{n-N_-}^-\big)^{-1} : P_n^- l^p \to P_{n-N_-}^+ l^p\end{aligned}$$

belong, respectively, to the subsets $P_{n+N_+}^+\mathcal{W}_p P_n^+$ and $P_{n-N_-}^-\mathcal{W}_p P_n^-$. Then, by analogy with the proof of Theorem 6.4, we infer that the left (resp., right) invertibility of the discrete operator $\mathcal{A}(t)$ on the space l^p implies that there exists its left (resp., right) inverse $(\mathcal{A}(t))^{(-1)}$ which belongs to the Wiener-type Banach algebra $\mathcal{W}_p$. □

Theorem 7.8. *A functional operator $A \in \mathfrak{A}_W$ is left (resp., right) invertible on the space $L^p(\mathbb{R}_+)$ for some $p \in (1,\infty)$ if and only if for all $t \in \gamma$ (equivalently, for all $t \in \mathbb{R}_+$) the discrete operators $\mathcal{A}(t) \in \mathcal{W}_p$ are left (resp., right) invertible on the space l^p.*

Proof. Fix $p \in (1,\infty)$. If a functional operator $A \in \mathfrak{A}_W$ is left (resp., right) invertible on the space $L^p(\mathbb{R}_+)$, then from Theorem 7.5 it follows that for all $t \in \gamma$ the discrete operators $\mathcal{A}(t) \in \mathcal{W}_p$ are left (resp., right) invertible on the space l^p.

Conversely, if for all $t \in \gamma$ the discrete operators $\mathcal{A}(t)$ are left (resp., right) invertible on the space l^p, then by Theorem 7.7 for all $t \in \gamma$ the operators $\mathcal{A}(t) \in$

$\mathcal{W}_p$ have some of their left (resp., right) inverses in the Banach algebra $\mathcal{W}_p$. This in view of Theorem 7.6 implies the left (resp., right) invertibility of the functional operator $A \in \mathfrak{A}_W$ on the space $L^p(\mathbb{R}_+)$, which completes the proof for $t \in \gamma$. The proof for $t \in \mathbb{R}_+$ is reduced to $t \in \gamma$ in view of (3.8). ☐

8. Criteria of two-sided and one-sided invertibility of operators $A \in \mathfrak{A}_W$

Combining Corollary 7.3, Theorem 3.4, Remark 3.5, and Theorem 6.9, we obtain the following invertibility criterion.

Theorem 8.1. *The functional operator* $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ *with coefficients* $a_k \in \mathrm{SO}(\mathbb{R}_+)$ *and a shift* $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$ *is invertible on the space* $L^p(\mathbb{R}_+)$ $(1 \le p \le \infty)$ *if and only if*

(i) $A_\xi(z) := \sum_{k\in\mathbb{Z}} a_k(\xi) z^k \neq 0$ *for every* $\xi \in \Delta$ *and every* $z \in \mathbb{T}$*;*
(ii) $N_- = N_+$*, where* $N_\pm := \mathrm{ind}\, A_\xi(\cdot)$ *for every* $\xi \in M_{\tau_\pm}(\mathrm{SO}(\mathbb{R}_+))$*;*
(iii) *there exists an* $n_0 \in \mathbb{N}$ *such that for every* $t \in \gamma$ *and every* $n > n_0$ *the operator* $\mathcal{D}_{n,\infty}(t)$ *given by (6.22) is invertible and* $\det \mathcal{D}_{n,0}(t) \neq 0$*, where the* $(2n-1)\times(2n-1)$ *matrices* $\mathcal{D}_{n,0}(t)$ *are given by (6.23).*

Proof. By Corollary 7.3, the operator $A \in \mathfrak{A}_W$ either is invertible on all the spaces $L^p(\mathbb{R}_+)$ for $p \in [1,\infty]$, or on none of them. Hence, we may consider only $p \in (1,\infty)$.

By Theorem 3.4 and Remark 3.5, the functional operator $A \in \mathfrak{A}_W$ is invertible on the space $L^p(\mathbb{R}_+)$ with $p \in (1,\infty)$ if and only if for all $t \in \gamma$ (equivalently, for all $t \in \overline{\gamma}$) the discrete operators $\mathcal{A}(t)$ are invertible on the space l^p. In view of Theorem 6.9, the invertibility of the operators $\mathcal{A}(t)$ on the space l^p for all $t \in \gamma$ is equivalent to conditions (i)–(iii) of the present theorem, where $n_0 \in \mathbb{N}$ is sufficiently large and independent of $t \in \gamma$ due to Theorem 6.7. ☐

Theorem 8.2. *The functional operator* $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ *with coefficients* $a_k \in \mathrm{SO}(\mathbb{R}_+)$ *and a shift* $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$ *is strictly left invertible on the space* $L^p(\mathbb{R}_+)$ $(1 < p < \infty)$ *if and only if*

(i) $A_\xi(z) := \sum_{k\in\mathbb{Z}} a_k(\xi) z^k \neq 0$ *for every* $\xi \in \Delta$ *and every* $z \in \mathbb{T}$*;*
(ii) $N_- > N_+$*, where* $N_\pm = \mathrm{ind}\, A_\xi(\cdot)$ *for every* $\xi \in M_{\tau_\pm}(\mathrm{SO}(\mathbb{R}_+))$*;*
(iii) *there exists an* $n_0 \in \mathbb{N}$ *such that for every* $t \in \gamma$ *and every* $n > n_0$ *the operator* $\mathcal{D}_{n,\infty}(t)$ *given by (6.22) is invertible and* $\mathrm{rank}\, \mathcal{D}_{n,0}(t) = 2n-1+N_+-N_-$*, where the* $(2n-1+N_+-N_-)\times(2n-1)$ *matrices* $\mathcal{D}_{n,0}(t)$ *are given by (6.23).*

Proof. Fix $p \in (1,\infty)$. By Theorems 7.8, 3.4, and Remark 3.5, the functional operator $A \in \mathfrak{A}_W$ is strictly left invertible on the space $L^p(\mathbb{R}_+)$ if and only if for all $t \in \gamma$ the discrete operators $\mathcal{A}(t)$ are strictly left invertible on the space l^p. According to Theorem 6.9, the strict left invertibility of the operators $\mathcal{A}(t)$ on the space l^p for all $t \in \gamma$ is equivalent to conditions (i)–(iii) of the present theorem, where $n_0 \in \mathbb{N}$ is sufficiently large and independent of $t \in \gamma$ (see Theorem 6.7). ☐

Analogously to Theorem 8.2 or passing to adjoint operators, one can prove the following criterion of the right invertibility of the operators $A \in \mathfrak{A}_W$.

Theorem 8.3. *The functional operator* $A = \sum_{k\in\mathbb{Z}} a_k U_\alpha^k \in \mathfrak{A}_W$ *with coefficients* $a_k \in \mathrm{SO}(\mathbb{R}_+)$ *and a shift* $\alpha \in \mathrm{SOS}(\mathbb{R}_+)$ *is strictly right invertible on the space* $L^p(\mathbb{R}_+)$ $(1 < p < \infty)$ *if and only if*

(i) $A_\xi(z) := \sum_{k\in\mathbb{Z}} a_k(\xi) z^k \neq 0$ *for every* $\xi \in \Delta$ *and every* $z \in \mathbb{T}$*;*

(ii) $N_- < N_+$*, where* $N_\pm = \mathrm{ind}\, A_\xi(\cdot)$ *for every* $\xi \in M_{\tau_\pm}(\mathrm{SO}(\mathbb{R}_+))$*;*

(iii) *there exists an* $n_0 \in \mathbb{N}$ *such that for every* $t \in \gamma$ *and every* $n > n_0$ *the operator* $\mathcal{D}_{n,\infty}(t)$ *given by* (6.22) *is invertible and* $\mathrm{rank}\, \mathcal{D}_{n,0}(t) = 2n - 1$*, where the* $(2n - 1 + N_+ - N_-) \times (2n - 1)$ *matrices* $\mathcal{D}_{n,0}(t)$ *are given by* (6.23)*.*

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