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HARDY-TYPE SPACE ESTIMATES FOR MULTILINEAR COMMUTATORS OF CALDERÓN–ZYGMUND OPERATORS ON NONHOMOGENEOUS METRIC MEASURE SPACES

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Abstract. Let $(\mathcal{X}, d, \mu)$ be a metric measure space satisfying the so-called *upper doubling condition* and the *geometrically doubling condition*. Let T be a Calderón–Zygmund operator and let $\vec{b} := (b_1, \ldots, b_m)$ be a finite family of $\widetilde{\mathrm{RBMO}}(\mu)$ functions. In this article, the authors establish the boundedness of the multilinear commutator $T_{\vec{b}}$, generated by T and $\vec{b}$ from the atomic Hardy-type space $\widetilde{H}^{1,q,m+1}_{\mathrm{fin},\vec{b},m,\rho}(\mu)$ into the Lebesgue space $L^1(\mu)$. The authors also prove that $T_{\vec{b}}$ is bounded from the atomic Hardy-type space $\widetilde{H}^{1,q,m+2}_{\mathrm{fin},\vec{b},m,\rho}(\mu)$ into the atomic Hardy space $\widetilde{H}^1(\mu)$ via the molecular characterization of $\widetilde{H}^1(\mu)$.

1. Introduction and preliminaries

The classical theory of Calderón–Zygmund operators originated from the study of the convolution operator with singular kernel on $\mathbb{R}$. From then on, it has become one of the core research areas in harmonic analysis. In 1976, Coifman, Rochberg, and Weiss [2] proved that the commutator $[b, T]$ of a Calderón–Zygmund operator T with a function $b \in \mathrm{BMO}(\mathbb{R}^d)$ defined by $[b, T](f)(x) := b(x)T(f)(x) - T(bf)(x)$, $x \in \mathbb{R}^d$, is bounded on $L^p(\mathbb{R}^d)$ for all $p \in (1, \infty)$. In 1995, Pérez [19] obtained a Hardy-type space estimate for $[b, T]$. Recently, Shu et al. [22] also considered some estimates for the commutators of Hardy operators.

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On the other hand, many results from real analysis and harmonic analysis on the classical Euclidean spaces have been extended to the space of the homogeneous type introduced by Coifman and Weiss [3], [4]. Recall that a quasimetric space $(\mathcal{X}, d)$ equipped with a nonnegative measure μ is called a *space of homogeneous type* in the sense of Coifman and Weiss [3], [4] if $(\mathcal{X}, d, \mu)$ satisfies the *measure doubling condition*: there exists a positive constant $C_{(\mu)}$ such that, for all balls $B(x,r) := \{y \in \mathcal{X} : d(x,y) < r\}$ with $x \in \mathcal{X}$ and $r \in (0,\infty)$,

$$\mu\big(B(x,2r)\big) \le C_{(\mu)}\mu\big(B(x,r)\big). \tag{1.1}$$

As was well known, the space of homogeneous type is a natural setting for Calderón–Zygmund operators and function spaces. Euclidean spaces equipped with Lebesgue measures, Euclidean spaces equipped with weighted Radon measures satisfying the doubling condition (1.1), and Heisenberg groups equipped with left-variant Haar measures are all the typical examples of spaces of homogeneous type.

Nevertheless, in the last two decades, many classical results concerning the Calderón–Zygmund operators and function spaces have been proved still valid for metric spaces equipped with *nondoubling measures* (see, e.g., [17], [18], [24]–[26], [28], [21], [16]). In particular, let μ be a nonnegative Radon measure on $\mathbb{R}^d$ which only satisfies the *polynomial growth condition* that there exist some positive constants C_0 and $n \in (0,d]$ such that, for all $x \in \mathbb{R}^d$ and $r \in (0,\infty)$,

$$\mu\big(B(x,r)\big) \le C_0 r^n, \tag{1.2}$$

where $B(x,r) := \{y \in \mathbb{R}^d : |x-y| < r\}$. Such a measure does not need to satisfy the doubling condition (1.1). The analysis on such nondoubling context plays a striking role in solving several long-standing problems related to the analytic capacity, like Vitushkin's conjecture or Painlevé's problem (see [26], [28]). Tolsa [24] introduced the atomic Hardy space $H^{1,q}_{\mathrm{atb}}(\mu)$ for $q \in (1,\infty]$ and its dual space, $\mathrm{RBMO}(\mu)$, the *space of functions with regularized bounded mean oscillation* with respect to μ as in (1.2), and he established the boundedness on $L^p(\mu)$ with $p \in (1,\infty)$ of commutators generated by Calderón–Zygmund operators and $\mathrm{RBMO}(\mu)$ functions. Tolsa [27] established a characterization of $H^{1,q}_{\mathrm{atb}}(\mu)$ in terms of the grand maximal operator. Meng and Yang [16] obtained the boundedness in some Hardy-type spaces of multilinear commutators generated by Calderón–Zygmund operators and $\mathrm{RBMO}(\mu)$ functions.

However, as was pointed out by Hytönen in [9], the measure μ satisfying the polynomial growth condition is different from, not more general than, the doubling measure. Hytönen [9] introduced a new class of metric measure spaces satisfying both the so-called *upper doubling condition* and the *geometrically doubling condition* (see, respectively, Definitions 1.1 and 1.3 below), which are also simply called *nonhomogeneous metric measure spaces*. This new class of metric measure spaces include both metric measure spaces of homogeneous type and metric measure spaces equipped with nondoubling measures as special cases.

From now on, we assume that $(\mathcal{X}, d, \mu)$ is a metric measure space of nonomogeneous type in the sense of Hytönen [9]. In this new setting, Hytönen [9] introduced the space $\mathrm{RBMO}(\mu)$ and established the corresponding John–Nirenberg inequality, and Hytönen and Martikainen [10] further established a version of the Tb theorem. Later, Hytönen et al. [12] and Bui and Duong [1] independently introduced the atomic Hardy space $H^{1,q}_{\mathrm{atb}}(\mu)$ and proved that the dual space of $H^{1,q}_{\mathrm{atb}}(\mu)$ is $\mathrm{RBMO}(\mu)$. Recently, Fu et al. [7] established the boundedness of multilinear commutators generated by Calderón–Zygmund operators and $\mathrm{RBMO}(\mu)$ functions. In addition, Fu et al. [6] introduced a version of the atomic Hardy space $\widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ ($\subset H^{1,q}_{\mathrm{atb}}(\mu)$ and simply denoted by $\widetilde{H}^1(\mu)$; see Definitions 1.10 and 1.11 below) and its corresponding dual space $\widetilde{\mathrm{RBMO}}(\mu)$ ($\supset \mathrm{RBMO}(\mu)$; see Definition 1.8 below) via the discrete coefficients $\widetilde{K}^{(\rho)}_{B,S}$, and they showed that the Calderón–Zygmund operator is bounded on $\widetilde{H}^1(\mu)$ via establishing a molecular characterization of $\widetilde{H}^1(\mu)$ in this context. (More research on function spaces and the boundedness of various operators on metric measure spaces of nonhomogeneous type can be found in [11], [14], [15], and the references therein. We refer the reader to the monograph [30] for more developments on harmonic analysis in this setting.)

Our main purpose here is to generalize the corresponding results in [16] to the present setting $(\mathcal{X}, d, \mu)$. We establish some Hardy-type space estimates for multilinear commutators generated by Calderón–Zygmund operators and $\widetilde{\mathrm{RBMO}}(\mu)$ functions. To state our main results, we recall some necessary notions and notation. We start with the following notion of upper doubling metric measure spaces originally introduced by Hytönen [9, Definition 2.6].

Definition 1.1. A metric measure space $(\mathcal{X}, d, \mu)$ is said to be *upper doubling* if μ is a Borel measure on $\mathcal{X}$ and there exist a *dominating function* $\lambda : \mathcal{X} \times (0,\infty) \to (0,\infty)$ and a positive constant $C_{(\lambda)}$ depending on λ such that, for each $x \in \mathcal{X}$, $r \to \lambda(x,r)$ is nondecreasing and, for all $x \in \mathcal{X}$ and $r \in (0,\infty)$,

$$\mu\big(B(x,r)\big) \le \lambda(x,r) \le C_{(\lambda)}\lambda(x,r/2). \tag{1.3}$$

Remark 1.2. (i) Obviously, a space of homogeneous type is a special case of upper doubling spaces, where we take the dominating function $\lambda(x,r) := \mu(B(x,r))$ for all $x \in \mathcal{X}$ and $r \in (0,\infty)$. On the other hand, the d-dimensional Euclidean space $\mathbb{R}^d$ with any Radon measure μ as in (1.2) is also an upper doubling space by taking $\lambda(x,r) := C_0 r^n$ for all $x \in \mathbb{R}^d$ and $r \in (0,\infty)$.

(ii) Let $(\mathcal{X}, d, \mu)$ be upper doubling with λ being the dominating function on $\mathcal{X} \times (0,\infty)$ as in Definition 1.1. It was proved in [12] that there exists another dominating function $\widetilde{\lambda}$ such that $\widetilde{\lambda} \le \lambda$, $C_{(\widetilde{\lambda})} \le C_{(\lambda)}$, and, for all $x, y \in \mathcal{X}$ with $d(x,y) \le r$,

$$\widetilde{\lambda}(x,r) \le C_{(\widetilde{\lambda})}\widetilde{\lambda}(y,r). \tag{1.4}$$

(iii) It was shown in [23] that the upper doubling condition is equivalent to the so-called *weak growth condition* (see [23, Definition 1.2 and Theorem 1.3]).

(iv) It was proved in [13] that the dominating function λ satisfying (1.4) has the following property: for any fixed ball $B \subset \mathcal{X}$, if $x_1, x_2 \in B$ and $y \in \mathcal{X}\backslash(kB)$ with $k \in [2,\infty)$, then $\lambda(x_1, d(x_1,y)) \sim \lambda(x_2, d(x_2,y))$; here and hereafter, the expression $A \sim B$ means that there exist positive constants C and $\widetilde{C}$ such that $A \le CB$ and $B \le \widetilde{C}A$ (see [13, Lemma 2.3]).

The following definition of the geometrically doubling condition is well known in analysis on metric spaces, which can be found in Coifman and Weiss [3, pp. 66-67], and is also known as the *metrically doubling condition* (see, e.g., [8, p. 81]). Moreover, spaces of homogeneous type are geometrically doubling, which was proved by Coifman and Weiss in [3, pp. 66-68]. In what follows, let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$.

Definition 1.3. A metric space $(\mathcal{X}, d)$ is said to be *geometrically doubling* if there exists some $N_0 \in \mathbb{N}$ such that, for any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, r/2)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most N_0.

Remark 1.4. For a metric space $(\mathcal{X}, d)$, Hytönen in [9] showed that geometrically doubling is equivalent to the following condition: for any $\epsilon \in (0, 1)$ and any ball $B(x, r) \subset \mathcal{X}$ with $x \in \mathcal{X}$ and $r \in (0, \infty)$, there exists a finite ball covering $\{B(x_i, \epsilon r)\}_i$ of $B(x, r)$ such that the cardinality of this covering is at most $N_0\epsilon^{-n_0}$; here and hereafter, N_0 is as in Definition 1.3 and $n_0 := \log_2 N_0$.

A metric measure space $(\mathcal{X}, d, \mu)$ is called a *nonhomogeneous metric measure space* if $(\mathcal{X}, d)$ is geometrically doubling and $(\mathcal{X}, d, \mu)$ is upper doubling. Based on Remark 1.2(ii), from now on, we *always assume* that $(\mathcal{X}, d, \mu)$ is a nonhomogeneous metric measure space with the dominating function λ satisfying (1.4).

Although the measure doubling condition is not assumed uniformly for all balls in the nonhomogeneous metric measure space $(\mathcal{X}, d, \mu)$, it was shown in [9] that there still exist many balls which have the following (α, β)-doubling property. In what follows, for any ball $B \subset \mathcal{X}$, we denote its *center* and *radius*, respectively, by c_B and r_B and, moreover, for any $\rho \in (0, \infty)$, we denote the ball $B(c_B, \rho r_B)$ by ρB.

Definition 1.5. Let $\alpha, \beta \in (1, \infty)$. A ball $B \subset \mathcal{X}$ is said to be (α, β)-*doubling* if $\mu(\alpha B) \le \beta\mu(B)$.

To be precise, it was proved in [9, Lemma 3.2] that, if a metric measure space $(\mathcal{X}, d, \mu)$ is upper doubling and $\alpha, \beta \in (1, \infty)$ with $\beta > [C_{(\lambda)}]^{\log_2 \alpha} =: \alpha^\nu$, then, for any ball $B \subset \mathcal{X}$, there exists some $j \in \mathbb{Z}_+$ such that $\alpha^j B$ is (α, β)-doubling. Moreover, let $(\mathcal{X}, d)$ be geometrically doubling, let $\beta > \alpha^{n_0}$ with $n_0 := \log_2 N_0$, and let μ be a Borel measure on $\mathcal{X}$ which is finite on bounded sets. Hytönen [9, Lemma 3.3] also showed that, for μ-almost every $x \in \mathcal{X}$, there exist arbitrary small (α, β)-doubling balls centered at x. Furthermore, the radii of these balls may be chosen to be of the form $\alpha^{-j} r$ for $j \in \mathbb{N}$ and any preassigned number $r \in (0, \infty)$. Throughout this article, for any $\alpha \in (1, \infty)$ and ball B, the *smallest* (α, β_α)-*doubling ball of the form* $\alpha^j B$ *with* $j \in \mathbb{Z}_+$ is denoted by $\widetilde{B}^\alpha$, where

$$\beta_\alpha := \alpha^{3(\max\{n_0,\nu\})} + \big[\max\{5\alpha, 30\}\big]^{n_0} + \big[\max\{3\alpha, 30\}\big]^{\nu} \tag{1.5}$$

(see [12] for the details). Also, for any ball B of $\mathcal{X}$, we denote by $\widetilde{B}$ the smallest $(2,\beta_2)$-doubling cube of the form $2^j B$ with $j\in\mathbb{Z}_+$, especially throughout this paper.

The following discrete coefficient $\widetilde{K}^{(\rho)}_{B,S}$ was first introduced by Bui and Duong [1] as analogous of the quantity introduced by Tolsa [24] (see also [25]) in the setting of nondoubling measures (see also [5], [6]). Before we recall the definition of $\widetilde{K}^{(\rho)}_{B,S}$, we first give an assumption: when we speak of a ball B in $(\mathcal{X},d,\mu)$, it is understood that it comes with a fixed center and radius, although these in general are not uniquely determined by B as a set (see [8, pp. 1–2]). In other words, for any two balls $B,S\subset\mathcal{X}$, if $B=S$, then $c_B=c_S$ and $r_B=r_S$. From this, we deduce that if $B\subset S$, then $r_B\le 2r_S$, which plays an essential role in the definition of $\widetilde{K}^{(\rho)}_{B,S}$ (see also Remark 1.7(i) and [5, pp. 314–315] for some details).

Definition 1.6. For any $\rho\in(1,\infty)$ and any two balls $B\subset S\subset\mathcal{X}$, let

$$\widetilde{K}^{(\rho)}_{B,S}:=1+\sum_{k=-\lfloor\log_\rho 2\rfloor}^{N^{(\rho)}_{B,S}}\frac{\mu(\rho^k B)}{\lambda(c_B,\rho^k r_B)},$$

where $N^{(\rho)}_{B,S}$ is the *smallest integer* satisfying $\rho^{N^{(\rho)}_{B,S}}r_B\ge r_S$ and, for arbitrary $a\in\mathbb{R}$, $\lfloor a\rfloor$ denotes the largest integer smaller than or equal to a.

Remark 1.7. (i) With the fact that $r_B\le 2r_S$, we deduce that $N^{(\rho)}_{B,S}\ge-\lfloor\log_\rho 2\rfloor$, which makes sense for the definition of $\widetilde{K}^{(\rho)}_{B,S}$.

(ii) By a change of variables and (1.3), we easily conclude that

$$\widetilde{K}^{(\rho)}_{B,S}\sim 1+\sum_{k=1}^{N^{(\rho)}_{B,S}+\lfloor\log_\rho 2\rfloor+1}\frac{\mu(\rho^k B)}{\lambda(c_B,\rho^k r_B)},$$

where the implicit equivalent positive constants are independent of balls $B\subset S\subset\mathcal{X}$, but depend on ρ.

(iii) For any two balls $B\subset S\subset\mathcal{X}$, let $K_{B,S}:=1+\int_{(2S)\setminus B}\frac{1}{\lambda(c_B,d(x,c_B))}\,d\mu(x)$. It was proved in [12, Lemma 2.2] that $K_{B,S}$ has all properties similar to those for $\widetilde{K}^{(\rho)}_{B,S}$ as in Lemma 2.1 below. Unfortunately, $K_{B,S}$ and $\widetilde{K}^{(\rho)}_{B,S}$ are usually not equivalent, but, for $(\mathbb{R}^d,|\cdot|,\mu)$ with μ as in (1.2), $K_{B,S}\sim\widetilde{K}^{(\rho)}_{B,S}$ with implicit equivalent positive constants independent of B and S (see [6] for more details on this).

Now we recall the $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$ space associated with $\widetilde{K}^{(\rho)}_{B,S}$, which was first introduced by Fu et al. in [6].

Definition 1.8. Let $\rho\in(1,\infty)$, and let $\gamma\in[1,\infty)$. A function $f\in L^1_{\mathrm{loc}}(\mu)$ is said to be in the space $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$ if there exist a positive constant $\widetilde{C}$ and, for any ball $B\subset\mathcal{X}$, a number f_B such that

$$\frac{1}{\mu(\rho B)}\int_B\left|f(x)-f_B\right|d\mu(x)\le\widetilde{C}\tag{1.6}$$

and, for any two balls B and B_1 such that $B \subset B_1$,

$$|f_B - f_{B_1}| \leq \widetilde{C}[\widetilde{K}^{(\rho)}_{B,B_1}]^{\gamma}. \tag{1.7}$$

The infimum of the positive constant $\widetilde{C}$ satisfying both (1.6) and (1.7) is defined to be the $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$ norm of f and is denoted by $\|f\|_{\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)}$.

Remark 1.9. (i) It was pointed out by Fu et al. [6] that the space $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$ is independent of $\rho \in (1,\infty)$ and $\gamma \in [1,\infty)$. In what follows, we denote $\widetilde{\mathrm{RBMO}}_{\rho,\gamma}(\mu)$ simply by $\widetilde{\mathrm{RBMO}}(\mu)$.

(ii) If we replace $\widetilde{K}^{(\rho)}_{B,S}$ by $K_{B,S}$ in Definition 1.8, then $\widetilde{\mathrm{RBMO}}(\mu)$ becomes the space $\mathrm{RBMO}(\mu)$ in [9]. Obviously, for $\rho \in (1,\infty)$ and $\gamma \in [1,\infty)$, $\mathrm{RBMO}(\mu) \subset \widetilde{\mathrm{RBMO}}(\mu)$. However, it is still unclear whether we always have $\mathrm{RBMO}(\mu) = \widetilde{\mathrm{RBMO}}(\mu)$ or not.

In the sequel, for $\tau \in \mathbb{N}$ and $i \in \{1,\dots,\tau\}$, we denote by C_i^τ the family of all finite subsets $\sigma := \{\sigma(1),\dots,\sigma(i)\}$ of $\{1,\dots,\tau\}$ with i different elements. For any $\sigma \in C_i^\tau$, the complementary sequence σ' is given by $\sigma' := \{1,\dots,\tau\} \setminus \sigma$. Let $\vec{b} := (b_1,\dots,b_\tau)$ be a finite family of locally integrable functions. For all $i \in \{1,\dots,\tau\}$ and $\sigma = \{\sigma(1),\dots,\sigma(i)\} \in C_i^\tau$, let $b_\sigma := b_{\sigma(1)} \cdots b_{\sigma(i)}$.

Definition 1.10. Let $\rho \in (1,\infty)$, let $q \in (1,\infty]$, and let $\gamma,\tau \in \mathbb{N}$. Suppose that $b_i \in \widetilde{\mathrm{RBMO}}(\mu)$ for $i \in \{1,\dots,\tau\}$. A function $h \in L^1(\mu)$ is called a $(\vec{b},\tau,q,\gamma,\rho)_\lambda$-atomic block if

(i) there exists a ball B such that $\operatorname{supp} h \subset B$;
(ii) $\int_{\mathcal{X}} h(y)\,d\mu(y) = 0$;
(iii) $\int_{\mathcal{X}} h(y)b_\sigma(y)\,d\mu(y) = 0$ for all $1 \leq i \leq \tau$ and $\sigma \in C_i^\tau$;
(iv) for any $j \in \{1,2\}$, there exist a function a_j supported on a ball $B_j \subset B$ and a number $\lambda_j \in \mathbb{C}$ such that $h = \lambda_1 a_1 + \lambda_2 a_2$ and

$$\|a_j\|_{L^q(\mu)} \leq \big[\mu(\rho B_j)\big]^{1/q-1}[\widetilde{K}^{(\rho)}_{B_j,B}]^{-\gamma}.$$

Moreover, let $|h|_{\widetilde{H}^{1,q,\gamma}_{\vec{b},\tau,\rho}(\mu)} := |\lambda_1| + |\lambda_2|$.

Definition 1.11. Let $\rho \in (1,\infty)$, let $q \in (1,\infty]$, and let γ, $\tau \in \mathbb{N}$. Suppose $b_i \in \widetilde{\mathrm{RBMO}}(\mu)$ for $i = 1,2,\dots,\tau$.

(i) A function $f \in L^1(\mu)$ is said to belong to the atomic Hardy-type space $\widetilde{H}^{1,q,\gamma}_{\vec{b},\tau,\rho}(\mu)$ if there exist $(\vec{b},\tau,q,\gamma,\rho)_\lambda$-atomic blocks $\{h_k\}_{k\in\mathbb{N}}$ such that $f = \sum_{k=1}^\infty h_k$ in $L^1(\mu)$ and $\sum_{k=1}^\infty |h_k|_{\widetilde{H}^{1,q,\gamma}_{\vec{b},\tau,\rho}(\mu)} < \infty$. The $\widetilde{H}^{1,q,\gamma}_{\vec{b},\tau,\rho}(\mu)$ norm of f is defined by

$$\|f\|_{\widetilde{H}^{1,q,\gamma}_{\vec{b},\tau,\rho}(\mu)} := \inf\Big\{\sum_{k=1}^\infty |h_k|_{\widetilde{H}^{1,q,\gamma}_{\vec{b},\tau,\rho}(\mu)}\Big\},$$

where the infimum is taken over all the possible decompositions of f as above.

(ii) The space $\widetilde{H}^{1,q,\gamma}_{\mathrm{fin},\vec{b},\tau,\rho}(\mu)$ is defined to be the set of all finite linear combinations of $(\vec{b},\tau,q,\gamma,\rho)_\lambda$-atomic blocks $\{h_k\}_{k\in\mathbb{N}}$. The norm of f in $\widetilde{H}^{1,q,\gamma}_{\mathrm{fin},\vec{b},\tau,\rho}(\mu)$ is defined by

$$\|f\|_{\widetilde{H}^{1,q,\gamma}_{\mathrm{fin},\vec{b},\tau,\rho}(\mu)} := \inf\Big\{\sum_{k=1}^{N}|h_k|_{\widetilde{H}^{1,q,\gamma}_{\vec{b},\tau,\rho}(\mu)} : f=\sum_{k=1}^{N}h_k,\, N\in\mathbb{N}\Big\}.$$

Remark 1.12. (i) If $\tau=0$, then the space $\widetilde{H}^{1,q,\gamma}_{\vec{b},\tau,\rho}(\mu)$ is just the atomic Hardy space $\widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ introduced by Fu et al. in [6]. It was pointed out by Fu et al. [6] that, for each $q\in(1,\infty]$, the atomic Hardy space $\widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ is independent of the choices of ρ and γ and that, for all $q\in(1,\infty)$, the spaces $\widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ and $\widetilde{H}^{1,\infty,\gamma}_{\mathrm{atb},\rho}(\mu)$ coincide with equivalent norms. Thus, in what follows, we denote $\widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ simply by $\widetilde{H}^1(\mu)$.

(ii) Let $\rho\in(1,\infty)$, let $p\in(1,\infty]$, and let $\gamma\in[1,\infty)$. It was pointed out by Fu et al. [6] that $[\widetilde{H}^{1,p,\gamma}_{\mathrm{atb},\rho}(\mu)]^*=\widetilde{\mathrm{RBMO}}(\mu)$.

(iii) It is easy to see that, for any $q\in(1,\infty]$, $\tau,\gamma\in\mathbb{N}$, and $\rho_1,\rho_2\in(1,\infty)$ with $1<\rho_1<\rho_2$,

$$\widetilde{H}^{1,q,\gamma}_{\vec{b},\tau,\rho_2}(\mu)\subset\widetilde{H}^{1,q,\gamma}_{\vec{b},\tau,\rho_1}(\mu)\subset\widetilde{H}^1(\mu),$$

and, for any $\rho\in(1,\infty)$, $q\in(1,\infty]$, $\tau\in\mathbb{N}$, and $\gamma_1,\gamma_2\in\mathbb{N}$ with $1\le\gamma_1<\gamma_2$,

$$\widetilde{H}^{1,q,\gamma_2}_{\vec{b},\tau,\rho}(\mu)\subset\widetilde{H}^{1,q,\gamma_1}_{\vec{b},\tau,\rho}(\mu)\subset\widetilde{H}^1(\mu),$$

and, for any $\rho\in(1,\infty)$, $\tau,\gamma\in\mathbb{N}$, and $q_1,q_2\in(1,\infty]$ with $1<q_1<q_2\le\infty$,

$$\widetilde{H}^{1,\infty,\gamma}_{\vec{b},\tau,\rho}(\mu)\subset\widetilde{H}^{1,q_2,\gamma}_{\vec{b},\tau,\rho}(\mu)\subset\widetilde{H}^{1,q_1,\gamma}_{\vec{b},\tau,\rho}(\mu)\subset\widetilde{H}^1(\mu).$$

However, it is still open if the spaces $\widetilde{H}^{1,q,\gamma}_{\vec{b},\tau,\rho}(\mu)$ are equivalent for any fixed $\tau\in\mathbb{N}$ and different $\rho\in(1,\infty)$, $\gamma\in\mathbb{N}$, and $q\in(1,\infty]$.

Definition 1.13. A function $K\in L^1_{\mathrm{loc}}(\{\mathcal{X}\times\mathcal{X}\}\backslash\{(x,x):x\in\mathcal{X}\})$ is called a *Calderón–Zygmund kernel* if there exists a positive constant $C_{(K)}$ such that

(i) for all $x,y\in\mathcal{X}$ with $x\neq y$,

$$\big|K(x,y)\big|\le C_{(K)}\frac{1}{\lambda(x,d(x,y))};\tag{1.8}$$

(ii) there exist positive constants $\delta\in(0,1]$ and $c_{(K)}$ depending on K such that, for all $x,\widetilde{x},y\in\mathcal{X}$ with $d(x,y)\ge c_{(K)}d(x,\widetilde{x})$,

$$\big|K(x,y)-K(\widetilde{x},y)\big|+\big|K(y,x)-K(y,\widetilde{x})\big|\le C_{(K)}\frac{[d(x,\widetilde{x})]^\delta}{[d(x,y)]^\delta\lambda(x,d(x,y))}.\tag{1.9}$$

Let $L^\infty_b(\mu)$ be the set of all $L^\infty(\mu)$ functions with bounded support. A linear operator T is called a *Calderón–Zygmund operator* with kernel K satisfying (1.8) and (1.9) if, for all $f\in L^\infty_b(\mu)$,

$$Tf(x):=\int_{\mathcal{X}}K(x,y)f(y)\,d\mu(y),\quad x\notin\operatorname{supp}(f).\tag{1.10}$$

We remark that a new example of the operator with the kernel satisfying (1.8) and (1.9) is the so-called *Bergman-type operator* appearing in [29, p. 950] (see also [10, Section 12] for an explanation).

Let $m \in \mathbb{N}$ and $b_i \in \widetilde{\mathrm{RBMO}}(\mu)$, $i = 1, 2, \ldots, m$. The multilinear commutator $T_{\vec{b}}$ generated by the Calderón–Zygmund operator T and $\vec{b} = (b_1, \ldots, b_m)$ is defined by setting, for all suitable functions f and $x \in \mathcal{X}$,

$$T_{\vec{b}}(f)(x) := \big[b_m, \big[b_{m-1}, \ldots, [b_1, T] \cdots\big]\big](f)(x), \tag{1.11}$$

where $[b_1, T]f(x) := b_1(x)Tf(x) - T(b_1 f)(x)$. The multilinear commutator $T_{\vec{b}}$ in the setting of $\mathbb{R}^d$ with the d-dimensional Lebesgue measure was first introduced by Pérez and Trujillo-González in [20]; it was introduced in the setting of $\mathbb{R}^d$ with the measure as in (1.2) by Meng and Yang in [16]; and it was introduced in the present setting $(\mathcal{X}, d, \mu)$ by Fu et al. in [6].

Now we state the main results of this article as follows.

Theorem 1.14. *Let $\rho \in (1, \infty)$, let $q \in (1, \infty]$, let $m \in \mathbb{N}$, and let $b_i \in \widetilde{\mathrm{RBMO}}(\mu)$ for all $i \in \{1, \ldots, m\}$. Let T and $T_{\vec{b}}$ be as in (1.10) and (1.11), respectively. Suppose that T is bounded on $L^2(\mu)$. Then the multilinear commutator $T_{\vec{b}}$ is bounded from $\widetilde{H}^{1,q,m+1}_{\mathrm{fin},\vec{b},m,\rho}(\mu)$ into $L^1(\mu)$.*

Remark 1.15. It is still unclear whether the boundedness of linear operators on the atomic Hardy-type space $\widetilde{H}^{1,q,\gamma}_{\vec{b},\tau,\rho}(\mu)$ can be deduced only from their behaviors on atoms. Thus, under the assumption of Theorem 1.14, it is unclear whether the multilinear commutator $T_{\vec{b}}$ is bounded from $\widetilde{H}^{1,q,m+1}_{\vec{b},m,\rho}(\mu)$ into $L^1(\mu)$ or not.

In what follows, the multilinear commutator $T_{\vec{b}}$ is said to satisfy $T^*_{\vec{b}}(1) = 0$ if, for all $h \in L^\infty_b(\mu)$ satisfying (ii) and (iii) of Definition 1.1, $\int_{\mathcal{X}} T_{\vec{b}}(h)(x)\, d\mu(x) = 0$. Observe that, by Theorem 1.14, we have $T_{\vec{b}}(h) \in L^1(\mu)$.

Theorem 1.16. *Let $\rho \in (2, \infty)$, let $q \in (1, \infty]$, let $m \in \mathbb{N}$, and let $b_i \in \widetilde{\mathrm{RBMO}}(\mu)$ for all $i \in \{1, \ldots, m\}$. Let T and $T_{\vec{b}}$ be as in (1.10) and (1.11), respectively. Suppose T is bounded on $L^2(\mu)$ and $T^*_{\vec{b}}(1) = 0$. Then the multilinear commutator $T_{\vec{b}}$ is bounded from $\widetilde{H}^{1,q,m+2}_{\mathrm{fin},\vec{b},m,\rho}(\mu)$ into $\widetilde{H}^1(\mu)$.*

We remark that, under the assumption of Theorem 1.16, it is unclear whether the multilinear commutator $T_{\vec{b}}$ is bounded from $\widetilde{H}^{1,q,m+2}_{\vec{b},m,\rho}(\mu)$ into $\widetilde{H}^1(\mu)$ or not.

This paper is organized as follows. In Section 2, we proved Theorem 1.14 by borrowing some ideas from [16, Theorem 1.1]. Section 3 is devoted to proving Theorem 1.16. We point out that although Theorem 1.16 is similar to [16, Theorem 1.3], its proof is different. In the proof of [16, Theorem 1.3] the authors used the characterization of the atomic Hardy space in terms of the grand maximal operator, which is still unknown in the present setting. Hence we prove Theorem 1.16 via the molecular characterization of $\widetilde{H}^1(\mu)$.

Finally, we make some conventions on notation. Throughout this paper, we always denote by C, $\widetilde{C}$, c, or $\widetilde{c}$ a *positive constant* which is independent of the main parameters, but they may vary from line to line. We use $C_{(\alpha)}$ to denote a positive constant depending on the parameter α. The expression $Y \lesssim Z$ means that there exists a positive constant C such that $Y \le CZ$. Given any $q \in (0,\infty)$, let $q' := q/(q-1)$ denote its *conjugate index*. Also, for any subset $E \subset \mathcal{X}$, χ_E denotes its *characteristic function*. For any $f \in L^1_{\rm loc}(\mu)$ and any measurable set E of $\mathcal{X}$, $m_E(f)$ denotes its mean over E, namely, $m_E(f) := \frac{1}{\mu(E)}\int_E f(x)\,d\mu(x)$.

2. Proof of Theorem 1.14

We begin with some necessary lemmas. The following useful properties of $\widetilde{K}^{(\rho)}_{B,S}$ were proved in [5, Lemmas 2.8, 2.9].

Lemma 2.1. *Let $(\mathcal{X}, d, \mu)$ be a nonhomogeneous metric measure space.*

- (i) *For any $\rho \in (1,\infty)$, there exists a positive constant $C_{(\rho)}$ depending on ρ such that, for all balls $B \subset R \subset S$, $\widetilde{K}^{(\rho)}_{B,R} \le C_{(\rho)}\widetilde{K}^{(\rho)}_{B,S}$.*
- (ii) *For any $\alpha \in [1,\infty)$ and $\rho \in (1,\infty)$, there exists a positive constant $C_{(\alpha,\rho)}$ depending on α and ρ such that, for all balls $B \subset S$ with $r_S \le \alpha r_B$, $\widetilde{K}^{(\rho)}_{B,S} \le C_{(\alpha,\rho)}$.*
- (iii) *For any $\rho \in (1,\infty)$, there exists a positive constant $C_{(\rho,\nu)}$ depending on ρ and ν such that, for all balls B, $\widetilde{K}^{(\rho)}_{B,\widetilde{B}^\rho} \le C_{(\rho,\nu)}$. Moreover, letting $\alpha, \beta \in (1,\infty)$, $B \subset S$ be any two concentric balls such that there exists no (α,β)-doubling ball in the form of $\alpha^k B$ with $k \in \mathbb{N}$ satisfying $B \subset \alpha^k B \subset S$, then there exists a positive constant $C_{(\alpha,\beta,\nu)}$ depending on α, β, and ν such that $\widetilde{K}^{(\rho)}_{B,S} \le C_{(\alpha,\beta,\nu)}$.*
- (iv) *For any $\rho \in (1,\infty)$, there exists a positive constant $c_{(\rho,\nu)}$ depending on ρ and ν such that, for all balls $B \subset R \subset S$, $\widetilde{K}^{(\rho)}_{B,S} \le \widetilde{K}^{(\rho)}_{B,R} + c_{(\rho,\nu)}\widetilde{K}^{(\rho)}_{R,S}$.*
- (v) *For any $\rho \in (1,\infty)$, there exists a positive constant $\widetilde{c}_{(\rho,\nu)}$ depending on ρ and ν such that, for all balls $B \subset R \subset S$, $\widetilde{K}^{(\rho)}_{R,S} \le \widetilde{c}_{(\rho,\nu)}\widetilde{K}^{(\rho)}_{B,S}$.*

Lemma 2.2. *Let $(\mathcal{X}, d, \mu)$ be a nonhomogeneous metric measure space and ρ_1, $\rho_2 \in (1,\infty)$. Then there exist positive constants $c_{(\rho_1,\rho_2,\nu)}$ and $C_{(\rho_1,\rho_2,\nu)}$ depending on ρ_1, ρ_2, and ν such that, for all balls $B \subset S$,*

$$c_{(\rho_1,\rho_2,\nu)}\widetilde{K}^{(\rho_1)}_{B,S} \le \widetilde{K}^{(\rho_2)}_{B,S} \le C_{(\rho_1,\rho_2,\nu)}\widetilde{K}^{(\rho_1)}_{B,S}.$$

To prove Theorem 1.14, we also need the following equivalent characterization of the space $\widetilde{\rm RBMO}(\mu)$ established in [13, Lemma 2.15] and the John–Nirenberg inequality for $\widetilde{\rm RBMO}(\mu)$ established in [13, Proposition 2.16].

Lemma 2.3. *Let $\eta, \rho \in (1,\infty)$, and let β_ρ be as in (1.5). For $f \in L^1_{\rm loc}(\mu)$, the following statements are equivalent:*

- (i) *$f \in \widetilde{\rm RBMO}(\mu)$;*
- (ii) *there exists a positive constant C such that, for all balls B,*

$$\frac{1}{\mu(\eta B)}\int_B \left|f(x) - m_{\widetilde{B}^\rho}(f)\right| d\mu(x) \le C$$

and, for all (ρ,β_ρ)-doubling balls $B\subset S$,

$$\big|m_B(f)-m_S(f)\big|\le C\widetilde{K}^{(\rho)}_{B,S}. \tag{2.1}$$

Moreover, the infimum of the above constant C is equivalent to $\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}$.

Proposition 2.4. *Let $(\mathcal{X},d,\mu)$ be a nonhomogeneous metric measure space. Then, for every $\rho\in(0,\infty)$, there exists a positive constant c such that, for all $f\in\widetilde{\mathrm{RBMO}}(\mu)$, balls B_0, and $t\in(0,\infty)$,*

$$\mu\big(\big\{x\in B_0:\big|f(x)-f_{B_0}\big|>t\big\}\big)\le 2\mu(\rho B_0)e^{-ct/\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}},$$

where f_{B_0} is as in Definition 1.8 with B replaced by B_0.

Lemma 2.5. *Let $m\in\mathbb{N}$, $b_i\in\widetilde{\mathrm{RBMO}}(\mu)$ for $i\in\{1,\dots,m\}$, $\rho,\eta\in(1,\infty)$, and $q\in[1,\infty)$. Then there exists a positive constant C such that, for any ball B,*

$$\bigg\{\frac{1}{\mu(\rho B)}\int_B\prod_{i=1}^m\big|b_i(x)-m_{\widetilde{B}^\eta}(b_i)\big|^q\,d\mu(x)\bigg\}^{1/q}\le C\prod_{i=1}^m\|b_i\|_{\widetilde{\mathrm{RBMO}}(\mu)}.$$

When $m=1$, Lemma 2.5 is a simple corollary of the John–Nirenberg inequality for $\widetilde{\mathrm{RBMO}}(\mu)$. From this and the Hölder inequality, it is easy to prove Lemma 2.5 for any $m\in\mathbb{N}$. We omit the details here.

Lemma 2.6. *Let $f\in\widetilde{\mathrm{RBMO}}(\mu)$, and let $\rho\in(1,\infty)$. Then, for all two balls $B\subset S\subset\mathcal{X}$, we have*

$$\big|m_{\widetilde{B}^\rho}(f)-m_{\widetilde{S}^\rho}(f)\big|\lesssim\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}\widetilde{K}^{(\rho)}_{B,S}.$$

Proof. For any fixed two balls $B\subset S$, we consider the following three cases of the relation of $\widetilde{B}^\rho$ and $\widetilde{S}^\rho$:

Case (I): $\widetilde{B}^\rho\subset\widetilde{S}^\rho$. In this case, $B\subset\widetilde{B}^\rho\subset\widetilde{S}^\rho$ and $B\subset S\subset\widetilde{S}^\rho$. By Lemma 2.1(v), (iv), and (iii), we have $\widetilde{K}^{(\rho)}_{\widetilde{B}^\rho,\widetilde{S}^\rho}\lesssim\widetilde{K}^{(\rho)}_{B,\widetilde{S}^\rho}\lesssim\widetilde{K}^{(\rho)}_{B,S}+\widetilde{K}^{(\rho)}_{S,\widetilde{S}^\rho}\lesssim\widetilde{K}^{(\rho)}_{B,S}$, which, together with (2.1), implies that $|m_{\widetilde{B}^\rho}(f)-m_{\widetilde{S}^\rho}(f)|\le\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}\widetilde{K}^{(\rho)}_{\widetilde{B}^\rho,\widetilde{S}^\rho}\lesssim\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}\widetilde{K}^{(\rho)}_{B,S}$.

Case (II): $\widetilde{S}^\rho\subset\widetilde{B}^\rho$. In this case, $B\subset S\subset\widetilde{S}^\rho\subset\widetilde{B}^\rho$. Similar to Case (I), it is easy to see that Lemma 2.6 holds true in this case.

Case (III): $\widetilde{B}^\rho\not\subset\widetilde{S}^\rho$ and $\widetilde{S}^\rho\not\subset\widetilde{B}^\rho$. In this case, $\widetilde{B}^\rho\cap(\widetilde{S}^\rho)^C\neq\emptyset$. Then we have $\widetilde{B}^\rho\subset3\widetilde{S}^\rho$. In fact, there exists $y\in\widetilde{S}^\rho$ such that $d(y,c_B)>r_{\widetilde{B}^\rho}$. Thus $r_{\widetilde{B}^\rho}\le d(y,c_B)\le d(y,c_S)+d(c_S,c_B)<r_{\widetilde{S}^\rho}+r_S\le2r_{\widetilde{S}^\rho}$. Furthermore, for any $w\in\widetilde{B}^\rho$, we have $d(w,c_S)\le d(w,c_B)+d(c_B,c_S)\le r_{\widetilde{B}^\rho}+r_{\widetilde{S}^\rho}<3r_{\widetilde{S}^\rho}$, which implies that $\widetilde{B}^\rho\subset3\widetilde{S}^\rho$. It then follows that $B\subset\widetilde{B}^\rho\subset\widetilde{3\widetilde{S}^\rho}^\rho$ and $B\subset S\subset\widetilde{3\widetilde{S}^\rho}^\rho$. From this, together with (2.1) and Lemma 2.1, we deduce that $|m_{\widetilde{B}^\rho}(f)-m_{\widetilde{S}^\rho}(f)|\le|m_{\widetilde{B}^\rho}(f)-m_{\widetilde{3\widetilde{S}^\rho}^\rho}(f)|+|m_{\widetilde{3\widetilde{S}^\rho}^\rho}(f)-m_{\widetilde{S}^\rho}(f)|\le\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}(\widetilde{K}^{(\rho)}_{\widetilde{B}^\rho,\widetilde{3\widetilde{S}^\rho}^\rho}+\widetilde{K}^{(\rho)}_{\widetilde{S}^\rho,\widetilde{3\widetilde{S}^\rho}^\rho})\lesssim\|f\|_{\widetilde{\mathrm{RBMO}}(\mu)}\widetilde{K}^{(\rho)}_{B,S}$, which completes the proof of Lemma 2.6. ☐

Let $m\in\mathbb{N}$ and $\vec{b}=(b_1,\ldots,b_m)$ be a finite family of $\widetilde{\mathrm{RBMO}}(\mu)$ functions. For all $i\in\{1,\ldots,m\}$ and $\sigma=\{\sigma(1),\ldots,\sigma(i)\}\in C_i^m$, we let $\vec{b}_\sigma:=(b_{\sigma(1)},\ldots,b_{\sigma(i)})$, $\|\vec{b}_\sigma\|_{\widetilde{\mathrm{RBMO}}(\mu)}:=\|b_{\sigma(1)}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\cdots\|b_{\sigma(i)}\|_{\widetilde{\mathrm{RBMO}}(\mu)}$, and, for any $x,y\in\mathcal{X}$ and any balls B and S in $\mathcal{X}$,

$$\big[b(x)-m_B(b)\big]_\sigma:=\big[b_{\sigma(1)}(x)-m_B(b_{\sigma(1)})\big]\cdots\big[b_{\sigma(i)}(x)-m_B(b_{\sigma(i)})\big],$$

and

$$\big[m_S(b)-m_B(b)\big]_\sigma:=\big[m_S(b_{\sigma(1)})-m_B(b_{\sigma(1)})\big]\cdots\big[m_S(b_{\sigma(i)})-m_B(b_{\sigma(i)})\big].$$

For any $\vec{b}:=(b_1,\ldots,b_m)$, write $\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}:=\|b_1\|_{\widetilde{\mathrm{RBMO}}(\mu)}\cdots\|b_m\|_{\widetilde{\mathrm{RBMO}}(\mu)}$. The following lemma is a special case of [6, Theorem 1.9].

Lemma 2.7. *Let $q\in(1,\infty)$, let $m\in\mathbb{N}$, and let $b_i\in\widetilde{\mathrm{RBMO}}(\mu)$ for $i\in\{1,\ldots,m\}$. Let T and $T_{\vec{b}}$ be as in (1.10) and (1.11), respectively. Suppose that T is bounded on $L^2(\mu)$. Then $T_{\vec{b}}$ is bounded on $L^q(\mu)$ with the norm no more than $C\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}$, where C is a positive constant.*

Now we can show Theorem 1.14 as follows.

Proof of Theorem 1.14. The argument is similar to the one in the proof of [16, pp. 38–39]. We will repeat it for the sake of completeness. By Definition 1.11(ii) and Remark 1.12(iii), it suffices to verify that, for any $(\vec{b},m,q,m+1,\rho)_\lambda$-atomic block h as in Definition 1.10 with $\rho\in(1,\infty)$ and $q\in(1,\infty)$, $\|T_{\vec{b}}h\|_{L^1(\mu)}\le C\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}|h|_{\widetilde{H}^{1,q,m+1}_{\vec{b},m,\rho}(\mu)}$, where C is a positive constant independent of h. For the sake of simplicity, we take $\rho=2$. Let all the notation be the same as in Definition 1.10. Then, for any $j\in\{1,2\}$, we have

$$\|a_j\|_{L^q(\mu)}\le\mu(2B_j)^{1/q-1}[\widetilde{K}^{(2)}_{B_j,B}]^{-(m+1)}. \tag{2.2}$$

Write

$$\int_{\mathcal{X}}\big|T_{\vec{b}}(h)(x)\big|\,d\mu(x)=\int_{2B}\big|T_{\vec{b}}(h)(x)\big|\,d\mu(x)+\int_{\mathcal{X}\setminus 2B}\big|T_{\vec{b}}(h)(x)\big|\,d\mu(x)=:\mathrm{M}+\mathrm{N}.$$

We first estimate the term M. To do this, we further decompose

$$\mathrm{M}\le\sum_{j=1}^2|\lambda_j|\int_{2B_j}\big|T_{\vec{b}}(a_j)(x)\big|\,d\mu(x)+\sum_{j=1}^2|\lambda_j|\int_{2B\setminus 2B_j}\big|T_{\vec{b}}(a_j)(x)\big|\,d\mu(x)=:\mathrm{M}_1+\mathrm{M}_2.$$

By the Hölder inequality, Lemma 2.7, (2.2), and $\widetilde{K}^{(2)}_{B_j,B}\ge1$, we have

$$\mathrm{M}_1\lesssim\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sum_{j=1}^2|\lambda_j|.$$

To estimate M_2, let $N_j = N^{(2)}_{2B_j,2B} + 2$. Notice that, for any $x, y \in \mathcal{X}$,

$$\prod_{i=1}^{m}\big[b_i(x) - b_i(y)\big] = \sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\big[b(x) - m_{\widetilde{B_j}}(b)\big]_\sigma\big[m_{\widetilde{B_j}}(b) - b(y)\big]_{\sigma'}, \tag{2.3}$$

where if $i = 0$, then $\sigma' = \{1, \ldots, m\}$ and $\sigma = \emptyset$. From this, (1.8), the Hölder inequality, Lemmas 2.5 and 2.1, Remark 1.2(iv), and (2.2), we deduce that

$$\begin{aligned}
\mathrm{M}_2 &\le \sum_{j=1}^{2}|\lambda_j|\sum_{k=1}^{N_j}\int_{2^{k+1}B_j\setminus 2^kB_j}\Big|\int_{B_j}\prod_{i=1}^{m}\big[b_i(x) - b_i(y)\big]K(x,y)a_j(y)\,d\mu(y)\Big|\,d\mu(x)\\
&\lesssim \sum_{j=1}^{2}|\lambda_j|\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\Big\{\int_{B_j}|a_j(y)|^q\,d\mu(y)\Big\}^{1/q}\\
&\quad\times\Big\{\int_{B_j}\big|\big[m_{\widetilde{B_j}}(b) - b(y)\big]_{\sigma'}\big|^{q'}\,d\mu(y)\Big\}^{1/q'}\Big\{\sum_{k=1}^{N_j}\sum_{l=0}^{i}\sum_{\eta(\sigma)\in C_l^i}\frac{1}{\lambda(c_{B_j}, 2^kr_{B_j})}\\
&\quad\times\int_{2^{k+1}B_j}\big|\big[b(x) - m_{\widetilde{2^{k+1}B_j}}(b)\big]_{\eta(\sigma)}\big[m_{\widetilde{2^{k+1}B_j}}(b) - m_{\widetilde{B_j}}(b)\big]_{\eta'(\sigma)}\big|\,d\mu(x)\Big\}\\
&\lesssim \sum_{j=1}^{2}|\lambda_j|\|a_j\|_{L^q(\mu)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\big[\mu(2B_j)\big]^{1/q'}\|\vec{b}_{\sigma'}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\\
&\quad\times\sum_{k=1}^{N_j}\sum_{l=0}^{i}\sum_{\eta(\sigma)\in C_l^i}\Big\{\frac{\mu(2^{k+2}B_j)}{\lambda(c_{B_j}, 2^kr_{B_j})}\|\vec{b}_{\eta(\sigma)}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\|\vec{b}_{\eta'(\sigma)}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\\
&\quad\times\big[\widetilde{K}^{(2)}_{\widetilde{B_j},\widetilde{2^{k+1}B_j}}\big]^{i-l}\Big\}\\
&\lesssim \sum_{j=1}^{2}|\lambda_j|\|a_j\|_{L^q(\mu)}\big[\mu(2B_j)\big]^{1/q'}\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\sum_{k=1}^{N_j}\frac{\mu(2^{k+2}B_j)}{\lambda(c_{B_j}, 2^{k+2}r_{B_j})}\big[\widetilde{K}^{(2)}_{\widetilde{B_j},\widetilde{2^{k+1}B_j}}\big]^m\\
&\lesssim \sum_{j=1}^{2}|\lambda_j|\big[\mu(2B_j)\big]^{1/q-1}\big[\widetilde{K}^{(2)}_{B_j,B}\big]^{-(m+1)}\big[\mu(2B_j)\big]^{1/q'}\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\big[\widetilde{K}^{(2)}_{B_j,B}\big]^{(m+1)}\\
&\lesssim \sum_{j=1}^{2}|\lambda_j|\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)},
\end{aligned}$$

where, in the penultimate inequality, we have used the fact that, for any $1 \le k \le N_j$, $\widetilde{K}^{(2)}_{\widetilde{B_j},\widetilde{2^{k+1}B_j}} \lesssim \widetilde{K}^{(2)}_{B_j,B}$.

It remains to estimate the term N. Recall that, for a ball B, c_B denotes its center. By Definition 1.10, (2.3), (1.9), the Hölder inequality, Lemmas 2.5, 2.6, and 2.1, and (2.2), we conclude that

$$\mathrm{N} = \int_{\mathcal{X}\setminus 2B}\Big|\int_B\prod_{i=1}^{m}\big[b_i(x) - b_i(y)\big]\big[K(x,y) - K(x,c_B)\big]h(y)\,d\mu(y)\Big|\,d\mu(x)$$

$$
\begin{aligned}
&\lesssim \sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\int_B \left|\left[b(y)-m_{\widetilde{B}}(b)\right]_{\sigma'}\right|\left|h(y)\right|d\mu(y)\\
&\quad\times\sum_{k=1}^{\infty}\int_{2^{k+1}B\setminus 2^kB}\left|\left[b(x)-m_{\widetilde{B}}(b)\right]_{\sigma}\right|\frac{[d(y,c_B)]^\delta}{[d(x,c_B)]^\delta\lambda(c_B,d(x,c_B))}\,d\mu(x)\\
&\lesssim \sum_{j=1}^{2}|\lambda_j|\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\Bigg\{\sum_{l=0}^{m-i}\sum_{\eta(\sigma')\in C_l^{m-i}}\int_{B_j}\left|a_j(y)\right|\left|\left[b(y)-m_{\widetilde{B_j}}(b)\right]_{\eta(\sigma')}\right.\\
&\quad\times\left.\left[m_{\widetilde{B_j}}(b)-m_{\widetilde{B}}(b)\right]_{\eta'(\sigma')}\right|d\mu(y)\Bigg\}\Bigg\{\sum_{k=1}^{\infty}\sum_{s=0}^{i}\sum_{\theta(\sigma)\in C_s^i}\frac{(r_B)^\delta}{(2^kr_B)^\delta\lambda(c_B,2^kr_B)}\\
&\quad\times\int_{2^{k+1}B}\left|\left[b(x)-m_{\widetilde{2^{k+1}B}}(b)\right]_{\theta(\sigma)}\left[m_{\widetilde{2^{k+1}B}}(b)-m_{\widetilde{B}}(b)\right]_{\theta'(\sigma)}\right|d\mu(x)\Bigg\}\\
&\lesssim \sum_{j=1}^{2}|\lambda_j|\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\Bigg\{\sum_{l=0}^{m-i}\sum_{\eta(\sigma')\in C_l^{m-i}}\|a_j\|_{L^q(\mu)}\\
&\quad\times\left\{\int_{B_j}\left|\left[b(y)-m_{\widetilde{B_j}}(b)\right]_{\eta(\sigma')}\left[m_{\widetilde{B_j}}(b)-m_{\widetilde{B}}(b)\right]_{\eta'(\sigma')}\right|^{q'}d\mu(y)\right\}^{1/q'}\Bigg\}\\
&\quad\times\sum_{k=1}^{\infty}\frac{\mu(2^{k+2}B)}{2^{k\delta}\lambda(c_B,2^kr_B)}[\widetilde{K}^{(2)}_{B,2^{k+1}B}]^i\|\vec{b}_\sigma\|_{\widetilde{\mathrm{RBMO}}(\mu)}\\
&\lesssim \sum_{j=1}^{2}|\lambda_j|\|a_j\|_{L^q(\mu)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\|\vec{b}_{\sigma'}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\left[\mu(2B_j)\right]^{1/q'}[\widetilde{K}^{(2)}_{B_j,B}]^{m-i}\\
&\lesssim \sum_{j=1}^{2}|\lambda_j|\|a_j\|_{L^q(\mu)}\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\left[\mu(2B_j)\right]^{1/q'}[\widetilde{K}^{(2)}_{B_j,B}]^{m}\\
&\lesssim \sum_{j=1}^{2}|\lambda_j|\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}[\widetilde{K}^{(2)}_{B_j,B}]^{-1}\lesssim\sum_{j=1}^{2}|\lambda_j|\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)},
\end{aligned}
$$

which, together with the estimate for M, completes the proof of Theorem 1.14. □

3. Proof of Theorem 1.16

To prove Theorem 1.16, we need the molecular characterization of the atomic Hardy space $\widetilde{H}^1(\mu)$ established in [6, Definition 1.10, Theorem 1.11].

Definition 3.1. Let $\rho\in(1,\infty)$, let $q\in(1,\infty]$, let $\gamma\in[1,\infty)$, and let $\epsilon\in(0,\infty)$. A function $b\in L^1(\mu)$ is called a $(q,\gamma,\epsilon,\rho)_\lambda$-*molecular block* if

(i) $\int_{\mathcal{X}} b(x)\,d\mu(x)=0$;

(ii) there exist some balls $B:=B(c_B,r_B)$ with $c_B\in\mathcal{X}$ and $r_B\in(0,\infty)$, and exist some constants $\widetilde{M},M\in\mathbb{N}$ such that, for all $k\in\mathbb{Z}_+$ and $j\in\{1,\dots,M_k\}$ with $M_k:=\widetilde{M}$ if $k=0$ and $M_k:=M$ if $k\in\mathbb{N}$, there exist functions $m_{k,j}$ supported on some balls $B_{k,j}\subset U_k(B)$ for all $k\in\mathbb{Z}_+$, where $U_0(B):=$

$\rho^2 B$ and $U_k(B) := \rho^{k+2}B \setminus \rho^{k-2}B$ with $k \in \mathbb{N}$, and $\lambda_{k,j} \in \mathbb{C}$ such that $b = \sum_{k=0}^{\infty}\sum_{j=1}^{M_k} \lambda_{k,j} m_{k,j}$ in $L^1(\mu)$,

$$\|m_{k,j}\|_{L^q(\mu)} \le \rho^{-k\epsilon}\big[\mu(\rho B_{k,j})\big]^{1/q-1}\big[\widetilde{K}^{(\rho)}_{B_{k,j},\rho^{k+2}B}\big]^{-\gamma}$$

and $|b|_{\widetilde{H}^{1,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)} := \sum_{k=0}^{\infty}\sum_{j=1}^{M_k} |\lambda_{k,j}| < \infty$.

A function $f \in L^1(\mu)$ is said to belong to the molecular Hardy space $\widetilde{H}^{1,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ if there exist a sequence of $(q,\gamma,\epsilon,\rho)_\lambda$-molecular blocks, $\{b_i\}_{i=1}^{\infty}$, such that $f = \sum_{i=1}^{\infty} b_i$ in $L^1(\mu)$ and $\sum_{i=1}^{\infty} |b_i|_{\widetilde{H}^{1,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)} < \infty$. Moreover, define

$$\|f\|_{\widetilde{H}^{1,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)} := \inf\Big\{\sum_{i=1}^{\infty} |b_i|_{\widetilde{H}^{1,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)}\Big\},$$

where the infimum is taken over all the possible decompositions of f as above.

The following equivalence between $\widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu)$ and $\widetilde{H}^{1,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ was established in [6, Theorem 1.11].

Lemma 3.2. *Let $\rho \in (1,\infty)$, let $q \in (1,\infty]$, let $\gamma \in [1,\infty)$, and let $\epsilon \in (0,\infty)$. Then $\widetilde{H}^{1,q,\gamma}_{\mathrm{atb},\rho}(\mu) = \widetilde{H}^{1,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ with equivalent norms.*

Remark 3.3. As a consequence of Lemma 2.2, we see that the space $\widetilde{H}^{1,q,\gamma,\epsilon}_{\mathrm{mb},\rho}(\mu)$ is independent of the choices of the parameters q, ρ, γ, and ϵ.

Bearing in mind the molecular characterization of the atomic Hardy space, we are ready to prove Theorem 1.16.

Proof of Theorem 1.16. Without loss of generality, we may assume that $\rho = 2\widetilde{\alpha}$ in Definition 1.10 with $\widetilde{\alpha} \in (1,2)$, and $\rho = 2$, $\gamma = 1$, and $\epsilon = \delta/2$ in Definition 3.1, where δ is as in (1.9). By Definition 1.11(ii) and Remark 1.12(iii), and Lemma 2.2 and Remark 3.3, we see that, to show Theorem 1.16, it suffices to prove that the multilinear commutator $T_{\vec{b}}$ map a $(\vec{b}, m, q, m+2, 2\widetilde{\alpha})_\lambda$-atomic block h into a $(q, 1, \frac{\delta}{2}, 2)_\lambda$-molecular block with $|T_{\vec{b}}h|_{\widetilde{H}^{1,q,1,\frac{\delta}{2}}_{\mathrm{mb},2}(\mu)} \le C\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}|h|_{\widetilde{H}^{1,q,m+2}_{\vec{b},m,2\widetilde{\alpha}}(\mu)}$, where C is a positive constant independent of h.

Indeed, let h be a $(\vec{b}, m, q, m+2, 2\widetilde{\alpha})_\lambda$-atomic block. Then $h := \sum_{j=1}^{2} \lambda_j a_j$, where, for any $j \in \{1,2\}$, $\mathrm{supp}(a_j) \subset B_j \subset B$ for some balls B_j and B as in Definition 1.10. Let $B_0 := 4\alpha B$, where $1 < \alpha < \widetilde{\alpha}$. Write

$$T_{\vec{b}}h = (T_{\vec{b}}h)\mathcal{X}_{B_0} + \sum_{k=1}^{\infty}(T_{\vec{b}}h)\mathcal{X}_{2^kB_0\setminus 2^{k-1}B_0} =: \mathrm{A}_1 + \mathrm{A}_2.$$

We first deal with the term A_1. Since $B_j \subset B$, we have $\alpha B_j \subset 4\alpha B = B_0$. Let $N_j := N^{(\alpha)}_{\alpha B_j, \frac{B_0}{2}}$. Obviously, $N_j \ge 0$. Without loss of generality, we may assume that $N_j \ge 3$. For the case of $N_j \in [0,3)$, we easily observe that $\alpha B_j \subset B_0 \subset 3\alpha^3 B_j$, which can be reduced to the case $N_j \ge 3$. Notice that $\alpha^{N_j-1}B_j \subset B_0$. We further decompose

$$\begin{aligned}\mathrm{A}_1 &= \sum_{j=1}^{2}\lambda_j(T_{\vec{b}}a_j)\mathcal{X}_{\alpha B_j} + \sum_{j=1}^{2}\sum_{i=1}^{N_j-2}\lambda_j(T_{\vec{b}}a_j)\mathcal{X}_{\alpha^{i+1}B_j\setminus\alpha^i B_j} + \sum_{j=1}^{2}\lambda_j(T_{\vec{b}}a_j)\mathcal{X}_{B_0\setminus\alpha^{N_j-1}B_j}\\ &=: \mathrm{A}_{1,1}+\mathrm{A}_{1,2}+\mathrm{A}_{1,3}.\end{aligned}$$

For $\mathrm{A}_{1,1}$, by Lemma 2.7, Definition 1.10, Lemmas 2.1 and 2.2, and the fact that $\widetilde{K}^{(2)}_{\alpha B_j,4B_0}\geq 1$, for any $j\in\{1,2\}$, we have

$$\begin{aligned}\left\|(T_{\vec{b}}a_j)\mathcal{X}_{\alpha B_j}\right\|_{L^q(\mu)} &\leq \|T_{\vec{b}}a_j\|_{L^q(\mu)} \lesssim \|a_j\|_{L^q(\mu)}\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\\ &\lesssim \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\big[\mu(2\widetilde{\alpha}B_j)\big]^{1/q-1}[\widetilde{K}^{(2\widetilde{\alpha})}_{B_j,B}]^{-(m+2)}\\ &\leq c_1\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\big[\mu(2\widetilde{\alpha}B_j)\big]^{1/q-1}[\widetilde{K}^{(2)}_{\widetilde{\alpha}B_j,4B_0}]^{-1},\end{aligned}$$

where c_1 is a positive constant independent of a_j and j. Let $\tau_{j,1}:=c_1\lambda_j\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}$, and let $n_{j,1}:=\tau_{j,1}^{-1}\lambda_j(T_{\vec{b}}a_j)\mathcal{X}_{\alpha B_j}$. Then $\mathrm{A}_{1,1}=\sum_{j=1}^{2}\tau_{j,1}n_{j,1}$, $\mathrm{supp}(n_{j,1})\subset\widetilde{\alpha}B_j\subset B_0$, and $\|n_{j,1}\|_{L^q(\mu)}\leq[\mu(2(\widetilde{\alpha}B_j))]^{1/q-1}[\widetilde{K}^{(2)}_{\widetilde{\alpha}B_j,4B_0}]^{-1}$.

To estimate $\mathrm{A}_{1,3}$, since $\alpha^{N_j-1}B_j\subset B_0\subset 3\alpha^{N_j+1}B_j$, it is easy to see that $r_{B_0}\sim r_{\alpha^{N_j-1}B_j}$. For any $j\in\{1,2\}$, let x_j and r_j be the center and the radius of B_j, respectively. From (1.8), (2.3), the Hölder's inequality, Remark 1.2(iv), Lemmas 2.5 and 2.6 with $\rho=2$, Definition 1.10, the fact that $\widetilde{K}^{(2)}_{B_j,B_0}\geq 1$, and Lemma 2.2, we deduce that

$$\begin{aligned}&\left\|(T_{\vec{b}}a_j)\mathcal{X}_{B_0\setminus\alpha^{N_j-1}B_j}\right\|_{L^q(\mu)}\\ &\quad\leq\left\{\int_{B_0\setminus\alpha^{N_j-1}B_j}\Big[\int_{B_j}\prod_{i=1}^{m}|b_i(x)-b_i(y)|\frac{|a_j(y)|}{\lambda(x,d(x,y))}\,d\mu(y)\Big]^q\,d\mu(x)\right\}^{1/q}\\ &\quad\lesssim\frac{1}{\lambda(x_j,\alpha^{N_j-1}r_j)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\int_{B_j}\big|\big[m_{\widetilde{B_j}}(b)-b(y)\big]_{\sigma'}\big||a_j(y)|\,d\mu(y)\\ &\qquad\times\left\{\int_{B_0\setminus\alpha^{N_j-1}B_j}\big|\big[b(x)-m_{\widetilde{B_j}}(b)\big]_{\sigma}\big|^q\,d\mu(x)\right\}^{1/q}\\ &\quad\lesssim\frac{\|a_j\|_{L^q(\mu)}}{\lambda(x_j,\alpha^{N_j-1}r_j)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\left\{\int_{B_j}\big|\big[m_{\widetilde{B_j}}(b)-b(y)\big]_{\sigma'}\big|^{q'}\,d\mu(y)\right\}^{1/q'}\\ &\qquad\times\sum_{l=0}^{i}\sum_{\eta(\sigma)\in C_l^i}\left\{\int_{B_0}\big|\big[b(x)-m_{\widetilde{B_0}}(b)\big]_{\eta(\sigma)}\big[m_{\widetilde{B_0}}(b)-m_{\widetilde{B_j}}(b)\big]_{\eta'(\sigma)}\big|^q\,d\mu(x)\right\}^{1/q}\\ &\quad\lesssim\frac{\|a_j\|_{L^q(\mu)}}{\lambda(x_j,\alpha^{N_j-1}r_j)}\sum_{i=0}^{m}\sum_{\sigma\in C_i^m}\big[\mu(2B_j)\big]^{1/q'}\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\big[\mu(2B_0)\big]^{1/q}[\widetilde{K}^{(2)}_{B_j,B_0}]^i\\ &\quad\lesssim\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\frac{[\mu(2B_j)]^{1/q'}}{[\mu(2\widetilde{\alpha}B_j)]^{1/q'}}\frac{[\mu(4B_0)]^{1/q}}{\lambda(x_j,\alpha^{N_j-1}r_j)}[\widetilde{K}^{(2\widetilde{\alpha})}_{B_j,B}]^{-(m+2)}[\widetilde{K}^{(2)}_{B_j,B_0}]^m\\ &\quad\leq c_3\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\big[\mu(4B_0)\big]^{1/q-1}[\widetilde{K}^{(2)}_{2B_0,4B_0}]^{-1},\end{aligned}$$

where c_3 is a positive constant independent of a_j and j. Let $\tau_{j,3} := c_3\lambda_j\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}$, and let $n_{j,3} := \tau_{j,3}^{-1}\lambda_j(T_{\vec{b}}a_j)\mathcal{X}_{B_0\setminus\alpha^{N_j-1}B_j}$. Then $\mathrm{A}_{1,3} = \sum_{j=1}^{2}\tau_{1,3}n_{1,3}$, $\mathrm{supp}(n_{j,3}) \subset 2B_0$, and $\|n_{j,3}\|_{L^q(\mu)} \le [\mu(2(2B_0))]^{1/q-1}[\widetilde{K}^{(2)}_{2B_0,4B_0}]^{-1}$.

Now we turn to estimate $\mathrm{A}_{1,2}$. From (1.8), (2.3), the Hölder's inequality, Remark 1.2(iv), Lemmas 2.5, 2.6, 2.1, and 2.2, and Definition 1.10, it follows that, for any $i \in \{1,\dots,N_j-2\}$,

$$\begin{aligned}
&\big\|(T_{\vec{b}}a_j)\mathcal{X}_{\alpha^{i+1}B_j\setminus\alpha^iB_j}\big\|_{L^q(\mu)}\\
&\quad\le \Big\{\int_{\alpha^{i+1}B_j\setminus\alpha^iB_j}\Big[\int_{B_j}\prod_{s=1}^{m}|b_s(x)-b_s(y)|\frac{|a_j(y)|}{\lambda(x,d(x,y))}\,d\mu(y)\Big]^q\,d\mu(x)\Big\}^{1/q}\\
&\quad\lesssim \frac{1}{\lambda(x_j,\alpha^ir_j)}\sum_{s=0}^{m}\sum_{\sigma\in C_s^m}\int_{B_j}\big|\big[m_{\widetilde{B_j}}(b)-b(y)\big]_{\sigma'}\big||a_j(y)|\,d\mu(y)\\
&\qquad\times\Big\{\int_{\alpha^{i+1}B_j\setminus\alpha^iB_j}\big|\big[b(x)-m_{\widetilde{B_j}}(b)\big]_{\sigma}\big|^q\,d\mu(x)\Big\}^{1/q}\\
&\quad\lesssim \frac{\|a_j\|_{L^q(\mu)}}{\lambda(x_j,\alpha^ir_j)}\sum_{s=0}^{m}\sum_{\sigma\in C_s^m}\Big\{\int_{B_j}\big|\big[m_{\widetilde{B_j}}(b)-b(y)\big]_{\sigma'}\big|^{q'}\,d\mu(y)\Big\}^{1/q'}\\
&\qquad\times\sum_{l=0}^{s}\sum_{\eta(\sigma)\in C_l^s}\Big\{\int_{\alpha^{i+1}B_j}\big|\big[b(x)-m_{\widetilde{\alpha^{i+1}B_j}}(b)\big]_{\eta(\sigma)}\\
&\qquad\times\big[m_{\widetilde{\alpha^{i+1}B_j}}(b)-m_{\widetilde{B_j}}(b)\big]_{\eta'(\sigma)}\big|^q\,d\mu(x)\Big\}^{1/q}\\
&\quad\lesssim \frac{\|a_j\|_{L^q(\mu)}}{\lambda(x_j,\alpha^ir_j)}\sum_{s=0}^{m}\sum_{\sigma\in C_s^m}\big[\mu(2B_j)\big]^{1/q'}\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\big[\mu(2\alpha^{i+1}B_j)\big]^{1/q}[\widetilde{K}^{(2)}_{B_j,\alpha^{i+1}B_j}]^s\\
&\quad\lesssim \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\frac{\|a_j\|_{L^q(\mu)}}{\lambda(x_j,\alpha^ir_j)}\big[\mu(2B_j)\big]^{1/q'}\big[\mu(2\alpha^{i+2}B_j)\big]^{1/q}[\widetilde{K}^{(2)}_{B_j,B}]^m\\
&\quad\le c_2\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\frac{\mu(2\alpha^{i+2}B_j)}{\lambda(x_j,\alpha^ir_j)}[\widetilde{K}^{(2)}_{B_j,B_0}]^{-1}\big[\mu(2\alpha^{i+2}B_j)\big]^{(1/q)-1}[\widetilde{K}^{(2)}_{\alpha^{i+2}B_j,4B_0}]^{-1},
\end{aligned}$$

where c_2 is a positive constant independent of a_j, j, and i. Let

$$\tau_{j,2}^{(i)} := c_2\lambda_j\|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\frac{\mu(2\alpha^{i+2}B_j)}{\lambda(x_j,\alpha^ir_j)}[\widetilde{K}^{(2)}_{B_j,B_0}]^{-1}$$

and

$$n_{j,2}^{(i)} := (\tau_{j,2}^{(i)})^{-1}\lambda_j(T_{\vec{b}}a_j)\mathcal{X}_{\alpha^{i+1}B_j\setminus\alpha^iB_j}.$$

Then $\mathrm{A}_{1,2} = \sum_{j=1}^{2}\sum_{i=1}^{N_j-2}\tau_{j,2}^{(i)}n_{j,2}^{(i)}$, $\mathrm{supp}(n_{j,2}^{(i)}) \subset \alpha^{i+2}B_j \subset 4B_0$, and

$$\|n_{j,2}^{(i)}\|_{L^q(\mu)} \le \big[\mu\big(2(\alpha^{i+2}B_j)\big)\big]^{1/q-1}[\widetilde{K}^{(2)}_{\alpha^{i+2}B_j,4B_0}]^{-1}.$$

Finally, we deal with the term A_2. For any $k \in \mathbb{N}$, by the geometrically doubling condition, there exists a ball covering $\{B_{k,j}\}_{j=1}^{M_0}$ with uniform radius $2^{k-3}r_{B_0}$ of

$\widetilde{U}_k(B_0) := 2^k B_0 \setminus 2^{k-1}B_0$ such that the cardinality $M_0 \le N_0 8^n$. Without loss of generality, we may assume that the centers of the balls in the covering belong to $\widetilde{U}_k(B_0)$.

Let $C_{k,1} := B_{k,1}$, let $C_{k,l} := B_{k,l} \setminus \bigcup_{m=1}^{l-1} B_{k,m}$, let $l \in \{2, 3, \dots, M_0\}$, and let $D_{k,l} := C_{k,l} \cap \widetilde{U}_k(B_0)$ for all $l \in \{1, 2, \dots, M_0\}$. Then we know that $\{D_{k,l}\}_{l=1}^{M_0}$ is pairwise disjoint, $\widetilde{U}_k(B_0) = \bigcup_{l=1}^{M_0} D_{k,l}$, and, for any $l \in \{1, 2, \dots, M_0\}$,

$$D_{k,l} \subset 2B_{k,l} \subset U_k(B_0) := 2^{k+2}B_0 \setminus 2^{k-2}B_0.$$

Write

$$\mathrm{A}_2 = \sum_{k=1}^{\infty}(T_{\vec{b}}h)\sum_{l=1}^{M_0} D_{k,l} = \sum_{k=1}^{\infty}\sum_{l=1}^{M_0}(T_{\vec{b}}h)\mathcal{X}_{D_{k,l}}.$$

Definition 1.10, together with (1.9), the Hölder's inequality, (2.3), Lemmas 2.5, 2.6, 2.1, and 2.2, and the fact that $\widetilde{K}^{(2)}_{B_j,B} > 1$, implies that, for any $k \in \mathbb{N}$ and $l \in \{1, \dots, M_0\}$,

$$\begin{aligned}
&\left\|(T_{\vec{b}}h)\mathcal{X}_{D_{k,l}}\right\|_{L^q(\mu)}\\
&\quad\le \left\{\int_{D_{k,l}}\left[\int_B \prod_{i=1}^m |b_i(x)-b_i(y)||h(y)||K(x,y)-K(x,c_B)|\,d\mu(y)\right]^q d\mu(x)\right\}^{1/q}\\
&\quad\lesssim \left\{\int_{D_{k,l}}\left[\int_B \prod_{i=1}^m |b_i(x)-b_i(y)||h(y)|\frac{[d(y,c_B)]^\delta[d(x,c_B)]^{-\delta}}{\lambda(c_B,d(x,c_B))}\,d\mu(y)\right]^q d\mu(x)\right\}^{1/q}\\
&\quad\lesssim \frac{(r_B)^\delta(2^{k-1}r_{B_0})^{-\delta}}{\lambda(c_B,2^{k-1}r_{B_0})}\left\{\int_{D_{k,l}}\left[\int_B \prod_{i=1}^m |b_i(x)-b_i(y)||h(y)|\,d\mu(y)\right]^q d\mu(x)\right\}^{1/q}\\
&\quad\lesssim \sum_{j=1}^2 |\lambda_j|2^{-k\delta}\frac{\|a_j\|_{L^q(\mu)}}{\lambda(c_B,2^{k-1}r_{B_0})}\sum_{i=0}^m\sum_{\sigma\in C_i^m}\left\{\int_{B_j}\left|\left[b(y)-m_{\widetilde{B_j}}(b)\right]_{\sigma'}\right|^{q'}d\mu(y)\right\}^{1/q'}\\
&\qquad\times\left\{\int_{D_{k,l}}\left|\left[b(x)-m_{\widetilde{B_j}}(b)\right]_\sigma\right|^q d\mu(x)\right\}^{1/q}\\
&\quad\lesssim \sum_{j=1}^2 |\lambda_j|2^{-k\delta}\frac{\|a_j\|_{L^q(\mu)}}{\lambda(c_B,2^{k-1}r_{B_0})}\sum_{i=0}^m\sum_{\sigma\in C_i^m}\left[\mu(2B_j)\right]^{1/q'}\|\vec{b}_{\sigma'}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\\
&\qquad\times\sum_{\zeta=0}^i\sum_{\eta(\sigma)\in C_\zeta^i}\left\{\left\{\int_{2B_{k,l}}\left|\left[b(x)-m_{\widetilde{2B_{k,l}}}(b)\right]_{\eta'(\sigma)}\right|^q d\mu(x)\right\}^{1/q}\right.\\
&\qquad\left.\times\left|\left[m_{\widetilde{2B_{k,l}}}(b)-m_{\widetilde{B_j}}(b)\right]_{\eta(\sigma)}\right|\right\}\\
&\quad\lesssim \sum_{j=1}^2 |\lambda_j|2^{-k\delta}\frac{\|a_j\|_{L^q(\mu)}}{\lambda(c_B,2^{k-1}r_{B_0})}\sum_{i=0}^m\sum_{\sigma\in C_i^m}\left[\mu(2B_j)\right]^{1/q'}\|\vec{b}_{\sigma'}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\\
&\qquad\times\left\{\sum_{\zeta=0}^i\sum_{\eta(\sigma)\in C_\zeta^i}\left[\mu(4B_{k,l})\right]^{1/q}\|\vec{b}_{\eta'(\sigma)}\|_{\widetilde{\mathrm{RBMO}}(\mu)}\right.
\end{aligned}$$

$$\times \Big\{ \sum_{\xi=0}^{\zeta} \sum_{\theta(\eta)\in C_\xi^\zeta} \big| \big[m_{\widetilde{2B_{k,l}}}(b) - m_{\widetilde{2^{k+2}B_0}}(b) \big]_{\theta'(\eta)} \big[m_{\widetilde{B_j}}(b) - m_{\widetilde{2^{k+2}B_0}}(b) \big]_{\theta(\eta)} \big| \Big\} \Big\}$$

$$\lesssim \sum_{j=1}^{2} |\lambda_j| 2^{-k\delta} \frac{\|a_j\|_{L^q(\mu)}}{\lambda(c_B, 2^{k-1}r_{B_0})} \sum_{i=0}^{m} \sum_{\sigma\in C_i^m} \big[\mu(2B_j)\big]^{1/q'} \|\vec{b}_{\sigma'}\|_{\widetilde{\mathrm{RBMO}}(\mu)}$$

$$\times \big[\mu(4B_{k,l})\big]^{1/q} \|\vec{b}_\sigma\|_{\widetilde{\mathrm{RBMO}}(\mu)} [\widetilde{K}^{(2)}_{B_j,2^{k+2}B_0}]^i$$

$$\lesssim \sum_{j=1}^{2} |\lambda_j| 2^{-k\delta} \frac{\|a_j\|_{L^q(\mu)}}{\lambda(c_B, 2^{k-1}r_{B_0})} \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)} \big[\mu(2B_j)\big]^{1/q'} \big[\mu(4B_{k,l})\big]^{1/q} [\widetilde{K}^{(2)}_{B_j,2^{k+2}B_0}]^m$$

$$\le c_4 \sum_{j=1}^{2} |\lambda_j| 2^{-k\delta/2} k^m \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)} 2^{-k\delta/2} \big[\mu(4B_{k,l})\big]^{1/q-1} [\widetilde{K}^{(2)}_{2B_{k,l},2^{k+2}B_0}]^{-1},$$

where c_4 is a positive constant independent of h and k. Let

$$\lambda_{k,l} := c_4 2^{-k\delta/2} k^m \sum_{j=1}^{2} |\lambda_j| \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}$$

and $m_{k,l} := \lambda_{k,l}^{-1}(T_{\vec{b}}h)\mathcal{X}_{D_{k,l}}$. Then $\mathrm{A}_2 = \sum_{k=1}^{\infty}\sum_{l=1}^{M_0} \lambda_{k,l} m_{k,l}$, $\operatorname{supp}(m_{k,l}) \subset 2B_{k,l} \subset U_k(B_0)$, and $\|m_{k,l}\|_{L^q(\mu)} \le 2^{-k\delta/2}[\mu(2(2B_{k,l}))]^{1/q-1}[\widetilde{K}^{(2)}_{2B_{k,l},2^{k+2}B_0}]^{-1}$.

Combining the estimates of A_1 and A_2, we see that $T_{\vec{b}}h$ is a $(q,1,\delta/2,2)_\lambda$-molecular block, which, together with Definition 1.6 and Lemmas 2.1 and 2.2, implies that

$$|T_{\vec{b}}h|_{\widetilde{H}^{1,q,1,\frac{\delta}{2}}_{\mathrm{mb},2}(\mu)} = \sum_{j=1}^{2} |\tau_{j,1}| + \sum_{j=1}^{2} \sum_{i=1}^{N_j-1} |\tau_{j,2}^{(i)}| + \sum_{j=1}^{2} |\tau_{j,3}| + \sum_{k=1}^{\infty} \sum_{l=1}^{M_0} |\lambda_{k,l}|$$

$$\lesssim \sum_{j=1}^{2} |\lambda_j| \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)} + \sum_{j=1}^{2} \sum_{i=1}^{N_j-1} |\lambda_j| \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}$$

$$\times \frac{\mu(2\alpha^{i+2}B_j)}{\lambda(x_i, \alpha^{i+2}r_j)} [\widetilde{K}^{(\alpha)}_{B_j,B}]^{-1} + \sum_{k=1}^{\infty} \sum_{l=1}^{M_0} 2^{-k\delta/2} k^m \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)} \sum_{j=1}^{2} |\lambda_j|$$

$$\lesssim \sum_{j=1}^{2} |\lambda_j| \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)} + M_0 \sum_{k=1}^{\infty} 2^{-k\delta/2} k^m \sum_{j=1}^{2} |\lambda_j| \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)}$$

$$\lesssim \sum_{j=1}^{2} |\lambda_j| \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)} \lesssim \|\vec{b}\|_{\widetilde{\mathrm{RBMO}}(\mu)} |h|_{\widetilde{H}^{1,q,m+2}_{\vec{b},m,2\widetilde{\alpha}}}.$$

This finishes the proof of Theorem 1.16. □

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References

1. T. A. Bui and X. T. Duong, *Hardy spaces, regularized BMO spaces and the boundedness of Calderón-Zygmund operators on non-homogeneous spaces*, J. Geom. Anal. **23** (2013), no. 2, 895–932. Zbl 1267.42013. MR3023861. DOI 10.1007/s12220-011-9268-y. 479, 481
2. R. R. Coifman, R. Rochberg, and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. (2) **103** (1976), no. 3, 611–635. Zbl 0326.32011. MR0412721. DOI 10.2307/1970954. 477
3. R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Math. **242**, Springer, Berlin, 1971. Zbl 0224.43006. MR0499948. 478, 480
4. R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645. Zbl 0358.30023. MR0447954. DOI 10.1090/S0002-9904-1977-14325-5. 478
5. X. Fu, H. Lin, Da. Yang, and Do. Yang, *Hardy spaces H^p over non-homogeneous metric measure spaces and their applications*, Sci. China Math. **58** (2015), no. 2, 309–388. Zbl 1311.42054. MR3301064. DOI 10.1007/s11425-014-4956-2. 481, 485
6. X. Fu, Da. Yang, and Do. Yang, *The molecular characterization of the Hardy space H^1 on non-homogeneous metric measure spaces and its application*, J. Math. Anal. Appl. **410** (2014), no. 2, 1028–1042. Zbl 1312.42027. MR3111887. DOI 10.1016/j.jmaa.2013.09.021. 479, 481, 482, 483, 484, 487, 489, 490
7. X. Fu, D. Yang, and W. Yuan, *Boundedness of multilinear commutators of Calderón-Zygmund operators on Orlicz spaces over non-homogeneous spaces*, Taiwanese J. Math. **16** (2012), no. 6, 2203–2238. Zbl 1275.47079. MR3001844. 479
8. J. Heinonen, *Lectures on Analysis on Metric Spaces*, Universitext, Springer, New York, 2001. Zbl 0985.46008. MR1800917. DOI 10.1007/978-1-4613-0131-8. 480, 481
9. T. Hytönen, *A framework for non-homogeneous analysis on metric spaces, and the RBMO space of Tolsa*, Publ. Mat. **54** (2010), no. 2, 485–504. Zbl 1246.30087. MR2675934. DOI 10.5565/PUBLMAT_54210_10. 478, 479, 480, 482
10. T. Hytönen and H. Martikainen, *Non-homogeneous Tb theorem and random dyadic cubes on metric measure spaces*, J. Geom. Anal. **22** (2012), no. 4, 1071–1107. Zbl 1261.42017. MR2965363. DOI 10.1007/s12220-011-9230-z. 479, 484
11. T. Hytönen and H. Martikainen, *Non-homogeneous $T1$ theorem for bi-parameter singular integrals*, Adv. Math. **261** (2014), 220-273. Zbl 1301.42030. MR3213300. DOI 10.1016/j.aim.2014.02.011. 479
12. T. Hytönen, D. Yang, and D. Yang, *The Hardy space H^1 on non-homogeneous metric spaces*, Math. Proc. Cambridge Philos. Soc. **153** (2012), no. 1, 9–31. Zbl 1250.42076. MR2943664. DOI 10.1017/S0305004111000776. 479, 481
13. H. Lin, S. Wu, and D. Yang, *Boundedness of certain commutators over non-homogeneous metric measure spaces*, Anal. Math. Phys. (2016). DOI 10.1007/s13324-016-0136-6. 479, 480, 485
14. H. Lin and D. Yang, *An interpolation theorem for sublinear operators on non-homogeneous metric measure spaces*, Banach J. Math. Anal. **6** (2012), 168–179. Zbl 1252.42025. MR2945995. 479
15. H. Lin and D. Yang, *Equivalent boundedness of Marcinkiewicz integrals on non-homogeneous metric measure spaces*, Sci. China Math. **57** (2014), no. 1, 123–144. Zbl 1304.42038. MR3146521. DOI 10.1007/s11425-013-4754-2. 479

16. Y. Meng and D. Yang, *Multilinear commutators of Calderón-Zygmund operators on Hardy-type spaces with non-doubling measures*, J. Math. Anal. Appl. **317** (2006), no. 1, 228–244. Zbl 1092.42007. MR2205323. DOI 10.1016/j.jmaa.2005.12.003. 478, 479, 484, 487
17. F. Nazarov, S. Treil, and A. Volberg, *Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators on nonhomogeneous spaces*, Internat. Math. Res. Notices **1998**, no. 9, 463–487. Zbl 0918.42009. MR1626935. 478
18. F. Nazarov, S. Treil, and A. Volberg, *The Tb-theorem on non-homogeneous spaces*, Acta Math. **190** (2003), no. 2, 151–239. Zbl 1065.42014. MR1998349. DOI 10.1007/BF02392690. 478
19. C. Pérez, *Endpoint estimates for commutators of singular integral operators*, J. Funct. Anal. **128** (1995), no. 1, 163–185. Zbl 0831.42010. MR1317714. DOI 10.1006/jfan.1995.1027. 477
20. C. Pérez and R. Trujillo-González, *Sharp weighted estimates for multilinear commutators*, London Math. Soc. **65** (2002), no. 3, 672–692. Zbl 1012.42008. MR1895740. DOI 10.1112/S0024610702003174. 484
21. Y. Sawano and H. Tanaka, *Morrey spaces for non-doubling measures*, Acta Math. Sin. (Engl. Ser.) **21** (2005), no. 6, 1535–1544. Zbl 1129.42403. MR2190025. DOI 10.1007/s10114-005-0660-z. 478
22. L. Shu, M. Wang, and M. Qu, *Commutators of Hardy type operators on Herz spaces with variable exponents*, Acta Math. Sin. (Chin. Ser.) **58** (2015), no. 1, 29–40. Zbl 1340.42039. MR3410572. 477
23. C. Tan and J. Li, *Littlewood-Paley theory on metric measure spaces with non doubling measures and its applications*, Sci. China Math. **58** (2015), no. 5, 983–1004. Zbl 1320.42012. MR3336357. DOI 10.1007/s11425-014-4950-8. 479
24. X. Tolsa, *BMO, H^1, and Calderón-Zygmund operators for non doubling measures*, Math. Ann. **319** (2001), no. 1, 89–149. Zbl 0974.42014. MR1812821. DOI 10.1007/s002080000144. 478, 481
25. X. Tolsa, *Littlewood-Paley theory and the $T(1)$ theorem with non-doubling measures*, Adv. Math. **164** (2001), no. 1, 57–116. Zbl 1015.42010. MR1870513. DOI 10.1006/aima.2001.2011. 478, 481
26. X. Tolsa, *Painlevé's problem and the semiadditivity of analytic capacity*, Acta Math. **190** (2003), no. 1, 105–149. Zbl 1060.30031. MR1982794. DOI 10.1007/BF02393237. 478
27. X. Tolsa, *The space H^1 for nondoubling measures in terms of a grand maximal operator*, Trans. Amer. Math. Soc. **355** (2003), no. 1, 315–348. Zbl 1021.42010. MR1928090. DOI 10.1090/S0002-9947-02-03131-8. 478
28. X. Tolsa, *Analytic Capacity, the Cauchy Transform, and Non-homogeneous Calderón-Zygmund Theory*, Progr. Math. **307**, Birkhäuser, Cham, 2014. Zbl 1290.42002. MR3154530. 478
29. A. Volberg and B. D. Wick, *Bergman-type singular operators and the characterization of Carleson measures for Besov-Sobolev spaces on the complex ball*, Amer. J. Math. **134** (2012), no. 4, 949–992. Zbl 1252.42020. MR2956255. DOI 10.1353/ajm.2012.0028. 484
30. D. Yang, D. Yang, and G. Hu, *The Hardy Space H^1 with Non-doubling Measures and Their Applications*, Lecture Notes in Math. **2084**, Springer, Cham, 2013. Zbl 1316.42002. MR3157341. 479

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