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DISJOINT HYPERCYCLIC WEIGHTED TRANSLATIONS ON GROUPS

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Abstract. Let $1 \leq p < \infty$, and let G be a locally compact group. We characterize disjoint hypercyclic weighted translation operators on the Lebesgue space $L^p(G)$ in terms of the weight, the Haar measure, and the group element. Disjoint supercyclic, disjoint mixing, and dual disjoint hypercyclic weighted translation operators are also characterized.

1. Introduction

Recently, we characterized chaotic, hypercyclic, and supercyclic weighted translation operators on the Lebesgue space of a locally compact group in [6], [7], [8], and [9], extending some works on hypercyclicity for weighted shifts on $\ell^p(\mathbb{Z})$ in [10], [12], [13], and [16]. In 2007, Bernal-Gonález, Bès, and Peris introduced the study of disjoint (or diagonal) hypercyclicity in [1] and [5], respectively. Since then, disjoint hypercyclicity was investigated intensively, and the results on disjoint hypercyclicity have been richly rewarding (see [2], [3], [4], [14], [17], and [18] for recent works on this subject). Among many important results, the characterization for weighted shifts on $\ell^p(\mathbb{Z})$ to be disjoint hypercyclic and disjoint supercyclic were given in [5] and [14], respectively. Inspired by [5] and [14], we continue our research on weighted translation operators on locally compact groups in the present article, while giving sufficient and necessary conditions for such operators to be disjoint hypercyclic and disjoint supercyclic.

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First, we recall that an operator T on a Banach space X is *supercyclic* if there exists an element $x \in X$ such that the orbit

$$\{\alpha T^n x : \alpha \in \mathbb{C}, n \geq 0\} = \bigcup_{n \geq 0} \mathbb{C} T^n x$$

is dense in X, where T^n denotes the nth iterate of T. Without the help of a complex multiple, T is called *hypercyclic* if $\overline{\{T^n x : n \geq 0\}} = X$. It is known that the notion of hypercyclicity in linear dynamics is related to some notions in topological dynamics. Indeed, hypercyclicity is equivalent to topological transitivity. An operator T is *topologically transitive* if, given two nonempty open subsets $U, V \subset X$, there is some $n \in \mathbb{N}$ such that $T^n(U) \cap V \neq \emptyset$. If $T^n(U) \cap V \neq \emptyset$ from some n onward, then T is called *topologically mixing*.

One can also compare the notions above to the setting of a sequence of operators. We recall that a sequence of operators (T_n) is called *topologically transitive* if, for any nonempty open sets U and V, we have $T_n(U) \cap V \neq \emptyset$ for some $n \in \mathbb{N}$. Similarly, if (T_n) satisfies the stronger condition that $T_n(U) \cap V \neq \emptyset$ for some n onward, then (T_n) is said to be *topologically mixing*. Also, a sequence of operators (T_n) on a Banach space X is *hypercyclic* if there exists a vector $x \in X$ such that the orbit $\{T_n x : n \geq 0\}$ is dense in X. If (T_n) has a dense set of hypercyclic vectors, then (T_n) is called *densely hypercyclic*. It is well known in [11] that, for a sequence of operators, dense hypercyclicity and topological transitivity are equivalent.

In [1, Definition 2.1] and [5, Definition 1.1], Bernal-Gonález, Bès, and Peris independently studied the new notion, namely *disjoint hypercyclicity*, as in the following.

Definition 1.1. Given $N \geq 2$, the operators $T_1, T_2, \ldots, T_N$ acting on a Banach space X are *disjoint hypercyclic*, or *diagonally hypercyclic* (in short, *d-hypercyclic*), if there is some vector $(x, x, \ldots, x)$ in the diagonal of $X^N = X \times X \times \cdots \times X$ such that

$$\big\{(x, x, \ldots, x), (T_1 x, T_2 x, \ldots, T_N x), (T_1^2 x, T_2^2 x, \ldots, T_N^2 x), \ldots\big\}$$

is dense in X^N, where $x \in X$ is a d-hypercyclic vector associated to the operators $T_1, T_2, \ldots, T_N$.

We can extend the definition naturally to notions of d-supercyclicity and d-chaoticity. For topological dynamics, several new notions were given accordingly in [5], as follows.

Definition 1.2. Given $N \geq 2$, the operators $T_1, T_2, \ldots, T_N$ on a Banach space X are *d-topologically transitive* if, given nonempty open sets $U, V_1, \ldots, V_N \subset X$, there is some $n \in \mathbb{N}$ such that

$$\emptyset \neq U \cap T_1^{-n}(V_1) \cap T_2^{-n}(V_2) \cap \cdots \cap T_N^{-n}(V_N).$$

If the above condition is satisfied from some n onward, then $T_1, T_2, \ldots, T_N$ are called *d-mixing*.

We note that in [5, Proposition 2.3], the operators $T_1, T_2, \dots, T_N$ are d-topologically transitive if and only if $T_1, T_2, \dots, T_N$ have a dense set of d-hypercyclic vectors. Likewise, the notions and results noted above can be extended to $N \geq 2$ sequences of operators $(T_{1,n})_{n=1}^{\infty}, (T_{2,n})_{n=1}^{\infty}, \dots, (T_{N,n})_{n=1}^{\infty}$. Indeed, d-topological transitivity and dense d-hypercyclicity for $(T_{1,n})_{n=1}^{\infty}, (T_{2,n})_{n=1}^{\infty}, \dots,$ $(T_{N,n})_{n=1}^{\infty}$ are also equivalent in [5, Proposition 2.3]. We will make use of these equivalences to obtain our main result in Sections 2 and 3. We conclude this section with a remark from [5, p. 299], where the author pointed out that T and cT cannot be d-hypercyclic where c is a scalar. Indeed, assume that T and cT have a d-hypercyclic vector $x \in X$. Then there exist sequences (n_k) and (m_k) such that $(T^{n_k}x, (cT)^{n_k}x) \to (x, 0)$ and $(T^{m_k}x, (cT)^{m_k}x) \to (0, x)$ as $k \to \infty$. But the first limit forces $|c| < 1$, and the second one implies that $|c| > 1$, which is a contradiction.

2. Disjoint hypercyclicity

In this section, we will characterize disjoint hypercyclic weighted translation operators on the Lebesgue space of a locally compact group and will give some examples of disjoint hypercyclicity on various groups.

In what follows, let G be a locally compact group with identity e and a right-invariant Haar measure λ. We denote by $L^p(G)$ $(1 \leq p < \infty)$ the complex Lebesgue space with respect to λ. A bounded function $w : G \to (0, \infty)$ is called a *weight* on G. Let $a \in G$, and let δ_a be the unit point mass at a. A *weighted translation* on G is a weighted convolution operator $T_{a,w} : L^p(G) \to L^p(G)$ defined by

$$T_{a,w}(f) = wT_a(f) \quad \big(f \in L^p(G)\big),$$

where w is a weight on G and $T_a(f) = f * \delta_a \in L^p(G)$ is the convolution

$$(f * \delta_a)(x) = \int_G f(xy^{-1})\, d\delta_a(y) = f(xa^{-1}) \quad (x \in G).$$

We also define a self-map $S_{a,w}$ on the subspace $L^p_c(G)$ of functions in $L^p(G)$ with compact support by

$$S_{a,w}(h) = \frac{h}{w} * \delta_{a^{-1}} \quad \big(h \in L^p_c(G)\big)$$

so that

$$T_{a,w}S_{a,w}(h) = h \quad \big(h \in L^p_c(G)\big).$$

Since the weighted translation $T_{a,w}$ is generated by the group element a and the weight function w, we first observe that some elements $a \in G$ and weights w should be excluded. For example, if $\|w\|_\infty < 1$, then $\|T_{a,w}\| < 1$, and $T_{a,w}$ is never hypercyclic. Also, it is demonstrated in [9, Lemma 1.1] that $T_{a,w}$ is not hypercyclic if a is a torsion element of G. Based on this result, the following lemma reveals that weighted translation operators cannot be disjoint hypercyclic if they are generated by a torsion element.

An element a in a group G is called a *torsion element* if it is of finite order. In a locally compact group G, an element $a \in G$ is called *periodic* (see [9, p. 2840] if the closed subgroup $G(a)$ generated by a is compact. We call an element in G *aperiodic* if it is not periodic. For discrete groups, periodic and torsion elements are identical.

Lemma 2.1. *Let G be a locally compact group, and let $a \in G$ be a torsion element. Let $1 \le p < \infty$. Given some $N \ge 2$, let $T_l = T_{a,w_l}$ be a weighted translation on $L^p(G)$ generated by a and a positive weight w_l for $1 \le l \le N$. Then the operators $T_1, T_2, \ldots, T_N$ are not disjoint hypercyclic.*

In the following, we will focus on the aperiodic group element $a \in G$, using the condition of aperiodicity. For aperiodic elements, [9] shows that an element $a \in G$ is aperiodic if and only if, for any compact set $K \subset G$, there exists some $N \in \mathbb{N}$ such that $K \cap K a^{\pm n} = \emptyset$ for all $n > N$. We note that in many familiar nondiscrete groups, including the additive group $\mathbb{R}^d$, the Heisenberg group, and the affine group, all elements except the identity are aperiodic (see [9, Remark 2.2]). Now we are ready to state the main result.

Theorem 2.2. *Let G be a locally compact group, and let a be an aperiodic element in G. Let $1 \le p < \infty$. Given some $N \ge 2$, let $T_l = T_{a,w_l}$ be a weighted translation on $L^p(G)$ generated by a and a positive weight w_l for $1 \le l \le N$. For $1 \le r_1 < r_2 < \cdots < r_N$, the following conditions are equivalent:*

(i) *$T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ have a dense set of d-hypercyclic vectors.*

(ii) *For each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets (E_k) in K such that $\lambda(K) = \lim_{k\to\infty} \lambda(E_k)$ and both sequences*

$$\varphi_{l,n} := \prod_{j=1}^{n} w_l * \delta_{a^{-1}}^{j} \qquad \textit{and} \qquad \widetilde{\varphi}_{l,n} := \Big(\prod_{j=0}^{n-1} w_l * \delta_{a}^{j}\Big)^{-1}$$

satisfy (for $1 \le l \le N$)

$$\lim_{k\to\infty} \|\varphi_{l,r_l n_k}|_{E_k}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{l,r_l n_k}|_{E_k}\|_\infty = 0$$

and (for $1 \le s < l \le N$)

$$\lim_{k\to\infty}\Big\|\frac{\widetilde{\varphi}_{s,(r_l-r_s)n_k}\cdot\widetilde{\varphi}_{l,r_l n_k}}{\widetilde{\varphi}_{s,r_l n_k}}\Big|_{E_k}\Big\|_\infty = \lim_{k\to\infty}\Big\|\frac{\varphi_{l,(r_l-r_s)n_k}\cdot\widetilde{\varphi}_{s,r_s n_k}}{\widetilde{\varphi}_{l,r_s n_k}}\Big|_{E_k}\Big\|_\infty = 0$$

for some subsequence $(n_k) \subset \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii). Let $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ be disjoint hypercyclic. Let $K \subset G$ be a compact set with $\lambda(K) > 0$. By aperiodicity of a, there is some M such that $K \cap K a^{\pm n} = \emptyset$ for all $n > M$.

Let $\chi_K \in L^p(G)$ be the characteristic function of K. Let $\varepsilon \in (0,1)$, and choose $0 < \delta < \frac{\varepsilon}{1+\varepsilon}$. By density of the disjoint hypercyclic vectors, there exist a disjoint hypercyclic vector $f \in L^p(G)$ and some $m > M$ such that

$$\|f - \chi_K\|_p < \delta^2 \qquad \text{and} \qquad \|T_l^{r_l m} f - \chi_K\|_p < \delta^2$$

for $l = 1, 2, \ldots, N$. Let

$$A = \big\{x \in K : \big|f(x) - 1\big| \ge \delta\big\},$$

and let

$$B = \{x \in G \setminus K : |f(x)| \geq \delta\}.$$

Then we have

$$|f(x)| > 1 - \delta \quad (x \in K \setminus A)$$

and

$$|f(x)| < \delta \quad \text{for } x \in (G \setminus K) \setminus B.$$

Moreover, by the inequality

$$\begin{aligned}\delta^{2p} > \|f - \chi_K\|_p^p &= \int_G |f(x) - \chi_K(x)|^p \, d\lambda(x) \\ &\geq \int_A |f(x) - 1|^p \, d\lambda(x) \geq \delta^p \lambda(A),\end{aligned}$$

we have $\lambda(A) < \delta^p$. Similarly, one has $\lambda(B) < \delta^p$. On the other hand, we have

$$\begin{aligned}\delta^{2p} > \|T_l^{r_l m} f - \chi_K\|_p^p &= \int_G |T_l^{r_l m} f(x) - \chi_K(x)|^p \, d\lambda(x) \\ &= \int_G |\widetilde{\varphi}_{l,r_l m}(x)^{-1} f(xa^{-r_l m}) - \chi_K(x)|^p \, d\lambda(x) \\ &= \int_G |\varphi_{l,r_l m}(x) f(x) - \chi_K(xa^{-r_l m})|^p \, d\lambda(x)\end{aligned}$$

by the right invariance of the Haar measure. Hence let

$$C_{l,m} = \{x \in K : |\widetilde{\varphi}_{l,r_l m}(x)^{-1} f(xa^{-r_l m}) - 1| \geq \delta\}$$

and

$$D_{l.m} = \{x \in K : |\varphi_{l,r_l m}(x) f(x)| \geq \delta\}.$$

Then

$$\widetilde{\varphi}_{l,r_l m}(x)^{-1} |f(xa^{-r_l m})| > 1 - \delta \quad (x \in K \setminus C_{l,m})$$

and

$$\varphi_{l,r_l m}(x) |f(x)| < \delta \quad (x \in K \setminus D_{l,m}).$$

Again, applying the inequality

$$\begin{aligned}\delta^{2p} > \|T_l^{r_l m} f - \chi_K\|_p^p &= \int_G |T_l^{r_l m} f(x) - \chi_K(x)|^p \, d\lambda(x) \\ \geq \int_{C_{l,m}} &|w_l(x) w_l(xa^{-1}) \cdots w_l(xa^{-(r_l m - 1)}) f(xa^{-r_l m}) - 1|^p \, d\lambda(x) \geq \delta^p \lambda(C_{l,m}),\end{aligned}$$

we have $\lambda(C_{l,m}) < \delta^p$. Using a similar argument and the right invariance of the Haar measure, one can deduce $\lambda(D_{l,m}) < \delta^p$. Since $K \cap Ka^{\pm m} = \emptyset$, we have

$$\widetilde{\varphi}_{l,r_l m}(x) < \frac{|f(xa^{-r_l m})|}{1 - \delta} < \frac{\delta}{1 - \delta} < \varepsilon \quad \text{on } K \setminus (C_{l,m} \cup Ba^{r_l m})$$

and

$$\varphi_{l,r_l m}(x) < \frac{\delta}{|f(x)|} < \frac{\delta}{1 - \delta} < \varepsilon \quad \text{on } K \setminus (D_{l,m} \cup A).$$

Next, we will show the other two weight conditions for $1 \le s < l \le N$. First, by the definition of $C_{l,m}$, we have

$$\left|w_l(x)w_l(xa^{-1})\cdots w_l(xa^{-(r_l m-1)})f(xa^{-r_l m})-1\right| < \delta \quad \text{on } K \setminus C_{l,m}.$$

Now, let $F_{s,m} = \{x \in G \setminus K : |\widetilde{\varphi}_{s,r_s m}(x)^{-1} f(xa^{-r_s m})| \ge \delta\}$ for $1 \le s < l \le N$. Then we have

$$\left|w_s(x)w_s(xa^{-1})\cdots w_s(xa^{-(r_s m-1)})f(xa^{-r_s m})\right| < \delta \quad \text{on } Ka^{-(r_l - r_s)m} \setminus F_{s,m} \subset G \setminus K,$$

which implies that

$$\left|w_s(xa^{-(r_l-r_s)m})\cdots w_s(xa^{-(r_l m-1)})f(xa^{-r_l m})\right| < \delta \quad \text{on } K \setminus F_{s,m}a^{(r_l-r_s)m}.$$

Moreover, $\lambda(F_{s,m}) < \delta^p$, which follows from the fact that

$$\begin{aligned}
\delta^{2p} &> \|T_s^{r_s m} f - \chi_K\|_p^p \ge \int_{F_{s,m}} \left|T_s^{r_s m} f(x)\right|^p d\lambda(x) \\
&= \int_{F_{s,m}} \left|w_s(x)w_s(xa^{-1})\cdots w_s(xa^{-(r_s m-1)})f(xa^{-r_s m})\right|^p d\lambda(x) \ge \delta^p \lambda(F_{s,m}).
\end{aligned}$$

Therefore, for $1 \le s < l \le N$ and $x \in K \setminus (F_{s,m}a^{(r_l-r_s)m} \cup C_{l,m})$, we have

$$\begin{aligned}
\frac{w_s(xa^{-(r_l-r_s)m})\cdots w_s(xa^{-(r_l m-1)})}{w_l(x)w_l(xa^{-1})\cdots w_l(xa^{-(r_l m-1)})} &= \frac{w_s(xa^{-(r_l-r_s)m})\cdots w_s(xa^{-(r_l m-1)})|f(xa^{-r_l m})|}{w_l(x)w_l(xa^{-1})\cdots w_l(xa^{-(r_l m-1)})|f(xa^{-r_l m})|} \\
&< \frac{\delta}{1-\delta} < \varepsilon;
\end{aligned}$$

that is,

$$\frac{\widetilde{\varphi}_{s,(r_l-r_s)m}(x)\cdot\widetilde{\varphi}_{l,r_l m}(x)}{\widetilde{\varphi}_{s,r_l m}(x)} < \varepsilon \quad \text{on } K \setminus (F_{s,m}a^{(r_l-r_s)m} \cup C_{l,m}).$$

Again, we have

$$\left|w_l(x)w_l(xa^{-1})\cdots w_l(xa^{-(r_l m-1)})f(xa^{-r_l m})\right| < \delta \quad \text{on } Ka^{-(r_s-r_l)m} \setminus F_{l,m},$$

which says

$$\left|w_l(xa^{-(r_s-r_l)m})\cdots w_l(xa^{-(r_s m-1)})f(xa^{-r_s m})\right| < \delta \quad \text{on } K \setminus F_{l,m}a^{(r_s-r_l)m}.$$

Therefore, for $1 \le s < l \le N$ and $x \in K \setminus (F_{l,m}a^{(r_s-r_l)m} \cup C_{s,m})$, we have

$$\begin{aligned}
\frac{w_l(xa^{-(r_s-r_l)m})\cdots w_l(xa^{-(r_s m-1)})}{w_s(x)w_s(xa^{-1})\cdots w_s(xa^{-(r_s m-1)})} &= \frac{w_l(xa^{-(r_s-r_l)m})\cdots w_l(xa^{-(r_s m-1)})|f(xa^{-r_s m})|}{w_s(x)w_s(xa^{-1})\cdots w_s(xa^{-(r_s m-1)})|f(xa^{-r_s m})|} \\
&< \frac{\delta}{1-\delta} < \varepsilon;
\end{aligned}$$

that is,

$$\frac{\varphi_{l,(r_l-r_s)m}(x)\cdot\widetilde{\varphi}_{s,r_s m}(x)}{\widetilde{\varphi}_{l,r_s m}(x)} < \varepsilon \quad \text{on } K \setminus (F_{l,m}a^{(r_s-r_l)m} \cup C_{s,m}).$$

Finally, let

$$E_m = (K \setminus A) \setminus \bigcup_{1\le l\le N} (Ba^{r_l m} \cup C_{l,m} \cup D_{l,m}) \setminus \bigcup_{1\le s<l\le N} (F_{s,m}a^{(r_l-r_s)m} \cup F_{l,m}a^{(r_s-r_l)m}).$$

Then we have

$$\lambda(K \setminus E_m) < 6N^2\delta^p, \qquad \|\varphi_{l,r_l m}|_{E_m}\|_\infty < \varepsilon, \qquad \|\widetilde{\varphi}_{l,r_l m}|_{E_m}\|_\infty < \varepsilon,$$

and

$$\Big\|\frac{\widetilde{\varphi}_{s,(r_l-r_s)m}\cdot\widetilde{\varphi}_{l,r_l m}}{\widetilde{\varphi}_{s,r_l m}}\Big|_{E_m}\Big\|_\infty < \varepsilon, \qquad \Big\|\frac{\varphi_{l,(r_l-r_s)m}\cdot\widetilde{\varphi}_{s,r_s m}}{\widetilde{\varphi}_{l,r_s m}}\Big|_{E_m}\Big\|_\infty < \varepsilon,$$

which proves the condition (ii).

(ii) $\Rightarrow$ (i). We show that $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ are d-topologically transitive. For $1 \le l \le N$, let U and V_l be nonempty open subsets of $L^p(G)$. Since the space $C_c(G)$ of continuous functions on G with compact support is dense in $L^p(G)$, we can pick $f, g_l \in C_c(G)$ with $f \in U$ and $g_l \in V_l$ for $l = 1, 2, \ldots, N$. Let K be the union of the compact supports of f and all g_l. Let $E_k \subset K$, and let the sequences $(\varphi_{l,n}), (\widetilde{\varphi}_{l,n})$ satisfy condition (ii).

By aperiodicity of a, there exists $M \in \mathbb{N}$ such that $K \cap Ka^{\pm n} = \emptyset$ for all $n > M$.

First, for $1 \le l \le N$, we have

$$\begin{aligned}
\|T_l^{r_l n_k} f\chi_{E_k}\|_p^p &= \int_{E_k a^{r_l n_k}} \big|w_l(x)w_l(xa^{-1})\cdots w_l(xa^{-(r_l n_k-1)})\big|^p \big|f(xa^{-r_l n_k})\big|^p \,d\lambda(x) \\
&= \int_{E_k} \big|w_l(xa^{r_l n_k})w_l(xa^{r_l n_k-1})\cdots w_l(xa)\big|^p \big|f(x)\big|^p \,d\lambda(x) \\
&= \int_{E_k} \varphi_{l,r_l n_k}^p(x)\big|f(x)\big|^p \,d\lambda(x) \to 0
\end{aligned}$$

as $k \to \infty$. Here we denote S_{a,w_l} by S_l. Applying similar arguments to the iterates $S_l^{r_l n_k}$, and using the sequence $(\widetilde{\varphi}_{l,r_l n_k})$, we have

$$\begin{aligned}
\lim_{k\to\infty} \|S_l^{r_l n_k} g_l\chi_{E_k}\|_p^p &= \lim_{k\to\infty} \int_{E_k a^{-r_l n_k}} \frac{1}{|w_l(xa)w_l(xa^2)\cdots w_l(xa^{r_l n_k})|^p} \\
&\qquad \times \big|g_l(xa^{r_l n_k})\big|^p \,d\lambda(x) \\
&= 0
\end{aligned}$$

for $1 \le l \le N$. Moreover, for $1 \le s < l \le N$, we have

$$\begin{aligned}
&\big\|T_l^{r_l n_k}(S_s^{r_s n_k} g_s\chi_{E_k})\big\|_p^p \\
&\quad = \int_G \big|w_l(x)w_l(xa^{-1})\cdots w_l(xa^{-(r_l n_k-1)})\big|^p \big|S_s^{r_s n_k} g_s\chi_{E_k}(xa^{-r_l n_k})\big|^p \,d\lambda(x) \\
&\quad = \int_G \frac{|w_l(x)w_l(xa^{-1})\cdots w_l(xa^{-(r_l n_k-1)})|^p}{|w_s(xa^{-r_l n_k+1})w_s(xa^{-r_l n_k+2})\cdots w_s(xa^{-r_l n_k+r_s n_k})|^p} \\
&\quad\quad \times \big|g_s\chi_{E_k}(xa^{-r_l n_k+r_s n_k})\big|^p \,d\lambda(x)
\end{aligned}$$

$$= \int_{E_k} \frac{|w_l(xa^{-(r_s-r_l)n_k})w_l(xa^{-(r_s-r_l)n_k-1})\cdots w_l(xa^{-(r_sn_k-1)})|^p}{|w_s(xa^{-(r_sn_k-1)})w_s(xa^{-(r_sn_k-2)})\cdots w_s(x)|^p}\big|g_s(x)\big|^p\,d\lambda(x)$$
$$= \int_{E_k} \frac{\varphi_{l,(r_l-r_s)n_k}(x)\cdot\widetilde{\varphi}_{s,r_sn_k}(x)}{\widetilde{\varphi}_{l,r_sn_k}(x)}\big|g_s(x)\big|^p\,d\lambda(x) \to 0$$

as $k \to \infty$. Similarly, we have

$$\begin{aligned}
&\big\|T_s^{r_sn_k}(S_l^{r_ln_k}g_l\chi_{E_k})\big\|_p^p \\
&\quad= \int_G \big|w_s(x)w_s(xa^{-1})\cdots w_s(xa^{-(r_sn_k-1)})\big|^p\big|S_l^{r_ln_k}g_l\chi_{E_k}(xa^{-r_sn_k})\big|^p\,d\lambda(x) \\
&\quad= \int_G \frac{|w_s(x)w_s(xa^{-1})\cdots w_s(xa^{-(r_sn_k-1)})|^p}{|w_l(xa^{-r_sn_k+1})w_l(xa^{-r_sn_k+2})\cdots w_l(xa^{-r_sn_k+r_ln_k})|^p} \\
&\qquad\times \big|g_l\chi_{E_k}(xa^{-r_sn_k+r_ln_k})\big|^p\,d\lambda(x) \\
&\quad= \int_{E_k} \frac{|w_s(xa^{-(r_l-r_s)n_k})w_s(xa^{-(r_l-r_s)n_k-1})\cdots w_s(xa^{-(r_ln_k-1)})|^p}{|w_l(xa^{-(r_ln_k-1)})w_l(xa^{-(r_ln_k-2)})\cdots w_l(x)|^p} \\
&\qquad\times \big|g_l(x)\big|^p\,d\lambda(x) \\
&\quad= \int_{E_k} \frac{\widetilde{\varphi}_{s,(r_l-r_s)n_k}(x)\cdot\widetilde{\varphi}_{l,r_ln_k}(x)}{\widetilde{\varphi}_{s,r_ln_k}(x)}\big|g_l(x)\big|^p\,d\lambda(x) \to 0
\end{aligned}$$

as $k \to \infty$.

Now, for each $k \in \mathbb{N}$, we let

$$v_k = f\chi_{E_k} + S_1^{r_1n_k}g_1\chi_{E_k} + S_2^{r_2n_k}g_2\chi_{E_k} + \cdots + S_N^{r_Nn_k}g_N\chi_{E_k} \in L^p(G).$$

Then, by $K \cap Ka^{\pm(r_l-r_s)n_k} = \emptyset$, we have

$$\|v_k - f\|_p^p \le \|f\|_\infty^p\lambda(K \setminus E_k) + \sum_{l=1}^N \|S_l^{r_ln_k}g_l\chi_{E_k}\|_p^p$$

and

$$\begin{aligned}
\|T_l^{r_ln_k}v_k - g_l\|_p^p \le{}& \|T_l^{r_ln_k}f\chi_{E_k}\|_p^p + \|T_l^{r_ln_k}S_1^{r_1n_k}g_1\chi_{E_k}\|_p^p + \cdots \\
&+ \|T_l^{r_ln_k}S_{l-1}^{r_{l-1}n_k}g_{l-1}\chi_{E_k}\|_p^p \\
&+ \|g_l\chi_{E_k} - g_l\|_p^p + \|T_l^{r_ln_k}S_{l+1}^{r_{l+1}n_k}g_{l+1}\chi_{E_k}\|_p^p + \cdots \\
&+ \|T_l^{r_ln_k}S_N^{r_Nn_k}g_N\chi_{E_k}\|_p^p.
\end{aligned}$$

Hence $\lim_{k\to\infty} v_k = f$ and $\lim_{k\to\infty} T_l^{r_ln_k}v_k = g_l$ for $l = 1, 2, \ldots, N$, which implies that

$$\emptyset \ne U \cap T_1^{-r_1n_k}(V_1) \cap T_2^{-r_2n_k}(V_2) \cap \cdots \cap T_N^{-r_Nn_k}(V_N). \qquad \square$$

We note that, if G is discrete, then $K = E_k$ for each $k \in \mathbb{N}$ in the proof of Theorem 2.2. Hence each E_k in the condition (ii) of Theorem 2.2 will be replaced by K if G is discrete.

Example 2.3. Let $G = \mathbb{Z}$, and let $a = -1 \in \mathbb{Z}$. Given $N \geq 2$, let $w_l * \delta_{-1}$ be a weight on $\mathbb{Z}$ for $l = 1, 2, \ldots, N$. Then the weighted translation operator $T_{-1,w_l*\delta_{-1}}$ is defined by

$$T_{-1,w_l*\delta_{-1}} f(i) = w_l(i+1) f(i+1) \quad \big(f \in \ell^p(\mathbb{Z})\big).$$

Also, the operator $T_{-1,w_l*\delta_{-1}}$ is just the bilateral weighted backward shift T_l given by $T_l e_i = w_{l,i} e_{i-1}$ with $w_{l,i} = w_l(i)$; that is, $T_l = T_{-1,w_l*\delta_{-1}}$ for $1 \leq l \leq N$. Here $(e_i)_{i\in\mathbb{Z}}$ is the canonical basis of $\ell^p(\mathbb{Z})$, and $(w_{l,i})_{i\in\mathbb{Z}}$ is a sequence of positive real numbers. Hence, by Theorem 2.2, for $1 \leq r_1 < r_2 < \cdots < r_N$, the operators $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ are disjoint hypercyclic if, given $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists a positive integer n such that, for all $|i| < q$, we have for $1 \leq l \leq N$

$$\varphi_{l,r_l n}(i) = \prod_{j=1}^{r_l n} (w_l * \delta_{-1}) * \delta_1^j(i) = \prod_{j=0}^{r_l n - 1} w_l(i-j) = \prod_{j=i-r_l n+1}^{i} w_l(j) < \varepsilon$$

and

$$\widetilde{\varphi}_{l,r_l n}^{-1}(i) = \prod_{j=0}^{r_l n-1} (w_l * \delta_{-1}) * \delta_{-1}^j(i) = \prod_{j=1}^{r_l n} w_l(i+j) = \prod_{j=i+1}^{i+r_l n} w_l(j) > \frac{1}{\varepsilon},$$

and for $1 \leq s < l \leq N$,

$$\begin{aligned} \frac{\widetilde{\varphi}_{s,(r_l-r_s)n}(i) \cdot \widetilde{\varphi}_{l,r_l n}(i)}{\widetilde{\varphi}_{s,r_l n}(i)} &= \frac{\prod_{j=i+1}^{i+r_l n} w_s(j)}{\prod_{j=i+1}^{i+(r_l-r_s)n} w_s(j) \cdot \prod_{j=i+1}^{i+r_l n} w_l(j)} \\ &= \frac{\prod_{j=i+(r_l-r_s)n+1}^{i+r_l n} w_s(j)}{\prod_{j=i+1}^{i+r_l n} w_l(j)} < \varepsilon \end{aligned}$$

and

$$\begin{aligned} \frac{\varphi_{l,(r_l-r_s)n}(x) \cdot \widetilde{\varphi}_{s,r_s n}(x)}{\widetilde{\varphi}_{l,r_s n}(x)} &= \frac{\prod_{j=i-(r_l-r_s)n+1}^{i} w_l(j) \cdot \prod_{j=i+1}^{i+r_s n} w_l(j)}{\prod_{j=i+1}^{i+r_s n} w_s(j)} \\ &= \frac{\prod_{j=i-(r_l-r_s)n+1}^{i+r_s n} w_l(j)}{\prod_{j=i+1}^{i+r_s n} w_s(j)} < \varepsilon, \end{aligned}$$

which are the conditions in [5, Theorem 4.7]. If we define $w_l : \mathbb{Z} \to (0, \infty)$ by

$$w_l(i) = \begin{cases} \frac{1}{2} & \text{if } i < 0, \\ 2 & \text{if } i \geq 0 \end{cases}$$

for $l = 1, 2, \ldots, N$, then the weights $w_1, w_2, \ldots, w_N$ satisfy the four weight conditions above.

Example 2.4. Let $G = \mathbb{R}$, and let $a = 2$. Given $N \geq 2$, let w_l be a weight on $\mathbb{R}$ for $l = 1, 2, \ldots, N$. Then the weighted translation T_{2,w_l} on $L^p(\mathbb{R})$ is defined by

$$T_{2,w_l} f(x) = w(x) f(x-2) \quad \big(f \in L^p(\mathbb{R})\big).$$

By Theorem 2.2, for $1 \le r_1 < r_2 < \cdots < r_N$, the operators $T_{2,w_1}^{r_1}, T_{2,w_2}^{r_2}, \ldots, T_{2,w_N}^{r_N}$ are disjoint hypercyclic if, given $\varepsilon > 0$ and a compact subset K of $\mathbb{R}$, there exists a positive integer n such that, for $x \in K$, we have for $1 \le l \le N$

$$\varphi_{l,r_l n}(x) = \prod_{j=1}^{r_l n} w_l * \delta_{-2}^{j}(x) = \prod_{j=1}^{r_l n} w_l(x+2j) < \varepsilon$$

and

$$\widetilde{\varphi}_{l,r_l n}^{-1}(x) = \prod_{j=0}^{r_l n-1} w_l * \delta_{2}^{j}(x) = \prod_{j=0}^{r_l n-1} w_l(x-2j) > \frac{1}{\varepsilon},$$

and, for $1 \le s < l \le N$,

$$\begin{aligned}\frac{\widetilde{\varphi}_{s,(r_l-r_s)n}(x)\cdot\widetilde{\varphi}_{l,r_l n}(x)}{\widetilde{\varphi}_{s,r_l n}(x)} &= \frac{\prod_{j=0}^{r_l n-1} w_s(x-2j)}{\prod_{j=0}^{(r_l-r_s)n-1} w_s(x-2j)\cdot\prod_{j=0}^{r_l n-1} w_l(x-2j)}\\ &= \frac{\prod_{j=(r_l-r_s)n}^{r_l n-1} w_s(x-2j)}{\prod_{j=0}^{r_l n-1} w_l(x-2j)} < \varepsilon\end{aligned}$$

and

$$\begin{aligned}\frac{\varphi_{l,(r_l-r_s)n}(x)\cdot\widetilde{\varphi}_{s,r_s n}(x)}{\widetilde{\varphi}_{l,r_s n}(x)} &= \frac{\prod_{j=1}^{(r_l-r_s)n} w_l(x+2j)\cdot\prod_{j=0}^{r_s n-1} w_l(x-2j)}{\prod_{j=0}^{r_s n-1} w_s(x-2j)}\\ &= \frac{\prod_{j=-(r_l-r_s)n}^{r_s n-1} w_l(x-2j)}{\prod_{j=0}^{r_s n-1} w_s(x-2j)} < \varepsilon.\end{aligned}$$

We may define $w_l : \mathbb{R} \to (0,\infty)$ by

$$w_l(x) = \begin{cases} \frac{1}{2} & \text{if } x \ge 1, \\ \frac{1}{2^x} & \text{if } -1 < x < 1, \\ 2 & \text{if } x \le -1 \end{cases}$$

for $l = 1, 2, \ldots, N$. Then the four weight conditions above are satisfied by the weights $w_1, w_2, \ldots, w_N$.

Example 2.5. Let

$$G = \mathbb{H} := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

be the Heisenberg group which is neither abelian nor compact. For convenience, an element in G is written as (x, y, z). Let $(x, y, z), (x', y', z') \in \mathbb{H}$. Then the multiplication is given by

$$(x, y, z) \cdot (x', y', z') = (x + x', y + y', z + z' + xy')$$

and

$$(x, y, z)^{-1} = (-x, -y, xy - z).$$

Let $a = (1, 0, 2) \in \mathbb{H}$, which is aperiodic. Given $N \geq 2$, let w_l be a weight on $\mathbb{H}$ for $l = 1, 2, \ldots, N$. Then $a^{-1} = (-1, 0, -2)$, and the weighted translation $T_{(1,0,2),w_l}$ on $L^p(\mathbb{H})$ is defined by

$$T_{(1,0,2),w_l} f(x, y, z) = w_l(x, y, z) f(x - 1, y, z - 2) \quad \big(f \in L^p(\mathbb{H})\big).$$

By Theorem 2.2, we have, for $1 \leq r_1 < r_2 < \cdots < r_N$, that the operators $T^{r_1}_{(1,0,2),w_1}, T^{r_2}_{(1,0,2),w_2}, \ldots, T^{r_N}_{(1,0,2),w_N}$ are disjoint hypercyclic if, given $\varepsilon > 0$ and a compact subset K of $\mathbb{H}$, there exists a positive integer n such that, for $(x, y, z) \in K$, we have for $1 \leq l \leq N$

$$\varphi_{l,r_l n}(x, y, z) = \prod_{j=1}^{r_l n} w_l * \delta^j_{(1,0,2)^{-1}}(x, y, z) = \prod_{j=1}^{r_l n} w_l(x + j, y, z + 2j) < \varepsilon$$

and

$$\widetilde{\varphi}^{-1}_{l,r_l n}(x, y, z) = \prod_{j=0}^{r_l n-1} w_l * \delta^j_{(1,0,2)}(x) = \prod_{j=0}^{r_l n-1} w_l(x - j, y, z - 2j) > \frac{1}{\varepsilon},$$

and, for $1 \leq s < l \leq N$,

$$\begin{aligned}
&\frac{\widetilde{\varphi}_{s,(r_l - r_s)n}(x, y, z) \cdot \widetilde{\varphi}_{l,r_l n}(x, y, z)}{\widetilde{\varphi}_{s,r_l n}(x, y, z)} \\
&\quad = \frac{\prod_{j=0}^{r_l n-1} w_s(x - j, y, z - 2j)}{\prod_{j=0}^{(r_l - r_s)n-1} w_s(x - j, y, z - 2j) \cdot \prod_{j=0}^{r_l n-1} w_l(x - j, y, z - 2j)} \\
&\quad = \frac{\prod_{j=(r_l - r_s)n}^{r_l n-1} w_s(x - j, y, z - 2j)}{\prod_{j=0}^{r_l n-1} w_l(x - j, y, z - 2j)} < \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\varphi_{l,(r_l - r_s)n}(x, y, z) \cdot \widetilde{\varphi}_{s,r_s n}(x, y, z)}{\widetilde{\varphi}_{l,r_s n}(x, y, z)} \\
&\quad = \frac{\prod_{j=1}^{(r_l - r_s)n} w_l(x + j, y, z + 2j) \cdot \prod_{j=0}^{r_s n-1} w_l(x - j, y, z - 2j)}{\prod_{j=0}^{r_s n-1} w_s(x - j, y, z - 2j)} \\
&\quad = \frac{\prod_{j=-(r_l - r_s)n}^{r_s n-1} w_l(x - j, y, z - 2j)}{\prod_{j=0}^{r_s n-1} w_s(x - j, y, z - 2j)} < \varepsilon.
\end{aligned}$$

Similarly, one can obtain the required weight conditions by defining $w : \mathbb{H} \to (0, \infty)$ as

$$w_l(x, y, z) = \begin{cases} \frac{1}{2} & \text{if } z \geq 1, \\ \frac{1}{2^z} & \text{if } -1 < z < 1, \\ 2 & \text{if } z \leq -1 \end{cases}$$

for $1 \leq l \leq N$.

Now we turn our attention to single weighted translation $T := T_{a,w}$ generated by an aperiodic element $a \in G$ and a weight w on G, and we denote $S_{a,w}$ by S. By a similar argument as in the proof of Theorem 2.2, we characterize hypercyclicity for the direct sum of weighted translation operators. Indeed, if we consider open sets U_l and V_l ($l = 1, 2, \ldots, N$) in $L^p(G)$, and if we pick $f_l, g_l \in C_c(G)$ with $f_l \in U_l$ and $g_l \in V_l$, then the same argument can be applied to obtain a sequence $(v_{l,k})$ for each l satisfying $\lim_{k\to\infty} v_{l,k} = f_l$ and $\lim_{k\to\infty} T^{r_l n_k} v_{l,k} = g_l$ for $l = 1, 2, \ldots, N$. We have the result below.

Proposition 2.6. *Let G be a locally compact group, and let a be an aperiodic element in G. Let $1 \le p < \infty$, and let $T = T_{a,w}$ be a weighted translation on $L^p(G)$ generated by a and a positive weight w. Given $N \ge 2$ and $r_1, r_2, \ldots, r_N \in \mathbb{N}$, the following conditions are equivalent:*

(i) *$T^{r_1} \oplus T^{r_2} \oplus \cdots \oplus T^{r_N}$ are hypercyclic on $(L^p(G))^N$.*

(ii) *For each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets (E_k) in K such that $\lambda(K) = \lim_{k\to\infty} \lambda(E_k)$ and both sequences*

$$\varphi_n := \prod_{j=1}^{n} w * \delta^j_{a^{-1}} \qquad \textit{and} \qquad \widetilde{\varphi}_n := \Big(\prod_{j=0}^{n-1} w * \delta^j_a\Big)^{-1}$$

satisfy (for $1 \le l \le N$)

$$\lim_{k\to\infty} \|\varphi_{r_l n_k}|_{E_k}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{r_l n_k}|_{E_k}\|_\infty = 0$$

for some subsequence $(n_k) \subset \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii). By the density of hypercyclic vectors of T^{r_l} for each l, there exist a hypercyclic vector $f_l \in L^p(G)$ and some $m > M$ such that

$$\|f_l - \chi_K\|_p < \delta^2 \qquad \text{and} \qquad \|T^{r_l m} f_l - \chi_K\|_p < \delta^2$$

for $l = 1, 2, \ldots, N$. Repeating a similar argument as in the proof of Theorem 2.2, we can obtain the weight conditions.

(ii) $\Rightarrow$ (i). As in the proof of Theorem 2.2, for each $k \in \mathbb{N}$, we let

$$v_{l,k} = f_l \chi_{E_k} + S^{r_l n_k} g_l \chi_{E_k} \in L^p(G).$$

Then, by $K \cap K a^{\pm(r_l - r_s)n_k} = \emptyset$, we have

$$\|v_{l,k} - f_l\|_p^p \le \|f_l\|_\infty^p \lambda(K \setminus E_k) + \|S^{r_l n_k} g_l \chi_{E_k}\|_p^p$$

and

$$\|T^{r_l n_k} v_{l,k} - g_l\|_p^p \le \|T^{r_l n_k} f_l \chi_{E_k}\|_p^p + \|g_l\|_\infty^p \lambda(K \setminus E_k).$$

Hence $\lim_{k\to\infty} v_{l,k} = f_l$ and $\lim_{k\to\infty} T^{r_l n_k} v_{l,k} = g_l$ for $l = 1, 2, \ldots, N$, which implies that

$$T^{r_1 n_k}(U_l) \cap V_l \ne \emptyset$$

for some k and $l = 1, 2, \ldots, N$. $\square$

Using Proposition 2.6 together with Theorem 2.2, we immediately have the following corollary.

Corollary 2.7. *Let G be a locally compact group, and let a be an aperiodic element in G. Let $1 \leq p < \infty$, and let $T = T_{a,w}$ be a weighted translation on $L^p(G)$ generated by a and a positive weight w. Given $N \geq 2$, for $r_0 = 0 < 1 \leq r_1 < r_2 < \cdots < r_N$, the following conditions are equivalent:*

(i) $T^{r_1}, T^{r_2}, \ldots, T^{r_N}$ *are disjoint hypercyclic.*

(ii) $\bigoplus_{0 \leq s < l \leq N} T^{r_l - r_s}$ *are hypercyclic on* $(L^p(G))^{\frac{N(N+1)}{2}}$.

(iii) *For each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets (E_k) in K such that $\lambda(K) = \lim_{k\to\infty} \lambda(E_k)$ and both sequences*

$$\varphi_n := \prod_{j=1}^{n} w * \delta_{a^{-1}}^{j} \qquad \text{and} \qquad \widetilde{\varphi}_n := \left(\prod_{j=0}^{n-1} w * \delta_a^j\right)^{-1}$$

satisfy (for $1 \leq l \leq N$)

$$\lim_{k\to\infty} \|\varphi_{r_l n_k}|_{E_k}\|_\infty = \lim_{k\to\infty} \|\widetilde{\varphi}_{r_l n_k}|_{E_k}\|_\infty = 0$$

and (for $1 \leq s < l \leq N$)

$$\lim_{k\to\infty} \|\widetilde{\varphi}_{(r_l - r_s)n_k}|_{E_k}\|_\infty = \lim_{k\to\infty} \|\varphi_{(r_l - r_s)n_k}|_{E_k}\|_\infty = 0$$

for some subsequence $(n_k) \subset \mathbb{N}$.

In particular, $T^1, T^2, \ldots, T^N$ are disjoint hypercyclic if and only if $T^1 \oplus T^2 \oplus \cdots \oplus T^N$ are hypercyclic.

Example 2.8. Let $G = \mathbb{Z}$, let $a = -1 \in \mathbb{Z}$, and let $w * \delta_{-1}$ be a weight on $\mathbb{Z}$. Then the weighted translation operator $T_{-1,w*\delta_{-1}}$ is the bilateral weighted backward shift on $\ell^p(\mathbb{Z})$. By Corollary 2.7, for $1 \leq r_1 < r_2 < \cdots < r_N$, the following are equivalent:

(i) $T^{r_1}_{-1,w*\delta_{-1}}, T^{r_2}_{-1,w*\delta_{-1}}, \ldots, T^{r_N}_{-1,w*\delta_{-1}}$ are disjoint hypercyclic.

(ii) $\bigoplus_{0 \leq s < l \leq N} T^{r_l - r_s}_{-1,w*\delta_{-1}}$ are hypercyclic on $(\ell^p(\mathbb{Z}))^{\frac{N(N+1)}{2}}$.

Hence we recover a result in [5, Corollary 4.9].

3. Other disjoint notions

We will extend some results in this section from d-hypercyclicity to d-mixing, d-supercyclicity, and dual d-hypercyclicity. First of all, the characterization of d-mixing is given below by replacing the subsequence with the full sequence in the proof of Theorem 2.2.

Theorem 3.1. *Let G be a locally compact group, and let a be an aperiodic element in G. Let $1 \leq p < \infty$. Given $N \geq 2$, let $T_l = T_{a,w_l}$ be a weighted translation on $L^p(G)$ generated by a and a positive weight w_l for $1 \leq l \leq N$. For $1 \leq r_1 < r_2 < \cdots < r_N$, the following conditions are equivalent:*

(i) $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ *are d-mixing.*

(ii) *For each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets (E_n) in K such that $\lambda(K) = \lim_{n\to\infty} \lambda(E_n)$ and both sequences*

$$\varphi_{l,n} := \prod_{j=1}^{n} w_l * \delta_{a^{-1}}^{j} \qquad \textit{and} \qquad \widetilde{\varphi}_{l,n} := \Big(\prod_{j=0}^{n-1} w_l * \delta_{a}^{j}\Big)^{-1}$$

satisfy (for $1 \le l \le N$)

$$\lim_{n\to\infty} \|\varphi_{l,r_l n}|_{E_n}\|_\infty = \lim_{n\to\infty} \|\widetilde{\varphi}_{l,r_l n}|_{E_n}\|_\infty = 0$$

and (for $1 \le s < l \le N$)

$$\lim_{n\to\infty} \Big\| \frac{\widetilde{\varphi}_{s,(r_l-r_s)n} \cdot \widetilde{\varphi}_{l,r_l n}}{\widetilde{\varphi}_{s,r_l n}}\Big|_{E_n} \Big\|_\infty = \lim_{n\to\infty} \Big\| \frac{\varphi_{l,(r_l-r_s)n} \cdot \widetilde{\varphi}_{s,r_s n}}{\widetilde{\varphi}_{l,r_s n}}\Big|_{E_n} \Big\|_\infty = 0.$$

Proof. (i) $\Rightarrow$ (ii). By the assumption of d-mixing, there exists a function $f_l \in L^p(G)$ for each $l = 1, 2, \ldots, N$ such that

$$\|f_l - \chi_K\|_p < \delta^2 \qquad \text{and} \qquad \|T_l^{r_l n} f_l - \chi_K\|_p < \delta^2$$

from some n onwards. Following the same argument as in the proof of Theorem 2.2, the weight conditions can be obtained.

(ii) $\Rightarrow$ (i). For each $n \in \mathbb{N}$, we let

$$v_n = f\chi_{E_n} + S_1^{r_1 n} g_1 \chi_{E_n} + S_2^{r_2 n} g_2 \chi_{E_n} + \cdots + S_N^{r_N n} g_N \chi_{E_n}.$$

Then we can deduce

$$\emptyset \neq U \cap T_1^{-r_1 n}(V_1) \cap T_2^{-r_2 n}(V_2) \cap \cdots \cap T_N^{-r_N n}(V_N),$$

which says $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ d-mixing. ◻

Let $T_{l,n} = \alpha_{l,n} T_l^{r_l n}$ for $1 \le l \le N$. In the result below, we will show that N sequences of operators $(T_{1,n})_{n=1}^\infty, (T_{2,n})_{n=1}^\infty, \ldots, (T_{N,n})_{n=1}^\infty$ are d-topologically transitive, which implies that $(\alpha_{1,n} T_1^{r_1 n})_{n=1}^\infty, (\alpha_{2,n} T_2^{r_2 n})_{n=1}^\infty, \ldots, (\alpha_{N,n} T_N^{r_N n})_{n=1}^\infty$ are d-hypercyclic, and therefore that $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ are d-supercyclic.

Theorem 3.2. *Let G be a locally compact group, and let a be an aperiodic element in G. Let $1 \le p < \infty$. Given $N \ge 2$, let $T_l = T_{a,w_l}$ be a weighted translation on $L^p(G)$ generated by a and a positive weight w_l for $1 \le l \le N$. For $1 \le r_1 < r_2 < \cdots < r_N$, the following conditions are equivalent:*

(i) *$T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ have a dense set of d-supercyclic vectors.*

(ii) *For each compact subset $K \subset G$ with $\lambda(K) > 0$, there is a sequence of Borel sets (E_k) in K, and there exist sequences $(\alpha_{l,n}) \subset \mathbb{C} \setminus \{0\}$ such that $\lambda(K) = \lim_{k\to\infty} \lambda(E_k)$ and both sequences*

$$\varphi_{l,n} := \prod_{j=1}^{n} w_l * \delta_{a^{-1}}^{j} \qquad \textit{and} \qquad \widetilde{\varphi}_{l,n} := \Big(\prod_{j=0}^{n-1} w_l * \delta_{a}^{j}\Big)^{-1}$$

satisfy (for $1 \le l \le N$)

$$\lim_{k\to\infty} \|\alpha_{l,n_k} \varphi_{l,r_l n_k}|_{E_k}\|_\infty = \lim_{k\to\infty} \Big\| \frac{1}{\alpha_{l,n_k}} \widetilde{\varphi}_{l,r_l n_k}\Big|_{E_k} \Big\|_\infty = 0$$

and (for $1 \leq s < l \leq N$*)*

$$\lim_{k\to\infty}\left\|\frac{\widetilde{\varphi}_{s,(r_l-r_s)n_k}\cdot\widetilde{\varphi}_{l,r_ln_k}}{\widetilde{\varphi}_{s,r_ln_k}}\bigg|_{E_k}\right\|_\infty = \lim_{k\to\infty}\left\|\frac{\varphi_{l,(r_l-r_s)n_k}\cdot\widetilde{\varphi}_{s,r_sn_k}}{\widetilde{\varphi}_{l,r_sn_k}}\bigg|_{E_k}\right\|_\infty = 0$$

for some sequence $(n_k) \subset \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii). By density of the disjoint supercyclic vectors, there exist a disjoint supercyclic vector $f \in L^p(G)$, some $m > M$, and $\alpha_l \in \mathbb{C}\setminus\{0\}$ such that

$$\|f-\chi_K\|_p < \delta^2 \qquad \text{and} \qquad \|\alpha_l T_l^{r_l m} f - \chi_K\|_p < \delta^2$$

for $l = 1, 2, \ldots, N$. Repeating a similar argument as in the proof of Theorem 2.2, one can obtain the weight conditions.

(ii) $\Rightarrow$ (i). By the same procedure as in the proof of Theorem 2.2, for each $k \in \mathbb{N}$, we let

$$v_k = f\chi_{E_k} + \frac{1}{\alpha_{1,n_k}}S_1^{r_1n_k}g_1\chi_{E_k} + \frac{1}{\alpha_{2,n_k}}S_2^{r_2n_k}g_2\chi_{E_k} + \cdots + \frac{1}{\alpha_{N,n_k}}S_N^{r_Nn_k}g_N\chi_{E_k} \in L^p(G).$$

Then, again, by $K \cap Ka^{\pm(r_l-r_s)n_k} = \emptyset$, we have

$$\|v_k - f\|_p^p \leq \|f\|_\infty^p \lambda(K\setminus E_k) + \sum_{l=1}^N \left\|\frac{1}{\alpha_{l,n_k}}S_l^{r_ln_k}g_l\chi_{E_k}\right\|_p^p$$

and

$$\begin{aligned}
&\|\alpha_{l,n_k}T_l^{r_ln_k}v_k - g_l\|_p^p \\
&\leq \|\alpha_{l,n_k}T_l^{r_ln_k}f\chi_{E_k}\|_p^p + \left\|\frac{\alpha_{l,n_k}}{\alpha_{1,n_k}}T_l^{r_ln_k}S_1^{r_1n_k}g_1\chi_{E_k}\right\|_p^p + \cdots \\
&\quad + \left\|\frac{\alpha_{l,n_k}}{\alpha_{l-1,n_k}}T_l^{r_ln_k}S_{l-1}^{r_{l-1}n_k}g_{l-1}\chi_{E_k}\right\|_p^p \\
&\quad + \|g_l\chi_{E_k} - g_l\|_p^p + \left\|\frac{\alpha_{l,n_k}}{\alpha_{l+1,n_k}}T_l^{r_ln_k}S_{l+1}^{r_{l+1}n_k}g_{l+1}\chi_{E_k}\right\|_p^p + \cdots \\
&\quad + \left\|\frac{\alpha_{l,n_k}}{\alpha_{N,n_k}}T_l^{r_ln_k}S_N^{r_Nn_k}g_N\chi_{E_k}\right\|_p^p.
\end{aligned}$$

Hence $\lim_{k\to\infty} v_k = f$ and $\lim_{k\to\infty}\alpha_{l,n_k}T_l^{r_ln_k}v_k = g_l$ for $l = 1, 2, \ldots, N$, which implies that

$$\emptyset \neq U \cap \frac{1}{\alpha_{1,n_k}}T_1^{-r_1n_k}(V_1) \cap \frac{1}{\alpha_{2,n_k}}T_2^{-r_2n_k}(V_2) \cap \cdots \cap \frac{1}{\alpha_{N,n_k}}T_N^{-r_Nn_k}(V_N).$$

Hence $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ are d-supercyclic with a dense set of d-supercyclic vectors. $\square$

Remark 3.3. We note that the following two conditions are equivalent upon applying a similar argument as in the proof of [7, Proposition 2.2].

(i) For $1 \leq l \leq N$, we have

$$\lim_{k\to\infty}\|\alpha_{l,n_k}\varphi_{l,r_ln_k}|_{E_k}\|_\infty = \lim_{k\to\infty}\left\|\frac{1}{\alpha_{l,n_k}}\widetilde{\varphi}_{l,r_ln_k}\bigg|_{E_k}\right\|_\infty = 0.$$

(ii) For $1 \le l, s \le N$, we have

$$\lim_{k\to\infty} \|\varphi_{l,r_l n_k}|_{E_k}\|_\infty \cdot \|\widetilde{\varphi}_{s,r_s n_k}|_{E_k}\|_\infty = 0.$$

Example 3.4. Let $G = \mathbb{Z}$, and let $a = -1 \in \mathbb{Z}$. Given $N \ge 2$, let $w_l * \delta_{-1}$ be a weight on $\mathbb{Z}$ for $l = 1, 2, \ldots, N$. Then the weighted translation operator $T_{-1,w_l*\delta_{-1}}$ is defined by

$$T_{-1,w_l*\delta_{-1}} f(i) = w_l(i+1) f(i+1) \quad \big(f \in \ell^p(\mathbb{Z})\big),$$

and $T_{-1,w_l*\delta_{-1}}$ is the bilateral weighted backward shift T_l given by $T_l e_i = w_{l,i} e_{i-1}$ with $w_{l,i} = w_l(i)$. By Theorem 3.2, for $1 \le r_1 < r_2 < \cdots < r_N$, the operators $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ are d-supercyclic if, given $\varepsilon > 0$ and $q \in \mathbb{N}$, there exists a positive integer n such that, for all $|i| < q$ and $|k| < q$, we have for $1 \le l, s \le N$

$$\varphi_{l,r_l n}(i) \cdot \widetilde{\varphi}_{s,r_s n}(k) = \frac{\prod_{j=0}^{r_l n-1} w_l(i-j)}{\prod_{j=1}^{r_s n} w_s(k+j)} = \frac{\prod_{j=i-r_l n+1}^{i} w_l(j)}{\prod_{j=k+1}^{k+r_s n} w_s(j)} < \varepsilon$$

and, for $1 \le s < l \le N$,

$$\begin{aligned}\frac{\widetilde{\varphi}_{s,(r_l-r_s)n}(i) \cdot \widetilde{\varphi}_{l,r_l n}(i)}{\widetilde{\varphi}_{s,r_l n}(i)} &= \frac{\prod_{j=i+1}^{i+r_l n} w_s(j)}{\prod_{j=i+1}^{i+(r_l-r_s)n} w_s(j) \cdot \prod_{j=i+1}^{i+r_l n} w_l(j)} \\ &= \frac{\prod_{j=i+(r_l-r_s)n+1}^{i+r_l n} w_s(j)}{\prod_{j=i+1}^{i+r_l n} w_l(j)} < \varepsilon\end{aligned}$$

and

$$\begin{aligned}\frac{\varphi_{l,(r_l-r_s)n}(x) \cdot \widetilde{\varphi}_{s,r_s n}(x)}{\widetilde{\varphi}_{l,r_s n}(x)} &= \frac{\prod_{j=i-(r_l-r_s)n+1}^{i} w_l(j) \cdot \prod_{j=i+1}^{i+r_s n} w_l(j)}{\prod_{j=i+1}^{i+r_s n} w_s(j)} \\ &= \frac{\prod_{j=i-(r_l-r_s)n+1}^{i+r_s n} w_l(j)}{\prod_{j=i+1}^{i+r_s n} w_s(j)} < \varepsilon,\end{aligned}$$

which are the conditions in [14, Theorem 4.2.1].

Recall that a hypercyclic operator T whose dual T^* is also hypercyclic is called a *dual hypercyclic* operator. In [15, p. 766], Salas constructed a dual hypercyclic weighted shift on a Hilbert space. Similarly, the d-hypercyclic operators $T_1, T_2, \ldots, T_N$ are said to be *dual d-hypercyclic* in [17] if $T_1^*, T_2^*, \ldots, T_N^*$ are also d-hypercyclic, where T_l^* is the dual of T_l for $1 \le l \le N$. In fact, that a separable Banach space supports dual d-hypercyclic operators was proved in [17, Theorem 3.4] and [18, Theorem S] independently. We note that Bès and Peris considered a weighted bilateral forward shift A with dual A^* on $\ell^2(\mathbb{Z})$, and they showed that both the operators $A, A^2, \ldots, A^N$ and the operators $A^*, A^{*2}, \ldots, A^{*N}$ are d-hypercyclic under some weight sequence in [5, Theorem 4.11].

Finally, we will characterize the disjoint hypercyclicity for the dual of weighted translation operators by applying Theorem 2.2 directly. Let $p \in [1, \infty]$ with conjugate exponent q, and let $\langle \cdot, \cdot \rangle : L^p(G) \times L^q(G) \to \mathbb{C}$ be the duality. The simple computation gives

$$\langle T_{a,w}f, g\rangle = \big\langle f, T_{a^{-1}}(wg)\big\rangle \quad \big(f \in L^p(G), g \in L^q(G)\big).$$

Therefore the dual map $T^*_{a,w} : L^q(G) \to L^q(G)$ is given by

$$T^*_{a,w}(g) = T_{a^{-1}}(wg) = T_{a^{-1},w*\delta_{a^{-1}}}(g) \quad \big(g \in L^q(G)\big),$$

which is also a weighted translation operator on $L^q(G)$. Moreover, the dual of $T^n_{a,w}$ is $T^{*n}_{a,w}$ by a computation; that is, $T^{n*}_{a,w} = T^{*n}_{a,w}$ for all $n \in \mathbb{N}$.

Corollary 3.5. *Let G be a locally compact group, and let a be an aperiodic element in G. Let $1 \le p < \infty$. Given some $N \ge 2$, let $T_l = T_{a,w_l}$ be a weighted translation on $L^p(G)$ generated by a and a positive weight w_l for $1 \le l \le N$. Let $T^*_l = T_{a^{-1},w_l*\delta_{a^{-1}}}$ be the dual of T_l. Then for $1 \le r_1 < r_2 < \cdots < r_N$, the operators $T_1^{*r_1}, T_2^{*r_2}, \ldots, T_N^{*r_N}$ have a dense set of d-hypercyclic vectors if each weight $w_l * \delta_{a^{-1}}$ satisfies conditions (ii) for a^{-1} in Theorem 2.2; that is, both sequences*

$$\varphi^*_{l,n} := \prod_{j=1}^{n} (w_l * \delta_{a^{-1}}) * \delta^j_{(a^{-1})^{-1}} \qquad \text{and} \qquad \widetilde{\varphi}^*_{l,n} := \Big(\prod_{j=0}^{n-1} (w_l * \delta_{a^{-1}}) * \delta^j_{a^{-1}}\Big)^{-1}$$

satisfy the weight conditions in Theorem 2.2.

The Corollary 3.5 says that there exist d-hypercyclic operators $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ such that the operators $T_1^{*r_1}, T_2^{*r_2}, \ldots, T_N^{*r_N}$ are also d-hypercyclic. However, this is not the case for d-mixing. We conclude noting that the operators $T_1^{r_1}, T_2^{r_2}, \ldots, T_N^{r_N}$ and the operators $T_1^{*r_1}, T_2^{*r_2}, \ldots, T_N^{*r_N}$ can never be d-mixing simultaneously since $T^*_l = T_{a^{-1},w_l*\delta_{a^{-1}}}$, and, for $a^{-1} \in G$, the sequences for the weight $w_l * \delta_{a^{-1}}$ in the condition (ii) of Theorem 2.2 are defined by

$$\varphi^*_{l,r_ln} := \prod_{j=1}^{r_ln} (w_l * \delta_{a^{-1}}) * \delta^j_{(a^{-1})^{-1}} = \prod_{j=0}^{r_ln-1} w_l * \delta^j_a = \widetilde{\varphi}^{-1}_{l,r_ln}$$

and

$$\widetilde{\varphi}^*_{l,r_ln} := \Big(\prod_{j=0}^{r_ln-1} (w_l * \delta_{a^{-1}}) * \delta^j_{a^{-1}}\Big)^{-1} = \Big(\prod_{j=1}^{r_ln} w_l * \delta^j_{a^{-1}}\Big)^{-1} = \varphi^{-1}_{l,r_ln}$$

for each l.

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