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ON SOME HARDY-TYPE INEQUALITIES FOR FRACTIONAL CALCULUS OPERATORS

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Abstract. In this article we present applications of Hardy-type and refined Hardy-type inequalities for a generalized fractional integral operator involving the Mittag-Leffler function in its kernel and for the Hilfer fractional derivative using convex and monotone convex functions.

1. Introduction

Let $(\Sigma_1, \Omega_1, \mu_1)$ and $(\Sigma_2, \Omega_2, \mu_2)$ be measure spaces with positive σ-finite measures. Let $U(f, k)$ denote the class of functions $g : \Omega_1 \to \mathbb{R}$ with the representation

$$g(x) = \int_{\Omega_2} k(x,t) f(t)\, d\mu_2(t),$$

and let A_k be an integral operator defined by

$$A_k f(x) := \frac{g(x)}{K(x)} = \frac{1}{K(x)} \int_{\Omega_2} k(x,t) f(t)\, d\mu_2(t), \tag{1.1}$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and a nonnegative kernel, $f : \Omega_2 \to \mathbb{R}$, is a measurable function, and

$$0 < K(x) := \int_{\Omega_2} k(x,t)\, d\mu_2(t), \quad x \in \Omega_1. \tag{1.2}$$

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The Hardy integral inequality is one of the most important inequalities in analysis. It has many applications; the most useful one is to stable degenerate stationary waves (see [12]). These considerations have led various researchers in the field of integral inequalities to explore certain extensions and generalizations by involving fractional calculus operators. Adeleke, Čižmešija, Oguntuase, Persson, Pokaz, Krulić, and Pečarić in [1] and [3] have added a lot in this regard. In recent papers, Iqbal et al. in [9] and [10] investigated certain applications of Hardy-type and refined Hardy-type inequalities involving Saigo, Riemann–Liouville, and Erdélyi–Kober fractional integral operators. But we give such types of inequalities for more general fractional integral and derivative operators using convex and monotone convex functions. Let us first recall the following basic definitions.

The first definition is presented in [15].

Definition 1.1. Let I be an interval in $\mathbb{R}$. A function $\Phi : I \to \mathbb{R}$ is considered to be convex if

$$\Phi\big(\lambda x + (1-\lambda)y\big) \leq \lambda\Phi(x) + (1-\lambda)\Phi(y) \tag{1.3}$$

for all points $x, y \in I$ and all $\lambda \in [0,1]$. The function Φ is strictly convex if inequality (1.3) holds strictly for all distinct points in I and $\lambda \in (0,1)$.

The following definition is characterized by Pečarić et al. in [13, p. 9].

Definition 1.2. Let $0 < a < b \leq \infty$. By $C^n[a,b]$, we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order n, and $\mathrm{AC}[a,b]$ is the space of all absolutely continuous functions on $[a,b]$. By $\mathrm{AC}^n[a,b]$, we denote the space of all functions $f \in C^{n-1}[a,b]$ with $f^{(n-1)} \in \mathrm{AC}[a,b]$.

The following theorem is given in [13].

Theorem 1.3. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, let u be a weight function on Ω_1, let k be a nonnegative measurable function on $\Omega_1 \times \Omega_2$, and let K be defined on Ω_1 by (1.2). Suppose that $K(x) > 0$ for all $x \in \Omega_1$, that the function $x \mapsto u(x)\frac{k(x,t)}{K(x)}$ is integrable on Ω_1 for each fixed $t \in \Omega_2$, and that v is defined on Ω_2 by*

$$v(t) := \int_{\Omega_1} u(x)\frac{k(x,t)}{K(x)}\, d\mu_1(x) < \infty. \tag{1.4}$$

If Φ is a convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} u(x)\Phi\big(A_k f(x)\big)\, d\mu_1(x) \leq \int_{\Omega_2} v(t)\Phi\big(f(t)\big)\, d\mu_2(t) \tag{1.5}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ such that $\operatorname{Im} f \subseteq I$, where A_k is defined by (1.1).

Substituting $k(x,t)$ by $k(x,t)f_2(t)$ and f by $\frac{f_1}{f_2}$, where $f_i : \Omega_2 \to \mathbb{R}$ $(i = 1, 2)$ are measurable functions in Theorem 1.3, we obtain the following result presented in [7].

Theorem 1.4. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, let u be a weight function on Ω_1, and let k be a nonnegative measurable function on $\Omega_1 \times \Omega_2$. Assume that the function $x \mapsto u(x)\frac{k(x,t)}{g_2(x)}$ is integrable on Ω_1 for each fixed $t \in \Omega_2$. Define p on Ω_2 by*

$$p(t) := f_2(t) \int_{\Omega_1} u(x) \frac{k(x,t)}{g_2(x)} \, d\mu_1(x) < \infty.$$

If $\Phi : I \to \mathbb{R}$ is a convex function and if $\frac{g_1(x)}{g_2(x)}, \frac{f_1(t)}{f_2(t)} \in I$, then the inequality

$$\int_{\Omega_1} u(x) \Phi\Big(\frac{g_1(x)}{g_2(x)}\Big) \, d\mu_1(x) \leq \int_{\Omega_2} p(t) \Phi\Big(\frac{f_1(t)}{f_2(t)}\Big) \, d\mu_2(t) \tag{1.6}$$

holds for all $g_i \in U(f_i, k)$ $(i = 1, 2)$ and for all measurable functions $f_i : \Omega_2 \to \mathbb{R}$ $(i = 1, 2)$.

Remark 1.5. If Φ is strictly convex on I and if $\frac{f_1(x)}{f_2(x)}$ is nonconstant, then the inequality given in (1.6) is strict.

Definition 1.6. Let $\Phi : I \to \mathbb{R}$ be a convex function. Then the subdifferential of Φ in x is denoted by $\partial\Phi(x)$ and is defined as

$$\partial\Phi(x) = \{y \in \mathbb{R} : y \text{ is the slope of a support line at } x\}.$$

The new refined general weighted Hardy-type inequality that has a nonnegative kernel and that is related to an arbitrary convex function is given in the following theorem (see [3]).

Theorem 1.7. *Let the assumptions of Theorem 1.3 be satisfied. Moreover, if Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and if $\varphi : I \to \mathbb{R}$ is any function such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality*

$$\begin{aligned}
&\int_{\Omega_2} v(t) \Phi\big(f(t)\big) \, d\mu_2(t) - \int_{\Omega_1} u(x) \Phi\big(A_k f(x)\big) \, d\mu_1(x) \\
&\geq \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,t) \Big| \big|\Phi\big(f(t)\big) - \Phi\big(A_k f(x)\big)\big| \\
&\quad - \big|\varphi\big(A_k f(x)\big)\big| \cdot \big|f(t) - A_k f(x)\big| \Big| \, d\mu_2(t) \, d\mu_1(x)
\end{aligned}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ such that $f(t) \in I$ for all $t \in \Omega_2$. If Φ is a monotone convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\begin{aligned}
&\int_{\Omega_2} v(t) \Phi\big(f(t)\big) \, d\mu_2(t) - \int_{\Omega_1} u(x) \Phi\big(A_k f(x)\big) \, d\mu_1(x) \\
&\geq \Big| \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} \operatorname{sgn}\big(f(t) - A_k f(x)\big) k(x,t) \big[\Phi\big(f(t)\big) - \Phi\big(A_k f(x)\big) \\
&\quad - \big|\varphi\big(A_k f(x)\big)\big| \cdot \big(f(t) - A_k f(x)\big)\big] \, d\mu_2(t) \, d\mu_1(x) \Big|
\end{aligned}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ such that $f(t) \in I$ for all fixed $t \in \Omega_2$, where $A_k f$ is defined by (1.1).

In the following theorem, we give a refinement of a Hardy-type inequality obtained by Kaijser et al. in [11].

Theorem 1.8. *Let $u:(0,b)\to\mathbb{R}$ be a weight function such that the functions $x\mapsto \frac{u(x)}{x}\cdot\frac{k(x,t)}{K(x)}$ are integrable on (t,b) for each fixed $t\in(0,b)$, and let the function $w:(0,b)\to\mathbb{R}$ be defined by*

$$w(t)=t\int_t^b \frac{k(x,t)}{K(x)}u(x)\frac{dx}{x},$$

where $0<b\le\infty$ and $k:(0,b)\times(0,b)\to\mathbb{R}$ is a nonnegative measurable function such that

$$K(x)=\int_0^x k(x,t)\,dt>0,\quad x\in(0,b).$$

If Φ is a convex function on an interval $I\subseteq\mathbb{R}$ and if $\varphi:I\to\mathbb{R}$ is such that $\varphi(x)\in\partial\Phi(x)$ for all $x\in\operatorname{Int} I$, then the inequality

$$\begin{aligned}\int_0^b w(t)\Phi\big(f(t)\big)\frac{dt}{t}&-\int_0^b u(x)\Phi\big(A_kf(x)\big)\frac{dx}{x}\\ &\ge\int_0^b\frac{u(x)}{K(x)}\int_0^x k(x,t)\Big|\big|\Phi\big(f(t)\big)-\Phi\big(A_kf(x)\big)\big|\\ &\quad-\big|\varphi\big(A_kf(x)\big)\big|\cdot\big|f(t)-A_kf(x)\big|\Big|\,dt\frac{dx}{x}\end{aligned}\tag{1.7}$$

holds for all measurable functions $f:(0,b)\to\mathbb{R}$ with values in I, where A_kf is defined by

$$A_kf(x)=\frac{1}{K(x)}\int_0^x k(x,t)f(t)\,dt,\quad x\in(0,b).$$

If the function Φ is concave, then the order of integrals on the left-hand side of (1.7) is reversed. If Φ is monotone convex on the interval $I\subseteq\mathbb{R}$, then the following inequality

$$\begin{aligned}\int_0^b w(t)\Phi\big(f(t)\big)\frac{dt}{t}&-\int_0^b u(x)\Phi\big(A_kf(x)\big)\frac{dx}{x}\\ &\ge\Big|\int_0^b\frac{u(x)}{K(x)}\int_0^x \operatorname{sgn}\big(f(t)-A_kf(x)\big)k(x,t)\big[\Phi\big(f(t)\big)-\Phi\big(A_kf(x)\big)\\ &\quad-\big|\varphi\big(A_kf(x)\big)\big|.\big(f(t)-A_kf(x)\big)\big]\,dt\frac{dx}{x}\Big|\end{aligned}$$

holds for all measurable functions $f:(0,b)\to\mathbb{R}$ with values in I.

The next mean value theorem is given in [4].

Theorem 1.9. *Let $(\Omega_1,\Sigma_1,\mu_1)$, $(\Omega_2,\Sigma_2,\mu_2)$ be measure spaces with σ-finite measures, and let $u:\Omega_1\to\mathbb{R}$ be a weight function. Let I be compact interval of $\mathbb{R}$, let*

$\tilde{h} \in C^2(I)$, and let $f : \Omega_2 \to \mathbb{R}$ a measurable function such that $\operatorname{Im} f \subseteq I$. Then there exists $\eta \in I$ such that

$$\begin{aligned}\int_{\Omega_2} v(t)\tilde{h}\big(f(t)\big)\,d\mu_2(t) - \int_{\Omega_1} u(x)\tilde{h}\big(A_k f(x)\big)\,d\mu_1(x) \\ = \frac{\tilde{h}''(\eta)}{2}\Big[\int_{\Omega_2} v(t)f^2(t)\,d\mu_2(t) - \int_{\Omega_1} u(x)\big(A_k f(x)\big)^2\,d\mu_1(x)\Big],\end{aligned}$$

where $A_k f$ and v are defined by (1.1) and (1.4), respectively.

2. Exponential convexity

We continue with the definition of an exponentially convex function as originally given in [2] by Bernstein.

Definition 2.1. A function $\Phi : (a,b) \to \mathbb{R}$ is *exponentially convex* if it is continuous and if

$$\sum_{i,j=1}^{n} t_i t_j \Phi(x_i + x_j) \geq 0$$

for all $n \in \mathbb{N}$ and all sequences $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ of real numbers such that $x_i + x_j \in (a,b)$, $1 \leq i,j \leq n$.

Lemma 2.2. *Let $s \in \mathbb{R}$, and let the function $\varphi_s : (0,\infty) \to \mathbb{R}$ be defined by*

$$\varphi_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0,1, \\ -\log x, & s = 0, \\ x \log x, & s = 1. \end{cases} \tag{2.1}$$

Then $\varphi_s''(x) = x^{s-2}$; that is, φ_s is a convex function.

The following theorem is presented in [4].

Theorem 2.3. *Let the conditions of Theorem 1.3 be satisfied, and let φ_s be defined by (2.1). Let f be a positive function. Then the function $\xi : \mathbb{R} \to [0,\infty)$ defined by*

$$\xi(s) = \int_{\Omega_2} v(t)\varphi_s\big(f(t)\big)\,d\mu_2(t) - \int_{\Omega_1} u(x)\varphi_s\big(A_k f(x)\big)\,d\mu_1(x)$$

is exponentially convex.

Theorem 2.4. *Let the conditions of Theorem 1.9 be satisfied. Moreover, let $k, \tilde{h} \in C^2(I)$ such that $\tilde{h}''(x) \neq 0$ for every $x \in I$ and*

$$\int_{\Omega_2} v(t)\tilde{h}\big(f(t)\big)\,d\mu_2(t) - \int_{\Omega_1} u(x)\tilde{h}\big(A_k f(x)\big)\,d\mu_1(x) \neq 0.$$

Then there exists $\eta \in I$ such that

$$\frac{k''(\eta)}{\tilde{h}''(\eta)} = \frac{\int_{\Omega_2} v(t)k(f(t))\,d\mu_2(t) - \int_{\Omega_1} u(x)k(A_k f(x))\,d\mu_1(x)}{\int_{\Omega_2} v(t)\tilde{h}(f(t))\,d\mu_2(t) - \int_{\Omega_1} u(x)\tilde{h}(A_k f(x))\,d\mu_1(x)}.$$

Using Theorem 1.3, and bearing in mind (1.5), we define the following positive linear functional:

$$\Delta_1(\Phi) = \int_{\Omega_2} v(t)\Phi\big(f(t)\big)\, d\mu_2(t) - \int_{\Omega_1} u(x)\Phi\big(A_k f(x)\big)\, d\mu_1(x). \tag{2.2}$$

We also define a linear functional by taking the positive difference of the left-hand side and the right-hand side of the inequality (1.6) given in Theorem 1.4 as

$$\Delta_2(\Phi) = \int_{\Omega_2} p(t)\Phi\Big(\frac{f_1(t)}{f_2(t)}\Big)\, d\mu_2(t) - \int_{\Omega_1} u(x)\Phi\Big(\frac{g_1(x)}{g_2(x)}\Big)\, d\mu_1(x). \tag{2.3}$$

First we give some necessary details about the divided differences. Let $I \subseteq \mathbb{R}$ be an interval, and let $f : I \to \mathbb{R}$ be a function. Then for distinct points $z_i \in I$, $i = 0, 1, 2$, the divided differences of first and second order are defined by

$$\begin{aligned} [z_i, z_{i+1}; f] &= \frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i} \quad (i = 0, 1), \\ [z_0, z_1, z_2; f] &= \frac{[z_1, z_2; f] - [z_0, z_1; f]}{z_2 - z_0}. \end{aligned} \tag{2.4}$$

The values of the divided differences are independent of the order of points z_0, z_1, z_2 and may be extended to include the cases when some or all points are equal; that is,

$$[z_0, z_0; f] = \lim_{z_1 \to z_0} [z_0, z_1; f] = f'(z_0),$$

provided that f' exists.

Now passing through the limit $z_1 \to z_0$ and replacing z_2 by z in (2.4), we have

$$[z_0, z_0, z; f] = \lim_{z_1 \to z_0} [z_0, z_1, z; f] = \frac{f(z) - f(z_0) - (z - z_0)f'(z_0)}{(z - z_0)^2} z \neq z_0,$$

provided that f' exists. Also, passing to the limit $z_i \to z$ $(i = 0, 1, 2)$ in (2.4), we have

$$[z, z, z; f] = \lim_{z_i \to z} [z_0, z_1, z_2; f] = \frac{f''(z)}{2},$$

provided that f'' exists. One can observe that, for all $z_0, z_1 \in I$, $[z_0, z_1, f] \geq 0$, if f is increasing on I, and if, for all $z_0, z_1, z_2 \in I$, $[z_0, z_1, z_2; f] \geq 0$, then f is convex on I.

Next, we recall the notion of n-exponential convexity given in [16].

Definition 2.5. For any open interval I of $\mathbb{R}$, the function $\Phi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} t_i t_j \Phi\Big(\frac{\zeta_i + \zeta_j}{2}\Big) \geq 0$$

holds for all choices of $t_i \in \mathbb{R}$, $\zeta_i \in I$, $i = 1, \ldots, n$. A function $\Phi : I \to \mathbb{R}$ is n-exponentially convex on I if it is n-exponentially convex in the Jensen sense and continuous on I.

The following theorem is given in [8].

Theorem 2.6. *Let $\Gamma = \{\Phi_p : p \in J\}$ be a family of functions defined on I such that the function $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is n-exponentially convex in the Jensen sense on J for every three distinct points $z_0, z_1, z_2 \in I$. Let Δ_i $(i = 1, 2)$ be linear functionals defined by (2.2) and (2.3). Then the function $p \mapsto \Delta_i(\Phi_p)$ $(i = 1, 2)$ is n-exponentially convex in the Jensen sense on J if it is continuous on J.*

The next section deals with applications of results given in Section 1 for the generalized fractional integral operator with the Mittag-Leffler function in its kernel.

3. Refined Hardy-type inequalities for the fractional integral operator with generalized Mittag-Leffler function in its kernel

In this section, first we give the definition of the Mittag-Leffler function (see [14]) and the fractional integral operator involving the generalized Mittag-Leffler function appearing in the kernel (see [19]).

Definition 3.1. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}; \min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0; p, q > 0$, and $q < \Re\alpha + p$. Then the generalized Mittag-Leffler function defined in [19] is given by

$$E^{\gamma,\delta,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{(\delta)_{pn}}, \tag{3.1}$$

where $(\gamma)_n$ represents the Pochhammer symbol, defined by $(\gamma)_n = \gamma(\gamma - 1) \times (\gamma - 2) \cdots (\gamma - n + 1)$. The function (3.1) represents all the previous generalizations of the Mittag-Leffler function by setting the following values.

- $p = q = 1$—This reduces to $E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{(\delta)_n}$ defined by Salim in [18].
- $\delta = p = 1$—This represents $E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}$, which was introduced by Shukla and Prajapati in [20]. In [21] Srivastava and Tomovski investigated the properties of this function and its existence for a wider set of parameters.
- $\delta = p = q = 1$—The operator (3.1) is defined by Prabhakar in [17] and is denoted as $E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!}$.
- $\gamma = \delta = p = q = 1$—It reduces to Wiman's function presented in [23], and moreover, if $\beta = 1$, then the Mittag-Leffler function $E_\alpha(z)$ will be the result.

Definition 3.2. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}; \min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\delta)\} > 0; p, q > 0$, and $q < \Re\alpha + p$. For all $g \in L(a, b)$, we introduce an integral operator

$$(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega;a^+} f)(x) = \int_a^x (x - t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}\big(\omega(x - t)^\alpha\big) f(t)\, dt, \tag{3.2}$$

which contains the generalized Mittag-Leffler function (3.1) in its kernel; this operator is investigated and its boundedness is proved under certain conditions.

Applying Theorem 1.3 for the integral operator given in (3.2), we obtain the following theorem.

Theorem 3.3. *Let $\alpha, \beta, \gamma, \delta, p, q$ be as in Definition 3.2, and let u be a weight function defined on (a,b). For each fixed $t \in (a,b)$, define a function $\tilde{v}$ by*

$$\tilde{v}(t) = \int_t^b u(x) \frac{(x-t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^\alpha)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\, dx < \infty. \tag{3.3}$$

If Φ is a convex function on the interval $I \in \mathbb{R}$, then the inequality

$$\int_a^b u(x) \Phi\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)} \Big)\, dx \le \int_a^b \tilde{v}(t) \Phi\big(f(t)\big)\, dt \tag{3.4}$$

holds true for all measurable functions $f \in L(a,b)$ such that $\operatorname{Im} f \subseteq I$.

Proof. Applying Theorem 1.3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, we get

$$\tilde{k}(x,t) = \begin{cases} (x-t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^\alpha), & a \le t \le x, \\ 0, & x < t \le b \end{cases} \tag{3.5}$$

(see Lemma 3.2 in [10]), and

$$\begin{aligned} \tilde{K}(x) &= \int_a^x (x-t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}\big(\omega(x-t)^\alpha\big) \\ &= (x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}\big(\omega(x-a)^\alpha\big). \end{aligned}$$

Then we get inequality (3.4). □

Next, we obtain the fractional inequality for the generalized fractional integral.

Theorem 3.4. *Let $\alpha, \beta, \gamma, \delta, p, q$ be as in Definition 3.2, and let u be a weight function defined on (a,b). For each fixed $t \in (a,b)$, define a function*

$$\hat{p}(t) := f_2(t) \int_t^b u(x) \frac{(x-t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^\alpha)}{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_2)(x)}\, dx < \infty.$$

If $\Phi : I \to \mathbb{R}$ is a convex function and $\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_1)(x)}{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_2)(x)}, \frac{f_1(t)}{f_2(t)} \in I$, then the inequality

$$\int_a^b u(x) \Phi\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_1)(x)}{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_2)(x)} \Big)\, dx \le \int_a^b \hat{p}(t) \Phi\Big(\frac{f_1(t)}{f_2(t)}\Big)\, dt \tag{3.6}$$

holds true.

Proof. Applying Theorem 1.4 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, $g_1(x) = \varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_1(x)$, $g_2(x) = \varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_2(x)$, and $k(x,t) = (x-t)^{\beta-1} \times E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^\alpha)$, we obtain inequality (3.6). □

Remark 3.5. If Φ is strictly convex on I and $\frac{f_1(x)}{f_2(x)}$ is nonconstant, then the inequality given in (3.6) is strict.

The new refined general weighted Hardy-type inequality which has a nonnegative kernel and is related to an arbitrary convex function given in [3] for the generalized fractional integral (3.2) follows in the next theorem.

Theorem 3.6. *Let the assumptions of Theorem 3.3 be satisfied. Moreover, if Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is any function such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality*

$$\begin{aligned}
&\int_a^b \tilde{v}(t)\Phi\big(f(t)\big)\,dt - \int_a^b u(x)\Phi\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big)\,dx \\
&\geq \int_a^b \frac{u(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)} \int_a^x (x-t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}\big(\omega(x-t)^\alpha\big) \\
&\quad \times \Big|\Big|\Phi\big(f(t)\big) - \Phi\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big)\Big| \\
&\quad - \Big|\varphi\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big)\Big| \\
&\quad \cdot \Big|f(t) - \frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big|\Big|\,dt\,dx
\end{aligned} \tag{3.7}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ such that $f(t) \in I$ for all $t \in (a,b)$. If Φ is a monotone convex function on an interval $I \subseteq \mathbb{R}$, then the inequality

$$\begin{aligned}
&\int_a^b \tilde{v}(t)\Phi\big(f(t)\big)\,dt - \int_a^b u(x)\Phi\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big)\,dx \\
&\geq \Big|\int_a^b \frac{u(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)} \\
&\quad \times \int_a^x \operatorname{sgn}\Big(f(t) - \frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big)(x-t)^{\beta-1} E^{\gamma,\delta,q}_{\alpha,\beta,p}\big(\omega(x-t)^\alpha\big) \\
&\quad \times \Big[\Phi\big(f(t)\big) - \Phi\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big) \\
&\quad - \Big|\varphi\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big)\Big| \\
&\quad \cdot \Big(f(t) - \frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big)\Big]\,dt\,dx\Big|
\end{aligned} \tag{3.8}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$ such that $f(t) \in I$ for all fixed $t \in (a,b)$.

Proof. Applying Theorem 1.7 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, and $\tilde{k}(x,t)$ given in (3.5), we get inequalities (3.7) and (3.8). ☐

The 1-dimensional setting gives refined Hardy- and Pólya–Knopp-type inequalities. In the following theorem, a refinement of a Hardy-type inequality obtained by Kaijser et al. in [11] is given for the generalized fractional integral operator.

Theorem 3.7. *Let $\alpha,\beta,\gamma,\delta,p,q$ be as in Definition 3.2, and let u be a weight function defined on (a,b). For each fixed $t \in (a,b)$, define a function w by*

$$w(t) = t\int_t^b \frac{(x-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}(\omega(x-t)^\alpha)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}u(x)\frac{dx}{x}.$$

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality

$$\begin{aligned}
&\int_a^b w(t)\Phi\big(f(t)\big)\frac{dt}{t} - \int_a^b u(x)\Phi\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big)\frac{dx}{x}\\
&\quad\geq \int_a^b \frac{u(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\int_a^x (x-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}\big(\omega(x-t)^\alpha\big)\\
&\qquad\times\bigg|\Big|\Phi\big(f(t)\big) - \Phi\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big)\Big|\\
&\qquad - \Big|\varphi\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big)\Big|\\
&\qquad\cdot\Big|f(t) - \frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big|\bigg|\,dt\frac{dx}{x}
\end{aligned}\tag{3.9}$$

holds for all measurable functions $f : (a,b) \to \mathbb{R}$ with values in I.

If the function Φ is concave, then the order of integrals on the left-hand side of (3.9) is reversed. If Φ is monotone convex on the interval $I \subseteq \mathbb{R}$, then the following inequality

$$\begin{aligned}
&\int_a^b w(t)\Phi\big(f(t)\big)\frac{dt}{t} - \int_a^b u(x)\Phi\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big)\frac{dx}{x}\\
&\quad\geq \Big|\int_a^b \frac{u(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\\
&\quad\times\int_a^x \operatorname{sgn}\Big(f(t) - \frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big)(x-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}\big(\omega(x-t)^\alpha\big)\\
&\quad\times\Big[\Phi\big(f(t)\big) - \Phi\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big)
\end{aligned}$$

$$-\Big|\varphi\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^{\beta}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^{\alpha})}\Big)\Big|$$
$$\cdot\Big(f(t)-\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^{\beta}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^{\alpha})}\Big)\Big]\,dt\frac{dx}{x}\Big| \tag{3.10}$$

holds for all measurable functions $f:(a,b)\to\mathbb{R}$ *with values in* I.

Proof. Applying Theorem 1.8 with $(0,b)=(a,b)$, $\tilde{k}(x,t)$ given in (3.5) and

$$A_k f(x)=\frac{1}{(x-a)^{\beta}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^{\alpha})}\int_a^x (x-t)^{\beta-1}E^{\gamma,\delta,q}_{\alpha,\beta,p}\big(\omega(x-t)^{\alpha}\big)f(t)\,dt,$$

we obtain equalities (3.9) and (3.10). □

Next we give the mean value theorems [4] for the generalized fractional integral (3.2).

Theorem 3.8. *Let the assumptions of Theorem 3.3 be satisfied. Let* I *be a compact interval of* $\mathbb{R}$, *let* $\tilde{h}\in C^2(I)$, *and let* $f:(a,b)\to\mathbb{R}$ *be a measurable function such that* $\operatorname{Im} f\subseteq I$. *Then there exists* $\eta\in I$ *such that*

$$\int_a^b \tilde{v}(t)\tilde{h}\big(f(t)\big)\,dt-\int_a^b u(x)\tilde{h}\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^{\beta}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^{\alpha})}\Big)\,dx$$
$$=\frac{\tilde{h}''(\eta)}{2}\Big[\int_a^b \tilde{v}(t)f^2(t)\,dt$$
$$-\int_a^b u(x)\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^{\beta}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^{\alpha})}\Big)^2\,dx\Big], \tag{3.11}$$

where $\tilde{v}$ *is defined by (3.3).*

Proof. Applying Theorem 1.9 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx$, $d\mu_2(t)=dt$, and $\tilde{k}(x,t)$ given in (3.5), we get equation (3.11). □

Theorem 3.9. *Let the assumptions of Theorem 3.8 be satisfied. Moreover,* $k,\tilde{h}\in C^2(I)$ *such that* $\tilde{h}''(x)\neq 0$ *for every* $x\in I$ *and*

$$\int_a^b \tilde{v}(t)\tilde{h}\big(f(t)\big)\,dt-\int_a^b u(x)\tilde{h}\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^{\beta}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^{\alpha})}\Big)\,dx\neq 0.$$

Then there exists $\eta\in I$ *such that*

$$\frac{k''(\eta)}{\tilde{h}''(\eta)}=\frac{\int_a^b \tilde{v}(t)k(f(t))\,dt-\int_a^b u(x)k\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^{\beta}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^{\alpha})}\Big)\,dx}{\int_a^b \tilde{v}(t)\tilde{h}(f(t))\,dt-\int_a^b u(x)\tilde{h}\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+}f)(x)}{(x-a)^{\beta}E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^{\alpha})}\Big)\,dx}.$$

We next present the linear functional given in [4] for the integral operator (3.2).

Theorem 3.10. *Let the conditions of Theorem 3.3 be satisfied, and let φ_s be defined by (2.1). Let f be a positive function. Then the function $\xi : \mathbb{R} \to [0, \infty)$ defined by*

$$\xi(s) = \int_a^b \tilde{v}(t)\varphi_s\big(f(t)\big)\, dt - \int_a^b u(x)\varphi_s\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big)\, dx \quad (3.12)$$

is exponentially convex.

Proof. Applying Theorem 2.3 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, and $\tilde{k}(x, t)$ given in (3.5), we get the linear functional (3.12). □

Under the assumptions of Theorem 3.3, we define a linear functional by taking the positive difference of the inequality stated in (3.4) as

$$\xi_1(\Phi) = \int_a^b \tilde{v}(t)\Phi\big(f(t)\big)\, dt - \int_a^b \Phi\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f)(x)}{(x-a)^\beta E^{\gamma,\delta,q}_{\alpha,\beta+1,p}(\omega(x-a)^\alpha)}\Big) u(x)\, dx. \quad (3.13)$$

We also define a linear functional by taking the positive difference of the left-hand side and right-hand side of the inequality (3.6) given in Theorem 3.4 for integral operator (3.2) as

$$\xi_2(\Phi) = \int_a^b \hat{p}(t)\Phi\Big(\frac{f_1(t)}{f_2(t)}\Big)\, dt - \int_a^b u(x)\Phi\Big(\frac{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_1)(x)}{(\varepsilon^{\gamma,\delta,q}_{\alpha,\beta,p,\omega,a^+} f_2)(x)}\Big)\, dx. \quad (3.14)$$

Theorem 3.11. *Let $\Gamma = \{\Phi_p : p \in J\}$ be a family of functions defined on I such that the function $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is n-exponentially convex in the Jensen sense on J for every three distinct points $z_0, z_1, z_2 \in I$. Let ξ_i $(i = 1, 2)$ be linear functionals defined by (3.13) and (3.14), respectively. Then the function $p \mapsto \xi_i(\Phi_p)$ $(i = 1, 2)$ is n-exponentially convex in the Jensen sense on J. If the function $p \mapsto \xi_i(\Phi_p)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. Applying Theorem 2.6 with $\Omega_1 = \Omega_2 = (a, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, and $k(x, t) = \tilde{k}(x, t)$, we complete the proof. □

Remark 3.12. In particular, if we choose $p = q = 1$ and $\omega = 0$, then we obtain Corollary 3 of [9].

4. Refined Hardy-type inequalities for the Hilfer fractional derivative

In this section, we first give the basic definition of the Hilfer fractional derivative. Then we present refined Hardy-type inequalities for the said derivative. Let us now recall the definition of the Hilfer fractional derivative which is presented in [22].

Definition 4.1. Let $f \in L^1[a, b], f * K_{(1-\nu)(1-\mu)} \in \mathrm{AC}^1[a, b]$. The fractional derivative operator $D^{\mu,\nu}_{a+}$ of order $0 < \mu < 1$ and type $0 < \nu \leq 1$ with respect to $x \in [a, b]$ is defined by

$$(D^{\mu,\nu}_{a+} f)(x) := I^{\nu(1-\mu)}_{a+} \frac{d}{dx}\big(I^{(1-\nu)(1-\mu)}_{a+} f(x)\big) \quad (4.1)$$

whenever the right-hand side exists. The derivative (4.1) is usually called the *Hilfer fractional derivative.*

The more general integral representation of equation (4.1) given in [6] is defined as follows. Let $f \in L^1[a,b], f * K_{(1-\nu)(n-\mu)} \in \mathrm{AC}^n[a,b], n-1<\mu<n, 0<\nu\leq 1$, $n \in \mathbb{N}$. Then the following equation holds true:

$$(D_{a+}^{\mu,\nu} f)(x) = \Big(I_{a+}^{\nu(n-\mu)} \frac{d^n}{dx^n} \big(I_{a+}^{(1-\nu)(n-\mu)} f(x) \big) \Big). \tag{4.2}$$

Especially for $\nu = 0$, $D_{a+}^{\mu,0} f = D_{a+}^{\mu} f$ is a Riemann–Liouvile fractional derivative of order μ, and for $\nu = 1$ it is a Caputo fractional derivative $D_{a+}^{\mu,1} f = ^C D_{a+}^{\mu} f$ of order μ. Applying the properties of the Riemann–Liouvile integral, the relation (4.2) can be rewritten in the form

$$\begin{aligned}(D_{a+}^{\mu,\nu} f)(x) &= \big(I_{a+}^{\nu(n-\mu)} \big((D_{a+}^{n-(1-\nu)(n-\mu)} f)(x) \big) \big) \\ &= \frac{1}{\Gamma(\nu(n-\mu))} \int_a^x (x-t)^{\nu(n-\mu)-1} \big((D_{a+}^{\mu+\nu(n-\mu)} f)(t) \big)\, dt. \end{aligned} \tag{4.3}$$

Our first result is an application of Theorem 1.3 given in [13] for the integral operator (4.3).

Theorem 4.2. *Let $f \in L^1[a,b]$, and let the fractional derivative operator be $D_{a+}^{\mu,\nu}$ of order $n-1<\mu<n$ and type $0<\nu\leq 1$, and let u be a weight function defined on (a,b). Then $\bar{v}$ is defined by*

$$\bar{v}(t) = \nu(n-\mu) \int_t^b u(x) \frac{(x-t)^{\nu(n-\mu)-1}}{(x-a)^{\nu(n-\mu)}}\, dx < \infty. \tag{4.4}$$

If Φ is a convex function on the interval I, then the inequality

$$\begin{aligned}&\int_a^b u(x) \Phi\Big(\frac{\Gamma(\nu(n-\mu)+1)}{(x-a)^{\nu(n-\mu)}} (D_{a+}^{\mu,\nu} f)(x) \Big)\, dx \\ &\leq \int_a^b \bar{v}(t) \Phi\big((D_{a+}^{\mu+\nu(n-\mu)} f)(t) \big)\, dt \end{aligned} \tag{4.5}$$

holds true.

Proof. Applying Theorem 1.3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$,

$$\bar{k}(x,t) = \begin{cases} \frac{(x-t)^{\nu(n-\mu)-1}}{\Gamma(\nu(n-\mu))}, & a \leq t \leq x; \\ 0, & x < t \leq b, \end{cases} \tag{4.6}$$

$$\bar{K}(x) = \frac{(x-a)^{\nu(n-\mu)}}{\Gamma(\nu(n-\mu)+1)}, \tag{4.7}$$

and $\bar{v}$ as in (4.4), we get inequality (4.5). □

Next, we obtain the fractional inequality for the generalized fractional integral.

Theorem 4.3. *Let $f_1, f_2 \in L^1[a,b]$, and let the fractional derivative operator be $D^{\mu,\nu}_{a+}$ of order $n-1<\mu<n$ and type $0<\nu\leq 1$. Moreover, let u be a weight function defined on (a,b), and for each fixed $t\in(a,b)$, define $\bar{p}$ on (a,b) as*

$$\bar{p}(t) := \frac{(D^{\mu+\nu(n-\mu)}_{a_+} f_2)(t)}{\Gamma(\nu(n-\mu))} \int_t^b u(x) \frac{(x-t)^{\nu(n-\mu)-1}}{(D^{\mu,\nu}_{a_+} f_2)(x)}\, dx < \infty. \tag{4.8}$$

If $\Phi : I \to \mathbb{R}$ is a convex function, then the inequality

$$\int_a^b u(x)\Phi\Big(\frac{(D^{\mu,\nu}_{a_+} f_1)(x)}{(D^{\mu,\nu}_{a_+} f_2)(x)}\Big)\, dx \leq \int_a^b \bar{p}(t)\Phi\Big(\frac{(D^{\mu+\nu(n-\mu)}_{a_+} f_1)(t)}{(D^{\mu+\nu(n-\mu)}_{a_+} f_2)(t)}\Big)\, dt \tag{4.9}$$

holds true for all $f_i \in L^1[a,b]$.

Proof. Applying Theorem 1.4 with $\Omega_1=\Omega_2=(a,b)$, $d\mu_1(x)=dx$, $d\mu_2(t)=dt$, and $\bar{k}(x)$ and $\bar{p}(t)$ given by (4.6) and (4.8), respectively, we obtain inequality (4.9). □

Remark 4.4. If Φ is strictly convex on I and $\frac{(D^{\mu+\nu(n-\mu)}_{a_+} f_1)(x)}{(D^{\mu+\nu(n-\mu)}_{a_+} f_2)(x)}$ is nonconstant, then the inequality given in (4.9) is strict.

The new refined general weighted Hardy-type inequality which has a nonnegative kernel and is related to an arbitrary convex function given in [3] for the generalized fractional integral (4.3) follows in the next theorem.

Theorem 4.5. *Let the fractional derivative operator $D^{\mu,\nu}_{a+}$ be of order $n-1<\mu<n$ and type $0<\nu\leq 1$, and let u be a weight function defined on (a,b). Moreover, if Φ is a convex function on an interval $I\subseteq\mathbb{R}$ and $\varphi : I\to\mathbb{R}$ is any function such that $\varphi(x)\in\partial\Phi(x)$ for all $x\in \operatorname{Int} I$ and $\bar{v}$ as in (4.4), then the inequality*

$$\begin{aligned}
&\int_a^b \bar{v}(t)\Phi\big((D^{\mu+\nu(n-\mu)}_{a_+} f)(t)\big)\, dt - \int_a^b u(x)\Phi\Big(\frac{\Gamma(\nu(n-\mu)+1)(D^{\mu,\nu}_{a_+} f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)\, dx \\
&\geq \nu(n-\mu)\int_a^b \frac{u(x)}{(x-a)^{\nu(n-\mu)}}\int_a^x (x-t)^{\nu(n-\mu)-1} \\
&\quad\times \bigg|\Big|\Phi\big((D^{\mu+\nu(n-\mu)}_{a_+} f)(t)\big) - \Phi\Big(\frac{\Gamma(\nu(n-\mu)+1)(D^{\mu,\nu}_{a_+} f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)\Big| \\
&\quad - \Big|\varphi\Big(\frac{\Gamma(\nu(n-\mu)+1)(D^{\mu,\nu}_{a_+} f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)\Big| \\
&\quad\cdot \Big|(D^{\mu+\nu(n-\mu)}_{a_+} f)(t) - \Big(\frac{\Gamma(\nu(n-\mu)+1)(D^{\mu,\nu}_{a_+} f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)\Big|\bigg|\, dt\, dx
\end{aligned} \tag{4.10}$$

holds for all measurable functions $D^{\mu+\nu(n-\mu)}_{a_+} f : (a,b)\to\mathbb{R}$ such that $(D^{\mu+\nu(n-\mu)}_{a_+} f)(t)\in I$ for all $t\in(a,b)$. If Φ is a monotone convex function on *an interval $I \subseteq \mathbb{R}$, then the inequality*

$$\int_a^b \bar{v}(t)\Phi\big((D_{a_+}^{\mu+\nu(n-\mu)}f)(t)\big)\,dt - \int_a^b u(x)\Phi\Big(\frac{\Gamma(\nu(n-\mu)+1)(D_{a_+}^{\mu,\nu}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)\,dx$$
$$\geq \Big|\nu(n-\mu)\int_a^b \frac{u(x)}{(x-a)^{\nu(n-\mu)}}$$
$$\times \int_a^x \operatorname{sgn}\Big((D_{a_+}^{\mu+\nu(n-\mu)}f)(t) - \frac{\Gamma(\nu(n-\mu)+1)(D_{a_+}^{\mu,\nu}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)(x-t)^{\nu(n-\mu)-1}$$
$$\times \Big[\Phi\big((D_{a_+}^{\mu+\nu(n-\mu)}f)(t)\big) - \Phi\Big(\frac{\Gamma(\nu(n-\mu)+1)(D_{a_+}^{\mu,\nu}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)$$
$$- \Big|\varphi\Big(\frac{\Gamma(\nu(n-\mu)+1)(D_{a_+}^{\mu,\nu}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)\Big|$$
$$\cdot \Big((D_{a_+}^{\mu+\nu(n-\mu)}f)(t) - \frac{\Gamma(\nu(n-\mu)+1)(D_{a_+}^{\mu,\nu}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)\Big]\,dt\,dx\Big| \tag{4.11}$$

holds for all measurable functions $D_{a_+}^{\mu+\nu(n-\mu)}f : (a,b) \to \mathbb{R}$ such that $(D_{a_+}^{\mu+\nu(n-\mu)}f)(t) \in I$ for all fixed $t \in (a,b)$.

Proof. Applying Theorem 1.7 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, and $\bar{k}(x,t)$, $\bar{K}(x)$ given by (4.6) and (4.7), respectively, we get inequalities (4.10) and (4.11). □

The 1-dimensional setting gives refined Hardy- and Pólya–Knopp-type inequalities. In the following theorem, a refinement of a Hardy-type inequality obtained by Kaijser et al. in [11] is given for the generalized fractional derivative operator.

Theorem 4.6. *Let $u : (a,b) \mapsto \mathbb{R}$ be a weight function, let $f \in L^1[a,b]$, and let the fractional derivative operator be $D_{a_+}^{\mu,\nu}$ of order $n-1 < \mu < n$ and type $0 < \nu \leq 1$. Then for each fixed $t \in (a,b)$, define $\bar{w}$ on (a,b) by*

$$\bar{w}(t) = \nu(n-\mu)t\int_t^b u(x)\frac{(x-t)^{\nu(n-\mu)-1}}{(x-a)^{\nu(n-\mu)}}\frac{dx}{x} < \infty,$$

where $\bar{K}(x)$ is given by (4.7) and $a > 0$.

If Φ is a convex function on an interval $I \subseteq \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x)$ for all $x \in \operatorname{Int} I$, then the inequality

$$\int_a^b \bar{w}(t)\Phi\big((D_{a_+}^{\mu+\nu(n-\mu)}f)(t)\big)\frac{dt}{t} - \int_a^b u(x)\Phi\Big(\frac{\Gamma(\nu(n-\mu)+1)(D_{a_+}^{\mu,\nu}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)\frac{dx}{x}$$
$$\geq \nu(n-\mu)\int_a^b \frac{u(x)}{(x-a)^{\nu(n-\mu)}}\int_a^x (x-t)^{\nu(n-\mu)-1}$$
$$\times \Big|\Big|\Phi\big((D_{a_+}^{\mu+\nu(n-\mu)}f)(t)\big) - \Phi\Big(\frac{\Gamma(\nu(n-\mu)+1)(D_{a_+}^{\mu,\nu}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)\Big|$$

$$
- \left|\varphi\Big(\frac{\Gamma(\nu(n-\mu)+1)(D^{\mu,\nu}_{a+}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)\right|
\cdot \left|(D^{\mu+\nu(n-\mu)}_{a+}f)(t) - \frac{\Gamma(\nu(n-\mu)+1)(D^{\mu,\nu}_{a+}f)(x)}{(x-a)^{\nu(n-\mu)}}\right|\Big| \, dt\frac{dx}{x} \tag{4.12}
$$

holds for all measurable functions $D^{\mu+\nu(n-\mu)}_{a+}f : (a,b) \to \mathbb{R}$ *with values in* I. *If the function* Φ *is concave, then the order of the integrals on the left-hand side of (4.12) is reversed. If* Φ *is monotone convex on the interval* $I \subseteq \mathbb{R}$, *then the following inequality*

$$
\begin{aligned}
&\int_a^b \bar{w}(t)\Phi\big((D^{\mu+\nu(n-\mu)}_{a+}f)(t)\big)\frac{dt}{t} \\
&\quad - \int_a^b u(x)\Phi\Big(\frac{\Gamma(\nu(n-\mu)+1)(D^{\mu,\nu}_{a+}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)\frac{dx}{x} \\
&\geq \Big|\nu(n-\mu)\int_a^b \frac{u(x)}{(x-a)^{\nu(n-\mu)}} \\
&\quad \times \int_a^x \operatorname{sgn}\Big((D^{\mu+\nu(n-\mu)}_{a+}f)(t) - \frac{\Gamma(\nu(n-\mu)+1)(D^{\mu,\nu}_{a+}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)(x-t)^{\nu(n-\mu)-1} \\
&\quad \times \Big[\Phi\big((D^{\mu+\nu(n-\mu)}_{a+}f)(t)\big) - \Phi\Big(\frac{\Gamma(\nu(n-\mu)+1)(D^{\mu,\nu}_{a+}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big) \\
&\quad - \left|\varphi\Big(\frac{\Gamma(\nu(n-\mu)+1)(D^{\mu,\nu}_{a+}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)\right| \\
&\quad \cdot \Big((D^{\mu+\nu(n-\mu)}_{a+}f)(t) - \frac{\Gamma(\nu(n-\mu)+1)(D^{\mu,\nu}_{a+}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)\Big] \, dt\frac{dx}{x}\Big|
\end{aligned} \tag{4.13}
$$

holds for all measurable functions $D^{\mu+\nu(n-\mu)}_{a+}f : (a,b) \to \mathbb{R}$ *with values in* I.

Proof. Applying Theorem 1.8 with $(0,b) = (a,b)$, $\bar{k}(x,t)$ given by (4.6) and

$$
\begin{aligned}
(A_k D^{\mu+\nu(n-\mu)}_{a+}f)(x) = {}& \frac{\nu(n-\mu)}{(x-a)^{\nu(n-\mu)}} \\
& \times \int_a^x (x-t)^{\nu(n-\mu)-1}(D^{\mu+\nu(n-\mu)}_{a+}f)(t)\, dt, \quad x \in (a,b),
\end{aligned}
$$

we obtain inequalities (4.12) and (4.13). □

Next we give the mean value theorems [4] for the Hilfer fractional derivative.

Theorem 4.7. *Let* $D^{\mu,\nu}_{a+}$ *be the fractional derivative operator of order* $n-1 < \mu < n$ *and type* $0 < \nu \leq 1$, *let* I *be a compact interval of* $\mathbb{R}$, *let* $\tilde{h} \in C^2(I)$, *and let* $D^{\mu+\nu(n-\mu)}_{a+}f : (a,b) \to \mathbb{R}$ *be a measurable function such that* $\operatorname{Im} D^{\mu+\nu(n-\mu)}_{a+}f \subseteq I$. *Then for the weight function* u *defined on* (a,b) *there exists*

$\eta \in I$ such that

$$\int_a^b \bar{v}(t)\tilde{h}\big((D_{a+}^{\mu+\nu(n-\mu)}f)(t)\big)\,dt - \int_a^b u(x)\tilde{h}\Big(\frac{\Gamma(\nu(n-\mu)+1)(D_{a+}^{\mu,\nu}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)\,dx$$
$$= \frac{\tilde{h}''(\eta)}{2}\Big[\int_a^b \bar{v}(t)(D_{a+}^{\mu+\nu(n-\mu)}f)^2(t)\,dt - \int_a^b u(x)\Big(\frac{\Gamma(\nu(n-\mu)+1)(D_{a+}^{\mu,\nu}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)^2\,dx\Big], \tag{4.14}$$

where $\bar{v}$ is defined by (4.4).

Proof. Applying Theorem 1.9 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, and $\bar{k}(x,t)$ and $\bar{K}(x)$ given by (4.6) and (4.7), respectively, we get equation (4.14). □

Theorem 4.8. *Let the fractional derivative operator be $D_{a+}^{\mu,\nu}$ of order $n-1 < \mu < n$ and type $0 < \nu \le 1$, and let I be a compact interval of $\mathbb{R}$, $k, \tilde{h} \in C^2(I)$ such that $\tilde{h}''(x) \neq 0$ for every $x \in I$. Moreover, $D_{a+}^{\mu+\nu(n-\mu)}f : (a,b) \to \mathbb{R}$ is a measurable function with $\operatorname{Im} D_{a+}^{\mu+\nu(n-\mu)}f \subseteq I$, u is a weight function, $\bar{v}$ is as in (4.4), and*

$$\int_a^b \bar{v}(t)\tilde{h}\big((D_{a+}^{\mu+\nu(n-\mu)}f)(t)\big)\,dt - \int_a^b u(x)\tilde{h}\Big(\frac{\Gamma(\nu(n-\mu)+1)(D_{a+}^{\mu,\nu}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)\,dx \neq 0.$$

Then there exists $\eta \in I$ such that the following equality holds true:

$$\frac{k''(\eta)}{\tilde{h}''(\eta)} = \frac{\int_a^b \bar{v}(t)k((D_{a+}^{\mu+\nu(n-\mu)}f)(t))\,dt - \int_a^b u(x)k\big(\frac{\Gamma(\nu(n-\mu)+1)(D_{a+}^{\mu,\nu}f)(x)}{(x-a)^{\nu(n-\mu)}}\big)\,dx}{\int_a^b \bar{v}(t)\tilde{h}((D_{a+}^{\mu+\nu(n-\mu)}f)(t))\,dt - \int_a^b u(x)\tilde{h}\big(\frac{\Gamma(\nu(n-\mu)+1)(D_{a+}^{\mu,\nu}f)(x)}{(x-a)^{\nu(n-\mu)}}\big)\,dx}.$$

The upcoming result represented in [4] is an application for the Hilfer fractional derivative.

Theorem 4.9. *Let the fractional derivative operator be $D_{a+}^{\mu,\nu}$ of order $n-1 < \mu < n$ and type $0 < \nu \le 1$, let $D_{a+}^{\mu+\nu(n-\mu)}f$ be a positive function, and let u be a weight function defined on (a,b), and let $\bar{v}$ be as in (4.4). Then the function $\xi : \mathbb{R} \to [0,\infty)$ defined by*

$$\xi(s) = \int_a^b \bar{v}(t)\varphi_s\big((D_{a+}^{\mu+\nu(n-\mu)}f)(t)\big)\,dt - \int_a^b u(x)\varphi_s\Big(\frac{\Gamma(\nu(n-\mu)+1)(D_{a+}^{\mu,\nu}f)(x)}{(x-a)^{\nu(n-\mu)}}\Big)\,dx, \tag{4.15}$$

is exponentially convex.

Proof. Applying Theorem 2.3 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, and $\bar{k}(x,t)$ and $\bar{K}(x)$ given by (4.6) and (4.7), respectively, we get the linear functional (4.15). □

Under the assumptions of Theorem 4.2, we define a linear functional by taking the positive difference of the inequality stated in (4.5) as

$$\zeta_1(\Phi) = \int_a^b \bar{v}(t)\Phi\big((D_{a+}^{\mu+\nu(n-\mu)} f)(t)\big)\, dt - \int_a^b \Phi\Big(\frac{\Gamma(\nu(n-\mu)+1)(D_{a+}^{\mu,\nu} f)(x)}{(x-a)^{\nu(n-\mu)}}\Big) u(x)\, dx. \tag{4.16}$$

We also define a linear functional by taking the positive difference of the left-hand side and right-hand side of the inequality (4.9) given in Theorem 4.3 for the Hilfer fractional derivative as

$$\zeta_2(\Phi) = \int_a^b \bar{p}(t)\Phi\Big(\frac{(D_{a+}^{\mu+\nu(n-\mu)} f_1)(t)}{(D_{a+}^{\mu+\nu(n-\mu)} f_2)(t)}\Big)\, dt - \int_a^b u(x)\Phi\Big(\frac{(D_{a+}^{\mu,\nu} f_1)(x)}{(D_{a+}^{\mu,\nu} f_2)(x)}\Big)\, dx, \tag{4.17}$$

where $f_i \in L^1[a,b]$ $(i = 1,2)$.

Theorem 4.10. *Let $\Gamma = \{\Phi_p : p \in J\}$ be a family of functions defined on I such that the function $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is n-exponentially convex in the Jensen sense on J for every three distinct points $z_0, z_1, z_2 \in I$. Let ξ_i $(i = 1,2)$ be linear functionals defined by (4.16) and (4.17), respectively. Then the function $p \mapsto \xi_i(\Phi_p)$ $(i = 1,2)$ is n-exponentially convex in the Jensen sense on J. If the function $p \mapsto \xi_i(\Phi_p)$ is continuous on J, then it is n-exponentially convex on J.*

Proof. Applying Theorem 2.6 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$, and $\bar{k}(x,t)$ and $\bar{K}(x)$ given by (4.6) and (4.7), respectively, we complete the proof. □

Remark 4.11. Similar Hardy-type inequalities can be obtained by using Prabhakar-type integral operators introduced in [5].

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