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ON A GENERALIZED ŠEMRL'S THEOREM FOR WEAK 2-LOCAL DERIVATIONS ON $B(H)$

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Abstract. We prove that, for every complex Hilbert space H, every weak 2-local derivation on $B(H)$ or on $K(H)$ is a linear derivation. We also establish that every weak 2-local derivation on an atomic von Neumann algebra or on a compact C*-algebra is a linear derivation.

1. Introduction

Let $\mathcal{S}$ be a subset of the space $L(X,Y)$ of all linear maps between Banach spaces X and Y. Following [2], [3], and [4], we will say that a (not necessarily linear or continuous) mapping $\Delta : X \to Y$ is a *weak 2-local* $\mathcal{S}$ *map* (resp., a *2-local* $\mathcal{S}$*-map*) if, for each $x, y \in X$ and $\phi \in Y^*$ (resp., for each $x, y \in X$), there exists $T_{x,y,\phi} \in \mathcal{S}$, depending on x, y, and ϕ (resp., $T_{x,y} \in \mathcal{S}$, depending on x and y), satisfying

$$\phi\Delta(x) = \phi T_{x,y,\phi}(x) \qquad \text{and} \qquad \phi\Delta(y) = \phi T_{x,y,\phi}(y)$$

(resp., $\Delta(x) = T_{x,y}(x)$ and $\Delta(y) = T_{x,y}(y)$).

When A is a Banach algebra and $\mathcal{S}$ is the set of derivations (resp., homomorphisms or automorphisms) on A, weak 2-local $\mathcal{S}$ maps on A are called *weak 2-local derivations* (resp., *weak 2-local homomorphisms* or *weak 2-local automorphisms*); *2-local* **-derivations* and *2-local* **-homomorphisms* on C*-algebras are similarly

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defined. We recall that a *-derivation on a C*-algebra A is a derivation $D : A \to A$ satisfying $D(a^*) = D(a)^*$ $(a \in A)$.

The notion of 2-local derivations goes back, formally, to 1997 when Šemrl [10] introduced the definition and proved that, for every infinite-dimensional separable Hilbert space H, every 2-local automorphism (resp., every 2-local derivation) on $B(H)$ is an automorphism (resp., a derivation). Ayupov and Kudaybergenov [2] proved that Šemrl's theorem also holds for arbitrary Hilbert spaces. In 2014, Ayupov and Kudaybergenov proved that every 2-local derivation on a von Neumann algebra is a derivation (see [3]).

The results on weak 2-local maps are even more recent. In a very recent contribution, Niazi and the second author proved the following generalization of the previously mentioned results.

Theorem 1.1 ([6, Theorem 3.10]). *Let H be a separable complex Hilbert space. Then every (not necessarily linear or continuous) weak 2-local *-derivation on $B(H)$ is linear and a *-derivation.*

The same authors proved that for finite-dimensional C*-algebras, the conclusions are stronger.

Theorem 1.2 ([6, Corollary 2.13]). *Every weak 2-local derivation on a finite-dimensional C*-algebra is a linear derivation.*

Let $\Delta : A \to B$ be a mapping between C*-algebras. We consider a new mapping $\Delta^\sharp : A \to B$ given by $\Delta^\sharp(x) := \Delta(x^*)^*$ $(x \in A)$. Obviously, $\Delta^{\sharp\sharp} = \Delta$, and $\Delta(A_{\text{sa}}) \subseteq B_{\text{sa}}$ for every Δ satisfying $\Delta = \Delta^\sharp$, where A_{sa} and B_{sa} denote the self-adjoint parts of A and B, respectively. The mapping Δ is linear if and only if $\Delta^\sharp$ enjoys the same property. The mapping Δ is called *symmetric* if $\Delta^\sharp = \Delta$ (equivalently, $\Delta(x^*) = \Delta(x)^*$, for all $x \in X$). Henceforth, the set of all symmetric linear maps from A into B will be denoted by $\mathcal{S}(A, B)$, weak 2-local $\mathcal{S}(A, B)$ maps between A and B will be called *weak 2-local symmetric maps*, while weak 2-local $L(A, B)$ maps between A and B will be called *weak 2-local linear maps*.

The study of weak 2-local maps has also been pursued in [4], where we obtained that every weak 2-local symmetric map between C*-algebras is linear (see [4, Theorem 2.5]). Among the consequences of this result, we also established that every weak 2-local *-derivation on a general C*-algebra is a (linear) *-derivation (see [4, Corollary 2.10]).

One of the main unsolved problems in this area reads as follows.

Problem 1.3. *Is every weak 2-local derivation on a general C*-algebra A a derivation?*

We will justify later that every weak 2-local derivation Δ on A writes as a linear combination $\Delta = \Delta_1 + \Delta_2$, where $\Delta_1 = \frac{\Delta+\Delta^\sharp}{2}$ and $\Delta_2 = \frac{\Delta-\Delta^\sharp}{2i}$ are weak 2-local derivations and symmetric maps. Thus, we will deduce that Problem 1.3 is equivalent to the following question.

Problem 1.4. *Let $\Delta : A \to A$ be a weak 2-local derivation on a C*-algebra which is also a symmetric map (i.e., $\Delta^\sharp = \Delta$). Is Δ a linear derivation?*

The above problems are questions that naturally arise in an attempt to generalize the abovementioned results by Šemrl [10] and Ayupov and Kudaybergenov [2], [3]. Both remain open, even in the intriguing case of $A = B(H)$.

In this article, we provide a complete, positive answer to both problems in several cases. In Theorem 3.1, we prove that every weak 2-local derivation on $A = B(H)$ is a linear derivation. This generalizes the results in [2], [3], [6], and [10]. We also establish that this *weak 2-local stability* of derivations is also true when A coincides with $K(H)$ (see Theorem 3.2), when A is an atomic von Neumann algebra (see Corollary 3.5), and when A is a compact C*-algebra (see Corollary 3.6).

The techniques and arguments provided in the present article are completely new compared with those in previous works. Our approach is divided into two main sections. In Section 2, we establish a certain boundedness principle showing that, for each weak 2-local derivation Δ on $B(H)$, or on $K(H)$, the mappings $a \mapsto p_F\Delta(p_F a p_F)p_F$ are uniformly bounded when p_F runs in the set of all finite-rank projections on H (see Theorems 2.15 and 2.17). In Section 3, we derive the main results of the article from an identity principle, which assures us that a weak 2-local derivation Δ on $B(H)$ with $\Delta^\sharp = \Delta$ coincides with a *-derivation D if and only if they coincide on every finite-rank projection in $B(H)$ (see Theorem 2.9).

2. Boundedness of weak 2-local derivations on the lattice of projections in $B(H)$

We recall some basic properties of weak 2-local maps which have been borrowed from [4] and [7].

Lemma 2.1 ([4, Lemma 2.1], [7, Lemma 2.1]). *Let X and Y be Banach spaces, and let $\mathcal{S}$ be a subset of the space $L(X,Y)$. Then the following properties hold.*

(a) *Every weak 2-local $\mathcal{S}$ map $\Delta : X \to Y$ is 1-homogeneous, that is, $\Delta(\lambda x) = \lambda\Delta(x)$, for every $x \in X$, $\lambda \in \mathbb{C}$.*

(b) *Suppose that there exists $C > 0$ such that every linear map $T \in \mathcal{S}$ is continuous with $\|T\| \leq C$. Then every weak 2-local $\mathcal{S}$ map $\Delta : X \to Y$ is C-Lipschitzian; that is, $\|\Delta(x) - \Delta(y)\| \leq C\|x - y\|$, for every $x, y \in X$.*

(c) *If $\mathcal{S}$ is a (real) linear subspace of $L(X,Y)$, then every (real) linear combination of weak 2-local $\mathcal{S}$ maps is a weak 2-local $\mathcal{S}$ map.*

(d) *Suppose that A and B are C*-algebras and that $\mathcal{S}$ is a real linear subspace of $L(A,B)$. If a mapping $\Delta : A \to B$ is a weak 2-local $\mathcal{S}$ map, then for each $\varphi \in B^*_{\mathrm{sa}}$ and every $x, y \in A$ there exists $T_{x,y,\varphi} \in \mathcal{S}$ satisfying $\varphi\Delta(x) = \varphi T_{x,y,\varphi}(x)$ and $\varphi\Delta(y) = \varphi T_{x,y,\varphi}(y)$.*

(e) *Suppose that A and B are C*-algebras and that $\mathcal{S}$ is a real linear subspace of $L(A,B)$ with $\mathcal{S}^\sharp = \mathcal{S}$ (in particular, when $\mathcal{S} = \mathcal{S}(A,B)$ is the set of all symmetric linear maps from A into B). Then a mapping $\Delta : A \to B$ is a weak 2-local $\mathcal{S}$ map if and only if $\Delta^\sharp$ is a weak 2-local $\mathcal{S}$ map.*

Henceforth, H will denote an arbitrary complex Hilbert space. The symbols $B(H)$ and $K(H)$ will denote the C*-algebras of all bounded and compact linear operators on H, respectively. If H is finite-dimensional, then every weak 2-local derivation on $B(H)$ is a linear derivation (see Theorem 1.2). We may therefore assume that H is infinite-dimensional.

Following standard notation, an element x in a C*-algebra A is said to be *finite* (resp., *compact*) in A if the wedge operator $x \wedge x : A \to A$, given by $x \wedge x(a) = xax$, is a finite-rank (resp., compact) operator on A. It is known that the ideal $\mathcal{F}(A)$ of finite elements in A coincides with $\mathrm{Soc}(A)$, the *socle* of A (i.e., the sum of all minimal right (equivalently left) ideals of A) and that $\mathcal{K}(A) = \overline{\mathrm{Soc}(A)}$ is the ideal of compact elements in A. Moreover, if H is a Hilbert space, then $\mathcal{F}(\mathcal{L}(H)) = \mathcal{F}(H)$ and $\mathcal{K}(\mathcal{L}(H)) = \mathcal{K}(H)$ are the ideals of finite-rank and compact elements in $B(H)$, respectively.

Suppose that $\Delta : B(H) \to B(H)$ is a weak 2-local derivation. By [6, Lemma 3.4], we know that $\Delta(K(H)) \subseteq K(H)$ and that $\Delta|_{K(H)} : K(H) \to K(H)$ is a weak 2-local derivation. Proposition 3.1 in [6] proves that $\Delta(a+b) = \Delta(a)+\Delta(b)$, for every $a, b \in \mathcal{F}(H)$.

Lemma 2.2 ([6, Lemma 3.4, Proposition 3.1])**.** *Let $\Delta : B(H) \to B(H)$ be a weak 2-local derivation. Then $\Delta|_{\mathcal{F}(H)} : \mathcal{F}(H) \to B(H)$ is linear.*

Let us revisit some basic facts on commutators. We recall that every derivation on a C*-algebra is continuous (see [9, Lemma 4.1.3]). A celebrated result of Sakai establishes that every derivation on a von Neumann algebra M is inner; that is, if $D : M \to M$ is a derivation, then there exists $z \in M$ such that $D(x) = [z, x] = zx - xz$ for every $x \in M$ (see [9, Theorem 4.1.6]). The element z given by Sakai's theorem is not unique; however, we can choose z satisfying $\|z\| \leq \|D\|$.

Let us consider two elements z, w in a C*-algebra A such that the derivations $[z, \cdot]$ and $[w, \cdot]$ coincide as linear maps on A. Since $[z, x] = [w, x]$ for every $x \in A$, we deduce that $z-w$ lies in the center of A. The reciprocal statement is also true; therefore, $[z, \cdot] = [w, \cdot]$ on A if and only if $z - w$ lies in the center, $Z(A)$, of A. It is known that a derivation of the form $[z, \cdot]$ is symmetric (i.e., a *-derivation) if and only if $z = w + c$, where $w = -w^*$ and where c lies in the center of A.

From now on, the set of all finite-dimensional subspaces of H will be denoted by $\mathfrak{F}(H)$. We consider in $\mathfrak{F}(H)$ the natural order given by inclusion. For each $F \in \mathfrak{F}(H)$, p_F will denote the orthogonal projection of H onto F.

Lemma 2.3. *Let $\Delta : B(H) \to B(H)$ be a weak 2-local derivation. For each $F \in \mathfrak{F}(H)$ there exists $z_F \in p_F B(H) p_F$ satisfying*

$$p_F \Delta(p_F a p_F) p_F = [z_F, p_F a p_F],$$

for every $a \in B(H)$. If Δ is symmetric (i.e., $\Delta^\sharp = \Delta$), then we can choose $z_F \in p_F B(H) p_F$ satisfying $z_F = -z_F^*$.

Proof. Let F be a finite-dimensional subspace of H. By [7, Proposition 2.7], the mapping $p_F \Delta p_F|_{p_F B(H) p_F} : p_F B(H) p_F \to p_F B(H) p_F$, $a \mapsto p_F \Delta(p_F a p_F) p_F$ is a weak 2-local derivation. Bearing in mind that $p_F B(H) p_F$ is a finite-dimensional C*-algebra, we deduce from Theorem 1.2 that $p_F \Delta p_F|_{p_F B(H) p_F}$ is a linear derivation. By Sakai's theorem, there exists $z_F \in p_F B(H) p_F$ satisfying the desired conclusion.

If Δ is symmetric, we can easily check that $p_F\Delta p_F|_{p_F B(H)p_F}$ also is symmetric, and hence a *-derivation on $p_F B(H) p_F$. In this case, we can obviously replace z_F with $\frac{z_F - z_F^*}{2}$ to get the final statement in the lemma. $\square$

Remark 2.4. Let $\Delta : B(H) \to B(H)$ be a weak 2-local derivation with $\Delta^\sharp = \Delta$, and let F be a subspace in $\mathfrak{F}(H)$. It is clear that the element z_F given by Lemma 2.3 above is not unique. We can consider the set

$$[z_F] := \big\{ z \in p_F B(H) p_F : z^* = -z \text{ and } p_F \Delta p_F|_{p_F B(H) p_F} = [z, \cdot] \big\}.$$

Given $z_1, z_2 \in [z_F]$, it follows that $z_1 - z_2 \in Z(p_F B(H) p_F) = \mathbb{C} p_F$, and since $(z_1 - z_2)^* = -(z_1 - z_2)$, it follows that there exists $\lambda \in \mathbb{R}$ such that $z_2 = z_1 + i\lambda p_F$. It is easy to check that there exists a unique $\widetilde{z}_F \in [z_F]$ satisfying

$$\|\widetilde{z}_F\| = \min\big\{ \|z\| : z \in [z_F] \big\}.$$

From now on, given an element a in a C*-algebra A, the spectrum of a will be denoted by $\sigma(a)$. Our next remark gathers some information about the norm of an inner *-derivation on $B(H)$.

Remark 2.5. Let z be an element in $B(H)$. Stampfli [11, Theorem 4] proves that

$$\big\| [z, \cdot] \big\| = \inf_{\lambda \in \mathbb{C}} \| z - \lambda \operatorname{Id}_H \|,$$

where $\|[z, \cdot]\|$ denotes the norm of the inner derivation $[z, \cdot]$ in $B(B(H))$.

For a compact subset $K \subset \mathbb{C}$, the radius, $\rho(K)$, of K is the radius of the smallest disk containing K. In general, two times the radius of a compact set K does not coincide with its diameter. Actually, $2\rho(K) \geq \operatorname{diam}(K)$. However, when $K \subset \mathbb{R}$ or $K \subset i\mathbb{R}$, we can easily see that $2\rho(K) = \operatorname{diam}(K)$.

When z is a normal operator in $B(H)$, we further know that $\|[z, \cdot]\| = 2\rho(\sigma(z))$ (see [11, Corollary 1]). In particular, for each z in $B(H)$ with $z = z^*$ or $z = -z^*$, we have

$$\big\| [z, \cdot] \big\| = 2\rho\big(\sigma(z)\big) = \operatorname{diam}\big(\sigma(z)\big) \leq 2\|z\|. \tag{1}$$

Let us observe that if $0 \in \sigma(z)$, then $\|z\| \leq \operatorname{diam}(\sigma(z))$ for every $z = \pm z^*$.

Given a projection p in a unital C*-algebra A, we will denote by $p^\perp$ the projection $1 - p$.

Lemma 2.6. *Let $\Delta : B(H) \to B(H)$ be a weak 2-local derivation with $\Delta^\sharp = \Delta$. Suppose that F_1 and F_2 are finite-dimensional subspaces of H with $F_1 \subseteq F_2$. We employ the notation given in Remark 2.4. Then for each $z_1 \in [z_{F_1}]$ and each $z_2 \in [z_{F_2}]$, we have*

$$[z_1, \cdot] = [p_{F_1} z_2 p_{F_1}, \cdot]$$

as operators on $p_{F_1} B(H) p_{F_1}$. Consequently, there exists a real λ (depending on z_1 and z_2) such that $z_1 + i\lambda p_{F_1} = p_{F_1} z_2 p_{F_1}$. In particular, $\operatorname{diam}(\sigma(z_1)) \leq \operatorname{diam}(\sigma(z_2))$ *and* $\operatorname{diam}(\sigma(z_1)) = \operatorname{diam}(\sigma(z'_1))$ *for every $z_1, z_1' \in [z_{F_1}]$.*

Proof. By Lemma 2.3 and Remark 2.4, we have

$$p_{F_1}\Delta(p_{F_1}ap_{F_1})p_{F_1} = [z_1, p_{F_1}ap_{F_1}]$$

and

$$p_{F_2}\Delta(p_{F_2}ap_{F_2})p_{F_2} = [z_2, p_{F_2}ap_{F_2}],$$

for every $a \in B(H)$. Since $p_{F_1} \leq p_{F_2}$, it follows that

$$\begin{aligned}[p_{F_1}z_2p_{F_1}, p_{F_1}ap_{F_1}] &= p_{F_1}[z_2, p_{F_1}ap_{F_1}]p_{F_1} = p_{F_1}p_{F_2}\Delta(p_{F_1}ap_{F_1})p_{F_2}p_{F_1} \\ &= p_{F_1}\Delta(p_{F_1}ap_{F_1})p_{F_1} = [z_1, p_{F_1}ap_{F_1}]\end{aligned}$$

for every $a \in B(H)$, which proves the first statement in the lemma. Since $Z(p_{F_1}B(H)p_{F_1}) = \mathbb{C}p_{F_1}$, $z_1^* = -z_1$, and $z_2^* = -z_2$, there exists $\lambda \in \mathbb{R}$ such that $z_1 + i\lambda p_{F_1} = p_{F_1}z_2p_{F_1}$ (see Remark 2.4). By Remark 2.5, we have

$$\begin{aligned}\operatorname{diam}\big(\sigma(z_1)\big) &= \big\|[z_1, \cdot]\big\| = \big\|[p_{F_1}z_2p_{F_1}, \cdot]|_{(p_{F_1}B(H)p_{F_1})}\big\| \\ &= \big\|p_{F_1}[z_2, p_{F_1}\cdot p_{F_1}]p_{F_1}\big\| \leq \big\|[z_2, \cdot]\big\| = \operatorname{diam}\big(\sigma(z_2)\big). \qquad \square\end{aligned}$$

Proposition 2.7. *Let $\Delta : B(H) \to B(H)$ be a weak 2-local derivation with $\Delta^\sharp = \Delta$. Suppose that the set $\mathcal{D}\mathrm{iam}(\Delta) = \{\operatorname{diam}(\sigma(w_F)) : w_F \in [z_F], F \in \mathfrak{F}(H)\}$ is unbounded. Then for each $G \in \mathfrak{F}(H)$, the set*

$$\mathcal{D}\mathrm{iam}_G^+ = \big\{\operatorname{diam}\big(\sigma(w_F)\big) : w_F \in [z_F], F \in \mathfrak{F}(H), F \supseteq G\big\}$$

is unbounded.

Proof. Let us fix an arbitrary $G \in \mathfrak{F}(H)$. For each $F \in \mathfrak{F}(H)$, we can find $K \in \mathfrak{F}(H)$ with $G, F \subseteq K$. Applying Lemma 2.6, we have

$$\operatorname{diam}\big(\sigma(w_F)\big), \operatorname{diam}\big(\sigma(w_G)\big) \leq \operatorname{diam}\big(\sigma(w_K)\big),$$

for every $w_F \in [z_F]$, $w_G \in [z_G]$, and $w_K \in [z_K]$. The unboundedness of $\mathcal{D}\mathrm{iam}$ implies the same property for $\mathcal{D}\mathrm{iam}_G^+$. $\square$

2.1. **An identity principle for weak 2-local derivations.** Let $\Delta : B(H) \to B(H)$ be a weak 2-local derivation with $\Delta^\sharp = \Delta$. Suppose that there exists $G \in \mathfrak{F}(H)$ such that the set

$$\mathcal{D}\mathrm{iam}_{G^\perp}^- = \big\{\operatorname{diam}\big(\sigma(w_F)\big) : w_F \in [z_F], F \in \mathfrak{F}(H), p_F \leq p_G^\perp\big\}$$

is bounded. For each $F \in \mathfrak{F}(H)$ with $p_F \leq p_G^\perp$, the element $\widetilde{z}_F$ has been chosen to satisfy

$$\|\widetilde{z}_F\| \leq \operatorname{diam}\big(\sigma(\widetilde{z}_F)\big) \leq 2\|\widetilde{z}_F\|.$$

Therefore, the net $(\widetilde{z}_F)_{F\in\mathfrak{F}(H), p_F\leq p_G^\perp}$ is bounded in $p_G^\perp B(H)p_G^\perp$. By Alaoglu's theorem, we can find $z_0 \in p_G^\perp B(H)p_G^\perp$ with $z_0 = -z_0^*$ and a subnet $(\widetilde{z}_F)_{F\in\Lambda}$ converging to z_0 in the weak*-topology of $p_G^\perp B(H)p_G^\perp$.

If the set $\mathcal{D}\mathrm{iam}(\Delta) = \mathcal{D}\mathrm{iam}_{\{0\}^\perp}^- = \{\operatorname{diam}(\sigma(w_F)) : w_F \in [z_F], F \in \mathfrak{F}(H)\}$ is bounded, we can similarly define, via Alaoglu's theorem, an element $z_0 = -z_0^* \in B(H)$ which is the weak*-limit of a convenient subnet of $(\widetilde{z}_F)_{F\in\mathfrak{F}(H)}$.

Proposition 2.8. *Let $\Delta : B(H) \to B(H)$ be a weak 2-local derivation with $\Delta^\sharp = \Delta$. Suppose that there exists $G \in \mathfrak{F}(H)$ such that the set*

$$\mathcal{D}\mathrm{iam}^-_{G^\perp} = \big\{\mathrm{diam}\big(\sigma(w_F)\big) : w_F \in [z_F], F \in \mathfrak{F}(H), p_F \leq p_G^\perp\big\}$$

is bounded, and let $z_0 \in p_G^\perp B(H) p_G^\perp$ ($z_0 = -z_0^*$) be the element determined in the preceding paragraph. Then $p_G^\perp \Delta(p) p_G^\perp = [z_0, p]$, for every projection $p \in \mathcal{F}(H)$ with $p \leq p_G^\perp$. If the set $\mathcal{D}\mathrm{iam}(\Delta) = \mathcal{D}\mathrm{iam}^-_{\{0\}^\perp} = \{\mathrm{diam}(\sigma(w_F)) : w_F \in [z_F], F \in \mathfrak{F}(H)\}$ is bounded, then $\Delta(p) = [z_0, p]$ for every projection $p \in \mathcal{F}(H)$.

Proof. Let us fix a finite-rank projection $p \in \mathcal{F}(H)$ with $p \leq p_G^\perp$. Since $(\widetilde{z}_F)_{F\in\Lambda}$ converges to z_0 in the weak*-topology of $p_G^\perp B(H) p_G^\perp$, where $(\widetilde{z}_F)_{F\in\Lambda}$ is the subnet fixed before Proposition 2.8, there exists $F_0 \in \Lambda$ such that $p \leq p_{F_0}$ (we observe that, under these hypotheses, there exists a monotone final function $h : \mathfrak{F}(H) \to \Lambda$ which defines the subnet). The subnet $(\widetilde{z}_F)_{F_0\subseteq F\in\Lambda}$ converges to z_0 in the weak*-topology of $B(H)$.

Clearly, the net $(p_F)_{F\in\mathfrak{F}(H)}$ converges to the projection $p_G^\perp$ in the strong*-topology of $B(H)$. Therefore, the subnet $(p_F)_{F_0\subseteq F\in\Lambda} \to p_G^\perp$ in the strong*-topology of $B(H)$. Since for each $F \in \Lambda$ with $F_0 \subseteq F$ we have $p \leq p_{F_0} \leq p_F$, we deduce, via [6, Lemma 3.2], Lemma 2.3, and Remark 2.4, that

$$p_F \Delta(p) p_F = p_F \Delta(p_F p p_F) p_F = p_F[\widetilde{z}_F, p_F p p_F] p_F = [\widetilde{z}_F, p]. \tag{2}$$

It is known that the product of every von Neumann algebra is jointly strong*-continuous on bounded sets (see [9, Proposition 1.8.12]), and we thus deduce that the net $(p_F \Delta(p) p_F)_{F_0\subseteq F\in\Lambda} \to p_G^\perp \Delta(p) p_G^\perp$ in the strong*-topology of $B(H)$, and hence $(p_F \Delta(p) p_F)_{F_0\subseteq F\in\Lambda} \to p_G^\perp \Delta(p) p_G^\perp$ also in the weak*-topology (see [9, Theorem 1.8.9]). This shows that the left-hand side in (2) converges to $p_G^\perp \Delta(p) p_G^\perp$ in the weak*-topology of $B(H)$.

Finally, the separate weak*-continuity of the product of $B(H)$ (see [9, Theorem 1.7.8]) shows that the right-hand side in (2) converges to $p_G^\perp [z_0, p] p_G^\perp = [z_0, p]$ in the weak*-topology. Therefore, $p_G^\perp \Delta(p) p_G^\perp = [z_0, p]$, as we desired. The second statement follows from the same arguments. □

We can state now an identity principle for weak 2-local derivations on $B(H)$.

Theorem 2.9. *Let $\Delta : B(H) \to B(H)$ be a weak 2-local derivation with $\Delta^\sharp = \Delta$. Let $p_0 \in \mathcal{F}(H)$ be a finite-rank projection. Suppose that z_0 is a skew-symmetric element in $(1-p_0)B(H)(1-p_0)$ such that $(1-p_0)\Delta(p)(1-p_0) = [z_0, p]$ for every finite-rank projection $p \in (1-p_0)B(H)(1-p_0)$. Then*

$$(1-p_0)\Delta\big((1-p_0)a(1-p_0)\big)(1-p_0) = \big[z_0, (1-p_0)a(1-p_0)\big],$$

for every $a \in B(H)$. If in addition $p_0 = 0$, then $\Delta = [z_0, \cdot]$ is a linear derivation on $B(H)$.

Proof. Let $D : B(H) \to B(H)$ denote the *-derivation defined by $D(a) = [z_0, a]$ ($a \in B(H)$). Lemma 2.2 (see also [6, Lemma 3.4, Proposition 3.1]) assures us that $\Delta|_{\mathcal{F}(H)} : \mathcal{F}(H) \to \mathcal{F}(H)$ is a linear mapping. Since every element in $\mathcal{F}(H)$

can be written as a finite linear combination of finite-rank projections in $B(H)$, it follows from our hypothesis that

$$\begin{aligned}(1-p_0)\Delta(1-p_0)|_{(1-p_0)\mathcal{F}(H)(1-p_0)} &= D|_{(1-p_0)\mathcal{F}(H)(1-p_0)}\\ &= [z_0,\cdot]|_{(1-p_0)\mathcal{F}(H)(1-p_0)}.\end{aligned} \tag{3}$$

Fix a in $(1-p_0)B(H)(1-p_0)$ and fix a finite-rank projection $p_1 \leq 1-p_0$. Having in mind that $p_1ap_1 + p_1ap_1^{\perp} + p_1^{\perp}ap_1 \in (1-p_0)\mathcal{F}(H)(1-p_0)$, and also recalling [6, Lemma 3.2] as well as (3), we conclude that

$$\begin{aligned}p_1\Delta(a)p_1 &= p_1\Delta(p_1ap_1 + p_1ap_1^{\perp} + p_1^{\perp}ap_1)p_1\\ &= p_1\big[z_0, (p_1ap_1 + p_1ap_1^{\perp} + p_1^{\perp}ap_1)\big]p_1.\end{aligned} \tag{4}$$

The net $(p_F)_{\substack{F\in\mathfrak{F}(H)\\ p_F\leq 1-p_0}}$ converges to $1-p_0$ in the strong*-topology of $B(H)$. We deduce from (4) that

$$p_F\Delta(a)p_F = p_F[z_0, p_Fa + p_F^{\perp}ap_F]p_F,$$

for every $F \in \mathfrak{F}(H)$ with $p_F \leq 1-p_0$. Taking strong*-limits in the above identity, it follows from the joint strong*-continuity of the product in $B(H)$ that

$$(1-p_0)\Delta(a)(1-p_0) = [z_0, a],$$

which finishes the proof. □

Our next result is a consequence of Proposition 2.8 and Theorem 2.9.

Corollary 2.10. *Let $\Delta : B(H) \to B(H)$ be a weak 2-local derivation with $\Delta^{\sharp} = \Delta$. Suppose that one of the following statements holds.*

(a) *The set $\mathcal{D}\mathrm{iam}(\Delta) = \{\mathrm{diam}(\sigma(w_F)) : w_F \in [z_F], F \in \mathfrak{F}(H)\}$ is bounded.*

(b) *The set $\{\|p_F\Delta p_F|_{p_FB(H)p_F}\| : F \in \mathfrak{F}(H)\}$ is bounded.*

Then Δ is a linear derivation.

Proof. If Δ satisfies (a), then the conclusion follows straightforwardly from Proposition 2.8 and Theorem 2.9. If we assume (b), we simply observe that for each $F \in \mathfrak{F}(H)$ we have

$$\|\widetilde{z}_F\| \leq \mathrm{diam}\big(\sigma(\widetilde{z}_F)\big) = \big\|[\widetilde{z}_F,\cdot]|_{p_FB(H)p_F}\big\| = \|p_F\Delta p_F|_{p_FB(H)p_F}\| \leq 2\|\widetilde{z}_F\|$$

(see Remarks 2.4 and 2.5). □

The following lemma states a simple property of derivations on M_n. The proof is left to the reader.

Lemma 2.11. *Let $D : M_n \to M_n$ be a *-derivation. Suppose that p_1 is a rank 1 projection in M_n. If $D(a) = 0$ for every $a = p_1^{\perp}ap_1^{\perp}$ in M_n, then there exists $\alpha \in i\mathbb{R}$ such that $D(x) = [\alpha p_1, x]$ for all $x \in M_n$.*

We state now an infinite-dimensional analogue of this lemma.

Proposition 2.12. *Let $\Delta : B(H) \to B(H)$ be a weak 2-local derivation with $\Delta^{\sharp} = \Delta$. Suppose that p_0 is a rank 1 projection in $B(H)$ such that $\Delta(a) = 0$ for every $a = p_0^{\perp}ap_0^{\perp}$ in $B(H)$. Then there exists $\alpha \in i\mathbb{R}$ such that $\Delta(x) = [\alpha p_0, x]$ for all $x \in B(H)$.*

Proof. Take a finite-rank projection $p \leq p_0^\perp$. Since

$$\Delta_p = (p+p_0)\Delta(p_0+p)|_{(p_0+p)B(H)(p_0+p)} : (p_0+p)B(H)(p_0+p) \to (p_0+p)B(H)(p_0+p)$$

is a weak 2-local derivation with $\Delta_p^\sharp = \Delta_p$ (see [7, Proposition 2.7]) and $(p_0 + p)B(H)(p_0+p) \cong M_m$ for a suitable m, we deduce from [6, Theorem 2.12] that Δ_p is a *-derivation. We also know that $\Delta_p(a) = 0$ for every $a \in (p_0+p)B(H)(p_0+p)$ with $a = pap$. Lemma 2.11 implies the existence of $\alpha(p) \in i\mathbb{R}$, depending on p, such that $\Delta_p(x) = [\alpha(p)p_0, x]$ for all $x \in (p_0 + p)B(H)(p_0 + p)$.

We claim that $\alpha(p)$ does not depend on p. Indeed, let p_1, p_2 be finite-rank projections with $p_j \leq p_0^\perp$. We can find a third finite-rank projection $p_3 \leq p_0^\perp$ such that $p_1, p_2 \leq p_3$. We know that $\Delta_{p_j}(x) = [\alpha(p_j)p_0, x]$ holds for all cases of $x \in (p_0 + p_j)B(H)(p_0 + p_j)$ for all $j = 1, 2, 3$. Since for each $j = 1, 2$,

$$(p_0 + p_j)\Delta_{p_3}(p_0 + p_j)|_{(p_0+p_j)B(H)(p_0+p_j)} = \Delta_{p_j},$$

we can easily see that $\alpha(p_j) = \alpha(p_3)$ for every $j = 1, 2$, which proves the claim. Therefore, there exists $\alpha \in i\mathbb{R}$ such that

$$(p + p_0)\Delta(x)(p_0 + p) = [\alpha p_0, x] \tag{5}$$

for all $x \in (p_0 + p)B(H)(p_0 + p)$ and every finite-rank projection $p \leq p_0^\perp$.

Let us fix $F \in \mathfrak{F}(H)$. We can find another finite-rank projection $p_1 \leq p_0^\perp$ such that $p_F \leq p_0+p_1$. We have shown that $\Delta_{p_1} = (p_0+p_1)\Delta(p_0+p_1)|_{(p_0+p_1)B(H)(p_0+p_1)} = [\alpha p_0, \cdot]|_{(p_0+p_1)B(H)(p_0+p_1)}$, and hence that $\|\Delta_{p_1}\| \leq 2|\alpha|$. Since $p_F\Delta_{p_1}p_F|_{p_FB(H)p_F} = p_F\Delta p_F|_{p_FB(H)p_F}$, we can also conclude that

$$\|p_F\Delta p_F|_{p_FB(H)p_F}\| \leq 2|\alpha|,$$

for every $F \in \mathfrak{F}(H)$. Corollary 2.10 implies that Δ is a linear *-derivation. The continuity and the linearity of Δ, combined with (5), give the desired statement. □

Theorem 2.13. *Let $\Delta : B(H) \to B(H)$ be a weak 2-local derivation with $\Delta^\sharp = \Delta$. Suppose that there exists $G \in \mathfrak{F}(H)$ such that the set*

$$\mathcal{D}\mathrm{iam}^-_{G^\perp} = \left\{\mathrm{diam}\big(\sigma(w_F)\big) : w_F \in [z_F], F \in \mathfrak{F}(H), p_F \leq p_G^\perp\right\}$$

is bounded. Then Δ is a linear *-derivation.

Proof. Combining Proposition 2.8 and Theorem 2.9, we deduce the existence of $z_0 = -z_0^*$ in $(1 - p_G)B(H)(1 - p_G)$ such that

$$(1 - p_G)\Delta(a)(1 - p_G) = \big[z_0, (1 - p_G)a(1 - p_G)\big],$$

for every $a \in B(H)$. The mapping $\Delta_1 = \Delta - [z_0, .]$ is a weak 2-local derivation on $B(H)$ with $\Delta_1 = \Delta_1^\sharp$, and it satisfies

$$(1 - p_G)\Delta_1(a)(1 - p_G) = 0 \tag{6}$$

for all $a \in (1 - p_G)B(H)(1 - p_G)$.

Let $q_1, \ldots, q_m$ be mutually orthogonal rank 1 projections such that $p_G = q_1 + \cdots + q_m$.

Let $\{\xi_j : j \in J\}$ be an orthonormal basis of $p_G^\perp(H)$. For each $j \in J$, we denote by p_j the rank projection corresponding to the orthogonal projection of H onto $\mathbb{C}\xi_j$. By Proposition 2.7 in [7], the mapping $(q_m+p_j)\Delta_1(q_m+p_j)|_{(q_m+p_j)B(H)(q_m+p_j)}$ is a linear *-derivation on $(q_m+p_j)B(H)(q_m+p_j)$. Therefore, there exists $z_j = \begin{pmatrix} \alpha_{00}^j & \alpha_{0j}^j \\ -\overline{\alpha_{0j}^j} & \alpha_{jj}^j \end{pmatrix} = -z_j^* \in M_2(\mathbb{C})$ such that $(q_m+p_j)\Delta_1(q_m+p_j)(a) = [z_j, a]$ for every $a \in (q_m+p_j)B(H)(q_m+p_j)$. We deduce from (6) that $\alpha_{jj}^j = 0$ (for every j). We have thus defined a family $(\alpha_{0j}^j) \subset \mathbb{C}$.

The same arguments given above show, via [7, Proposition 2.7] and (6), that for each finite subset $J_0 \subset J$, with $k_0 = \sharp J_0$, and $p_{J_0} = \sum_{j\in J_0} p_j$ that

$$(q_m + p_{J_0})\Delta_1(a)(q_m + p_{J_0}) = [z_{J_0}, a] \tag{7}$$

for all $a \in (q_m+p_{J_0})B(H)(q_m+p_{J_0})$, where z_{J_0} identifies with the $(k_0+1)\times(k_0+1)$ skew-symmetric matrix given by $z_{J_0} = \alpha_{00}q_{k_0} + \sum_{j\in J_0} \alpha_{0j}^j e_{0j} - \overline{\alpha_{0j}^j} e_{0j}^*$, where e_{0j} is the unique minimal partial isometry satisfying $e_{0j}e_{0j}^* = q_m$ and $e_{0j}^*e_{0j} = p_j$, and α_{00} is a suitable complex number.

We claim that the family $\sum_{j\in J} |\alpha_{0j}^j|^2$ is summable. Indeed, for each finite subset $J_0 \subset J$, we can show from (7) and [6, Lemma 3.2] that

$$\sum_{j\in J_0} \alpha_{0j}^j e_{0j} + \overline{\alpha_{0j}^j} e_{0j}^* = (q_m + p_{J_0})\Delta_1(p_{J_0})(q_m + p_{J_0}) = (q_m + p_{J_0})\Delta_1(p_G^\perp)(q_m + p_{J_0}),$$

and hence that

$$\sum_{j\in J_0} |\alpha_{0j}^j|^2 = \big\|(q_m + p_{J_0})\Delta_1(p_{J_0})(q_m + p_{J_0})\big\|^2 \leq \big\|\Delta_1(p_G^\perp)\big\|^2,$$

which assures the boundedness of the set $\{\sum_{j\in J} |\alpha_{0j}^j|^2 : J_0 \subset J \text{ finite}\}$ and proves the claim.

Thanks to the claim, the element $z_1 = \sum_{j\in J} \alpha_{0j}^j e_{0j} - \overline{\alpha_{0j}^j} e_{0j}^*$ is a well-defined skew-symmetric element in $B(H)$. We further know, from (7), that

$$(q_m + p_{J_0})\Delta_1(a)(q_m + p_{J_0}) = (q_m + p_{J_0})[z_1, a](q_m + p_{J_0}) \tag{8}$$

for every finite subset $J_0 \subset J$, $p_{J_0} = \sum_{j\in J_0} p_j$, and every element a in $p_{J_0}B(H)p_{J_0}$. In the case in which $a = p_{J_0}$, we get

$$(q_m + p_{J_0})\Delta_1(p_{J_0})(q_m + p_{J_0}) = (q_m + p_{J_0})[z_1, p_{J_0}](q_m + p_{J_0}).$$

Lemma 3.2 in [6] implies that

$$\begin{aligned}(q_m + p_{J_0})\Delta_1(p_G^\perp)(q_m + p_{J_0}) &= (q_m + p_{J_0})\Delta_1\big(p_{J_0} + (p_G^\perp - p_{J_0})\big)(q_m + p_{J_0})\\ &= (q_m + p_{J_0})\Delta_1(p_{J_0})(q_m + p_{J_0})\\ &= (q_m + p_{J_0})[z_1, p_{J_0}](q_m + p_{J_0})\\ &= (q_m + p_{J_0})[z_1, p_G^\perp](q_m + p_{J_0}).\end{aligned}$$

Letting $p_{J_0} \nearrow p_G^\perp$ in the strong*-topology, we get

$$(q_m+p_G^\perp)\Delta_1(p_G^\perp)(q_m+p_G^\perp) = (q_m+p_G^\perp)[z_1, p_G^\perp](q_m+p_G^\perp) = \widehat{z}_1 = \sum_{j\in J} \alpha_{0j}^j e_{0j} + \overline{\alpha_{0j}^j} e_{0j}^*.$$

Clearly, $z_1 = q_m\widehat{z}_1 p_G^\perp - p_G^\perp \widehat{z}_1 q_m$. Let $p \leq p_G^\perp$ be a finite-rank projection. We deduce from the last identity that

$$q_m[z_1,p]p = q_m\widehat{z}_1 p = q_m\Delta_1(p_G^\perp)p = q_m\Delta_1\big(p+(p_G^\perp - p)\big)p = q_m\Delta_1(p)p,$$

where the last equality follows from [6, Lemma 3.2]. We similarly prove $p[z_1, p]q_m = -p\widehat{z}_1 q_m = p\Delta_1(p)q_m$, and hence, by (6),

$$(q_m+p_G^\perp)\Delta_1(p)(q_m+p_G^\perp) = (q_m+p_G^\perp)[z_1,p](q_m+p_G^\perp).$$

Now, [6, Proposition 3.1] shows that Δ_1 is linear on $\mathcal{F}(H)$. We thus deduce from the above that

$$(q_m+p_G^\perp)\Delta_1(a)(q_m+p_G^\perp) = (q_m+p_G^\perp)[z_1,a](q_m+p_G^\perp), \tag{9}$$

for every $a \in p_G^\perp\mathcal{F}(H)p_G^\perp$.

We claim now that

$$(q_m+p_G^\perp)\Delta_1(a)(q_m+p_G^\perp) = (q_m+p_G^\perp)[z_1,a](q_m+p_G^\perp),$$

for every element a in $p_G^\perp B(H)p_G^\perp$. For this purpose, let us fix $a \in p_G^\perp B(H)p_G^\perp$, and a projection p_{J_0}, with J_0 a finite subset of J. Keeping in mind that $(q_m+p_{J_0})a + (q_m+p_{J_0})^\perp a(q_m+p_{J_0}) \in p_G^\perp\mathcal{F}(H)p_G^\perp$, a new application of [6, Lemma 3.2] proves that

$$\begin{aligned}
&(q_m+p_G^\perp)[z_1,a](q_m+p_G^\perp)\\
&\quad= (q_m+p_G^\perp)\big[z_1,(q_m+p_{J_0})a+(q_m+p_{J_0})^\perp a(q_m+p_{J_0})\big](q_m+p_G^\perp)\\
&\quad= (q_m+p_{J_0})\Delta_1\big((q_m+p_{J_0})a+(q_m+p_{J_0})^\perp a(q_m+p_{J_0})\big)(q_m+p_{J_0})\\
&\quad= (q_m+p_{J_0})\Delta_1(a)(q_m+p_{J_0}).
\end{aligned}$$

If in the previous identity we let $p_{J_0} \nearrow p_G^\perp$ in the strong*-topology, we obtain the equality stated in the claim.

The mapping $(q_m+p_G^\perp)\Delta_1(q_m+p_G^\perp)|_{(q_m+p_G^\perp)B(H)(q_m+p_G^\perp)}$ is a weak 2-local derivation on $(q_m+p_G^\perp)B(H)(q_m+p_G^\perp)$ (see [7, Proposition 2.7]). We know from (9) that $(q_m+p_G^\perp)\Delta_1(a)(q_m+p_G^\perp) = (q_m+p_G^\perp)[z_1,a](q_m+p_G^\perp)$ for every element a in $p_G^\perp B(H)p_G^\perp$. We set

$$\Delta_2 = (q_m+p_G^\perp)\Delta_1(q_m+p_G^\perp)|_{(q_m+p_G^\perp)B(H)(q_m+p_G^\perp)} - (q_m+p_G^\perp)[z_1,\cdot\,](q_m+p_G^\perp).$$

Then Δ_2 is a weak 2-local derivation on $(q_m+p_G^\perp)B(H)(q_m+p_G^\perp)$ and $\Delta_2(a)=0$ for every $a \in p_G^\perp B(H)p_G^\perp$. Proposition 2.12 proves that Δ_2 is a linear *-derivation on $(q_m+p_G^\perp)B(H)(q_m+p_G^\perp)$, which implies the same conclusion for the mapping $(q_m+p_G^\perp)\Delta(q_m+p_G^\perp)|_{(q_m+p_G^\perp)B(H)(q_m+p_G^\perp)}$.

If we set $G_1 = (\sum_{j=1}^{m-1} q_j)(H) \subsetneq G$, then we conclude that the set

$$\mathcal{D}\mathrm{iam}^-_{G_1^\perp} = \big\{\mathrm{diam}\big(\sigma(w_F)\big) : w_F \in [z_F], F \in \mathfrak{F}(H), p_F \leq p_{G_1}^\perp\big\}$$

is bounded (just apply the fact that $p_{G_1}^\perp\Delta p_{G_1}^\perp|_{p_{G_1}^\perp B(H)p_{G_1}^\perp}$ is a bounded linear *-derivation). If we apply the above reasoning to G_1, p_{m-1}, and Δ, we deduce that

$$(q_{m-1}+p_{G_1}^\perp)\Delta(q_{m-1}+p_{G_1}^\perp)|_{(q_{m-1}+p_{G_1}^\perp)B(H)(q_{m-1}+p_{G_1}^\perp)}$$

is a bounded linear *-derivation. Repeating these arguments a finite number of steps, we prove that Δ is a bounded linear *-derivation. $\square$

The key technical result needed in our arguments follows now as a direct consequence of Theorem 2.13.

Corollary 2.14. *Let $\Delta : B(H) \to B(H)$ be a weak 2-local derivation with $\Delta^\sharp = \Delta$. Suppose that the set $\mathcal{D}\mathrm{iam}(\Delta) = \{\mathrm{diam}(\sigma(w_F)) : w_F \in [z_F], F \in \mathfrak{F}(H)\}$ is unbounded. Then there exists a sequence $(F_n) \subset \mathfrak{F}(H)$ such that $p_{F_n} \perp p_{F_m}$ for every $n \neq m$, and $\mathrm{diam}(\sigma(\widetilde{z}_{F_n})) \geq 4^n$ for every natural n.*

Proof. If there exists $G \in \mathfrak{F}(H)$ such that

$$\mathcal{D}\mathrm{iam}^-_{G^\perp} = \big\{\mathrm{diam}\big(\sigma(w_F)\big) : w_F \in [z_F], F \in \mathfrak{F}(H), p_F \leq p_G^\perp\big\}$$

is bounded, then Theorem 2.13 implies that Δ is a linear *-derivation, which contradicts the unboundedness of the set

$$\mathcal{D}\mathrm{iam}(\Delta) = \big\{\mathrm{diam}\big(\sigma(w_F)\big) = \|p_F \Delta p_F|_{p_F B(H) p_F}\| : w_F \in [z_F], F \in \mathfrak{F}(H)\big\}.$$

We can therefore assume that $\mathcal{D}\mathrm{iam}^-_{G^\perp}$ is unbounded for every $G \in \mathfrak{F}(H)$.

We will argue by induction. Let us fix $F_1 \in \mathfrak{F}(H)$ with $\mathrm{diam}(\sigma(\widetilde{z}_{F_1})) \geq 4$. In the notation employed before, the set $\mathcal{D}\mathrm{iam}^-_{F_1{}^\perp}$ is unbounded. The mapping $p_{F_1}^\perp \Delta p_{F_1}^\perp|_{p_{F_1}^\perp B(H) p_{F_1}^\perp} : p_{F_1}^\perp B(H) p_{F_1}^\perp \to p_{F_1}^\perp B(H) p_{F_1}^\perp$ is a weak 2-local derivation and a symmetric mapping (see [7, Proposition 2.7]). Therefore, the set $\mathcal{D}\mathrm{iam}(p_{F_1}^\perp \Delta p_{F_1}^\perp|_{p_{F_1}^\perp B(H) p_{F_1}^\perp})$ must be unbounded. We can find $F_2 \in \mathfrak{F}(H)$ with $p_{F_2} \perp p_{F_1}$ and $\mathrm{diam}(\sigma(\widetilde{z}_{F_2})) \geq 4^2$.

Suppose that we have defined $F_1, \ldots, F_n$ satisfying the desired conditions. Set $K_n := F_1 \oplus^{\ell_2} \cdots \oplus^{\ell_2} F_n \in \mathfrak{F}(H)$. According to the arguments at the beginning of the proof, $\mathcal{D}\mathrm{iam}^-_{K_n{}^\perp}$ is unbounded. Therefore, we can find $F_{n+1} \in \mathfrak{F}(H)$ such that $p_{F_{n+1}} \perp p_{F_j}$ for every $j = 1, \ldots, n$ and $\mathrm{diam}(\sigma(\widetilde{z}_{F_{n+1}})) \geq 4^{n+1}$. $\square$

We show next that every weak 2-local derivation on $B(H)$ is bounded on the lattice of projections of $B(H)$.

Theorem 2.15. *Let $\Delta : B(H) \to B(H)$ be a weak 2-local derivation with $\Delta^\sharp = \Delta$. Then the following statements hold.*

(a) *The set $\mathcal{D}\mathrm{iam}(\Delta) = \{\mathrm{diam}(\sigma(w_F)) : w_F \in [z_F], F \in \mathfrak{F}(H)\}$ is bounded.*
(b) *The set $\{\|\widetilde{z}_F\| : F \in \mathfrak{F}(H)\}$ is bounded.*

Consequently, by Alaoglu's theorem, we can find $z_0 \in B(H)$ with $z_0 = -z_0^*$ and a subnet $(\widetilde{z}_F)_{F \in \Lambda}$ of $(\widetilde{z}_F)_{F \in \mathfrak{F}(H)}$ converging to z_0 in the weak*-topology of $B(H)$.

Proof. (a) Arguing by contradiction, we suppose that $\mathcal{D}\mathrm{iam}(\Delta)$ is unbounded. By Corollary 2.14, there exists a sequence $(F_n) \subset \mathfrak{F}(H)$ such that $p_{F_n} \perp p_{F_m}$ for every $n \neq m$, and $\mathrm{diam}(\sigma(\widetilde{z}_{F_n})) \geq 4^n$ for every natural n. We can pick a sequence of mutually orthogonal rank 1 projections $(p_k) \subseteq B(H)$ satisfying $p_{2n-1}, p_{2n} \leq p_{F_n}$, $\widetilde{z}_{F_n} = i\lambda_{2n-1} p_{2n-1} + i\lambda_{2n} p_{2n} + (p_{F_n} - p_{2n-1} - p_{2n}) \widetilde{z}_{F_n} (p_{F_n} - p_{2n-1} - p_{2n})$ $(\lambda_{2n-1}, \lambda_{2n} \in \mathbb{R})$, and $|\lambda_{2n-1} - \lambda_{2n}| = \lambda_{2n-1} - \lambda_{2n} = \mathrm{diam}(\sigma(\widetilde{z}_{F_n})) \geq 4^n$.

Let e_n be the unique rank 2 partial isometry in $B(H)$ defined by $e_n = \xi_{2n} \otimes \xi_{2n-1} + \xi_{2n-1} \otimes \xi_{2n}$, where ξ_{2n} and ξ_{2n-1} are norm 1 vectors in $p_{2n}(H)$ and

$p_{2n-1}(H)$, respectively. Since $e_n \perp e_m$, for every $n \neq m$, the series $\sum_{n=1}^{\infty} e_n$ converges to an element $a_0 \in B(H)$. Set $s_{2n} := \sum_{k=1}^{2n} p_k \leq p_{K_n}$, where $K_n = \bigoplus_{k=1}^{n} F_k$. Clearly, $a_0 = s_{2n} a_0 s_{2n} + s_{2n}^{\perp} a_0 s_{2n}^{\perp}$. Applying the properties of $\widetilde{z}_{F_n}$ (see Lemma 2.3, Remark 2.4, and Lemma 2.6) and [6, Lemma 3.2], we have

$$\begin{aligned} s_{2n}\Delta(a_0)s_{2n} &= s_{2n}\Delta(s_{2n}a_0s_{2n})s_{2n} = s_{2n}[\widetilde{z}_{K_n}, s_{2n}a_0s_{2n}]s_{2n} \\ &= \Big[i\sum_{k=1}^{n} \lambda_{2k-1}p_{2k-1} + \lambda_{2k}p_{2k}, s_{2n}a_0s_{2n}\Big]. \end{aligned}$$

Let us consider the functional $\phi_0 = \sum_{k=1}^{n} \frac{1}{2^k}\omega_{\xi_{2k-1},\xi_{2k}}$, where, following the standard notation, $\omega_{\xi_{2k-1},\xi_{2k}}(a) = \langle \xi_{2k-1}, a(\xi_{2k})\rangle$ $(a \in B(H))$. We deduce from the above that $\|\phi_0\| \leq 1$ and that

$$\begin{aligned} \big\|\Delta(a_0)\big\| \geq \big|\phi_0\big(s_{2n}\Delta(a_0)s_{2n}\big)\big| &= \sum_{k=1}^{n} \frac{1}{2^k}(\lambda_{2k-1} - \lambda_{2k}) \\ &= \sum_{k=1}^{n} \frac{1}{2^k}|\lambda_{2k-1} - \lambda_{2k}| > \sum_{k=1}^{n} \frac{1}{2^k}4^k = \sum_{k=1}^{n} 2^k, \end{aligned}$$

which is impossible.

(b) Take $F \in \mathfrak{F}(H)$ and any $z \in [z_F]$. If we choose $i\lambda \in \sigma(z_F)$, the inequalities

$$\|\widetilde{z}_F\| \leq \|z - i\lambda p_F\| \leq \operatorname{diam}\big(\sigma(z - i\lambda p_F)\big) = \operatorname{diam}\big(\sigma(z)\big) = \operatorname{diam}\big(\sigma(\widetilde{z}_F)\big)$$

hold because $0 \in \sigma(z - i\lambda p_F)$ and $(z - i\lambda p_F)^* = -(z - i\lambda p_F)$. Finally, the desired conclusion follows from statement (a). □

We can now provide a positive answer to Problem 1.4 in the case $A = B(H)$.

Theorem 2.16. *Let $\Delta : B(H) \to B(H)$ be a weak 2-local derivation with $\Delta^\sharp = \Delta$. Then Δ is a linear *-derivation.*

Proof. By Theorem 2.15, the set

$$\mathcal{D}\mathrm{iam}(\Delta) = \big\{\operatorname{diam}\big(\sigma(w_F)\big) : w_F \in [z_F], F \in \mathfrak{F}(H)\big\}$$

is bounded. The desired conclusion follows from Corollary 2.10. □

All the results from Lemma 2.3 to Proposition 2.14 remain valid when $\Delta : K(H) \to K(H)$ is a weak 2-local derivation with $\Delta^\sharp = \Delta$. Actually, the conclusion of Theorem 2.15 also holds for every such mapping Δ with practically the same proof, but replacing $a_0 = \sum_{n=1}^{\infty} e_n \in B(H)$ with $a_0 = \sum_{n=1}^{\infty} (\frac{2}{3})^n e_n \in K(H)$, because in that case we would have

$$\big\|\Delta(a_0)\big\| \geq \big|\phi_0\big(s_{2n}\Delta(a_0)s_{2n}\big)\big| = \sum_{k=1}^{n} \frac{1}{2^k}\Big(\frac{2}{3}\Big)^k |\lambda_{2k-1} - \lambda_{2k}| > \sum_{k=1}^{n} \Big(\frac{4}{3}\Big)^k,$$

obtaining the desired contradiction. We have thus obtained an appropriate version of Theorem 2.15 for weak 2-local derivations on $K(H)$.

Theorem 2.17. *Let $\Delta : K(H) \to K(H)$ be a weak 2-local derivation with $\Delta^\sharp = \Delta$. Then the following statements hold.*

(a) *The set* $\mathcal{D}\mathrm{iam}(\Delta) = \{\mathrm{diam}(\sigma(w_F)) : w_F \in [z_F], F \in \mathfrak{F}(H)\}$ *is bounded.*
(b) *The set* $\{\|\widetilde{z}_F\| : F \in \mathfrak{F}(H)\}$ *is bounded.*

Consequently, by Alaoglu's theorem, we can find $z_0 \in B(H)$ *with* $z_0 = -z_0^*$ *and a subnet* $(\widetilde{z}_F)_{F\in\Lambda}$ *of* $(\widetilde{z}_F)_{F\in\mathfrak{F}(H)}$ *converging to* z_0 *in the weak*-topology of* $B(H)$.

Applying a subtle adaptation of the previous arguments, we get the following.

Theorem 2.18. *Let* $\Delta : K(H) \to K(H)$ *be a weak 2-local derivation with* $\Delta^\sharp = \Delta$. *Then* Δ *is a linear* **-derivation.*

3. Weak 2-local derivations on $B(H)$

We can culminate now the study of weak 2-local derivations on $B(H)$ with the promised solution to Problem 1.3 in the case $A = B(H)$.

Theorem 3.1. *Let* H *be an arbitrary complex Hilbert space, and let* Δ *be a weak 2-local derivation on* $B(H)$. *Then* Δ *is a linear derivation.*

Proof. We have already commented that H can be assumed to be infinite-dimensional. Suppose that $\Delta : B(H) \to B(H)$ is a weak 2-local derivation. Since the set $\mathcal{S} = \mathrm{Der}(A)$, of all derivations on $B(H)$, is a linear subspace of $B(B(H))$, we deduce from Lemma 2.1(c) and (e) that $\Delta_1 = \frac{\Delta+\Delta^\sharp}{2}$ and $\Delta_2 = \frac{\Delta-\Delta^\sharp}{2i}$ are weak 2-local derivations on $B(H)$. Since $\Delta_1 = \Delta_1^\sharp$ and $\Delta_2 = \Delta_2^\sharp$, Theorem 2.9 proves that Δ_1 and Δ_2 are linear *-derivations on $B(H)$, and thus, $\Delta = \Delta_1 + i\Delta_2$ is a linear derivation on $B(H)$. ☐

According to Theorem 2.18, the arguments developed to prove Theorem 3.1 are also valid to obtain the following.

Theorem 3.2. *Let* H *be an arbitrary complex Hilbert space, and let* Δ *be a weak 2-local derivation on* $K(H)$. *Then* Δ *is a linear derivation.*

We begin with a suitable generalization of [6, Lemma 3.2].

Lemma 3.3. *Let* A_1 *and* A_2 *be C*-algebras, and let* $\Delta : A_1 \oplus^\infty A_2 \to A_1 \oplus^\infty A_2$ *be a weak 2-local derivation. Then* $\Delta(A_j) \subseteq A_j$ *for every* $j = 1, 2$. *Moreover, if* π_j *denotes the projection of* $A_1 \oplus^\infty A_2$ *onto* A_j, *we have* $\pi_j\Delta(a_1 + a_2) = \pi_j\Delta(a_j)$, *for every* $a_1 \in A_1$, $a_2 \in A_2$ *and* $j = 1, 2$.

Proof. Let us fix $a_1 \in A_1$. Every C*-algebra admits a bounded approximate unit (see [8, Theorem 1.4.2]), and thus, by Cohen's factorization theorem (see [5, Theorem VIII.32.22, Corollary VIII.32.26]), there exist $b_1, c_1 \in A_1$ satisfying $a_1 = b_1c_1$. We recall that $A^* = A_1^* \oplus^{\ell_1} A_2^*$. By hypothesis, for each $\phi \in A_2^*$, there exists a derivation $D_{a_1,\phi} : A_1 \oplus^\infty A_2 \to A_1 \oplus^\infty A_2$ satisfying

$$\phi\Delta_{a_1,\phi}(a_1) = \phi D_{a_1,\phi}(a_1) = \phi D_{a_1,\phi}(b_1c_1) = \phi\big(D_{a_1,\phi}(b_1)c_1\big) + \phi\big(b_1D_{a_1,\phi}(c_1)\big) = 0,$$

where in the last equalities we applied that $D_{a_1,\phi}(b_1)c_1$ and $b_1D_{a_1,\phi}(c_1)$ both lie in A_1 and that $\phi \in A_2^*$. We deduce, via the Hahn–Banach theorem, that $\Delta(a_1) \in A_1$.

The above arguments also show that, for each derivation $D : A_1 \oplus^\infty A_2 \to A_1 \oplus^\infty A_2$, we have $D(A_j) \subseteq A_j$ for every $j = 1, 2$. It follows from the hypothesis that, for each $\phi \in A_1^*$, $a_1 \in A_1$, and $a_2 \in A_2$, there exists a derivation $D_{\phi,a_1+a_2,a_1} : A_1 \oplus^\infty A_2 \to A_1 \oplus^\infty A_2$ satisfying

$$\phi\Delta(a_1) = \phi D_{\phi,a_1+a_2,a_1}(a_1) \qquad \text{and} \qquad \phi\Delta(a_1+a_2) = \phi D_{\phi,a_1+a_2,a_1}(a_1+a_2).$$

In particular, $\phi\Delta(a_1) = \phi\Delta(a_1+a_2)$, for every $\phi \in A_1^*$. Then it follows that $\pi_1\Delta(a_1) = \pi_1\Delta(a_1+a_2)$. ☐

For further purposes, we will also explore the stability of the above results under ℓ_∞- and c_0-sums.

Proposition 3.4. *Let (A_j) be an arbitrary family of C*-algebras. Suppose that, for each j, every weak 2-local derivation on A_j is a linear derivation. Then the following statements hold.*

(a) *Every weak 2-local derivation on $A = \bigoplus^{\ell_\infty} A_j$ is a linear derivation.*
(b) *Every weak 2-local derivation on $A = \bigoplus^{c_0} A_j$ is a linear derivation.*

Proof. (a) Let $\Delta : \bigoplus_{j\in J}^{\ell_\infty} A_j \to \bigoplus_{j\in J}^{\ell_\infty} A_j$ be a weak 2-local derivation. Let π_j denote the natural projection of A onto A_j. If we fix an index $j_0 \in J$, it follows from Lemma 3.3 that $\Delta(A_{j_0}) \subseteq A_{j_0}$ and that $\Delta(\bigoplus_{j_0\neq j\in J}^{\ell_\infty} A_j) \subseteq \bigoplus_{j_0\neq j\in J}^{\ell_\infty} A_j$. We deduce from the assumptions that $\Delta|_{A_j} : A_j \to A_j$ is a linear derivation for every j.

We will finish the proof by showing that $\{\|\Delta|_{A_j}\| : j \in J\}$ is a bounded set. Otherwise, there exist infinite sequences $(j_n) \subseteq J$, $(a_{j_n}) \subset A$, with $a_{j_n} \in A_{j_n}$, $\|a_{j_n}\| \leq 1$, and $\|\Delta(a_{j_n})\| > 4^n$, for every natural n. Let $a_0 = \sum_{n=1}^\infty a_{j_n} \in A$. For each natural n, $a_0 = a_{j_n} + (a_0 - a_{j_n})$ with $a_{j_n} \perp (a_0 - a_{j_n})$ in A. It follows from the above properties and the second statement in Lemma 3.3 that

$$\big\|\Delta(a_0)\big\| \geq \big\|\pi_{j_n}\Delta(a_0)\big\| = \big\|\Delta(a_{j_n})\big\| > 4^n,$$

for every $n \in \mathbb{N}$, which is impossible.

(b) The proof of (a), but replacing $a_0 = \sum_{n=1}^\infty a_{j_n}$ with $a_0 = \sum_{n=1}^\infty \frac{1}{2^n} a_{j_n} \in A$, remains valid in this case. ☐

Following standard notation, we say that a von Neumann algebra M is *atomic* if $M = \bigoplus^{\ell_\infty} B(H_\alpha)$, where each H_α is a complex Hilbert space. We recall that a Banach algebra is called *dual* or *compact* if, for every $a \in A$, the operator $A \to A$, $b \mapsto aba$ is compact. By [1], compact C*-algebras are precisely the algebras of the form $(\bigoplus_{i\in I} K(H_i))_{c_0}$, where each H_i is a complex Hilbert space.

We finish this article with a couple of corollaries which follow straightforwardly from Theorems 3.1 and 3.2 and from Proposition 3.4.

Corollary 3.5. *Every weak 2-local derivation on an atomic von Neumann algebra is a linear derivation.*

Corollary 3.6. *Every weak 2-local derivation on a compact C*-algebra is a linear derivation.*

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References

1. J. C. Alexander, *Compact Banach algebras*, Proc. Lond. Math. Soc. (3) **18** (1968), 1–18. Zbl 0184.16502. MR0229040. DOI 10.1112/plms/s3-18.1.1. 396
2. S. Ayupov and K. Kudaybergenov, *2-local derivations and automorphisms on $B(H)$*, J. Math. Anal. Appl. **395** (2012), no. 1, 15–18. Zbl 1275.47078. MR2943598. DOI 10.1016/j.jmaa.2012.04.064. 382, 383, 384
3. S. Ayupov and K. Kudaybergenov, *2-local derivations on von Neumann algebras*, Positivity **19** (2015), no. 3, 445–455. Zbl 1344.46046. MR3386119. DOI 10.1007/s11117-014-0307-3. 382, 383, 384
4. J. C. Cabello and A. M. Peralta, *Weak-2-local symmetric maps on C^*-algebras*, Linear Algebra Appl. **494** (2016), 32–43. Zbl 1337.47053. MR3455684. DOI 10.1016/j.laa.2015.12.024. 382, 383, 384
5. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis, II: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups*, Grundlehren Math. Wiss. **152**, Springer, New York, 1970. Zbl 0213.40103. MR0262773. 395
6. M. Niazi and A. M. Peralta, *Weak-2-local *-derivations on $B(H)$ are linear *-derivations*, Linear Algebra Appl. **487** (2015), 276–300. Zbl 1348.47029. MR3405562. DOI 10.1016/j.laa.2015.09.028. 383, 384, 385, 388, 389, 390, 391, 392, 394, 395
7. M. Niazi and A. M. Peralta, *Weak-2-local derivations on $\mathbb{M}_n$*, to appear in Filomat, preprint, arXiv:1503.01346 [math.OA]. 384, 385, 390, 391, 392, 393
8. G. K. Pedersen, *C^*-algebras and Their Automorphism Groups*, London Math. Soc. Monogr. Ser. **14**, Academic Press, London, 1979. Zbl 0416.46043. MR0548006. 395
9. S. Sakai, *C^*-algebras and W^*-algebras*, Ergeb. Math. Grenzgeb. (3) **60**, Springer, New York, 1971. Zbl 0219.46042. MR0442701. 385, 388
10. P. Šemrl, *Local automorphisms and derivations on $B(H)$*, Proc. Amer. Math. Soc. **125** (1997), no. 9, 2677–2680. Zbl 0887.47030. MR1415338. DOI 10.1090/S0002-9939-97-04073-2. 383, 384
11. J. G. Stampfli, *The norm of a derivation*, Pacific J. Math. **33** (1970), no. 3, 737–747. Zbl 0197.10501. MR0265952. DOI 10.2140/pjm.1970.33.737. 386

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