Notre Dame Journal of Formal Logic
Volume 56, Number 1, 2015

Extensionalizing Intensional Second-Order Logic

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Abstract Neo-Fregean approaches to set theory, following Frege, have it that sets are the *extensions of concepts*, where concepts are the values of second-order variables. The idea is that, given a second-order entity X, there may be an object εX, which is the extension of X. Other writers have also claimed a similar relationship between second-order logic and set theory, where sets arise from *pluralities*.

This paper considers two interpretations of second-order logic—as being either *extensional* or *intensional*—and whether either is more appropriate for this approach to the foundations of set theory. Although there seems to be a case for the extensional interpretation resulting from modal considerations, I show how there is no obstacle to starting with an intensional second-order logic. I do so by showing how the ε operator can have the effect of "extensionalizing" intensional second-order entities.

It is often thought that there is a close connection between sets and the denotation of the second-order variables under various interpretations of second-order logic. Even if it is denied that second-order entities just *are* sets (as, e.g., is famously claimed by Quine [14]), it might be thought that sets "arise" from second-order entities in some way.

So, for example, Frege's (inconsistent) set theory had it that sets are *extensions* of *concepts* (which are what his second-order variables range over). More recently, attempts to extend the neo-Fregean program of Bob Hale and Crispin Wright [7] to set theory have followed suit to some extent, albeit with restrictions on which concepts form sets (e.g., Boolos [2], Hale [6], Shapiro [15]). In addition, a number of articles which are less explicitly Fregean in motivation have claimed that sets arise from the denotation of second-order variables, where second-order quantification is interpreted as *plural* quantification (e.g., Burgess [3], Linnebo [12]).

Received May 9, 2012; accepted December 9, 2013
2010 Mathematics Subject Classification: Primary 03A05; Secondary 00A30
Keywords: modal set theory, abstraction, second-order logic, plural logic
 10.1215/00294527-2835092

Call such an approach to set theory the *abstractionist* approach.[1] The idea is that, for a second-order entity X, there may be (though will not always be, on pain of contradiction) an object εX, which is the *set of* X.[2] Central to an abstractionist approach to set theory will be a principle of extensionality similar to a restriction of Frege's Basic Law V. Where X and Y are second-order entities which do have a corresponding set,

$$\varepsilon X = \varepsilon Y \leftrightarrow \forall x(Xx \leftrightarrow Yx). \tag{Ext}$$

Now, a question arises concerning the interpretation of second-order logic which is appropriate for this approach to set theory. In particular, there is a choice between an *extensional* interpretation of the second-order quantifiers—for example, as *plural* quantification, so that sets arise from pluralities—and an interpretation of the second-order quantifiers as being over *intensional* entities, such as properties or Fregean concepts. In this paper, I shall look at considerations concerning the modal properties of various interpretations of second-order logic and of sets which seem to tell in favor of the plural interpretation. In particular, on this interpretation, sets can be seen as inheriting their extensional nature from the extensional nature of the second-order entities. I shall, however, argue that there is no obstacle to proceeding with intensional second-order logic and will show how one can proceed in such a way, by making use of transworld identity conditions which have the effect of "extensionalizing" the second-order entities.

The outline of the paper will be as follows. In Section 1, I shall flesh out in more detail the general shape that an abstractionist set theory may take in a nonmodal setting, and what the intended interpretation of the ε operator would be in this setting. My aim will be for this to be general enough so as to accommodate a wide range of theories or positions which can reasonably be called abstractionist. In Section 2, I shall discuss extending the theory of the previous section to modal contexts. In particular, I shall consider the modal behavior of the second-order quantifiers and the expected modal behavior of sets. At this point, it will become apparent that there is a prima facie argument in favor of an extensional interpretation of the second-order quantifiers, since then the modal properties of the second-order quantifiers will be very similar to the modal properties of sets.

In Section 3, I consider a way of avoiding the conclusion of this argument by restricting attention to only certain kinds of property, which I then go on to dismiss. Finally, in Sections 4 and 5 I present my own solution to the problem, whereby the ε operator has the effect of "extensionalizing" concepts. Section 4 concerns the semantics of such an operator, and Section 5 shows how the correct behavior can be described in the object language, by adopting a transworld principle of extensionality.

1 Abstractionist Set Theory

The key feature of an abstractionist set theory—as I shall understand it here—is the presence of a *type-lowering* function ε. By this I mean a function which takes a second-order term in its argument place and which results in a first-order term. Where X is a second-order term, εX will be a first-order term which should be read as "the set of X" or something similar.[3] Such operators are also often called *abstraction operators*; I shall for the most part follow this usage.

If any set theory is to be carried out in such a framework, it will obviously be necessary to define a membership relation. But this is easily done. We can simply take $x \in y$ to be an abbreviation of $\exists X(y = \varepsilon X \wedge Xx)$. With extensionality on board, it is then possible to prove that, where εX exists, $x \in \varepsilon X$ and Xx will be equivalent.[4]

Now, if a full impredicative comprehension scheme is part of the second-order logic, then it cannot be the case that for every X, "εX" denotes a set.[5] If that were the case, a naive comprehension principle for sets would result, and Russell's paradox would follow shortly thereafter—we simply take the second-order entity R defined by the formula $x \notin x$ and consider the resulting set.

So, set abstraction must be restricted somehow. Although I do not wish to dwell on this issue—my principle concern is with the behavior of the sets that *do* exist—I will note what may be done to achieve this. There are two questions concerning restriction: first, what the restrictions will be (i.e., which second-order entities form sets) and second, how these restrictions will be imposed.

On the first of these, one may wish to impose either necessary conditions, sufficient conditions, or necessary *and* sufficient conditions for a second-order entity to have a corresponding set. So, for example, [2] gives as a necessary and sufficient condition that a concept not be in a one-to-one correspondence with the universe. Or, the existence assumptions which form part of the standard ZF axioms can be seen as giving sufficient conditions for a concept to have a corresponding set.[6]

In terms of actually imposing a restriction, there are two options. The first is to adopt a free logic and so allow that not every function is total and not every singular term refers. Then we can allow that, for "bad" second-order entities X (such as the one defined by the formula $x \notin x$), "εX" does not refer to any object. Explicit existence assumptions can then be given in terms of an existence predicate $E!$, where $E!x$ is an abbreviation of $\exists y(y = x)$.

In this case, extensionality can be restricted as follows:

$$\forall X \forall Y \big[E!\varepsilon X \wedge E!\varepsilon Y \to \big(\varepsilon X = \varepsilon Y \leftrightarrow \forall x(Xx \leftrightarrow Yx)\big)\big]. \qquad (\mathrm{Ext}_1)$$

Then necessary, sufficient, or necessary and sufficient conditions for X to have a corresponding set can be given as

$$E!\varepsilon X \to \varphi(X), \qquad \varphi(X) \to E!\varepsilon X, \qquad \varphi(X) \leftrightarrow E!\varepsilon X,$$

respectively.[7]

A second way of restricting is to let all bad second-order entities map to the same "dummy" object. This way of restricting set abstraction, which is adopted by [2], has been the focus of much of the discussion concerning neo-Fregean set theory. It has the advantage of not requiring a free logic, but it has a disadvantage in that it does not easily allow for set existence assumptions which do not take the form of a necessary and sufficient condition statable as a formula of the language.

If we add a primitive predicate of second-order terms "Good(X)" to the language, then we can express extensionality as follows:

$$\forall X \forall Y \big[\varepsilon X = \varepsilon Y \leftrightarrow \big(\neg\,\mathrm{Good}(X) \wedge \neg\,\mathrm{Good}(Y) \vee \forall x(Xx \leftrightarrow Yx)\big)\big]. \qquad (\mathrm{Ext}_2)$$

This has the effect of mapping all bad second-order entities to the same object and mapping other second-order entities to objects extensionally. Necessary, sufficient, or necessary and sufficient conditions for a second-order entity to have a set can then be given as

$$\mathrm{Good}(X) \to \varphi(X), \qquad \varphi(X) \to \mathrm{Good}(X), \qquad \varphi(X) \leftrightarrow \mathrm{Good}(X).$$

If necessary and sufficient conditions for goodness are given by a formula $\varphi(X)$, then clearly we can dispense with Good as a primitive, by replacing Good(X) with $\varphi(X)$ in the above (as is the case in, e.g., [2]). But if we only have a sufficient condition or a necessary condition, we must retain the primitive predicate.

Which approach is taken does not have much in the way of consequence for the questions that I am considering. From now on, I shall adopt the free logic approach since, in easily accommodating a wider variety of restrictions, it is in many respects the more general of the two. It also simplifies the semantics somewhat, since there is no need for a distinguished dummy object for bad concepts to map to. But not much hangs on this.

1.1 Model theory and intended interpretations Just as it could be said that the intended interpretation of $\in$ in standard treatments first-order set theory is the *real* membership relation (or at least, the membership relation in the ambient set theory), so too can we single out the intended interpretation of the abstraction operator. First, it should be noted what a model will be and what form any interpretation of ε must take.

A model will be a triple $\mathcal{M} = \langle D, \mathbf{D}_2, I \rangle$. Here, D is the domain of the first-order variables, and $\mathbf{D}_2 \subseteq \mathcal{P}(D)$ is the domain of the second-order variables.[8] I shall only consider *full* second-order models, so that $\mathbf{D}_2$ is the full power set of D, but nothing of import hangs on this for my present purposes. I is an interpretation function, which maps each item of nonlogical vocabulary onto entities of the appropriate kind.

In the present case, there is only one item of nonlogical vocabulary—namely, the abstraction operator ε. Since ε is to denote a function from second-order entities to first-order entities, $I(\varepsilon)$ will be a (partial) function $I(\varepsilon) : \mathbf{D}_2 \to D$. Moreover, there is a natural intended interpretation. This will be, for $Y \in \mathbf{D}_2$,

$$I(\varepsilon)(Y) = \begin{cases} Y & Y \in D, \\ \text{undefined} & Y \notin D. \end{cases}$$

Thus, it takes each subset of D onto itself if it is also a member of D but remains undefined otherwise. This interpretation will only make sense (or rather, define a function which is at least partly defined) when there are some subsets of the domain D which are also members of D. But this will be the case for most natural models of set theory, such as transitive sets, where every member of D is also a subset of D, or models built up from a set of urelements by successively taking power sets.

Satisfaction in the model and relative to a variable assignment a will then just be given in the standard way. It should be noted that, since the language features functional expressions, it is necessary first to recursively define the denotation of terms, including complex terms, with respect to an assignment. This I shall denote as $t^{\mathcal{M},a}$. For a (first- or second-order) variable v, $v^{\mathcal{M},a} = a(v)$. For a nonlogical primitive ξ, $\xi^{\mathcal{M},a} = I(\xi)$. For a complex term $f(t_1, \ldots, t_n)$, $(f(t_1, \ldots, t_n))^{\mathcal{M},a} = f^{\mathcal{M},a}(t_1^{\mathcal{M},a}, \ldots, t_n^{\mathcal{M},a})$. In particular, for set abstract terms of the form εX, $(\varepsilon X)^{\mathcal{M},a} = \varepsilon^{\mathcal{M},a}(X^{\mathcal{M},a})$. Satisfaction of a formula in a model with respect to a variable assignment can then proceed in the usual way.

2 Modalizing

So far, there will be no difference—at least, no technical difference—between the abstractionist approach when taking the second-order quantifiers to be plural quantifiers and the same approach when taking the second-order quantifiers to range over something like properties or concepts.[9] In both cases, the semantics is the same, where subsets of D are taken as surrogates for second-order entities. A difference does however occur when developing the underlying logic to a *modal* logic, in which case the natural semantics differ.

2.1 Modal set-theoretic targets Before considering the modal logic of plural quantification and quantification over properties, it will be useful to survey the modal properties that could be expected of *sets*.[10] For the sake of simplicity, I shall consider only cases where the domain is constant across worlds, so that objects exist of necessity. I shall also assume that the modal logic satisfies the S5 axioms, so that the accessibility relation is an equivalence relation.

In such a case, we would expect the following rigidity principle to hold of sets:

$$\Box\forall x\forall y(\Diamond x \in y \to \Box x \in y). \qquad (\mathrm{Rigid}_{\in})$$

This can be thought of as the combination of two principles. First, that if an object is a member of a set, then necessarily it is a member of that set ($x \in y \to \Box x \in y$). Second, that if an object is *not* a member of a set, then necessarily it is not a member of that set ($x \notin y \to \Box x \notin y$ or, contraposed, $\Diamond x \in y \to x \in y$).

This principle can be seen as resulting from some plausible considerations concerning the nature of sets. One may, for example, claim that it is essential to a set that it has the elements that it actually has. Or, informal considerations concerning extensionality could play a role: given a set x, had there been a set with different members, it would not have been the same set as x. Although ($\mathrm{Rigid}_{\in}$) may be thought of as being motivated by extensionality, it does not follow from (Ext) nor from the more standard form of extensionality as expressed using a primitive membership relation. One of my main aims in Section 5 will be to show how a more general version of extensionality does entail ($\mathrm{Rigid}_{\in}$).

Another principle which has been suggested is the following:

$$\Box\forall x\big(\Diamond\exists X(x = \varepsilon X) \to \Box\exists X(x = \varepsilon X)\big). \qquad (\mathrm{Rigid}_{\mathrm{Set}})$$

This expresses the idea that nothing that actually is a set could have failed to be a set and that nothing which is not a set could have been a set. Again, this seems to be a natural principle to adopt.

These principles then should be the targets for any modal set theory. The obvious question to ask is: does either the plural version or the property version of the abstraction approach do better at hitting such a target? In order to evaluate this question, we need to look more carefully at the modal properties of each version.

2.2 Modal second-order logic In a modal setting, the interpretation of second-order quantification as plural quantification and its interpretation as quantification over properties are likely to come apart (see, e.g., Williamson [17], but see also Hewitt [9] for a dissenting view). The reason is that a plural variable will refer to *some objects*, and, being a variable, will do so rigidly. Hence, an assignment will assign a plural variable the same extension at every world. A concept variable will refer to the same concept in each world, but the extension of a concept may vary with each world. Hence an assignment will assign to each concept variable a function from worlds to the extension of the concept in that world.

More formally, we can extend the semantics of nonmodal second-order logic in the following way. As usual for a quantified modal logic, a model will be of the form $\mathcal{M} = \langle W, D, I \rangle$,[11] where W is a set of worlds, and D is the domain of objects (i.e., values of first-order variables). In contrast to the nonmodal case, we will now have different domains for second-order variables depending on whether we take them to be extensional (as in the case of plural variables) or intensional (as in the case of concept variables). The domain of the plural variables $\mathbf{D}_{\mathrm{pl}}$ will simply be $\mathcal{P}(D)$. For concept variables, by contrast, the domain $\mathbf{D}_{\mathrm{pr}}$ will be the set of functions $f : W \to \mathcal{P}(D)$.

Since we are now distinguishing between plural quantification and concept quantification, it will be useful to set up differing notation for the variables. For plural variables, I shall use repeated lower-case letters, xx, yy, zz, and so forth, and for concept variables, capital letters, F, G, H, and so forth. Then plural membership (x is one of xx) will be denoted with $\prec$ (so that $x \prec xx$ means that x is one of xx).

The interpretation function will differ slightly from the nonmodal case. Whereas there the interpretation function is simply a map from nonlogical vocabulary to entities of the appropriate kind, in the modal case it will be a binary function from worlds and nonlogical vocabulary to entities of the appropriate kind. The reason is that we may wish to allow for nonrigid designators—terms which denote different entities depending on the world at which they are evaluated. So, for example, we may wish to have a nonrigidly designating constant c, in which case for each $w \in W$, $I(c, w) \in D$, but, for some $w, w' \in W$, $I(c, w) \neq I(c, w')$. This differs from most approaches to quantified modal logic, where the interpretation function is monadic, as in the nonmodal case. Thus the nonlogical primitives are assumed to be rigid, and nonrigidity only arises (if at all) in the case of complex terms. I wish to consider nonrigidly designating primitives since doing so will provide an option concerning the interpretation of ε (albeit one which I shall reject). However, since for the most part I shall be considering rigidly designating nonlogical constants, I shall often suppress mention of the world and simply write $I(\xi)$ for the interpretation of ξ.

Given the two domains of the second-order variables and the interpretation function, the relevant part of the semantics can then be given. First, a variable assignment a will assign to each concept variable F an element $a(F) \in \mathbf{D}_{\mathrm{pr}}$ and to each plural variable xx an element $a(xx) \in \mathbf{D}_{\mathrm{pl}}$. Then, as before, a denotation can be given to each term with respect to an assignment. As was the case with the interpretation of the nonlogical vocabulary, denotations will be relative not just to a model and an assignment but also to a world. So, each term t has as a denotation $t^{\mathcal{M},w,a}$. Similarly to the nonmodal case, this can be defined inductively as follows: for variables v of various types, $v^{\mathcal{M},w,a} = a(v)$; for a nonlogical primitive ξ, $\xi^{\mathcal{M},w,a} = I(\xi, w)$; and for a complex term $f(t_1, \ldots, t_n)$, $(f(t_1, \ldots, t_n))^{\mathcal{M},w,a} = f^{\mathcal{M},w,a}(t_1^{\mathcal{M},w,a}, \ldots, t_n^{\mathcal{M},w,a})$. I shall leave it open, for the moment, how exactly this should apply to set abstract terms, since there will be a number of options, depending, for example, on whether it is taken to apply to plural terms or to predicate terms.

Then, satisfaction can be defined along usual lines. In particular, predication, plural membership, and the two kinds of second-order quantification will have as their semantic clauses

$$\begin{aligned}
&\mathcal{M}, w, a \models Ft \quad \text{iff } t^{\mathcal{M},w,a} \in a(F)(w),\\
&\mathcal{M}, w, a \models t \prec xx \quad \text{iff } t^{\mathcal{M},w,a} \in a(xx),\\
&\mathcal{M}, w, a \models \forall F\varphi \quad \text{iff for each } f \in \mathbf{D}_{\mathrm{pr}},\ \mathcal{M}, w, a^F_f \models \varphi,\\
&\mathcal{M}, w, a \models \forall xx\varphi \quad \text{iff for each } X \in \mathbf{D}_{\mathrm{pl}},\ \mathcal{M}, w, a^{xx}_X \models \varphi,
\end{aligned}$$

where a^F_f denotes the assignment which is identical with a with the exception of mapping the variable F to the function f (and similarly for a^{xx}_X).

With these semantic clauses in place, it is easy to see that plural quantification and concept quantification will differ. In addition, this difference will manifest itself in the object language[12] in the form of principles very much like (Rigid$_\in$). In particular, we have the following.

Proposition 2.1 *The following principle concerning plural quantification is valid:*

$$\Box\forall x\forall xx(\Diamond x \prec xx \leftrightarrow \Box x \prec xx), \qquad (\mathrm{Rigid}_{\mathrm{pl}})$$

whereas the corresponding principle for properties,

$$\Box\forall x\forall F(\Diamond Fx \leftrightarrow \Box Fx), \qquad (\mathrm{Rigid}_{\mathrm{pr}})$$

is not valid.

Proof That (Rigid$_{\mathrm{pl}}$) holds at any world in any model follows simply from the fact that, for any assignment a, whether $\mathcal{M}, w, a \models x \prec xx$ does not depend on w.

A countermodel of (Rigid$_{\mathrm{pr}}$) can be given by $W = \{w_1, w_2\}$, $D = \{0, 1\}$. Then a variable assignment a such that $a(F)$ is the function given by $a(F)(w_1) = \{0\}$, $a(F)(w_2) = \{1\}$ will clearly provide a counterexample. $\Box$

That (Rigid$_{\mathrm{pl}}$) is valid and (Rigid$_{\mathrm{pr}}$) is not valid is very natural. If something is one of some things, then it cannot but be one of those very things. But, if something has a property, it will not in general be the case that it necessarily has that property.

2.3 The prima facie case against properties Now, it may seem at this point that, in order to pursue an abstractionist approach to set theory, we must take the second-order quantifiers to be plural quantifiers; pluralities satisfy the same rigidity properties as sets, whereas concepts do not. Thus the extensional and rigid nature of sets could be seen to be inherited from the pluralities from which they arise. And, indeed, a number of authors have taken the intensional and nonrigid nature of concepts to mean that extensional entities—such as sets—may not arise from them in the way in which sets may arise from pluralities. Instead, corresponding to concepts will be some kind of intensional objects.

So, for example, in [11] and [12] Linnebo claims that pluralities give rise to sets in a broadly abstractionist manner but that concepts give rise to objects which he calls "properties." (It should be noted that this use of "property" differs from mine. In particular, I have used the word to refer to values of second-order variables, whereas for Linnebo, properties are a kind of object and thus a possible value of first-order variables.) Moreover, the reasoning that he gives for this distinction derives from the differing modal properties of pluralities and concepts, so that "the intensional nature of concepts often prevents concepts from defining sets" [12, p. 150]. In [11], he gives more detail. First, he notes, as I have done, that concepts do not behave rigidly:

> The identity of a concept ... is tied to its condition of application .... Had there been other objects satisfying the condition ... than there actually are, then these objects too would have fallen under the concept .... And had some of the objects which actually satisfy the condition ... not done so, then they would not have fallen under this concept. ([12, p. 159])

Then, introducing properties as "nominalized concepts" he asserts that

> Because properties are just nominalized concepts, they inherit their essential properties, namely their conditions of application. This means that properties and sets have completely different essential properties. ([12, p. 160])

Other writers have entertained similar claims, albeit not corresponding as closely to the present framework (i.e., that of considering two forms of second-order modal logic and taking sets to be abstracted from the second-order entities). So, [13, p. 303] considers distinguishing *sets*, which are "constituted by their elements" with the result that "set membership is rigid," from *classes*, which are "constituted by predicates" with a possible consequence that we "reject the rigidity of membership for classes"; Barcan Marcus [1] makes a similar distinction between classes (which correspond to predicates and are nonrigid) and "assortments," which are rigid.

I wish to claim that the poor fit between the modal behavior of properties and sets need not rule out an abstractionist set theory based on the intensional interpretation of second-order logic. To get clearer on the obstacles that face the concept-based approach, it will be useful to consider an argument which purports to show that intensional logic cannot be used for such a purpose. That argument is as follows. First, instantiate y in ($\mathrm{Rigid}_{\in}$) with εF (where we assume that εF exists). Then we get

$$\Box\forall x(\Diamond x \in \varepsilon F \rightarrow \Box x \in \varepsilon F).$$

But, since, when εF exists, $x \in \varepsilon F$ and Fx are equivalent, we have

$$\Box\forall x(\Diamond Fx \rightarrow \Box Fx).$$

But this is just an instance of ($\mathrm{Rigid}_{\mathrm{pr}}$), which we saw was not valid.

This argument is not, as it stands, valid. For the argument to be valid—and, in particular, for the substitution of εF for y to be legitimate—we need an additional claim to the effect that the set abstract term εF is a rigid designator. Thus, for the defender of the concept-based approach to avoid the conclusion of the argument, they must deny that set abstract terms, so formed, are rigid designators.

Now, of course, set abstract terms as commonly used in natural language—such as "the set of inhabitants of England"—are nonrigid designators; such a term will plainly refer to different sets under different circumstances. But it is not sufficient for the defender of the concept approach to simply assert this. They need to be able to claim that these nonrigid terms can be conceived of as arising from abstraction on concepts—in this case, on the concept *inhabitant of England*—rather than as abstraction on pluralities—in this case, on the inhabitants on England. In the plural case, there is a simple explanation as to how a nonrigid set term can result; the constituent plural term "the occupants of England" is *itself* a nonrigid designator. That is, it would refer to a different plurality of people under different circumstances. But the same cannot be said of the concept approach. For if "inhabitant of England" is taken to denote a concept or property, then it will do so rigidly; regardless of the circumstances, it will refer to the same property of being an inhabitant of England. It thus might be thought that a consequence of the concept-based approach would be that abstract terms, based as they are on rigidly denoting concept terms, are themselves rigid.[13]

So, the defender of the concept is left with the following task: to explain how an abstractionist account of set theory can account for this nonrigidity when set terms are formed out of second-order variables, which are themselves rigidly designating.

The carrying out of such a task will be the aim of Sections 4 and 5 of this paper. In Section 4, I shall show that there is a natural interpretation of the ε operator which allows set abstract terms to be nonrigidly designating. Thus, the intensional nature of second-order variables may be preserved (i.e., ($\mathrm{Rigid}_{\mathrm{pr}}$) fails), while allowing for the sets that arise out of them to be extensional (i.e., ($\mathrm{Rigid}_{\in}$) is true). In Section 5, I show how such behavior can be enforced in the object language in a natural—and, moreover, distinctly abstractionist—manner.

Before doing so, however, I wish to briefly consider an alternative way out for the defender of the concept-based approach, which involves placing further restrictions on set formation.[14] This will be the topic of the next chapter.

3 Interlude: Rigid Properties

The problems that seem to arise when sets are abstracted from properties only arise for properties which are themselves nonrigid, that is, for properties, such as *inhabitant of England*, for which the relevant instance of ($\mathrm{Rigid}_{\mathrm{pr}}$) is false. But there are some properties—such as *natural number less than* 17—for which the relevant instance of ($\mathrm{Rigid}_{\mathrm{pr}}$) is true. This suggests that we might avoid the problems by restricting set formation to those properties which are rigid. That is, in addition to any restrictions on Basic Law V which have been put in place so as to avoid paradox (as in Section 1), the following necessary condition is put in place:

$$E!\varepsilon F \to \Box\forall x(\Diamond Fx \to \Box Fx).$$

It is then trivial that this restriction will succeed in avoiding any problems that are not faced by the plural account. For all intents and purposes, rigid properties will behave exactly like pluralities. In particular, ($\mathrm{Rigid}_{\mathrm{pr}}$) will simply be a tautology when restricted to rigid properties.

Despite the fact that this proposal will avoid the problems faced by the unrestricted intensional account, it is unsatisfactory for a number of reasons. First, the proposal does not so much *solve* the mismatch between properties and sets but merely skirts around it. The challenge is to show how an intensional logic may be reconciled with an abstractionist approach to set theory. But the present proposal does not do this, since by restricting to rigid properties, the logic may as well be extensional; all that is intensional about it has been thrown out or ignored. As such, the present approach fails to be explanatory in a way in which I claim my proposal is. It would desirable to explain how it is that nonrigid set abstract terms may arise from rigid terms for properties. But this approach simply *ignores* nonrigid set abstract terms.

A second reason why the present approach is undesirable is that it seems to rule out set abstract terms that would otherwise seem to be perfectly acceptable. The restriction, if it is to be imposed in the same way as any other restriction, should apply equally well in modal and nonmodal contexts alike. But then, even if we just consider how things actually are, it will deny that the property "inhabitant of England" has an extension, even though this concept is clear, precise, and does not threaten paradox in the way in which the Russell set and other paradoxical sets do. Worse, although there will be no set of inhabitants of England, there will be a set which has as members all and only those objects that are inhabitants of England, assuming some rigid property can be picked out under which all inhabitants of England actually fall (perhaps some long disjunction making use of rigid designators for each individual involved).

This then leads on to the third problem with the proposal; it results in a kind of failure of extensionality which is perhaps worse than the one which it seeks to avoid. In contrast to the kind of restrictions which might be motivated by a desire to avoid the paradoxes, the restriction in question here fails to be a congruence with respect to coextensiveness. That is, two concepts may be coextensive but differ with respect to whether they are rigid. As a result, just as in the above example, one may have two concepts which, although coextensive, do not define the same set (since one does not define a set at all).

As such, the present approach should be rejected. Instead, I shall show how extensions of nonrigid properties may be accommodated rather than ignored.

4 Interpretations of ε in a Modal Context

What is the intended interpretation of the set operator when we consider possible-worlds models for modal logic? Is there an interpretation which can reasonably be called "intended" and which allows for the intensional interpretation of second-order logic?

In the case of the plural interpretation, the same interpretation as in the nonmodal case will do the job. We just let $\varepsilon^{\mathcal{M}}$ be the function $\varepsilon^{\mathcal{M}} : \mathcal{P}(D) \to D$ given by

$$I(\varepsilon)(X) = \begin{cases} X & X \in D, \\ \text{undefined} & X \notin D. \end{cases} \tag{1}$$

It is then simple to check that, with the standard translation of $\in$, this will satisfy (Rigid$_\in$) and (Ext).

Proposition 4.1 *For any $\mathcal{M} = \langle D, W, I\rangle$ where I is defined as in (1), $\mathcal{M} \models$ (Rigid$_\in$) and $\mathcal{M} \models$ (Rigid$_{\text{Set}}$).*

Proof First, we can prove that, for any assignment a,

$$\mathcal{M}, w, a \models x \in y \quad \text{iff } a(x) \in a(y) \text{ and } a(y) \subseteq D \tag{$*$}$$

and

$$\mathcal{M}, w, a \models \exists xx(x = \varepsilon xx) \quad \text{iff } a(x) \subseteq D. \tag{$**$}$$

For ($*$), suppose that $\mathcal{M}, w, a \models x \in y$. Then, by the definition of $\in$, $\mathcal{M}, w, a \models \exists xx(y = \varepsilon xx \wedge x \prec xx)$. So, there is $X \subseteq D$ such that $a(y) = I(\varepsilon)(X)$ and $a(x) \in X$. By (1), we then have $a(y) = X$, so $a(x) \in a(y)$ and $a(y) \subseteq D$, as required.

For the converse direction, suppose that $a(x) \in a(y)$ and $a(y) \subseteq D$. That $\mathcal{M}, w, a \models \exists xx(y = \varepsilon xx \wedge x \prec xx)$ follows simply by considering an assignment which maps xx to $a(y)$ (which, being a subset of D, is eligible as such an interpretation).

For ($**$), $\mathcal{M}, w, a \models \exists xx(x = \varepsilon xx)$, if and only if for some $X \subseteq D$, $a(x) = X$, if and only if $a(x) \subseteq D$.

Now we can show that (Rigid$_\in$) and (Rigid$_{\text{Set}}$) hold. For the first, suppose that $\mathcal{M}, w, a \models \Diamond(x \in y)$. So, for some $w' \in W$, $\mathcal{M}, w', a \models (x \in y)$, and thus $a(x) \in a(y)$ by the left-to-right direction of ($*$). Now, for any arbitrary world $w'' \in W$, $\mathcal{M}, w'', a \models x \in y$, by the right-to-left direction of ($*$). Hence $\mathcal{M}, w, a \models \Box(x \in y)$, as required.

For ($\text{Rigid}_{\text{Set}}$) the proof is essentially the same, using ($**$). $\Box$

But what about the intensional case? Since ε denotes a function from intensions to objects, we would perhaps expect the interpretation of ε to be a function $\mathbf{D}_{\text{pr}} \to D$. Indeed, given what I have said about how the denotation of a complex term is to be defined, it must be. But, when considering that εF must be a nonrigid designator, the value of $(\varepsilon F)^{\mathcal{M},w,a}$ must depend on w. In particular, we want the following (bearing in mind that this is not a proposal for an interpretation of ε but the outcome of such an interpretation for the denotation of εF):

$$(\varepsilon F)^{\mathcal{M},w,a} = \begin{cases} a(F)(w) & a(F)(w) \in D, \\ \text{undefined} & a(F)(w) \notin D. \end{cases}$$

The most immediate suggestion of how to achieve this outcome would be for the interpretation of ε itself to depend on a world. So, we assign ε a different interpretation $I(\varepsilon, w) : \mathbf{D}_{\text{pr}} \to D$, depending on the world w:

$$I(\varepsilon, w)(f) = \begin{cases} f(w) & f(w) \in D, \\ \text{undefined} & f(w) \notin D. \end{cases} \tag{2}$$

However, on closer inspection this proposal may be thought to be unsatisfactory. Since it assigns ε a different interpretation at different worlds, the result is that ε becomes in some sense a nonrigid designator. That is, at each world it denotes a *different* type-lowering function, just as in the example of a nonrigid constant term, c denotes a different *object* at each world. But this is rather implausible. We do not mean to denote a different function at each world; we want to signify the *set of* operator, no matter at which world we are.

There is, however, a different way to look at the matter which is more plausible. That is to treat ε as a rigid designator (by it having the same interpretation at every world) but with its interpretation being an intensional entity. The appropriate comparison is that, instead of being like a nonrigid singular constant, ε is like a predicate, such as "red," which denotes the same *property* or *intension* at every world, but that this intension acts like a function from worlds to extensions. So, the interpretation of ε is a function $I(\varepsilon) : W \times \mathbf{D}_2 \to D$, which may be given by

$$I(\varepsilon)(w, f) = \begin{cases} f(w) & f(w) \in D, \\ \text{undefined} & f(w) \notin D. \end{cases} \tag{3}$$

Equations (2) and (3) are obviously very similar; indeed all that has changed is that the parameter w has moved from one bracket to another. But they are nonetheless importantly different. This difference is perhaps clearer if the suppressed world parameter of I is displayed, even in the case where, as in (3), it makes no difference. Then there is no change for (2), but (3) becomes

$$I(\varepsilon, w)(w', f) = \begin{cases} f(w') & f(w') \in D, \\ \text{undefined} & f(w') \notin D. \end{cases} \tag{3$'$}$$

In (2), ε is assigned a function $\mathbf{D}_{\text{pr}} \to D$, whereas in (3$'$), ε is assigned a completely different kind of function, which, instead of taking one argument (from $\mathbf{D}_2$), takes two arguments (one from $\mathbf{D}_2$ and one from W).

That $I(\varepsilon)$ is now a function from worlds and intensions to objects means that a slight change is required in how the denotation of a complex term is defined. Rather than being defined as $(\varepsilon F)^{\mathcal{M},w,a} = I(\varepsilon)(a(F))$, it must instead be defined as $(\varepsilon F)^{\mathcal{M},w,a} = I(\varepsilon)(w, a(F))$, to take into account the fact that $I(\varepsilon)$ is now a binary function.[15]

The interpretation of ε given in (3) and (3′) will also then satisfy ($\text{Rigid}_\in$) and ($\text{Rigid}_{\text{Set}}$).

Proposition 4.2 *For any $\mathcal{M} = \langle D, W, I\rangle$ where I is defined as in (3), $\mathcal{M} \models$ ($\text{Rigid}_\in$) and $\mathcal{M} \models$ ($\text{Rigid}_{\text{Set}}$).*

Proof First we can see that, as for the plural case, we have

$$M, w, a \models x \in y \quad \text{iff } a(x) \in a(y) \text{ and } a(y) \subseteq D$$

and

$$M, w, a \models \exists F(y = \varepsilon F) \quad \text{iff } a(y) \subseteq D.$$

For the first, suppose that $\mathcal{M}, w \models x \in y$. Then, by the definition of $\in$, $\mathcal{M}, w \models \exists F(y = \varepsilon F \wedge Fy)$, so there is an intension $f : W \to \mathcal{P}(D)$ such that $a(y) = I(\varepsilon)(w, f)$ and $a(x) \in f(w) \subseteq D$. By (3), we thus have $a(y) = f(w)$, and so $a(x) \in a(y)$ and $a(y) \subseteq D$, as required. For the converse direction, suppose that $a(x) \in a(y) \subseteq D$. Then, let $f \in \mathbf{D}_{\text{pr}}$ be given by $f(w) = a(y)$ for all $w \in W$. This then provides a suitable assignment to F to make $(y = \varepsilon F \wedge Fx)$ true.

For the second, suppose that $\mathcal{M}, w \models \exists F(x = \varepsilon F)$. So, there is $f : W \to \mathcal{P}(D)$ such that $a(x) = I(\varepsilon)(w, f)$. So, by (3), $a(x) = f(w) \subseteq D$, as required. For the converse direction, suppose that $a(x) \subseteq D$. Let $f(w) : W \to \mathcal{P}(D)$ be the constant function $f(w) = a(x)$, which will then witness $\exists F(x = \varepsilon F)$ at any world.

With these, the proof of ($\text{Rigid}_\in$) and ($\text{Rigid}_{\text{Set}}$) then can proceed exactly as for the plural case. □

5 Transworld Extensionality

So, there is no obstacle as far as semantics go to having extensional sets arising from intensional entities. All that is required is that ε itself be intensional in such a way as to cancel out the intensionality in the second-order variables. Such an operator might be called an *extensionalizing* operator. But, we might ask, how can we force the interpretation of ε to have such behavior in the object language? That is, what additional axioms are needed in order to guarantee that the operator has the appropriate extensionalizing behavior?

Of course, one possible answer is that ($\text{Rigid}_\in$) and ($\text{Rigid}_{\text{Set}}$) will suffice. But it would be desirable if these were not required, and if instead they were to follow from suitably modalized versions of the nonmodal axioms. After all, it is considerations concerning extensionality which seem to motivate the two rigidity constraints in the first place. To reiterate these, given a set, the reason it necessarily has the members that it does is that, in some counterfactual situation where some set has different members, this other set cannot be the same set, by extensionality. Thus, it would be natural if these principles were to follow from a principle of extensionality.

There may be additional reasons to favor a way of recovering rigidity from modalizations of the nonmodal axioms, but which depend on more substantial background commitments. For example, neo-Fregeans like Hale and Wright wish to claim that the identity conditions for abstract terms such as εF can serve to fix the meaning of the abstraction operator in question, with the appropriate properties of associated abstract objects following. In this case, it would be desirable that the modal properties also follow from the identity conditions, without having to resort to extra axioms.

5.1 Necessitation of extensionality The obvious axiom to look to provide rigidity is the extensionality axiom; the other axioms just specify what sets exist but not how they are related to each other. The obvious way to modalize extensionality is simply to take its necessitation:

$$\Box\forall F\forall G\big(E!\varepsilon F \wedge E!\varepsilon G \to \big(\varepsilon F = \varepsilon G \leftrightarrow \forall x(Fx \leftrightarrow Gx)\big)\big). \qquad (\text{Ext}\Box)$$

A model $\mathcal{M} = \langle W, D, I\rangle$ will satisfy (Ext$\Box$) just in case, for every w, and every $f, g : W \to \mathcal{P}(D)$ where $I(\varepsilon)(w, f)$ and $I(\varepsilon)(w, g)$ are defined,

$$I(\varepsilon)(w, f) = I(\varepsilon)(w, g) \quad \text{iff } f(w) = g(w) \qquad (4)$$

(making use of extensionality in the metatheory).

But it can be seen that this will not force $I(\varepsilon)$ to satisfy the rigidity requirements.

Proposition 5.1 *We have (Ext$\Box$) $\not\models$ (Rigid$_\in$).*

Proof Let $W = \{w_1, w_2\}$, and let D be a transitive set. Let $\pi : D \to D$ be a (nontrivial) permutation of D, and define $I(\varepsilon)$ as follows:

$$I(\varepsilon)(w, f) = \begin{cases} f(w) & w = w_1, f(w) \in D, \\ \pi(f(w)) & w = w_2, f(w) \in D, \\ \text{undefined} & f(w) \notin D. \end{cases} \qquad (5)$$

It will then be the case that $\mathcal{M} \models$ (Ext$\Box$). Let $f, g \in \mathbf{D}_{\text{pr}}$. At w_1, $I(f, w) = I(g, w)$ iff $f(w) = g(w)$, by (5), as required. At w_2, $I(f, w) = I(g, w)$ iff $\pi(f(w)) = \pi(g(w))$ (by (5)), iff $f(w) = g(w)$ (by injectivity of π), as required.

But $\mathcal{M} \not\models$ (Rigid$_\in$). The aim is to find an assignment a such that $\mathcal{M}$, $a \not\models \Diamond x \in y \to \Box x \in y$. First, we can show that $\mathcal{M}, w_1 \models x \in y$ iff $a(x) \in a(y)$, and $\mathcal{M}, w_2 \models x \in y$ iff $a(x) \in \pi^{-1}(a(y))$. For the first,

$$\begin{aligned} \mathcal{M}, w_1, a \models x \in y \quad & \text{iff } \mathcal{M}, w_1, a \models \exists F(y = \varepsilon F \wedge Fx) \\ & \text{iff } \exists f \in \mathbf{D}_{\text{pr}} \text{ such that } a(y) = f(w) \text{ and } a(x) \in f(w) \\ & \text{iff } a(x) \in a(y), \end{aligned}$$

as required.

For the second,

$$\begin{aligned} \mathcal{M}, w_2, a \models x \in y \quad & \text{iff } \mathcal{M}, w_2, a \models \exists F(y = \varepsilon F \wedge Fx) \\ & \text{iff } \exists f \in \mathbf{D}_{\text{pr}} \text{ such that } a(y) = \pi\big(f(w)\big) \text{ and } a(x) \in f(w) \\ & \text{iff } a(x) \in \pi^{-1}\big(a(y)\big), \end{aligned}$$

as required.

Now, let $\beta \in D$ such that $\pi^{-1}(\beta) \neq \beta$. So, either (a) $\exists\alpha \in D$ such that $\alpha \in \beta$ but $\alpha \notin \pi^{-1}(\beta)$, or (b) $\exists\alpha \in D$ such that $\alpha \notin \beta$ and $\alpha \in \pi^{-1}(\beta)$.

Suppose that (a) is the case. Then let $a(x) = \alpha$, and let $a(y) = \beta$. So, $\mathcal{M}, w_1, a \models x \in y$, so $\mathcal{M}, w_1, a \models \Diamond x \in y$. But $\mathcal{M}, w_2, a \not\models x \in y$, so $\mathcal{M}, w_1, a \not\models \Box(x \in y)$. Hence $\mathcal{M}, w_1, a \not\models$ (Rigid$_\in$). A similar argument can be given in case (b).

Hence, $\mathcal{M}$ is a countermodel, as required. $\Box$

5.2 Transworld extensionality The considerations which told in favor of rigidity concerned what might be called *transworld* extensionality. That is, they involved comparing the extension of a set at one world, with the extension of a set in some counterfactual situation. The problem with (Ext$\Box$) and its model-theoretic counterpart is that they do not give criteria of identity *across* worlds but rather always within one world. What is required instead is something like the following. For any $w_1, w_2 \in W$, and any $f, g \in \mathbf{D}_2$:

$$I(\varepsilon)(f, w_1) = I(\varepsilon)(g, w_2) \quad \text{iff } f(w_1) = g(w_2). \tag{6}$$

Any interpretation of ε which satisfies (6) will then satisfy (Rigid$_\in$).

Proposition 5.2 *For $\mathcal{M} = \langle D, W, I\rangle$, if I satisfies (6), then $\mathcal{M} \models$ (Rigid$_\in$).*

Proof Consider a model $\mathcal{M}$ and an assignment a. Suppose that $\mathcal{M}$ satisfies $\Diamond(x \in y)$; so, for some $w \in W$ and some $f : W \to \mathcal{P}(D)$, $a(y) = I(\varepsilon)(w, f)$ and $a(x) \in f(w)$. We need to show that for every $w' \in W$ there is $f' : W \to \mathcal{P}(D)$ such that $a(y) = I(\varepsilon)(w', f')$ and $a(x) \in f'(w')$. We do this as follows. Let f' be defined by, for any $w' \in W$, $f'(w') = f(w)$ (so f has the same extension at every world—namely, the same one that it has at the originally considered world).

Now consider an arbitrary world w'. We have $f'(w') = f(w)$, so by the transworld identity condition, $I(\varepsilon)(w', f') = I(\varepsilon)(w, f) = a(y)$, and so $a(x) \in f'(w')$; hence $\mathcal{M}$ satisfies $\Box(x \in y)$, and so $\mathcal{M}$ satisfies (Rigid$_\in$). $\Box$

So, since (Ext$\Box$) does not achieve the goal of giving transworld extensionality conditions, is there another principle which does? If we are prepared to introduce a few more expressive devices into the modal language, then there will be. What we want to be able to do is answer questions along the lines of "had things been different from how they actually are, would F have had the same extension that it actually has?" or, more generally, "had things been different, would F have had the same extension as G actually has?" This makes use of an actuality operator in order to bring some part of the claim out of the scope of a modal operator. So, we may be able to express transworld extensionality by introducing an actuality operator @ to the language. However, as well as acting as an operator on formulas, as is usual, it will be required that @ can act as an operator on *terms*, having the effect that in "$@t$", the term t must be evaluated as if it were exempt from the scope of modal operators. The reason is that we want to express terms such as "the actual set of F." This is a natural use of the actuality operator if we are to consider languages that may feature nonrigid designators.

Then, the following principle can be laid down as an improvement of (Ext$\Box$):

$$\Box\forall F \forall G\big[E!(@\varepsilon F) \wedge E!\varepsilon G \to @\varepsilon F = \varepsilon G \leftrightarrow \forall x(@Fx \leftrightarrow Gx)\big]. \tag{Ext@}$$

The semantics of an actuality operator will depend on designating an actual world $w_@$ as part of a model. Then it can be easily be seen that (Ext@) would have the following effect. A model $\mathcal{M}$ with designated actual world $w_@$ will satisfy (Ext$\Box$) just in the case where, for any $w \in W$,

$$I(\varepsilon)(w_@, f) = I(\varepsilon)(w, g) \quad \text{iff } f(w_@) = g(w).$$

Now, this comes close as a solution but is still not quite acceptable. What (Ext@) allows one to do is evaluate questions of identity between the extension that a concept actually has and extensions that concepts may have had. But it does not allow one to evaluate identity claims between an extension that a concept may have had in one situation with the extension that another concept may have had in another situation.

To attain the required level of generality, we can introduce more nuanced scoping operators into the language. So, for example, Hodes [10] develops a modal logic which contains an operator $\downarrow$. This has the effect that what follows it is to be exempt from just the *innermost* modal operator.[16] Again, it will be required to extend the use of $\downarrow$ somewhat so that it may apply to terms as well as to formulas. Then, we may express transworld extensionality as follows:

$$\Box\Box\forall F\forall G\big[E!{\downarrow}\varepsilon F \wedge E!\varepsilon G \rightarrow \big({\downarrow}\varepsilon F = \varepsilon G \leftrightarrow \forall x({\downarrow}Fx \leftrightarrow Gx)\big)\big]. \tag{Ext$\downarrow$}$$

Now (although I shall not go into detail of the semantics of $\downarrow$ here), it can be seen that the effect of (Ext$\downarrow$) is precisely (6), as required. The two necessity operators mean that two worlds w_1 and w_2 must be considered, and the use of $\downarrow$ ensures that the evaluation of εF and Fx occurs at w_1, while the evaluation of εG and Gx occur at w_2.

6 Conclusions

What then are we to make of this? My main aim was to show that, if one wishes to develop an abstractionist set theory, there is no more reason—or at least, none arising from modal considerations—to see sets as arising from pluralities than there is to see sets as arising from intensional entities. There need be no general requirement that abstraction operators themselves be extensional, and this allows for a natural way in which intensional second-order entities may result in purely extensional abstracts. This conclusion is supported further by the observation that a transworld principle of extensionality can secure just the appropriate extensionalizing behavior of the abstraction operator.

But the conclusions can go further than just abstractionist set theory, either to the theory of abstraction in general (as plays a key role in neo-Fregean philosophy), or to modal set theory in general. I will briefly consider these in turn and hint at how similar considerations at work here may also play a role.

Neo-Fregeans want to make use of operators similar to ε to build a logicist foundation of mathematics and, more generally, to explain reference to abstract objects. One key example is that of the *number of* operator, which is said to be governed by a criterion of identity known as *Hume's principle*. This says that, for concepts F and G, the number of F's $=$ the number of G's iff there is a relation which puts the F's and G's into one-to-one correspondence. Like set abstract terms, number terms such as "the number of F's" will in general be nonrigid. For example, the term "the number of human inhabitants of Earth" refers to a natural number in the vicinity of 7 billion, but it might have referred to a different natural number. A similar treatment of the transworld identity conditions would likewise be required in order to capture this.[17]

The approach here is also likely to be of use in motivating modal principles of sets in general; the method of expressing transworld extensionality by use of a scoping device is not specific to the abstractionist approach. Discussions of modal set theory, such as [4] and [13] typically assert that rigidity principles such as (Rigid$_\in$) and (Rigid$_{\mathrm{Set}}$) hold of sets, and also typically informally motivate these with considerations of extensionality. By adopting a modal principle of extensionality along similar lines to (Ext$\downarrow$), these motivations could be internalized to the theory. Then, instead of being an addition to the theory, ($\mathrm{Rigid}_{\in}$) and ($\mathrm{Rigid}_{\mathrm{Set}}$) will become theorems of the theory.

Notes

1. This is a somewhat wider use of the term "abstractionist" than is common, where it is used to refer to an explicitly Fregean and neo-Fregean approach to mathematics in general.

2. This way of putting things does not do justice to the plural interpretation of second-order logic nor, arguably, to the interpretation of second-order logic over concepts.
   Under the plural interpretation, it would not be right to call the denotation of X a single *entity*. Instead, for plural quantifiers, this should read as "for any objects xx, there may be a set εxx, which is the set of them." And it may not be correct to refer to the values of second-order variables as *entities* on the concept reading, since this may suggest that they are a kind of object and thus values of first-order variables.
   For simplicity's sake, I shall continue to write of the value of a plural variable or a concept variable as being a single entity, with the understanding that this could be transformed to whichever strictly correct reading may be preferred.

3. It would also be possible to proceed instead with a relation between second-order entities and first-order entities, $\mathrm{Set}(X, x)$, where this is to be read as "x is the set of X" or something similar. This is the approach taken by [3] and [12]. There are a few reasons why I have chosen to proceed with a type-lowering function rather than a relation: (1) There is a long history, going back to Frege [5], of examining principles of a form similar to (Ext)—called *abstraction principles*—which make use of type-lowering functions. (2) My aim in this paper will be to argue against some reasons for thinking that, for the purposes of abstractionist set theory, sets should be thought of as arising from pluralities rather than properties. It seems to me that these reasons are stronger when considering the functional approach. So, by concentrating on the functional approach, I am addressing a (prima facie) stronger, rather than weaker, argument.
   In any case, is should be apparent that both approaches are in all other respects equivalent.

4. *Proof*: For the left-to-right direction, suppose that $x \in \varepsilon X$. Then, by the definition, $\exists Y(Yx \wedge \varepsilon X = \varepsilon Y)$. By extensionality, since $\varepsilon X = \varepsilon Y$, $\forall y(Xy \leftrightarrow Yy)$. So, since Yx, Xx, as required.
   For the right-to-left direction, suppose Xx. Then, trivially (but making use of the existence of εX), $\varepsilon X = \varepsilon X \wedge Xx$. Thus $\exists Y(\varepsilon X = \varepsilon Y \wedge Yx)$, which is $x \in \varepsilon X$, as required. □

5. I will not consider approaches that place restrictions on the second-order comprehension principle.

6. That this is the case allows one to easily develop a theory within this framework which interprets all of ZFC. Simply add to (Ext_1) set existence axioms such as

$$\neg\exists x Xx \to E!\varepsilon X, \tag{Empty set}$$

$$\forall x(Xx \leftrightarrow x = u \vee x = v) \to E!\varepsilon X, \tag{Pairing}$$

$$\forall x\big(Xx \leftrightarrow \forall y(y \in x \to y \in u)\big) \to E!\varepsilon X, \tag{Power set}$$

and so on for other existence axioms. Then choice can be taken to be a principle in the second-order logic, and foundation can be achieved simply by restricting quantifiers to well-founded sets.

7. This way of restricting extensionality and stating existence conditions will likely not be acceptable from a neo-Fregean point of view. The reasons are: (1) it makes use of the abstraction operator ε in the restriction (see, e.g., Wright [18, pp. 9–10] for reasons why neo-Fregeans find this unacceptable) and (2) existence assumptions here are explicit, whereas neo-Fregeans would like existence claims to follow from the abstraction principle itself.

   However, in certain circumstances, there will be equivalent ways of restriction which are acceptable for the neo-Fregean. When $\varphi(X)$ is a sufficient condition for X to have a set, then (Ext_1) together with the restriction will be equivalent to the following:

$$\forall X \forall Y \big(\varphi(X) \wedge \varphi(Y) \rightarrow \varepsilon X = \varepsilon Y \leftrightarrow \forall x (Xx \leftrightarrow Yx)\big).$$

   (This is essentially the same as the form of restriction (A) discussed by [6].)

   When $\varphi(X)$ is a necessary and sufficient condition for X to have a set, the following will be equivalent to (Ext_1) together with the restriction:

$$\forall X \forall Y \big(\varepsilon X = \varepsilon Y \leftrightarrow \varphi(X) \wedge \varphi(Y) \wedge \forall x (Xx \leftrightarrow Yx)\big).$$

   In either of these cases, if φ is a sentence which does not involve the abstraction operator, the resulting principles avoid the problems which face (Ext_1).

8. By considering only one second-order domain, this will only accommodate monadic second-order logic. But since the target of study is the abstraction operator ε, which acts only on monadic second-order entities, this will suffice. (In addition, in the case of plural logic, it is quite unclear what polyadic second-order variables might refer to.)

9. One difference in the nonmodal case might be achieved by restricting plural comprehension to cases where a formula is instantiated by at least one (or perhaps at least two) objects. The rationale behind this would be that it is incorrect to say, for example "there are some things which are all and only the non-self-identicals"—there is not even *one* non-self-identical, let alone *some* non-self-identicals. I am following [3] and [12] in ignoring such a consideration. Ultimately, this may be a feature of plural logic which tells against it as a basis for abstractionist set theory (since then it may not be able to accommodate the empty set). My discussion will however only concern differences between the interpretations with respect to modal behavior.

10. See, for example, Fine [4], Parsons [13], Williamson [16] for arguments that sets should be expected to obey such properties.

11. Since the underlying modal logic will be S5, I am for simplicity omitting mention of an accessibility relation; every world will be accessible from every other world.

12. Unlike, say, a difference between the second-order domain being *all* subsets of D versus just the second-order definable subsets of D.

13. This may be what [13, p. 303] means to suggest when he writes that "classes are essentially extensions of predicates . . . however it might be with sets, when we speak of classes in modal context, we should regard class abstracts as rigid designators."

14. Thank you to the anonymous referees for highlighting this alternative proposal and for some suggestions on why it may be less than optimal.

15. It should be noted that, although (2) and (3) are different ways of interpreting the abstraction operator, this difference does not manifest itself in the object language, at least, not without the addition of extra expressive power to the language. The reason is that the abstraction operator only ever appears as a component of a term of the form εF, and (2) and (3) result in the same interpretation for such a term.

   The difference would manifest itself in the object language if we were to add some expressive resources, such as quantification and identity over the appropriate kinds of function. Then, the rigidity of ε could be expressed as

$$\Box\forall f\big(\Diamond(f = \varepsilon) \to \Box(f = \varepsilon)\big).$$

   So, if one is only interested with differences at the level of the object language, the difference between (2) and (3) will be moot.

   Thank you to the anonymous referees for highlighting this matter to me.

16. See also [13, Appendix] for similar scoping devices, which instead indicate within how many modal scopes a formula *is* contained.

17. Hale and Wright [8, p. 358] explicitly invoke the notion of "transworld parallelism" when discussing the identity conditions for the directions of lines. It is unclear whether an approach similar to the one taken here could capture such a relation, but perhaps this is due partly to it being less than completely clear what transworld parallelism would, in general, amount to.

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Acknowledgments

Many thanks to Bob Hale, Rosanna Keefe, and Øystein Linnebo for useful discussions on earlier versions of this paper. I would also like to thank the two anonymous referees. Author's work supported by an Arts and Humanities Research Council doctoral award.

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