

## Bounded Martin's Maximum with an Asterisk

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**Abstract** We isolate natural strengthenings of Bounded Martin's Maximum which we call  $\text{BMM}^*$  and  $A\text{-BMM}^{*,++}$  (where  $A$  is a universally Baire set of reals), and we investigate their consequences. We also show that if  $A\text{-BMM}^{*,++}$  holds true for every set of reals  $A$  in  $L(\mathbb{R})$ , then Woodin's axiom  $(*)$  holds true. We conjecture that  $\text{MM}^{++}$  implies  $A\text{-BMM}^{*,++}$  for every  $A$  which is universally Baire.

W. H. Woodin, P. Larson, I. Farah, and M. Magidor asked the second author whether the method developed in [2] and [4] can be applied to show other  $\Pi_2$  statements which are discussed in [13]. In particular, they asked if the statements from Definition 1.2 below can be shown from Bounded Martin's Maximum,  $\text{BMM}$ , together with the precipitousness of  $\text{NS}_{\omega_1}$ . This led the second author to the formulation of the "maximality" principle  $\text{BMM}^*$  (see Definition 1.9), which says that if a  $\Sigma_1$  statement  $\varphi$  (with parameters from  $H_{\omega_2}$ ) is "honestly consistent," then  $\varphi$  holds true in  $V$ .

A scenario for proving  $\text{BMM}^*$  from  $\text{BMM}$  plus  $\text{NS}_{\omega_1}$  is precipitous appears naturally: one would have to show that if a  $\Sigma_1$  statement is "honestly consistent," then it can be forced by a stationary set preserving forcing. It has been conjectured (see, e.g., Magidor [10, Conjecture 6.8]) that Martin's Maximum<sup>++</sup> implies Woodin's axiom  $(*)$ . Showing that every "honestly consistent"  $\Sigma_1$  statement can be forced by a stationary set preserving forcing would verify this conjecture, but the present paper has to leave this conjecture unanswered.

We are able to show, though, that a strengthening of  $\text{BMM}^*$  implies  $(*)$ . This strengthening allows  $\text{NS}_{\omega_1}$  as well as universally Baire sets  $A$  as parameters and will be written as  $A\text{-BMM}^{*,++}$  (see Definition 2.6). Our Theorem 2.7 says that in the presence of large cardinals,  $(*)$  follows from  $A\text{-BMM}^{*,++}$  for all sets of reals  $A$

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in  $L(\mathbb{R})$ . We conjecture that  $MM^{++}$  implies  $A\text{-BMM}^{*,++}$  for every universally Baire set  $A$ .

We assume that the reader has some familiarity with forcing axioms as well as with Woodin’s  $\mathbb{P}_{\max}$ . Classical texts on forcing axioms are Foreman, Magidor, and Shelah [5] and Goldstern and Shelah [6] (see also [10]). The forcing  $\mathbb{P}_{\max}$  was introduced in [13] (see also Larson [9]).

Throughout this paper, we let  $NS = NS_{\omega_1}$  denote the nonstationary ideal on  $\omega_1$ . The Bounded Proper Forcing Axiom, BPFA (see [6]), says that for every proper poset  $\mathbb{P}$  and every  $\mathbb{P}$ -generic filter  $G$  over  $V$ ,

$$((H_{\omega_2})^V; \in) <_{\Sigma_1} ((H_{\omega_2})^{V[G]}; \in).$$

The formulation of Bounded Martin’s Maximum, BMM, results from that of BPFA by replacing “proper” with “stationary set preserving.” Given a universally Baire set  $A \subset \mathbb{R}$ , *A Bounded Martin’s Maximum*<sup>++</sup> (see [13, Definition 10.91]) says that for every stationary set preserving poset  $\mathbb{P}$  and every  $\mathbb{P}$ -generic filter  $G$  over  $V$ ,

$$((H_{\omega_2})^V; \in, (NS_{\omega_1})^V, A) <_{\Sigma_1} ((H_{\omega_2})^{V[G]}; \in, (NS_{\omega_1})^{V[G]}, A^*),$$

where  $A^*$  is  $V[G]$ ’s version of  $A$ ; that is, if the trees  $T$  and  $U$  witness that  $A$  is  $|\mathbb{P}|^+$ -universally Baire with  $A = p[T]$ , then  $A^* = p[T] \cap V[G]$ .

A  $\mathbb{P}_{\max}$ -condition is a countable transitive structure  $p = (M; \in, I, a)$  such that  $M$  is a model of a fragment of ZFC plus  $MA_{\omega_1}$ ,  $p \models \text{“}I \text{ is a normal uniform ideal on } \omega_1\text{,”}$   $a \in \mathcal{P}(\omega_1^M) \cap M$  is such that  $\omega_1^M = \omega_1^{L[a,x]}$  for some  $x \in \mathbb{R} \cap M$ , and  $p$  is generically iterable (see [13, Definition 3.5]). If  $p = (M; \in, I, a)$  and  $q = (N; \in, J, b)$  are in  $\mathbb{P}_{\max}$ , then  $q <_{\mathbb{P}_{\max}} p$  if and only if there is a generic iteration of  $p$  which gives rise to an embedding

$$j: p = (M; \in, I, a) \rightarrow (M^*; \in, I^*, j(a))$$

such that  $j(a) = b$ ,  $\{M^*, j\} \in N$ , and  $J \cap M^* = I^*$ . *Woodin’s Axiom* (\*) (see [13, Definition 5.1]) says that AD, the *Axiom of Determinacy*, holds in  $L(\mathbb{R})$  and  $L(\mathcal{P}(\omega_1))$  is a  $\mathbb{P}_{\max}$ -extension of  $L(\mathbb{R})$ ; that is, there is some  $G$  which is  $\mathbb{P}_{\max}$ -generic over  $L(\mathbb{R})$  and

$$L(\mathcal{P}(\omega_1)) = L(\mathbb{R})[G].$$

### 1 Bounded Martin’s Maximum\*

Let us start with some examples.

**Definition 1.1** Let  $B \subset \omega_1$ . We say that  $B$  is *amenablely closed* if and only if for all  $D \subset \omega_1$ , if  $D \cap \xi \in L[B]$  for all  $\xi < \omega_1$ , then  $D \in L[B]$ .

By Baumgartner [1], “ $B$  is amenablely closed” may be formulated in the presence of BPFA in a  $\Sigma_1$  fashion as follows.

Let  $B \subset \omega_1$  be amenablely closed. The set of all cofinal branches through the tree  $T = {}^{<\omega_1}\omega_1 \cap L[B]$  is then contained in  $L[B]$  and has cardinality  $\aleph_1$  in  $V$  since, under BPFA,  $\omega_2^V$  is inaccessible (in fact,  $\Sigma_2$ -reflecting) in every inner model of the form  $L[X]$  for  $X \subset \omega_1$  (see [6]). If BPFA holds true, then  $T$  is *weakly special*; that is, there is a function  $f: T \rightarrow \omega$  such that for all  $s, t, t' \in T$ , if  $f(s) = f(t) = f(t')$ ,  $s \subset t$ , and  $s \subset t'$ , then  $t \subset t'$  or  $t' \subset t$  (see [1]). For each cofinal branch  $b$  through  $T$  there is then some  $s \in T$  such that

$$b = \{t \in T: \exists t' \supset s(t \subset t' \wedge f(t') = f(s))\}.$$

We then have that under BPFA a given  $B \subset \omega_1$  is amenable closed if and only if there is some  $\alpha < \omega_2$  and some  $f: {}^{<\omega_1}\omega_1 \cap J_{\omega_1}[B] \rightarrow \omega$  witnessing that  ${}^{<\omega_1}\omega_1 \cap J_{\omega_1}[B]$  is weakly special and such that for all  $s \in T$ ,

$$\{t \in T: \exists t' \supset s (t \subset t' \wedge f(t') = f(s))\} \in J_\alpha[B].$$

**Definition 1.2** We will be concerned with the following two statements.

- (1) Let  $S \subset \omega_1$  be stationary and costationary. There is then some  $x \in \mathbb{R}$  and some  $G$  which is  $\text{Col}(\omega, < \omega_1^V)$ -generic over  $L[x]$  such that  $L[x, S] = L[x, G]$  (see [13, Theorem 5.74(5)]).
- (2) Let  $A \subset \omega_1$ . There is then some amenable closed  $B \subset \omega_1$  with  $A \in L[B]$  (see [13, Theorem 6.108 (5)]).

It is not hard to see that, for example, if BPFA holds true, then both (1) and (2) may be formulated as  $\Pi_2^{H\omega_2}$  sentences. For (2), this uses the remark after Definition 1.1.

The following observation is very easy.

**Lemma 1.3** *If (1) holds, then  $\mathbb{R}$  is closed under  $\sharp$ 's, and  $\delta_2^1 = \omega_2$ .*

**Proof** Let  $z \subset \omega$ . In order to show that  $z^\sharp$  exists it suffices to see that every  $X \in \mathcal{P}(\omega_1) \cap L[z]$  either contains a club or is disjoint from a club, as then the club filter on  $\omega_1$ , restricted to  $L[z]$ , is an  $L[z]$ -ultrafilter. Suppose that  $S' \in \mathcal{P}(\omega_1) \cap L[z]$  is stationary and costationary in  $V$ . Then  $S = (S' \setminus \omega) \cup z$  is also stationary and costationary. By (1), there is some  $x \in \mathbb{R}$  and some  $G$  which is  $\text{Col}(\omega, < \omega_1)$ -generic over  $L[x]$  with  $L[x, S] = L[x, G]$ . But  $L[x, S] = L[x, z]$ , so that there is some  $\bar{G}$  which is  $\text{Col}(\omega, < \omega_1)$ -generic over  $L[x, z]$  with  $L[x, G] = L[x, z, \bar{G}]$ . But then  $L[x, z] = L[x, S] = L[x, G] = L[x, z, \bar{G}]$ , which contradicts the fact that every real  $z \in L[x, G]$  is in  $L[x, G \upharpoonright \alpha]$  for some  $\alpha < \omega_1$ .

To see that  $\delta_2^1 = \omega_2$ , let  $\beta < \omega_2$ , and let  $A \subset \omega_1$  be such that  $\beta < (\omega_1^V)^{+L[A]}$ . Let  $S' \subset \omega_1$  be stationary and costationary, and let

$$S = \{\omega \cdot \alpha: \alpha \in S'\} \cup \{\omega \cdot \alpha + 1: \alpha \in A\}.$$

Then  $S$  is again stationary and costationary, and if  $x \in \mathbb{R}$  and  $G \text{ Col}(\omega, < \omega_1)$ -generic over  $L[x]$  are such that  $L[x, S] = L[x, G]$ , then

$$(\omega_1^V)^{+L[x]} = (\omega_1^V)^{+L[x, G]} = (\omega_1^V)^{+L[x, S]} \geq (\omega_1^V)^{+L[A]} > \beta,$$

so that  $\beta < \delta_2^1$ . □

In particular, (1) by itself implies  $\neg\text{CH}$ , the negation of the Continuum Hypothesis. On the other hand, in  $L$ , every subset of  $\omega_1$  is trivially amenable closed, so that (2) holds in  $L$  and does not by itself imply  $\neg\text{CH}$ . The situation is a bit more tricky under forcing axioms. As we said, under BPFA,  $\omega_2^V$  is inaccessible in every inner model of the form  $L[B]$  for  $B \subset \omega_1$ . Suppose that (2) holds, and suppose that  $\omega_2^V$  is inaccessible in every inner model of the form  $L[B]$  for  $B \subset \omega_1$ . If  $W \subset V$  is an inner model of GCH, then we may pick some  $A \in W$ ,  $A \subset \omega_1$ , such that  $HC \cap W = HC \cap L[A]$ . If  $A \in L[B]$ , where  $B \subset \omega_1$  is amenable closed, then  $\mathcal{P}(\omega_1) \cap W \subset L[B]$ , so that  $(\omega_1^V)^{+W} < \omega_2$ . In particular, we have the following.

**Lemma 1.4** *If (2) holds and  $H_{\omega_2}$  is closed under  $\sharp$ 's, then CH fails.*

Whereas Lemma 1.3 shows that (1) by itself is a fairly strong principle, (2) is only strong in the presence of, for example, a precipitous ideal on  $\omega_1$ .

**Lemma 1.5** *If (2) holds and there is a precipitous ideal on  $\omega_1$ , then there is an inner model with a Woodin cardinal.*

**Proof** If there is a precipitous ideal on  $\omega_1$ , then  $H_{\omega_2}$  is closed under  $\sharp$ 's. Suppose that Lemma 1.5 fails, and let  $K$  denote the core model below a Woodin cardinal. By the remarks before the statement of Lemma 1.4,  $(\omega_1^V)^{+K} < \omega_2$ . On the other hand, by Claverie and Schindler [3, Theorem 0.3], if there is a precipitous ideal on  $\omega_1$ , then  $(\omega_1^V)^{+K} = \omega_2$ , a contradiction.  $\square$

We are now about to propose our strengthening of BMM (Bounded Martin's Maximum). Recall that BMM says that if  $A \in H_{\omega_2}$ ,  $\varphi(x)$  is a  $\Sigma_1$  formula and  $\mathbb{P} \in V$  is a poset which preserves stationary subsets of  $\omega_1$ , then

$$V^{\mathbb{P}} \models \varphi(A) \implies V \models \varphi(A).$$

We might strengthen this statement by saying that if  $\varphi(A)$  is “consistent,” then  $\varphi(A)$  is true, where we might try to spell out “consistent” as in the following version of BMM.

Let us write  $\text{BMM}^{\circ}$  for the statement that if  $A \in H_{\omega_2}$ , if  $\varphi(x)$  is a  $\Sigma_1$  formula, and if there is some transitive model  $\mathfrak{A}$  such that

- (a)  $\mathfrak{A} \in V^{\text{Col}(\omega, 2^{\aleph_1})}$ ,
- (b)  $(H_{\omega_2})^V \subset \mathfrak{A}$ ,
- (c) if  $T \subset \omega_1^V$ ,  $T \in V$ , and  $V \models T$  is stationary, then  $\mathfrak{A} \models T$  is stationary, and
- (d)  $\mathfrak{A} \models \text{ZFC}^- + \varphi(A)$ ,

then  $\varphi(A)$  is true in  $V$ .

If in (a) we demand that  $\mathfrak{A}$  be in  $V$  rather than just  $V^{\text{Col}(\omega, 2^{\aleph_1})}$ , then the hypothesis would already say that  $\varphi(A)$  is true in  $V$ . If we dropped (c), then a counterexample would be given by  $\varphi(A) \equiv “A$  is disjoint from a club” for some  $A \subset \omega_1$  which is stationary in  $V$  but not in  $\mathfrak{A}$ .

Clearly,  $\text{BMM}^{\circ}$  is a strengthening of BMM. By Schindler [11],  $\text{BMM}^{\circ}$  thus implies that  $V$  is closed under  $\sharp$ 's. This may be used to show that  $\text{BMM}^{\circ}$  is in fact inconsistent. Let us consider the statement  $\varphi(\omega_1) \equiv “\text{there is some } x \in \mathbb{R} \text{ such that } \omega_1 = \omega_1^{L[x]}.”$  Let  $V_{\alpha}$  be a model of a sufficiently rich finite fragment of ZFC. We may force over  $V_{\alpha}$  by Jensen coding to add some  $G$  which is class generic over  $V_{\alpha}$  such that in  $V_{\alpha}[G]$ , there is some real  $x$  with  $V_{\alpha}[G] = J_{\alpha}[x]$ . As Jensen coding preserves stationary subsets of  $\omega_1$  (see [11]), Shoenfield absoluteness yields that there is some  $\mathfrak{A}$  with (a), (b), (c), and (d) for  $A = \omega_1$  and  $\varphi(\omega_1) \equiv “\text{there is some } x \in \mathbb{R} \text{ such that } \omega_1 = \omega_1^{L[x]}.”$  Then  $\text{BMM}^{\circ}$  would imply that in  $V$  there is a real  $x$  such that  $\omega_1 = \omega_1^{L[x]}$ , which contradicts the existence of  $x^{\sharp}$ .

The problem with  $\text{BMM}^{\circ}$  is that it ignores that the model  $\mathfrak{A}$  has to be “as closed as”  $V$ . For BMM this is automatic, as every set generic extension of  $V$  is “as closed as”  $V$ . We need to make this requirement explicit if we aim to arrive at a consistent weakening of  $\text{BMM}^{\circ}$  that strengthens BMM. We will spell out the necessary closure of  $\mathfrak{A}$  in terms of universally Baire sets of reals, basically as in [13].

We call a function  $F: \mathbb{R} \rightarrow \mathbb{R}$  *universally Baire* if and only if its graph  $F = \{(x, F(x)): x \in \mathbb{R}\}$  is a universally Baire subset of  $\mathbb{R}^2$ . Let  $U: \mathbb{R} \rightarrow \mathbb{R}$  be universally Baire, as being witnessed by the class-sized trees  $T$  and  $U$  with  $F = p[T]$  and  $V^{\mathbb{P}} \models p[U] = {}^{\omega}\omega \setminus p[T]$  for all  $\mathbb{P} \in V$ . Then if  $\mathbb{P} \in V$  is any poset and if  $G$  is  $\mathbb{P}$ -generic over  $V$ ,  $F^G$  denotes the (possibly partial) function  $p[T]^{V[G]}$ .

It is easy to see that  $F^G$  is indeed a function. Also, this function is independent from the choice of  $T$  and  $U$ , so the notation  $F^G$  is unambiguous.

**Definition 1.6** Let  $F: \mathbb{R} \rightarrow \mathbb{R}$  be universally Baire. Let  $\Omega$  be an uncountable cardinal, and let  $G$  be  $\text{Col}(\omega, \Omega)$ -generic over  $V$ . Let  $\mathfrak{A} \in V[G]$  be a transitive model of  $\text{ZFC}^-$  which is countable in  $V[G]$ . We say that  $\mathfrak{A}$  is *closed under  $F$*  (or,  *$F$ -closed*) if and only if for all posets  $\mathbb{P} \in \mathfrak{A}$  and for all  $g \in V[G]$  which are  $\mathbb{P}$ -generic over  $\mathfrak{A}$ ,  $\mathfrak{A}[g]$  is closed under  $F^G$ ; that is,  $F^G(x) \in \mathfrak{A}[g]$  for all  $x \in \mathbb{R} \cap \mathfrak{A}[g]$  in the domain of  $F^G$ .

The following lemma can be proved easily by an absoluteness argument.

**Lemma 1.7** Let  $F: \mathbb{R} \rightarrow \mathbb{R}$  be universally Baire. Let  $\mathbb{P} \in V$  be a poset, and let  $H$  be  $\mathbb{P}$ -generic over  $V$ . If  $V[H']$  is a set generic extension of  $V[H]$ , then  $F^{H'} \upharpoonright \mathbb{R}^{V[H']} = F^H$ .

Here is an example, of which the case  $n = 1$  will be important later. Let  $n < \omega$ , and let  $V$  be closed under  $X \mapsto M_n^\#(X)$ . Then  $F: x \mapsto M_n^\#(x)$ , construed as a function from  $\mathbb{R}$  to  $\mathbb{R}$ , is universally Baire (see [3, Lemma 2.9]). If  $\mathfrak{A}$  is closed under  $F$  in the sense of Definition 1.6, then  $\mathfrak{A}$  must be closed under  $X \mapsto M_n^\#(X)$  in the ordinary sense. The same is of course true for mouse operators other than  $M_n^\#$ .

**Definition 1.8** Let  $X \in H_{\omega_2}$ , and let  $\varphi(x)$  be a  $\Sigma_1$  formula in the language of set theory. We say that  $\varphi(X)$  is *honestly consistent* if and only if for every  $F: \mathbb{R} \rightarrow \mathbb{R}$  which is universally Baire there is an  $F$ -closed transitive model  $\mathfrak{A}$  such that

- (a)  $\mathfrak{A} \in V^{\text{Col}(\omega, 2^{\aleph_1})}$ ;
- (b)  $(H_{\omega_2})^V \subset \mathfrak{A}$ ;
- (c) if  $T \subset \omega_1^V$ ,  $T \in V$ , and  $V \models T$  is stationary, then  $\mathfrak{A} \models T$  is stationary; and
- (d)  $\mathfrak{A} \models \text{ZFC}^- + \varphi(X)$ .

**Definition 1.9** By *Bounded Martin's Maximum\**,  $\text{BMM}^*$ , we mean the conjunction of the following two statements:

- (a)  $\text{NS}_{\omega_1}$  is precipitous, and
- (b) if  $X \in H_{\omega_2}$  and if  $\varphi(x)$  is a  $\Sigma_1$  formula such that  $\varphi(X)$  is honestly consistent, then  $\varphi(X)$  holds true in  $V$ .

**Theorem 1.10** If  $\text{BMM}^*$  holds true, then so does (1).

**Proof** Let  $\theta = 2^{\aleph_1}$  and  $\rho = (2^\theta)^+$ , and let  $H$  be  $\text{Col}(\omega, < \rho)$ -generic over  $V$ . Note that  $\rho = \omega_1^{V[H]}$ . Let  $x \in \mathbb{R} \cap V[H]$  be a real coding the structure  $(H_{(2^{\aleph_1})^+}; \in, \text{NS})^V$ . There is some  $G$  which is  $\text{Col}(\omega, < \rho)$ -generic over  $V[x]$  with the property that  $V[x, G] = V[H]$ . We have  $\bigcup G: \omega \times \rho \rightarrow \rho$ , and for each  $\eta < \rho$ ,  $\bigcup G(\cdot, \eta): \omega \rightarrow \eta$  is a surjection. Setting

$$\bar{S}_\xi = \left\{ \eta < \rho: \bigcup G(0, \eta) = \xi \right\}$$

for  $\xi < \rho$ ,  $(\bar{S}_\xi: \xi < \rho)$  is a family of pairwise disjoint subsets of  $\rho = \omega_1^{V[H]}$  such that each  $\bar{S}_\xi$  is stationary in  $V[H]$ .

Let  $e: \rho \rightarrow [\rho]^{<\rho} \cap L[x, G]$ ,  $e \in L[x, G]$  be an enumeration of all the bounded subsets of  $\rho$  which exist in  $L[x, G]$ .

Let  $\bar{D} = \{\alpha < \rho: J_\alpha[x] \models \text{ZFC}^-\}$ , let  $D' \subset \rho$  be the club of all limit points of  $\bar{D}$ , and let  $D = \bar{D} \setminus D'$ . Then  $D$  is an unbounded nonstationary subset of  $\rho$ . We let

$d: \omega \times \rho \times \rho \rightarrow D$  be some bijection which exists in  $L[x]$ . Setting  $S_\xi = \bar{S}_\xi \cap D'$  for  $\xi < \rho$ , we have that  $(S_\xi: \xi < \rho)$  is a family of pairwise disjoint subsets of  $\rho$  each of which is stationary in  $V[x, G]$  and such that  $S_\xi \cap D = \emptyset$  for all  $\xi < \rho$ .

We now fix  $S \in V$ ,  $S \subset \omega_1^V$ , stationary and costationary in  $V$ . Working inside  $L[x, G]$ , we may construct a generic iteration

$$((\mathcal{M}_i, \pi_{ij}: i \leq j \leq \rho), (G_i: i < \rho))$$

of  $\mathcal{M}_0 = (H_{(2^{\aleph_1})^+}; \in, \text{NS})^V$  with the following properties.

- (i) If  $\xi, i < \rho$  and  $e(\xi) \in \mathcal{M}_i \setminus \bigcup_{k < i} \text{ran}(\pi_{ki})$  is stationary in  $\mathcal{M}_i$ , then  $S_\xi \setminus \text{crit}(G_i) \subset \pi_{i\rho}(e(\xi))$ .
- (ii) For  $n < \omega$  and  $\eta, \xi < \rho$ ,  $G(n, \eta) = \xi$  iff  $d(n, \eta, \xi) \in \pi_{0\rho}(S)$ .

In particular, if  $T \subset \rho$ ,  $T \in M_\rho$ ,  $M_\rho \models T$  is stationary, then  $T$  is stationary in  $V[H]$ .

Also,  $L[x, \pi_{0\rho}(S)] = L[x, G]$ . This is true as  $D, d \in L[x]$ , so that  $G$  may be read off from  $d$  and  $\pi_{0\rho}(S)$  inside  $L[x, \pi_{0\rho}(S)]$ , that is,  $L[x, G] \subset L[x, \pi_{0\rho}(S)]$ . On the other hand, the generic iteration  $((\mathcal{M}_i, \pi_{ij}: i \leq j \leq \rho), (G_i: i < \rho))$  is inside  $L[x, G]$ , so that we certainly have  $\pi_{0\rho}(S) \in L[x, G]$ , so that  $L[x, \pi_{0\rho}(S)] = L[x, G]$ .

We may lift the iteration maps to act on  $V$ ; that is, there is a unique generic iteration

$$((N_i, \tilde{\pi}_{ij}: i \leq j \leq \rho), (G_i: i < \rho))$$

of  $(V; \in, \text{NS})$  such that  $\mathcal{M}_i = (H_{(2^{\aleph_1})^+})^{N_i}$  for  $i \leq \rho$  and  $\pi_{ij} = \tilde{\pi}_{ij} \upharpoonright \mathcal{M}_i$  for  $i \leq j \leq \rho$ . Let us write  $N = N_\rho$ .

Now let  $F: \mathbb{R} \rightarrow \mathbb{R}$  be universally Baire, and let  $T_0, U_0$  be the class-sized trees witnessing that  $F$  is universally Baire (with  $F = p[T_0]$ ). Set  $T_\rho = \tilde{\pi}_{0\rho}(T_0)$  and  $U_\rho = \tilde{\pi}_{0\rho}(U_0)$ , so that  $p[T_\rho] = p[T_0]$  and  $p[U_\rho] = p[U_0]$ .

By Lemma 1.7, every rank initial segment of  $V[H]$  is closed under  $F$ . Hence in  $V[H]$ , there is some transitive  $F$ -closed  $\mathfrak{A}$  with  $(H_{(2^{\aleph_1})^+})^N \subset \mathfrak{A}$ ,  $\mathfrak{A} \models T$  is stationary for all  $T \subset \rho$ ,  $T \in \mathcal{M}_\rho$ , such that  $\mathcal{M}_\rho \models T$  is stationary and such that  $\mathfrak{A}$  is a model of  $\text{ZFC}^-$  plus “there is some real  $x$  and some  $G$  which is  $\text{Col}(\omega, < \rho)$ -generic over  $L[x]$  with  $L[x, \pi_{0\rho}(S)] = L[x, G]$ .” (Just take an appropriate rank initial segment of  $V[H]$  as  $\mathfrak{A}$ .)

We may use the tree  $T_\rho$  to witness the fact that  $\mathfrak{A}$  is  $F$ -closed. By absoluteness then, in  $N^{\text{Col}(\omega, \tilde{\pi}_{0,\rho}(2^{\aleph_1}))}$  there is some transitive  $F$ -closed (as being witnessed by  $T_\rho$ )  $\mathfrak{A}$  with the above properties. Pulling this back via  $\tilde{\pi}_{0\rho}$  we get that in  $V^{\text{Col}(\omega, 2^{\aleph_1})}$  there is some transitive  $F$ -closed (as being witnessed by  $T_0$ )  $\mathfrak{A}$  with  $(H_{\omega_2})^V \subset \mathfrak{A}$ ,  $\mathfrak{A} \models T$  is stationary for all  $T \subset \omega_1$ ,  $T \in V$ , such that  $V \models T$  is stationary, and such that  $\mathfrak{A}$  is a model of  $\text{ZFC}^-$  plus “there is some real  $x$  and some  $G$  which is  $\text{Col}(\omega, < \omega_1)$ -generic over  $L[x]$  with  $L[x, S] = L[x, G]$ .”

We have shown that (1) is honestly consistent. □

**Theorem 1.11** *If BMM\* holds true, then so does (2).*

**Proof** Let us again write  $\theta = 2^{\aleph_1}$  and  $\rho = (2^\theta)^+$ , and let  $G$  be  $\text{Col}(\omega, < \rho)$ -generic over  $V$ . Let  $H$  be  $\text{Col}(\rho, \rho)$ -generic over  $V[G]$ . We have that  $\rho = \omega_1^{V[G, H]}$  and  $\diamond$  holds in  $V[G, H]$ . Let  $e^*: \rho \rightarrow [\rho]^{<\rho} \cap V[G, H]$ ,  $e^* \in V[G, H]$ , be an enumeration of all the bounded subsets of  $\rho$  which exist in  $V[G, H]$ . Let  $(\tau_i: i < \rho)$

witness that  $\diamond$  holds in  $V[G, H]$ . As in the previous proof, we may set

$$\bar{S}_\xi = \left\{ \eta < \rho : \bigcup G(0, \eta) = \xi \right\}$$

for  $\xi < \rho$ , so that  $(\bar{S}_\xi : \xi < \rho)$  is a family of pairwise disjoint subsets of  $\rho$ , each  $\bar{S}_\xi$  being stationary in  $V[G, H]$ .

Let  $x \in \mathbb{R} \cap V[G]$  be such that the structure  $(H_{(2^{\aleph_1})^+}; \in, \text{NS})^V$  is in  $L[x]$  and is countable there. Let  $x^\# = (J_\alpha[x]; \in, U)$ , and let  $\kappa = \text{crit}(U)$ . Let  $g \in V[G]$  be  $\text{Col}(\omega, < \kappa)$ -generic over  $x^\#$  (equivalently, over  $L[x]$ ), and let

$$I = \{X \in \mathcal{P}(\kappa) \cap x^\#[g] : \exists Y \in UY \cap X = \emptyset\}.$$

Then  $(x^\#[g]; \in, I) \models I$  is a  $\sigma$ -complete uniform normal ideal on  $\kappa$ , and  $(x^\#[g]; \in, I)$  is generically iterable via  $I$  and its images in a way that every iteration map lifts an iteration map resulting from iterating the ground model  $x^\#$ .

We may let  $(W_\xi : \xi < \kappa) \in x^\#[g]$  be a partition of  $\kappa$  into  $I$ -positive sets. We may also let  $e : \kappa \rightarrow [\kappa]^{<\kappa} \cap L[x, g]$  be an enumeration of all the bounded subsets of  $\kappa$  which exist in  $L[x, g]$ .

Working inside  $(x^\#[g]; \in, I)$ , we may construct a generic iteration

$$((\mathcal{M}_i, \pi_{ij} : i \leq j \leq \kappa), (G_i : i < \kappa))$$

of  $\mathcal{M}_0 = (H_{(2^{\aleph_1})^+}; \in, \text{NS})^V$  with the following property.

- (i) If  $\xi, i < \kappa$  and  $e(\xi) \in \mathcal{M}_i \setminus \bigcup_{k < i} \text{ran}(\pi_{ki})$  is stationary in  $\mathcal{M}_i$ , then  $W_\xi \setminus \text{crit}(G_i) \subset \pi_{i\kappa}(e(\xi))$ .

In particular,  $\mathcal{M}_\kappa \in x^\#[g]$  and  $(\text{NS})^{\mathcal{M}_\kappa} = I \cap \mathcal{M}_\kappa$ . Working inside  $V[G, H]$ , we may then construct a generic iteration

$$((\mathcal{M}_i^*, \pi_{ij}^* : i \leq j \leq \rho), (G_i^* : i < \rho))$$

of  $\mathcal{M}_0^* = (x^\#[g]; \in, I)$  with the following properties.

- (ii) If  $\xi, i < \rho$  and  $e^*(\xi) \in \pi_{0i}^*(I^+) \setminus \bigcup_{k < i} \text{ran}(\pi_{ki})$ , then  $\bar{S}_\xi \setminus \text{crit}(G_i^*) \subset \pi_{i\kappa}(e^*(\xi))$ .
- (iii) If  $i < \rho$ , then  $G_i^*$  is generic over  $L[\mathcal{M}_i^*, (\tau_k : k \leq i)]$  (not just over  $\mathcal{M}_i^*$ ).

In particular,  $\pi_{0\rho}^*(I) = (\text{NS}_\rho)^{V[G, H]} \cap \mathcal{M}_\rho^*$ . Also, if  $k \leq i \leq \rho$ , then

$$\tau_k \in \mathcal{M}_k^* \iff \tau_k \in \mathcal{M}_i^*.$$

Let

$$((\mathcal{M}_i, \pi_{ij} : i \leq j \leq \rho), (G_i : i < \rho)) = \pi_{0\rho}^*((\mathcal{M}_i, \pi_{ij} : i \leq j \leq \kappa), (G_i : i < \kappa)),$$

which is a generic iteration of  $\mathcal{M}_0 = (H_{(2^{\aleph_1})^+}; \in, \text{NS})^V$ . We have  $\pi_{0,\rho}((\text{NS})^V) = \pi_{0,\rho}(I) \cap \mathcal{M}_\rho^* = (\text{NS}_\rho)^{V[G, H]} \cap \mathcal{M}_\rho$ , so that every  $T \subset \rho$ ,  $T \in \mathcal{M}_\rho$ , which is stationary in  $\mathcal{M}_\rho$  is also stationary in  $V[G, H]$ .

Let  $D \subset \rho$ ,  $D \in V[G, H]$ . Let  $S_D = \{i < \rho : \tau_i = D \cap i\}$  which is stationary in  $V[G, H]$ . Suppose that  $D \cap \xi \in \mathcal{M}_\rho^*$  for every  $\xi < \rho$ . There is then a club  $C \subset \rho$  such that  $D \cap i \in \mathcal{M}_i^*$  for all  $i < \rho$ . This gives some stationary  $\bar{S} \subset S_D \cap C$  and some  $i_0 < \rho$  and  $\bar{D} \in \mathcal{M}_{i_0}^*$  such that  $\pi_{i_0 j}^*(\bar{D}) = D \cap j$  for all  $j \in \bar{S}$ . But then  $D = \pi_{i_0 \rho}^*(\bar{D}) \in \mathcal{M}_\rho^*$ . Writing  $\mathcal{M}_\rho^* = (J_{\alpha^*}[x, g^*]; \in, \pi_{0\rho}(I))$  and letting  $B \subset \rho$  code  $x \oplus g^*$  in a simple way, we have shown that  $B$  is amenably closed in  $V[G, H]$ .

We may again lift  $((\mathcal{M}_i, \pi_{ij} : i \leq j \leq \rho), (G_i : i < \rho))$  to a generic iteration  $((N_i, \tilde{\pi}_{ij} : i \leq j \leq \rho), (G_i : i < \rho))$  of  $(V; \in, (\text{NS})^V)$ . Let us write  $N = N_\rho$ .

Let us fix some  $A \subset \omega_1$ ,  $A \in V$ . Also, let  $F: \mathbb{R} \rightarrow \mathbb{R}$  be universally Baire, and let  $T_0, U_0$  be the class-sized trees witnessing that  $F$  is universally Baire (with  $F = p[T_0]$ ). Set  $T_\rho = \tilde{\pi}_{0,\rho}(T_0)$  and  $U_\rho = \tilde{\pi}_{0,\rho}(U_0)$ , so that  $p[T_\rho] = p[T_0]$  and  $p[U_\rho] = p[U_0]$ .

By Lemma 1.7, every rank initial segment of  $V[G, H]$  is closed under  $F$ . In  $V[G, H]$ , there is thus some transitive  $F$ -closed (as being witnessed by  $T_\rho$ )  $\mathfrak{A}$  with  $(H_{(2^{\aleph_1})^+})^N \subset \mathfrak{A}$ ,  $\mathfrak{A} \models T$  is stationary for all  $T \subset \rho$ ,  $T \in N$ , such that  $N \models T$  is stationary, and such that  $\mathfrak{A}$  is a model of  $ZFC^-$  plus “there is some amenable closed  $B \subset \rho$  with  $\tilde{\pi}_{0,\rho}(A) \in L[B]$ .” (Take an appropriate rank initial segment of  $V[G, H]$  as  $\mathfrak{A}$ .)

Hence by absoluteness, in  $N^{\text{Col}(\omega, \tilde{\pi}_{0,\rho}(2^{\aleph_1}))}$  there is some transitive  $F$ -closed (as being witnessed by  $T_\rho$ )  $\mathfrak{A}$  with the above properties. Pulling this back via  $\tilde{\pi}_{0,\rho}$  we get that in  $V^{\text{Col}(\omega, 2^{\aleph_1})}$  there is thus a transitive  $F$ -closed (as being witnessed by  $T_0$ )  $\mathfrak{A}$  with  $(H_{\omega_2})^V \subset \mathfrak{A}$ ,  $\mathfrak{A} \models T$  is stationary for all  $T \subset \omega_1$ ,  $T \in V$ , such that  $V \models T$  is stationary, and such that  $\mathfrak{A}$  is a model of  $ZFC^-$  plus “there is some amenable closed  $B \subset \rho$  with  $A \in L[B]$ .”

We have shown that (2) is honestly consistent. □

There is an obvious question which we have to leave unanswered: Does  $\text{BMM}$  plus  $\text{NS}_{\omega_1}$  precipitously prove  $\text{BMM}^*$ ? We will explore this question further in the next section.

## 2 $\text{BMM}^*$ and Woodin’s Axiom $(*)$

We now aim to discuss the relationship between  $\text{BMM}^*$  and  $(*)$ . In order to do so, we shall need strengthenings of  $\text{BMM}^*$ , which we call  $\text{BMM}^{*,++}$  (in analogy with  $\text{MM}^{++}$  and  $\text{BMM}^{++}$ ; see Definition 2.2) and  $A\text{-BMM}^{*,++}$  (where  $A$  is a universally Baire set of reals; see Definition 2.6). We apologize for the awkward notation.

**Definition 2.1** Let  $X \in H_{\omega_2}$ , and let  $\varphi(x, \dot{I}_{\text{NS}})$  be a  $\Sigma_1$  formula in the language of set theory, augmented by a predicate  $\dot{I}_{\text{NS}}$  for the nonstationary ideal on  $\omega_1$ . We say that  $\varphi(X, \dot{I}_{\text{NS}})$  is *honestly consistent* if and only if for every  $F: \mathbb{R} \rightarrow \mathbb{R}$  which is universally Baire there is an  $F$ -closed transitive model  $\mathfrak{A}$  such that

- (a)  $\mathfrak{A} \in V^{\text{Col}(\omega, 2^{\aleph_1})}$ ;
- (b)  $(H_{\omega_2})^V \subset \mathfrak{A}$ ;
- (c) if  $T \subset \omega_1^V$ ,  $T \in V$ ,  $V \models T$  is stationary, then  $\mathfrak{A} \models T$  is stationary; and
- (d)  $\mathfrak{A} \models ZFC^- + \varphi(X, \dot{I}_{\text{NS}})$ .

**Definition 2.2** By *Bounded Martin’s Maximum* $^{*,++}$ ,  $\text{BMM}^{*,++}$ , we mean the conjunction of the following two statements.

- (a)  $\text{NS}_{\omega_1}$  is precipitous, and
- (b) if  $X \in H_{\omega_2}$  and if  $\varphi(x, \dot{I}_{\text{NS}_{\omega_1}})$  is a  $\Sigma_1$  formula in the language of set theory, augmented by a predicate for the nonstationary ideal on  $\omega_1$ , such that  $\varphi(X, \dot{I}_{\text{NS}_{\omega_1}})$  is honestly consistent, then  $\varphi(X, \dot{I}_{\text{NS}_{\omega_1}})$  holds true in  $V$ .

In Definitions 2.1 and 2.2 we understand that the predicate  $\dot{I}_{\text{NS}_{\omega_1}}$  is interpreted by  $(\text{NS}_{\omega_1})^{\mathfrak{A}}$  and  $(\text{NS}_{\omega_1})^V$  inside  $\mathfrak{A}$  and  $V$ , respectively. Of course,  $\text{BMM}^{*,++}$  strengthens both  $\text{BMM}^*$  as well as  $\text{BMM}^{++}$ .



After the first version of this paper had been written, J. Zapletal mentioned the following principle to us.

**Definition 2.3**

- (3) Let  $A \subset \omega_1$ . There is then some  $B \subset \omega_1$  with  $A \in L[B]$  such that for every  $D \in \mathcal{P}(\omega_1) \cap L[B]$ , if  $L[B] \models \text{“}D \text{ is stationary,“}$  then  $V \models \text{“}D \text{ is stationary.“}$  (see Zapletal [14]).

Our proof of Theorem 1.11 presented above also produces the following result.

**Theorem 2.4** *If  $\text{BMM}^{*,++}$  holds true, then so does (3).*

**Definition 2.5** Let  $X \in H_{\omega_2}$ , let  $A \subset \mathbb{R}$  be universally Baire, and let  $\varphi(x, \dot{A}, \dot{I}_{\text{NS}_{\omega_1}})$  be a  $\Sigma_1$  formula in the language of set theory, augmented by predicates  $\dot{A}$  and  $\dot{I}_{\text{NS}_{\omega_1}}$  for  $A$  and for the nonstationary ideal on  $\omega_1$ , respectively. We say that  $\varphi(X, \dot{A}, \dot{I}_{\text{NS}_{\omega_1}})$  is *honestly consistent* if and only if for every  $F: \mathbb{R} \rightarrow \mathbb{R}$  which is universally Baire there is an  $F$ -closed transitive model  $\mathfrak{M}$  such that

- (a)  $\mathfrak{M} \in V^{\text{Col}(\omega, 2^{\aleph_1})}$ ;
- (b)  $(H_{\omega_2})^V \subset \mathfrak{M}$ ;
- (c) if  $T \subset \omega_1^V$ ,  $T \in V$ ,  $V \models T$  is stationary, then  $\mathfrak{M} \models T$  is stationary; and
- (d)  $\mathfrak{M} \models \text{ZFC}^- + \varphi(X, \dot{A}, \dot{I}_{\text{NS}_{\omega_1}})$ .

**Definition 2.6** Let  $A \subset \mathbb{R}$  be universally Baire. By  $A$ -Bounded Martin's Maximum $^{*,++}$ ,  $A$ - $\text{BMM}^{*,++}$ , we mean the conjunction of the following two statements:

- (a)  $\text{NS}_{\omega_1}$  is precipitous, and
- (b) if  $X \in H_{\omega_2}$  and if  $\varphi(x, \dot{A}, \dot{I}_{\text{NS}_{\omega_1}})$  is a  $\Sigma_1$  formula in the language of set theory, augmented by predicates for  $A$  and for the nonstationary ideal on  $\omega_1$ , such that  $\varphi(X, \dot{A}, \dot{I}_{\text{NS}_{\omega_1}})$  is honestly consistent, then  $\varphi(X, \dot{A}, \dot{I}_{\text{NS}_{\omega_1}})$  holds true in  $V$ .

In Definitions 2.5 and 2.6 we again understand that the predicate  $\dot{I}_{\text{NS}_{\omega_1}}$  is interpreted by  $(\text{NS}_{\omega_1})^{\mathfrak{M}}$  and  $(\text{NS}_{\omega_1})^V$  inside  $\mathfrak{M}$  and  $V$ , respectively; moreover, if the trees  $T$  and  $U$  witness that  $A$  is universally Baire with  $A = p[T]$ , then  $\dot{A}$  is supposed to be interpreted by  $A$  inside  $V$  and by  $p[T] \cap \mathfrak{M} = A^* \cap \mathfrak{M}$  inside  $\mathfrak{M}$ , where  $A^* = p[T] \cap V^{\text{Col}(\omega, 2^{\aleph_1})}$  is the version of  $A$  inside  $V^{\text{Col}(\omega, 2^{\aleph_1})}$ .

We now prove the following result which is in the spirit of [13, Theorems 10.127–10.129, 10.137]. This result also shows that  $\text{BMM}^*$  is consistent, in case the reader may have wondered. This is true because if we let  $V$  be the least inner model of ZFC which has  $\omega$  Woodin cardinals  $\delta_0 < \delta_1 < \dots$  and is closed under  $X \mapsto M_{\omega}^{\#}(X)$ , if  $G$  is  $\text{Col}(\omega, < \sup_{n < \omega} \delta_n)$ -generic over  $V$ , and if

$$\mathbb{R}^* = \bigcup \{ \mathbb{R} \cap V[G \upharpoonright \delta_n] : n < \omega \},$$

then we may construct inside  $V[G]$  an inner model

$$L^{M_{\omega}^{\#}}(\mathbb{R}^*)$$

of ZF plus AD which is the least inner model whose set of reals is  $\mathbb{R}^*$  and is closed under  $X \mapsto M_{\omega}^{\#}(X)$ , and

$$L^{M_{\omega}^{\#}}(\mathbb{R}^*)^{\mathbb{P}_{\max}}$$

satisfies the hypotheses of Theorem 2.7 as well as (\*).

**Theorem 2.7** *Suppose that  $M_{\omega}^{\#}$  exists<sup>1</sup> and is fully iterable.<sup>2</sup> Suppose that  $\text{NS}_{\omega_1}$  is precipitous. Then the following statements are equivalent.*

- (A) (\*)
- (B) *For every set  $A$  of reals with  $A \in L(\mathbb{R})$ ,  $A$ -bounded Martin's Maximum<sup>\*,++</sup> holds true.*

**Proof** We first show (B)  $\implies$  (A). Let  $\text{sat}(\text{NS})$  denote the saturation of NS, that is, the least cardinal  $\mu$  such that every antichain in  $\mathcal{P}(\omega_1)/\text{NS}$  has cardinality less than  $\mu$ . In what follows, we shall write  $\kappa$  for  $2^{<\text{sat}(\text{NS})} = \text{Card}(H_{\text{sat}(\text{NS})})$ . If  $2^{\aleph_1} = \aleph_2$ , then  $\kappa = 2^{\aleph_1} = \aleph_2$  if NS is saturated, and  $\kappa = 2^{\aleph_2}$  otherwise.

(B)  $\implies$  (A) is now an immediate consequence of the following result.

**Theorem 2.8** *Let  $M$  be an inner model of ZF such that  $\mathbb{R} \subseteq M$ , and let  $\Gamma = \mathcal{P}(\mathbb{R}) \cap M$ . Let  $\kappa = 2^{<\text{sat}(\text{NS})}$ . Assume the following hypotheses.*

- (a) *NS is precipitous.*
- (b) *AD holds true in  $M$ .*
- (c) *Every set of reals in  $\Gamma$  is  $\kappa^+$ -universally Baire.*
- (d) *If  $A$  is a set of reals in  $\Gamma$ ,  $\varphi$  is a  $\Pi_2^1$  formula, and  $g$  is  $\text{Col}(\omega, \kappa)$ -generic over  $V$ , then*

$$\varphi(A) \iff \varphi(A^g),$$

*where  $A^g$  is  $V[g]$ 's version of  $A$ ; that is, if the trees  $T$  and  $U$  witness that  $A$  is  $\kappa^+$ -universally Baire with  $A = (p[T])^V$ , then  $A^g = (p[T])^{V[g]}$ .*

- (e) *For every set  $A$  of reals in  $\Gamma$ ,  $A$ -Bounded Martin's Maximum<sup>\*,++</sup> holds true.*

*Let  $A_0 \subset \omega_1$  be such that  $\omega_1^{L[A_0]} = \omega_1$ . Then there is some  $G \in V$  such that  $G$  is  $\mathbb{P}_{\max}$ -generic over  $M$  and*

$$L(\mathbb{R})[G] = L(\mathbb{R})[A_0] = L(\mathcal{P}(\omega_1)). \quad (1)$$

**Proof of Theorem 2.8** Let us fix  $M$  as in the statement of the theorem. Let us also fix, until the end of this proof, some  $A_0 \subset \omega_1$  such that  $\omega_1^{L[A_0]} = \omega_1$ . Let  $G$  be the set of all  $p = (M_0; \in, J_0, a_0) \in \mathbb{P}_{\max}$  such that there is some generic iteration

$$((\mathcal{M}_i, \pi_{i,j}: i \leq j \leq \omega_1), (G_i: i < \omega_1))$$

of  $M_0 = p$  such that  $\pi_{0,\omega_1}(a_0) = A_0$  and, writing  $M_{\omega_1} = (M_{\omega_1}; \in, J_{\omega_1}, A_0)$ , every set in

$$J_{\omega_1}^+ = (\mathcal{P}(\omega_1) \cap M_{\omega_1}) \setminus J_{\omega_1}$$

is stationary in  $V$ .

We claim that  $G$  is  $\mathbb{P}_{\max}$ -generic over  $M$  and that (1) holds true for  $G$ . In order to verify this, we shall need to prove the following three claims, which will be shown from the hypotheses of Theorem 2.8.

**Claim 2.9**  *$G$  is a filter.*

**Claim 2.10** *If  $D \in M$  is a dense subset of  $\mathbb{P}_{\max}$ , then  $D \cap G \neq \emptyset$ .*

By a standard  $\mathbb{P}_{\max}$ -argument, if  $p \in G$ , then there is a *unique* generic iteration

$$((\mathcal{M}_i, \pi_{i,j}: i \leq j \leq \omega_1), (G_i: i < \omega_1))$$

of  $\mathcal{M}_0 = p$  such that  $\pi_{0,\omega_1}(a_0) = A_0$ . Assuming Claims 2.9 and 2.10 and following [13], we shall then write  $\mathcal{P}(\omega_1)_G$  for the set of all  $X \subset \omega_1$  for which there is some  $p \in G$  such that if

$$((\mathcal{M}_i, \pi_{i,j}: i \leq j \leq \omega_1), (G_i: i < \omega_1))$$

is the generic iteration of  $\mathcal{M}_0 = p$  with  $\pi_{0\omega_1}(a_0) = A_0$ , then  $X \in \text{ran}(\pi_{i,\omega_1})$  for some  $i < \omega_1$ .

**Claim 2.11** *We have  $\mathcal{P}(\omega_1) = \mathcal{P}(\omega_1)_G$ .*

If NS were assumed to be saturated, then Claim 2.9 would be given by [13, Theorem 4.74] and Claim 2.11 would follow from [13, Lemma 3.12, Corollary 3.13]. Under the hypotheses (a) and (e) instead, one can prove Claims 2.9 and 2.11 by an easy application of the forcing developed in [2]. Using hypothesis (a), [2] designs a stationary set preserving forcing which (for a given regular cardinal  $\theta \geq \aleph_2$ ) adds a generic iteration

$$(\mathcal{M}_i, \pi_{i,j}: i \leq j \leq \omega_1)$$

of a countable model  $\mathcal{M}_0 = (M_0; \in, I_0)$  such that  $\mathcal{M}_{\omega_1} = (H_\theta; \in, \text{NS})$ . This immediately gives Claim 2.11 by Bounded Martin's Maximum<sup>++</sup>. Also, if  $p, q \in G$ , then we may assume without loss of generality that  $p, q, A_0 \cap \omega_1^{M_0} \in M_0$ , so that Bounded Martin's Maximum<sup>++</sup> also yields Claim 2.9.

It remains to verify Claim 2.10.

Let us fix  $D \subset \mathbb{P}_{\max}$ ,  $D \in M$ , a dense set in  $\mathbb{P}_{\max}$ , and let  $D^* \in \Gamma$  be a set of reals coding  $D$  according to some natural coding device. As  $D^*$  is  $\kappa^+$ -universally Baire, we may pick trees  $T$  and  $U$  on  $\omega \times 2^\kappa$  such that

$$D^* = p[T] \quad \text{and} \quad \Vdash_{\text{Col}(\omega, \kappa)} p[U] = {}^\omega \omega \setminus p[T].$$

The following is a variant of the argument for Theorems 1.10 and 1.11.

Let us pick some  $g$  which is  $\text{Col}(\omega, \kappa)$ -generic over  $V$ , so that  $(\kappa^+)^V = \omega_1^{V[g]}$  and  $H_{\text{sat}(\text{NS})}$  is countable in  $V[g]$ . By our hypothesis (a) and the proof of [13, Lemma 3.10],  $p_0 = ((H_{\text{sat}(\text{NS})})^V; \in, (\text{NS})^V, A_0)$  is then a  $\mathbb{P}_{\max}$  condition in  $V[g]$ . The statement

$$\forall p \in \mathbb{P}_{\max} \exists q \in \mathbb{P}_{\max} (q \leq_{\mathbb{P}_{\max}} p \wedge q \in D) \tag{2}$$

which expresses that  $D$  is dense in  $\mathbb{P}_{\max}$  is  $\Pi_2^1$  in  $\mathbb{P}_{\max} \oplus D$  in the codes, so that by hypothesis (d) there is some  $q = (N_0; \in, J_0, A'_0) \in V[g]$  belonging to the set of  $\mathbb{P}_{\max}$ -conditions coded by  $(D^*)^g$  and such that  $q <_{\mathbb{P}_{\max}} p_0$ . Let

$$j_0: ((H_{\text{sat}(\text{NS})})^V; \in, (\text{NS})^V, A_0) \rightarrow (N_0; \in, J_0, A'_0)$$

be such that  $p_0, j_0 \in N_0$  witness that  $q < p_0$ .

Let

$$(S_\xi: \xi < (\kappa^+)^V) \in V[g]$$

be a partition of  $(\kappa^+)^V$  into stationary sets. Working inside  $V[g]$ , we may then choose a generic iteration

$$(\mathcal{N}_i, \sigma_{i,j}: i \leq j \leq \kappa^+)$$

of  $\mathcal{N}_0 = (N_0; \in, J_0, A'_0) = q$  such that, writing  $\mathcal{N}_{(\kappa^+)^V} = (N; \in, J, A')$ ,

$$\forall S \in (\mathcal{P}((\kappa^+)^V) \cap N) \setminus J \exists \xi < (\kappa^+)^V \exists \beta < (\kappa^+)^V S_\xi \setminus \beta \subset S$$

(see, e.g., [2, proof of Lemma 5] and also the proofs of Theorems 1.10, 1.11). In particular,

$$J = (\text{NS})^{V[g]} \cap N. \tag{3}$$

Writing

$$\begin{aligned} j &= \sigma_{0,(\kappa^+)^V}(j_0): ((H_{\text{sat}(\text{NS})})^V; \in, (\text{NS})^V, A_0) \rightarrow \sigma_{0,(\kappa^+)^V}(p_0) \\ &= (M_{(\kappa^+)^V}; \in, I, A'), \end{aligned}$$

we thus also have

$$I = J \cap M_{(\kappa^+)^V} = (\text{NS})^{V[g]} \cap M_{(\kappa^+)^V}.$$

As  $V$  is  $(\kappa^+)$ -iterable in  $V[g]$  by our hypothesis (a) and the proof of [13, Lemma 3.10], we may lift the generic iteration of  $((H_{\text{sat}(\text{NS})})^V; \in, (\text{NS})^V, A_0)$  which gave rise to  $j_0$  to a generic iteration of  $(V; \in, (\text{NS})^V, A_0)$ . Let us write

$$\hat{j}: V \rightarrow M$$

for the induced iteration map, so that  $\hat{j} \supset j$ .

Now let  $x \in p[T] \cap V[g]$  code  $\mathcal{N}_0$ , and let  $(x, y) \in [T] \cap V[g]$ . This gives

$$(x, \hat{j}''y) \in [\hat{j}(T)]. \tag{4}$$

By  $D^*$ -Bounded Martin's Maximum $^{*,+++}$ , the proof of Claim 2.10 will be finished if we show that the natural  $\Sigma_1$  statement  $\varphi(A_0, \dot{D}^*, \dot{I}_{\text{NS}_{\omega_1}})$  expressing the existence of a  $\mathbb{P}_{\text{max}}$ -condition in  $G$  coded by a real in  $D^*$  is an honestly consistent statement, in the sense of Definition 2.5. The proof that  $\varphi(A_0, \dot{D}^*, \dot{I}_{\text{NS}_{\omega_1}})$  is honestly consistent in the sense of Definition 2.5 is essentially as in the proofs of Theorems 1.10 and 1.11.

Let  $F: \mathbb{R} \rightarrow \mathbb{R}$  be a universally Baire function in  $V$ , let  $\eta > \kappa$  be a cardinal, let  $\bar{T}$  and  $\bar{U}$  be a pair of trees on  $\omega \times 2^\eta$  witnessing the  $\eta^+$ -universal Baireness of  $F$  (with  $F = p[\bar{T}]$ ), and set  $T^* = \hat{j}(\bar{T})$  and  $U^* = \hat{j}(\bar{U})$ , so that  $p[\bar{T}] = p[T^*]$  and  $p[\bar{U}] = p[U^*]$ .

In  $V^{\text{Col}(\omega, 2^\eta)}$  there is a  $p[T^*]$ -closed model  $\mathfrak{A}$  such that  $H_{\omega_2}^M \subseteq \mathfrak{A}$ , every set in  $(\mathcal{P}(\omega_1) \setminus \text{NS}_{\omega_1})^M$  is stationary in  $\mathfrak{A}$ , and such that  $\mathfrak{A}$  satisfies  $\text{ZFC}^-$  together with  $\varphi(A_0, [\hat{j}(T)], \text{NS}_{\omega_1})$ . (The existence of  $\mathfrak{A}$  in  $\text{Col}(\omega, 2^\eta)$  is witnessed by some rank-initial segment of  $V[g]$ .) By absoluteness,  $\text{Col}(\omega, \hat{j}(2^\eta))$  forces over  $M$  that there is a  $p[\hat{j}(\bar{T})]$ -closed model  $\mathfrak{A}$  such that  $H_{\omega_2}^M \subseteq \mathfrak{A}$ , every set in  $(\mathcal{P}(\omega_1) \setminus \text{NS}_{\omega_1})^M$  is stationary in  $\mathfrak{A}$ , and such that  $\mathfrak{A}$  satisfies  $\text{ZFC}^-$  together with  $\varphi(A_0, [\hat{j}(T)], \text{NS}_{\omega_1})$ . Finally, by elementarity of  $\hat{j}(T)$  we get that  $V^{\text{Col}(\omega, 2^\eta)}$  forces over  $V$  that there is a  $p[\bar{T}]$ -closed model  $\mathfrak{A}$  such that  $H_{\omega_2}^V \subseteq \mathfrak{A}$ , every set in  $(\mathcal{P}(\omega_1) \setminus \text{NS}_{\omega_1})^V$  is stationary in  $\mathfrak{A}$ , and such that  $\mathfrak{A}$  satisfies  $\text{ZFC}^-$  together with  $\varphi(A_0, D^*, \text{NS}_{\omega_1})$ .  $\square$

We are now going to prove  $(A) \implies (B)$  of Theorem 2.7. This will be arranged by varying the argument for [13, Theorem 10.99] (see also the proof of [13, Theorem 10.127]).

We shall use the following lemma to produce  $A$ -iterable  $\mathbb{P}_{\text{max}}$ -conditions, where  $A$  is a set of reals (see [13, Definition 4.3] on the definition of the concept of

"A-iterability.") The proof of [13, Lemma 4.40] presents a different method for producing A-iterable structures, but we thought that writing up the method for proving Lemma 2.12 would be of independent interest.

**Lemma 2.12** *Suppose that  $M_\omega^{\#\#}$  exists and is fully iterable. Let  $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ . There is then some  $x \in \mathbb{R}$  and some  $\mathbb{Q} \in M_\omega^{\#\#}(x)$  which has the  $\delta$ -c.c. and is of size  $\delta$  in  $M_\omega^{\#\#}(x)$ , where  $\delta$  is the least Woodin cardinal of  $M_\omega^{\#\#}(x)$ , such that if  $g \in V$  is  $\mathbb{Q}$ -generic over  $M_\omega^{\#\#}(x)$ , then*

$$M_\omega^{\#\#}(x)[g]$$

*is an A-iterable  $\mathbb{P}_{\max}$ -condition.*

**Proof** Let  $A$  be definable from  $x \in \mathbb{R}$  and (finitely many)  $\mathbb{R}$ -indiscernibles inside  $L(\mathbb{R})$ . Let  $\mathbb{Q} \in M_\omega^{\#\#}(x)$  be a standard forcing iteration of length  $\delta$  to force both NS to be saturated as well as  $\text{MA}_{\omega_1}$ , where  $\delta$  is the least Woodin cardinal of  $M_\omega^{\#\#}(x)$ . We claim that if  $g \in V$  is  $\mathbb{Q}$ -generic over  $M_\omega^{\#\#}(x)$ , then  $M_\omega^{\#\#}(x)[g]$  is an A-iterable  $\mathbb{P}_{\max}$ -condition.

Let us write  $M = M_\omega^{\#\#}(x)[g]$ . We know from [13, Lemma 3.10] that  $M$  is generically iterable and is hence a  $\mathbb{P}_{\max}$ -condition. It thus remains to be seen that  $M$  is A-iterable.

The set  $A \cap \mathcal{N}$  is uniformly definable over any  $z$ -mouse  $\mathcal{N}$  with infinitely many Woodin cardinals and a top measure, where  $x$  is coded into  $z \in H_{\omega_1}$ , in the following way. Let  $y \in A$  iff  $L(\mathbb{R}) \models \varphi(y, x, \eta_0, \dots, \eta_{k-1})$ , where  $\eta_0 < \dots < \eta_k$  are  $\mathbb{R}$ -indiscernibles. Let  $\mathcal{N}'$  result from  $\mathcal{N}$  by iterating the top measure of  $\mathcal{N}$  and its images  $k + 1$  times, and let  $\kappa_0 < \dots < \kappa_k$  be the sequence of the critical points. Then

$$y \in A \cap \mathcal{N} \iff \Vdash_{\text{Col}(\omega, < \sup_n(\delta_n))}^{\mathcal{N}'} L_{\kappa_k}(\mathbb{R}^*) \models \varphi(y, x, \kappa_0, \dots, \kappa_{k-1}), \tag{5}$$

where  $\delta_0 < \delta_1 < \dots$  are the Woodin cardinals of  $\mathcal{N}$  (and thus also of  $\mathcal{N}'$ ) and  $\mathbb{R}^*$  denotes the collection of all reals which are added by proper initial segments of the forcing  $\text{Col}(\omega, < \sup_n(\delta_n))$  (see Steel [12, p. 1663]). In particular,  $A \cap \mathcal{N} \in \mathcal{N}$ , and thus  $A \cap M \in M$ .

It remains to be seen that if

$$j: M \rightarrow N$$

is a generic iteration of  $M$ , then  $j(A \cap M) = A \cap N$ . Suppose not. Let  $\zeta_1 < \zeta_2 < \dots$  be the Woodin cardinals of  $M$  (i.e., the Woodin cardinals of  $M_\omega^{\#\#}(x)$  above  $\delta + 1$ ). Let

$$j: M \mid (\delta^+M) \rightarrow N$$

be a generic iteration of  $M \mid (\delta^+M)$  with  $j(A \cap M) \neq A \cap N$ , and let  $M^*$  be an iterate of  $M$  via extenders with critical points and lengths between  $\delta$  and  $\zeta_1$  such that  $j$  is generic over  $M^*$  for the extender algebra at  $\zeta_1$ . Using (5),  $M^*[j]$  can see that  $j: M \mid (\delta^+M) \rightarrow N$  is a generic iteration with  $j(A \cap M) \neq A \cap N$ , and by pulling back the statement that there is such a generic iteration we thus get that in  $M^{\text{Col}(\omega, \zeta_1)}$  there is some generic iteration  $j: M \mid (\delta^+M) \rightarrow N$  with  $j(A \cap M) \neq A \cap N$ .

However, inside  $M$ ,  $A \cap M$  is  $\zeta_1^+$ -universally Baire, again using (5). Namely, we may let  $T \in M$  be a tree of height  $\omega$  searching for  $y, \bar{M}, k, h$  such that  $k: \bar{M} \rightarrow (M \mid \sup_n(\zeta_n))^{\#} \in M$  is elementary,  $h$  is  $\text{Col}(\omega, k^{-1}(\zeta_1))$ -generic over  $\bar{M}$  with  $y \in \bar{M}[h]$ , and  $y$  is in  $A \cap \bar{M}[h]$  as computed using the recipe (5) for

$N = \bar{M}[h]$ . If  $(y, \bar{M}, k, h) \in [T]$ , then we write  $y \in p[T]$ . We also let  $U \in M$  be defined in exactly the same way, except that “ $y$  is in  $A \cap \bar{M}[h]$ ” gets replaced by “ $y$  is not in  $A \cap \bar{M}[h]$ .” If  $(y, \bar{M}, k, h) \in [U]$ , then we write  $y \in p[U]$ . The trees  $T$  and  $U$  are easily seen to witness that  $A \cap M$  is  $\zeta_1^+$ -universally Baire inside  $M$ .

Now let  $j: M \mid (\delta^{+M}) \rightarrow N$  be a generic iteration inside  $M^{\text{Col}(\omega, \zeta_1)}$  with  $j(A \cap M) \neq A \cap N$ . We have  $A \cap M = p[T] \cap M$ , and thus  $j(A \cap M) = p[j(T)] \cap N = (p[j(T)] \cap M^{\text{Col}(\omega, \zeta_1)}) \cap N$ . However,  $(p[j(T)] \cap M^{\text{Col}(\omega, \zeta_1)}) = (p[T] \cap M^{\text{Col}(\omega, \zeta_1)})$  by the fact that  $T, U$  witness that  $A \cap M$  is  $\zeta_1^+$ -universally Baire in  $M$ . Therefore  $j(A \cap M) = p[T] \cap N = A \cap N$ , a contradiction!  $\square$

We have to prove  $(A) \implies (B)$  of Theorem 2.7.

We assume that  $M_\omega^{\#\#}$  exists and is fully iterable and also that  $(*)$  holds true. Let us fix a set  $B$  of reals in  $L(\mathbb{R})$ , and let also  $A \in H_{\omega_2}$ . Let  $\varphi(x, \check{B}, \check{I}_{\text{NS}_{\omega_1}})$  be a  $\Sigma_1$  formula in the language of set theory, augmented by predicates for  $B$  and for the nonstationary ideal on  $\omega_1$ . Suppose that  $\varphi(A, \check{B}, \check{I}_{\text{NS}_{\omega_1}})$  is honestly consistent in the sense of Definition 2.5. We aim to show that  $\varphi(A, \check{B}, \check{I}_{\text{NS}_{\omega_1}})$  holds true in  $V$ .

Suppose not. We may assume without loss of generality that  $A \subset \omega_1$  and in fact that  $A$  is  $\mathbb{P}_{\max}$ -generic over  $L(\mathbb{R})$  (see [13, Theorem 4.60]). Let  $\dot{A}$  be the canonical name for  $A$ . Now say that

$$p = (M, \in, I, a) \Vdash \neg \varphi(\dot{A}, \check{B}, \check{I}_{\text{NS}_{\omega_1}}), \tag{6}$$

where  $p \in G_A = \{q = (N, \in, I', a') \in \mathbb{P}_{\max} : a' = A \cap \omega_1^N\}$ . We shall derive a contradiction by finding some  $q <_{\mathbb{P}_{\max}} p$  with  $q \Vdash \varphi(\dot{A}, \check{B}, \check{I}_{\text{NS}_{\omega_1}})$ .

By our hypothesis, the function  $F: \mathbb{R} \rightarrow \mathbb{R}$  with  $F(x) =$  (the canonical real code for)  $M_\omega^{\#\#}(x)$ ,  $x \in \mathbb{R}$ , is universally Baire. Let  $\mathfrak{A}$  be an  $F$ -closed witness to the fact that  $\varphi(A, \check{B}, \check{I}_{\text{NS}_{\omega_1}})$  is honestly consistent.

Let  $M_\omega^{\#\#}(X) \in \mathfrak{A}$  be such that  $X$  is transitive and  $(\mathcal{P}(\omega_1) \cap \mathfrak{A}) \cup \{(\text{NS}_{\omega_1})^\mathfrak{A}\} \in X$ . Let  $\delta$  be the least Woodin cardinal of  $M_\omega^{\#\#}(X)$ , and let  $g$  be  $\mathbb{Q}$ -generic over  $M_\omega^{\#\#}(X)$ , where  $\mathbb{Q}$  is, in  $M_\omega^{\#\#}(X)$ , a standard forcing iteration of size  $\delta$  with the  $\delta$ -c.c. forcing both that NS is precipitous and that  $\text{MA}_{\omega_1}$  holds. By Lemma 2.12, inside  $V^{\text{Col}(\omega, 2^{\aleph_1})}$  we have that

$$q = (M_\omega^{\#\#}(X)[g]; \in, \text{NS}^{M_\omega^{\#\#}(X)[g]}, A)$$

is a  $B^*$ -iterable  $\mathbb{P}_{\max}$ -condition with  $q <_{\mathbb{P}_{\max}} p$ , and

$$q \Vdash \varphi(A, B^*, \text{NS}_{\omega_1}),$$

so that  $q \Vdash \varphi(\dot{A}, \check{B}, \check{I}_{\text{NS}_{\omega_1}})$ .

The assertion that there is such a  $q$  is now absolute between  $V$  and  $V^{\text{Col}(\omega, 2^{\aleph_1})}$ . We obtained a contradiction!  $\square$

It remains open whether  $(*)$  can be forced over models of choice containing large cardinals or whether  $(*)$  indeed follows from a forcing axiom. In [13, Theorem 10.70], Woodin proves that  $(*)$  does not follow from  $\text{MM}^{++}(2^{\aleph_0})$ . In [7] and [8], Larson shows that  $(*)$  does not follow from  $\text{MM}^{+\omega}$ , and he asks whether  $(*)$  follows from  $\text{MM}^{++}$  (see [8, Question 7.2]). Woodin asks whether  $(*)$  can be forced from large cardinals as [13, Question (18)(a), p. 924] (see also [9, p. 2158]).

Theorem 2.7 yields an obvious scenario for showing that  $\text{MM}^{++}$  implies  $(*)$ . Basically, one would have to show that if a  $\Sigma_1$  statement  $\varphi$  with parameters as in  $A\text{-BMM}^{*,++}$  is honestly consistent in the sense of Definition 2.5, then  $\varphi$  can be forced by a stationary set-preserving forcing. We do not know how to do that, though.

### Notes

1.  $M_\omega^{\#\#}$  is a mouse with  $\omega$  Woodin cardinals and a top measure which is closed under  $\#\text{'s}$ .
2. For example, suppose that there is a proper class of Woodin cardinals.

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