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Layered Posets and Kunen's Universal Collapse

Sean Cox

Abstract We develop the theory of *layered posets* and use the notion of layering to prove a new iteration theorem (Theorem 6): if κ is weakly compact, then *any* universal Kunen iteration of κ-cc posets (each possibly of size κ) is κ-cc, as long as direct limits are used sufficiently often. This iteration theorem simplifies and generalizes the various chain condition arguments for universal Kunen iterations in the literature on saturated ideals, especially in situations where finite support iterations are not possible. We also provide two applications:

1. For any $n \geq 1$, a wide variety of $< \omega_{n-1}$-closed, ω_{n+1}-cc posets of size ω_{n+1} can consistently be absorbed (as regular suborders) by quotients of saturated ideals on ω_n (see Theorem 7 and Corollary 8).
2. For any $n \in \omega$, the tree property at ω_{n+3} is consistent with Chang's conjecture $(\omega_{n+3}, \omega_{n+1}) \twoheadrightarrow (\omega_{n+1}, \omega_n)$ (Theorem 9).

1 Introduction

A classic theorem of Solovay and Tennenbaum reads as follows.

Theorem 1 (Solovay and Tennenbaum [25]) *If κ is regular uncountable, then any finite support iteration of κ-cc posets is κ-cc.*

For iterations which are not finite support, the situation is much trickier. A commonly used theorem in these more general situations is the following.

Theorem 2 (Jech [14, Theorem 16.30]) *Let κ be a regular uncountable cardinal, and let α be a limit ordinal. Let $\mathbb{P}_\alpha$ be an iteration such that, for each $\beta < \alpha$, $\mathbb{P}_\beta = \mathbb{P}_\alpha \restriction \beta$ satisfies the κ-chain condition. If $\mathbb{P}_\alpha$ is a direct limit and either*

- *$\mathrm{cf}(\alpha) \neq \kappa$ or*
- *$\mathrm{cf}(\alpha) = \kappa$ and there are stationarily many $\beta < \alpha$ such that $\mathbb{P}_\beta$ is a direct limit,*

then $\mathbb{P}_\alpha$ satisfies the κ-chain condition.

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The difficult part of applying Theorem 2 is typically the verification that $\mathbb{P}_\beta$ is κ-cc when $\mathbb{P}_\beta$ is an *inverse* limit. Usually this is taken care of simply by iterating *small* posets, as in the following corollary, which is heavily used throughout the set theory literature.

Corollary 3 (Baumgartner [1]; see also Cummings [5, Proposition 7.13]) *If κ is inaccessible and $\langle \mathbb{P}_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ is an iteration such that*

- *for all $\alpha < \kappa$, $\Vdash_{\mathbb{P}_\alpha} |\dot{Q}_\alpha| < \kappa$ and*
- *a direct limit is taken at κ and on a stationary set of limit stages below κ,*

then $\mathbb{P}_\kappa$ has the κ-cc (in fact, is κ-Knaster).

What if, in the statement of Corollary 3, we do not assume that each $\dot{Q}_\alpha$ is forced to have size $< \kappa$? This is in general a difficult problem, even if the posets $\dot{Q}_\beta$ are assumed to be highly closed; in fact, there is a countable support iteration of the length ω of countably closed, ω_2-cc posets which fails to be ω_2-cc (see Kunen [18, Exercise V.5.23]). A prominent family of iterations which do *not* use small posets are the so-called *universal Kunen iterations*. Kunen introduced the first version in [17] to prove that saturated ideals on ω_1 are consistent relative to huge cardinals. There he assumed that κ was a huge cardinal and defined a κ-length finite support iteration $\langle \mathbb{P}_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$, where, importantly, each $\dot{Q}_\alpha$ was forced to have size κ and was in fact from an inner model of $V^{\mathbb{P}_\alpha}$.[1] He used the Knaster property of the Silver collapse, together with Theorem 1 about finite support iterations, to prove that $\mathbb{P}_\kappa$ had the κ-cc. The poset $\mathbb{P}_\kappa$ is highly universal, even for many posets of size κ; contrast this with the Levy collapse $\mathrm{Col}(\omega, < \kappa)$, which is only universal for posets of size $< \kappa$. The strong universality property of $\mathbb{P}_\kappa$ is the key to obtaining master conditions which enable one to construct a saturated ideal in the final model.

Laver generalized Kunen's universal iteration to contexts where larger supports were used, in order to obtain saturated ideals on ω_2 and beyond (see Foreman [9] for a discussion of the history). However, in those settings the κ-cc preservation of the resulting universal iteration had to be checked on a case-by-case basis[2] and seemingly was believed to be a somewhat delicate issue (see Cummings [5, Remark 20.3]). We prove that, at least if κ is weakly compact, then the situation is not really delicate at all; *any* universal Kunen iteration of κ-cc posets—that is, where $\dot{Q}_\beta$ is forced by $V_\beta \cap \mathbb{P}_\beta$ to be κ-cc[3]—will be κ-cc, provided that direct limits are used often.

The proof heavily uses the notion of a *layered poset*. Layering has previously appeared in the literature in the form of so-called *layered ideals* (e.g., Foreman, Magidor, and Shelah [12]), but it is a general property of posets which is quite useful and interesting. A poset $\mathbb{P}$ is κ-stationarily layered if and only if there are stationarily many $\mathbb{Q} \in P_\kappa(\mathbb{P})$ which are regular suborders of $\mathbb{P}$; κ-club layering is defined similarly.

Among the basic facts we prove about layering is the following lemma, proved in Section 3.3.

Lemma 4 *If $\mathbb{P}$ is κ-stationarily layered, then $\mathbb{P}$ is κ-Knaster.*

The converse is, in general, false (see Cox and Lücke [3]). In [3] it is also proved that the weak compactness of κ is equivalent to "every κ-cc poset is κ-stationarily layered." In short,

$$\kappa\text{-stationarily layered} \implies \kappa\text{-Knaster} \implies \kappa\text{-cc},$$

and if κ is weakly compact, then the three notions are equivalent (and highly productive).

In this article, the notion of layering is used to prove the following general theorems about universal Kunen iterations. See Section 5 for a general discussion of universal Kunen iterations and precise statements of the theorems.

Theorem 5 (see Theorem 49 for the precise statement) *If κ is Mahlo, then any universal Kunen iteration of sufficiently κ-layered posets is κ-stationarily layered, provided direct limits were taken at all inaccessible $\gamma \leq \kappa$.*

Theorem 6 (see Corollary 50 for the precise statement) *If κ is weakly compact, then any universal Kunen iteration of κ-cc posets is κ-stationarily layered, provided direct limits were taken at all inaccessible $\gamma \leq \kappa$.*

Note that, in particular, by Lemma 4 the universal Kunen iteration is κ-Knaster in the conclusion of both theorems.

We give several applications of Theorem 6. The statement of the metamathematical Theorem 7 below is necessarily somewhat technical; the role of the parameter r in (1) is to allow, for example, the forcing to be defined inside $H_{\kappa^{++}}$ using some bookkeeping device; for example, a parameter which is a κ^+-to-one surjection from $\kappa^+ \to H_{\kappa^+}$ is used to define some iteration of length κ^+. See Corollary 8 for specific examples of posets which satisfy the requirements of Theorem 7.

Theorem 7 *Assume that $\phi(-,-)$ is a set-theoretic formula with two free variables and that ZFC proves the following: whenever κ is a successor of a regular cardinal, there exists an $r \in H_{\kappa^{++}}$ such that, letting μ be the cardinal predecessor of κ,*

$$\mathbb{S}^{H_{\kappa^{++}}}_{r,\phi} := \left\{z \in H_{\kappa^{++}} \;\middle|\; (H_{\kappa^{++}}, \in) \models \phi(z,r)\right\} \text{ is a } <\mu\text{-closed, } \kappa^+\text{-cc poset of size at most } \kappa^+. \tag{1}$$

Then, whenever V is a model of ZFC and $V \models$ "$\mu < \kappa$ are regular cardinals and κ is a huge cardinal," there is a $< \mu$-closed poset in V which forces the following statements:

- *$\kappa = \mu^+ = 2^\mu$ and $2^\kappa = \kappa^+$;*
- *there is a normal, saturated ideal $\mathcal{J}$ on κ;*
- *there exists some $r \in H_{\kappa^{++}}$ such that (1) holds, and there exists some regular embedding*

$$e : \mathbb{S}^{H_{\kappa^{++}}}_{r,\phi} \to \wp(\kappa)/\mathcal{J}.$$

The formula $\phi(-,-)$ in the assumptions of Theorem 7 can be designed in ways to specify a particular ω_n or to specify that certain other requirements are met, if desired.[4] The following corollary lists some specific examples which satisfy the requirements of Theorem 7. We say that $\mathbb{R}$ absorbs $\mathbb{Q}$ if there exists a regular embedding from $\mathbb{Q} \to \mathbb{R}$.

Corollary 8 *Fix $n \in \omega$, $n \geq 1$. Each of the following posets can consistently be absorbed by the quotient of a saturated ideal on ω_n:*

1. *Jensen's $< \omega_n$-closed forcing of size ω_{n+1} (under GCH) to add an ω_n-Kurepa tree (see [5]);*
2. *the forcing to add a single Hechler subset of ω_n (see Cummings and Shelah [6]);*

3. *a* $<\omega_{n-1}$*-closed,* ω_{n+1}*-Suslin tree;*
4. *the* ω_{n+1}*-length,* $<\omega_{n-1}$*-support iteration of adding Hechler subsets of* ω_{n-1} *(see [6]); more generally, any sufficiently definable*[5] $<\omega_{n-1}$*-support iteration of length* ω_{n+1} *which uses* $<\omega_{n-1}$*-closed posets of size* $\leq\omega_n$ *at each step;*
5. $\leq\omega_{n-1}$*-support,* ω_{n+1}*-length iteration of* $\mathrm{Sacks}(\omega_{n-1})$ *(as in Kanamori [15]);*
6. *the following poset can consistently be absorbed into the quotient of a saturated ideal on* ω_2*: the* σ*-closed,* ω_2*-cc poset of size* ω_3 *from Shelah [24], which he used to force* $2^{\omega_1}=\omega_3$ *together with a generalized version of Martin's axiom for a certain subclass of the* σ*-closed,* ω_2*-cc posets.*

Finally, we prove that a certain instance of Chang's conjecture is consistent with the tree property.

Theorem 9 *If* $\mu<\kappa$ *are both regular and* κ *is a huge cardinal, then there is a forcing extension which satisfies the following:*

- $\kappa=\mu^+$*;*
- *Chang's conjecture* $(\mu^{+3},\mu^+)\twoheadrightarrow(\mu^+,\mu)$ *holds; and*
- *the tree property holds at* μ^{+3}.

In particular, for any $n\in\omega$*, the tree property at* ω_{n+3} *is consistent with Chang's conjecture:*

$$(\omega_{n+3},\omega_{n+1})\twoheadrightarrow(\omega_{n+1},\omega_n).$$

The article is organized as follows:

- Section 2 includes some preliminaries.
- Section 3 introduces layered posets; proves that layering implies the Knaster condition; provides some examples; and proves a useful lemma about layering and elementary embeddings.
- Section 4 introduces the important notion of a *coherent, conservative* system of reduction operations and proves Lemma 43, which is crucial to passing limit stages in the proofs of Section 5.
- Section 5 provides the relevant background for universal Kunen-type iterations and proves (the precise versions of) Theorems 5 and 6.
- Section 6 proves Theorems 7 and 9.

2 Preliminaries

2.1 Regular embeddings and quotient forcing If p, q are conditions in a poset $\mathbb{P}$, we write $p\leq q$ to mean that p is stronger than q, $p\parallel q$ to mean that p and q are compatible, and $p\perp q$ to mean that p and q are incompatible. Given $\mathbb{P}$-names τ and τ', we will say that τ is equivalent to τ' (modulo $\mathbb{P}$) if and only if $\Vdash_{\mathbb{P}}\tau=\tau'$. The *maximality principle* is the fact that if $p\Vdash_{\mathbb{P}}\exists x\,\phi(x)$, where ϕ is a statement in the forcing language, then there is some $\mathbb{P}$-name τ such that $p\Vdash\phi(\tau)$.

A function $e:\mathbb{P}\to\mathbb{Q}$ is a *regular embedding* if and only if it is order- and incompatibility-preserving, and whenever A is a maximal antichain in $\mathbb{P}$, we have that $e[A]$ is a maximal antichain in $\mathbb{Q}$. If $\mathrm{id}:\mathbb{P}\to\mathbb{Q}$ is a regular embedding, then $\mathbb{P}$ is called a *regular suborder* of $\mathbb{Q}$. If $\mathrm{id}:\mathbb{P}\to\mathbb{Q}$ is order- and incompatibility-preserving—but not necessarily regular—then we say $\mathbb{P}$ is a *suborder* of $\mathbb{Q}$.

Definition 10 Let $\mathbb{P}$ be a suborder of $\mathbb{Q}$ and $q \in \mathbb{Q}$. We say that $p \in \mathbb{P}$ is a *reduct of q into $\mathbb{P}$* if and only if

$$\forall p' \leq_{\mathbb{P}} p \ p' \parallel_{\mathbb{Q}} q.$$

If p is also $\geq_{\mathbb{Q}} q$, then we say p is a *nice* reduct of q into $\mathbb{P}$.

The following lemma is standard, but we provide the short proof for convenience. Note that the reduction characterization yields a Σ_0-characterization of the regular suborder relation.

Lemma 11 *Let $\mathbb{P}$ be a suborder of $\mathbb{Q}$. The following are equivalent:*

1. *$\mathbb{P}$ is a regular suborder of $\mathbb{Q}$;*
2. *for every $q \in \mathbb{Q}$ there is a reduct of q into $\mathbb{P}$.*

Proof Assume that $\mathbb{P}$ is a regular suborder of $\mathbb{Q}$. So every maximal antichain in $\mathbb{P}$ is a maximal antichain in $\mathbb{Q}$. Assume for a contradiction that there is some $q \in \mathbb{Q}$ which has no reduct; this implies that

$$D_{\perp q} := \{p \in \mathbb{P} \mid p \perp_{\mathbb{Q}} q\} \text{ is dense in } \mathbb{P}.$$

Let G be $\mathbb{Q}$-generic with $q \in G$. Then $G \cap \mathbb{P}$ is $\mathbb{P}$-generic and so $G \cap D_{\perp q} \neq \emptyset$, contradicting that G is a filter on $\mathbb{Q}$ and that $q \in G$.

For the other direction, let A be a maximal antichain in $\mathbb{P}$. Let q be any condition in $\mathbb{Q}$, and let p_q be any reduct of q into $\mathbb{P}$. Let $a \in A$ be compatible in $\mathbb{P}$ with p_q, and let $p' \in \mathbb{P}$ witness this compatibility. Because $p' \leq p_q$, it follows that $p' \parallel_{\mathbb{Q}} q$. Then q is compatible with $a \in A$. Because q is arbitrary, A is maximal in $\mathbb{Q}$. □

Definition 12 Suppose that $e : \mathbb{P} \to \mathbb{Q}$ is a regular embedding, $p \in \mathbb{P}$, and $q \in \mathbb{Q}$. We write $q \leq_e p$ to mean

$$q \Vdash_{\mathbb{Q}} \check{p} \in \left(\check{e}^{-1}[\dot{G}_{\mathbb{Q}}]\right) \uparrow,$$

where $(\check{e}^{-1}[\dot{G}_{\mathbb{Q}}]) \uparrow$ denotes the set of all conditions in $\mathbb{P}$ which are weaker than some condition in $\check{e}^{-1}[\dot{G}_{\mathbb{Q}}]$.

If $q \in \mathbb{Q}$ and $p \in \mathbb{P}$, then we say that p is an *e-reduct of q* if and only if

$$\forall p' \leq_{\mathbb{P}} p \ \ e(p') \parallel_{\mathbb{Q}} q.$$

Note that if $e = \mathrm{id}$ and $\mathbb{Q}$ is separative, then $q \leq_e p$ is equivalent to $q \leq_{\mathbb{Q}} p$.

Lemma 13 *Suppose that $\mathbb{B}$ and $\mathbb{C}$ are complete Boolean algebras and that $e : \mathbb{B} \to \mathbb{C}$ is a regular embedding. Let $c \in \mathbb{C}$, and set*

$$b_c^e := \sup_{\mathbb{B}}\{b \in \mathbb{B} \mid b \textit{ is an } e\textit{-reduct of } c\}.$$

Then

1. *b_c^e is an e-reduct of c and moreover is the largest one;*
2. *$b_c^e \geq_e c$ (as in Definition 12)*

Proof To see that b_c^e is an e-reduct of c, let $b' \leq b_c^e$ with $b' \neq 0$. By the definition of b_c^e there is some b which is an e-reduct of c such that $b' \wedge b \neq 0$. Then $b' \wedge b \leq b$, and because b is an e-reduct of c, we have $e(b' \wedge b) \parallel c$. Because e is order-preserving and $b' \wedge b \leq b'$ we have $e(b') \parallel c$, which completes the proof that b_c^e is an e-reduct of c. Clearly it is the largest such, by its definition as the sup of all e-reducts of c.

To see that $b^e_c \leq_e c$, suppose toward a contradiction that this were false; then there is some $c' \leq c$ such that

$$c' \Vdash_{\mathbb{C}} b^e_c \notin e^{-1}[\dot{G}_{\mathbb{C}}].$$

It follows easily that

$$c' \perp e(b^e_c). \tag{2}$$

Let b' be any e-reduct of c'. Then $b' \vee b^e_c > b^e_c$; otherwise $b' \leq b^e_c$ and, because $c' \parallel e(b')$, this would contradict (2). Because $b' \vee b^e_c > b^e_c$ and b^e_c is defined as the supremum of all e-reducts of c, we have that $b' \vee b^e_c$ is *not* a reduct of c. We will prove that $b' \vee b^e_c$ is in fact an e-reduct of c, which will yield a contradiction and complete the proof. So let $r \leq b' \vee b^e_c$, $r \neq 0$. Then $r \parallel b'$ or $r \parallel b^e_c$.

- Case 1: $r \parallel b'$. Because b' is an e-reduct of c', we have that $e(r) \parallel c'$, and because $c' \leq c$ this implies that $e(r) \parallel c$.
- Case 2: $r \parallel b^e_c$. Then because b^e_c is an e-reduct of c (by the first part of the proof), we have $e(r) \parallel c$.

□

Corollary 14 *Suppose that $\mathbb{B}$ is a complete regular subalgebra of $\mathbb{C}$ and that $c \in \mathbb{C}$. Let*

$$b_c := \sup_{\mathbb{B}}\{b \in \mathbb{B} \mid b \text{ is a reduct of } c\}.$$

Then b_c is a nice reduct of c.

Proof This follows immediately from Lemma 13 by taking the e from that lemma to be the identity map from $\mathbb{B} \to \mathbb{C}$. □

The following fact is well known.

Lemma 15 *If $e : \mathbb{P} \to \mathbb{Q}$ is a regular embedding of separative posets, then e lifts to a regular embedding of their Boolean completions.*

If the map e from Lemma 15 is the identity—that is, if $\mathbb{P}$ is a regular suborder of $\mathbb{Q}$—then we can essentially take the lifting to the Boolean completions to be the identity map, which simplifies the notation a bit. In other words, if $\mathbb{P}$ is a regular suborder of $\mathbb{Q}$, then we can extend both to view $\mathrm{ro}(\mathbb{P})$ as a regular suborder of $\mathrm{ro}(\mathbb{Q})$. I am grateful to Joel David Hamkins for pointing this out to me on MathOverflow.

Lemma 16 (Hamkins [13]) *Suppose that $\mathbb{P}$ is a regular suborder of the separative poset $\mathbb{Q}$. In $\mathrm{ro}(\mathbb{Q})$ define*

$$\mathbb{B}(\mathbb{P}) := \left\{ \sup_{\mathrm{ro}(\mathbb{Q})}(X) \;\middle|\; X \subseteq \mathbb{P} \right\}.$$

Then $\mathbb{B}(\mathbb{P})$ is a complete regular subalgebra of $\mathrm{ro}(\mathbb{Q})$, and $\mathbb{P}$ is dense in $\mathbb{B}(\mathbb{P})$ (so $\mathbb{B}(\mathbb{P}) \simeq \mathrm{ro}(\mathbb{P})$).

Definition 17 Let $\mathbb{P}$ be a regular suborder of the separative poset $\mathbb{Q}$, and let $q \in \mathbb{Q}$. Set

$$\mathrm{WeakestReduct}_{\mathrm{ro}(\mathbb{P})}(q) = \sup\left\{b \in \mathrm{ro}(\mathbb{P}) \mid b \text{ is a reduct of } q\right\},$$

where we are viewing $\mathrm{ro}(\mathbb{P})$ as a regular complete subalgebra of $\mathrm{ro}(\mathbb{Q})$ as in Lemma 16.

We will often use the following convention.

Convention 18 Assume that $\mathbb{P}$ is a regular suborder of $\mathbb{Q}$, σ is a statement in the $\mathbb{P}$ forcing language, and $q \in \mathbb{Q}$. We will write $q \Vdash_{\mathbb{Q}} \sigma$ to mean the following: whenever G is $\mathbb{Q}$-generic with $q \in G$, we have that $V[G \cap \mathbb{P}] \models \sigma_{G\cap\mathbb{P}}$.

Lemma 19 *Let $\mathbb{P}$ be a regular suborder of $\mathbb{Q}$, let $q \in \mathbb{Q}$, and let σ be a statement in the forcing language of $\mathbb{P}$. Then the following are equivalent:*

1. *$q \Vdash_{\mathbb{Q}} \sigma$ (as in Convention 18);*
2. $\mathrm{WeakestReduct}_{\mathrm{ro}(\mathbb{P})}(q) \Vdash_{\mathrm{ro}(\mathbb{P})} \sigma$.

Proof By Corollary 14, $\mathrm{WeakestReduct}_{\mathrm{ro}(\mathbb{P})}(q)$ is a nice reduct of q into $\mathrm{ro}(\mathbb{P})$; that is, it is a reduct and, moreover, $\mathrm{WeakestReduct}_{\mathrm{ro}(\mathbb{P})}(q) \geq q$.

To see that (1) implies (2), if not, then there is some $p' \leq \mathrm{WeakestReduct}_{\mathrm{ro}(\mathbb{P})}(q)$ in $\mathrm{ro}(\mathbb{P})$ such that p' forces the negation of σ. But because $\mathrm{WeakestReduct}_{\mathrm{ro}(\mathbb{P})}(q)$ is a reduct, p' is compatible in $\mathrm{ro}(\mathbb{Q})$ with q, yielding a contradiction to assumption (1).

To see that (2) implies (1), this is simply because $\mathrm{WeakestReduct}_{\mathrm{ro}(\mathbb{P})}(q) \geq q$; any $\mathrm{ro}(\mathbb{Q})$-generic which includes q must also include $\mathrm{WeakestReduct}_{\mathrm{ro}(\mathbb{P})}(q)$, so assumption (2) ensures that σ holds in any such generic extension. □

If $\mathbb{P}$ is a regular suborder of $\mathbb{Q}$ and $G_{\mathbb{P}}$ is generic for $\mathbb{P}$, then in $V[G_{\mathbb{P}}]$ the quotient $\mathbb{Q}/G_{\mathbb{P}}$ is defined as the set of all $q \in \mathbb{Q}$ which are compatible (in $\mathbb{Q}$) with every member of $G_{\mathbb{P}}$. The ordering of $\mathbb{Q}/G_{\mathbb{P}}$ is the order inherited from $\mathbb{Q}$. The following fact is standard.

Fact 20 If $\mathbb{P}$ is a regular suborder of $\mathbb{Q}$, then

$$\mathbb{Q} \text{ is forcing-equivalent to } \mathbb{P} * \check{\mathbb{Q}}/\dot{G}_{\mathbb{P}}.$$

We will also use the following fact in the proof of Theorem 7.

Fact 21 If $\mathbb{P}$ is a regular suborder of $\mathbb{Q}$ and $\mathbb{Q}$ has the κ-cc, then

$$\Vdash_{\mathbb{P}} \quad \check{\mathbb{Q}}/\dot{G}_{\mathbb{P}} \text{ has the } \kappa\text{-cc}.$$

Proof Suppose for a contradiction that there is some $p \in \mathbb{P}$ forcing that there is a κ-sized antichain in the quotient. By the maximality principle there is a $\mathbb{P}$-name $\langle \dot{q}_\xi \mid \xi < \kappa \rangle$ which is forced by p to be an antichain in the quotient. But then $\{(p, \dot{q}_\xi) \mid \xi < \kappa\}$ is easily seen to be an antichain in $\mathbb{P} * \check{\mathbb{Q}}/\dot{G}_{\mathbb{P}}$. By Fact 20 this implies that $\mathbb{Q}$ fails to have the κ-cc, a contradiction. □

2.2 Generalized Sacks forcing The proof of Theorem 9 will make use of the following generalized Sacks forcing. If T is a subtree of ${}^{<\kappa}2$, $\alpha < \kappa$, and $s \in {}^{\alpha}2$, we say that *s splits in T* if and only if both $s^\frown 0$ and $s^\frown 1$ are in T.

Definition 22 (Kanamori [15, Definition 1.1]) Let κ be a regular uncountable cardinal. $\mathrm{Sacks}(\kappa)$ is the poset of all subtrees T of ${}^{<\kappa}2$ such that

1. if $\alpha < \kappa$ is a limit ordinal, $s \in {}^{\alpha}2$, and $s \restriction \beta \in T$ for every $\beta < \alpha$, then $s \in T$;
2. if $s \in T$, then there is a $t \in T$ such that $s \subset t$ and t splits in T;
3. if $\alpha < \kappa$ is a limit ordinal, $s \in {}^{\alpha}2$, and $s \restriction \beta$ splits in T for cofinally many $\beta < \alpha$, then s splits in T.

T is stronger than S if and only if $T \subseteq S$.

Lemma 23 *If $T \in \mathrm{Sacks}(\kappa)$ and $f : \kappa \to 2$ is a cofinal branch through T (i.e., $f \restriction \alpha \in T$ for all $\alpha < \kappa$), then*

$$C_{f,T} := \{\alpha < \kappa \mid f \restriction \alpha \textit{ splits in } T\} \textit{ is a closed, unbounded subset of } \kappa.$$

Proof By Definition 22(3) it suffices to show that $C_{f,T}$ is cofinal in κ. Fix any $\alpha_0 < \kappa$. Because $f \restriction \alpha_0 \in T$, by Definition 22(2) there is some $g \supseteq f \restriction \alpha_0$ which splits in T; that is, $g^\frown 0 \in T$ and $g^\frown 1 \in T$. Let $\beta_{f,g}$ be the maximal ordinal such that $g \restriction \beta_{f,g} = f \restriction \beta_{f,g}$. Note that

$$\beta_{f,g} \geq \alpha_0.$$

We claim that $f \restriction \beta_{f,g}$ splits in T. For conceptual clarity, consider two cases:

1. If $\beta_{f,g} = \mathrm{dom}(g)$, then $g^\frown 0 = (f \restriction \beta_{f,g})^\frown 0$ and $g^\frown 1 = (f \restriction \beta_{f,g})^\frown 1$. Because both of these functions are in T by the assumption that g splits in T, this shows that $f \restriction \beta_{f,g}$ splits in T.
2. If $\beta_{f,g} < \mathrm{dom}(g)$, then $g \restriction \beta_{f,g} = f \restriction \beta_{f,g}$ and $g(\beta_{f,g}) \neq f(\beta_{f,g})$. Because $g \restriction (\beta_{f,g} + 1)$ and $f \restriction (\beta_{f,g} + 1)$ are both in T, this again implies that $f \restriction \beta_{f,g}$ splits in T. □

Kanamori [15, Lemma 1.2] proved that $\mathrm{Sacks}(\kappa)$ is $< \kappa$-closed, but we will actually need a bit more. Recall that if D is a subset of a poset $\mathbb{P}$, we say D is *directed* if and only if, whenever $d_0, d_1 \in D$, there is a $d \in D$ such that $d \leq d_0$ and $d \leq d_1$. A poset $\mathbb{P}$ is $< \kappa$-directed closed if and only if, whenever D is a $< \kappa$-sized, directed subset of $\mathbb{P}$, we have that D has a lower bound in $\mathbb{P}$.

Lemma 24 *For all regular uncountable κ, the poset $\mathrm{Sacks}(\kappa)$ is $< \kappa$-directed closed.*

Proof Let D be a $< \kappa$-sized, directed subset of $\mathrm{Sacks}(\kappa)$. We prove that $\bigcap D$ inherits all requirements of conditionhood from Definition 22. The only nontrivial part is verifying Definition 22(2); all other requirements of Definition 22 are easily inherited by arbitrary intersections of conditions.

So suppose $s \in \bigcap D$. We need to find some $t \supset s$ which splits in $\bigcap D$.

Claim 24.1 *There is an $f : \kappa \to 2$ extending s such that f is a cofinal branch through every member of D.*

Proof We recursively define $f \restriction \alpha$ for all $\alpha \in [\mathrm{dom}(s), \kappa]$ and verify inductively that $f \restriction \alpha \in \bigcap D$ for all such α's. First set $f \restriction \mathrm{dom}(s) := s$. If α is a limit ordinal, then $f \restriction \alpha$ is defined as the union of $f \restriction \beta$ for all $\beta < \alpha$; the induction hypothesis and Definition 22(1) ensure that $f \restriction \alpha \in \bigcap D$.

Now suppose $f \restriction \alpha$ is defined; we want to define $f \restriction (\alpha + 1)$. We claim that at least one of $(f \restriction \alpha)^\frown 0$ or $(f \restriction \alpha)^\frown 1$ is an element of $\bigcap D$. If not, then there are $T_0, T_1 \in D$ such that $(f \restriction \alpha)^\frown 0 \notin T_0$ and $(f \restriction \alpha)^\frown 1 \notin T_1$. Because D is directed, there is a $T \in D$ such that $T \subseteq T_0 \cap T_1$. Now the induction hypothesis ensures that $f \restriction \alpha \in T$; and Definition 22(2) ensures that $(f \restriction \alpha)^\frown i \in T$ for at least one $i \in \{0, 1\}$. But then $(f \restriction \alpha)^\frown i \in T_i$, a contradiction. □

Let $f : \kappa \to 2$ be as given by the claim, and for every $T \in D$ let $C_{f,T}$ be the club from Lemma 23. Because $|D| < \kappa$, $C := \bigcap_{T \in D} C_{f,T}$ is club in κ. Pick some

$\beta \in C \cap (\mathrm{dom}(s), \kappa)$. Then $f \restriction \beta$ splits in every member of D; that is,

$$(f \restriction \beta)^\frown 0 \text{ and } (f \restriction \beta)^\frown 1 \text{ are both in } \bigcap D.$$

Because $f \restriction \beta \supset s$ this completes the proof. □

2.3 Other miscellaneous background A regular uncountable cardinal κ is:

- *weakly compact* if and only if κ is inaccessible and, whenever M is a transitive ZF^- model of size κ and M is closed under $< \kappa$ sequences, we have that there are a transitive set N and an elementary embedding $j : M \to N$ with critical point κ; there are many other well-known equivalent formulations;
- *almost huge* if and only if there is an elementary embedding $j : V \to N$ with critical point κ such that N is closed under $< j(\kappa)$ sequences;
- *huge* if and only if there is an elementary embedding $j : V \to N$ with critical point κ such that N is closed under $j(\kappa)$ sequences.

The *tree property* holds at κ if and only if, whenever T is a tree of height κ and each level of T is of size $< \kappa$, we have that T has a cofinal branch. If

$$\mu_3 > \mu_2 > \mu_1 > \mu_0$$

are cardinals then Chang's conjecture $(\mu_3, \mu_2) \twoheadrightarrow (\mu_1, \mu_0)$ means: whenever $\mathfrak{A}$ is a first-order structure on μ_3 in a countable language, there is an $X \prec \mathfrak{A}$ such that $|X| = \mu_1$ and $|X \cap \mu_2| = \mu_0$. Various versions of this are discussed at length in [9].

The proof of Theorem 7 makes use of Foreman's duality theorem, specifically regarding the "Magidor variation" of Kunen's original argument. All of these topics are covered extensively in [9]; but we only need a special case. To describe the special case, we recall (see Kanamori [16]) that a sequence $\vec{U} = \langle U_\gamma \mid \gamma < \delta \rangle$ is called a (κ, δ) *tower* if and only if each U_γ is a normal measure on $P_\kappa(\gamma)$, and if $\gamma < \gamma'$, then

$$U_\gamma = \big\{ \{x \cap V_\gamma : x \in A\} \;\big|\; A \in U_{\gamma'} \big\}$$

Given such a tower, if $\gamma < \gamma'$, then there is an elementary embedding $k_{\gamma,\gamma'} : \mathrm{ult}(V, U_\gamma) \to \mathrm{ult}(V, U_{\gamma'})$ such that $j_{\gamma'} = k_{\gamma,\gamma'} \circ j_\gamma$, where $j_{\gamma'}$ and j_γ are the ultrapower maps for $U_{\gamma'}$ and U_γ, respectively. This yields a directed system of ultrapower maps and a direct limit ultrapower embedding $j_{\vec{U}} : V \to_{\vec{U}} N_{\vec{U}}$ with critical point κ such that $N_{\vec{U}}$ is well founded and, in fact,

$$N_{\vec{U}} \text{ is closed under } < \delta \text{ sequences} \tag{3}$$

and in general $j_{\vec{U}}(\kappa) \geq \delta$. If $j_{\vec{U}}(\kappa) = \delta$, then $\vec{U}$ is called an *almost huge* (κ, δ) tower; in that case (by (3)) note that $N_{\vec{U}}$ is closed under $< j_{\vec{U}}(\mathrm{crit}(j_{\vec{U}}))$-sequences. We use the following fact.

Fact 25 Suppose that $j : V \to N$ is a huge embedding with critical point κ; that is, j is elementary and N is closed under $\delta := j(\kappa)$ sequences. Then there exists an almost huge (κ, δ)-tower $\vec{U}$ and an elementary embedding $k : N_{\vec{U}} \to N$ such that $j = k \circ j_{\vec{U}}$ and $\mathrm{crit}(k) = (\delta^+)^{N_{\vec{U}}}$. Note, in particular, this implies that k fixes δ and that $(\delta^+)^{N_{\vec{U}}} < (\delta^+)^N$.

An ideal $\mathcal{I}$ on a regular uncountable cardinal κ is called *normal* if it is closed under diagonal unions and *saturated* if and only if $\wp(\kappa)/\mathcal{I}$ has the κ^+-chain condition. The following theorem is a special instance of the duality theorem of Foreman [9] (see also Foreman and Komjath [10] or Cox and Zeman [4, Section 5.3] for more detailed proofs).

Theorem 26 *Suppose that $\vec{U}$ is an almost huge (κ, δ) tower and that $j_{\vec{U}} : V \to_{\vec{U}} N_{\vec{U}}$ is the corresponding embedding. Suppose also that $\mathbb{P} \subset V_\kappa$ is a κ-cc poset, that $\dot{\mathbb{C}}_{\mathrm{Silv}}(\kappa, < \delta)$ is the $\mathbb{P}$-name for the Silver collapse which turns δ into κ^+ (described in Section 3.2), and that, in $N_{\vec{U}}$, there exists a regular embedding*

$$\iota : \mathbb{P} * \dot{\mathbb{C}}_{\mathrm{Silv}}(\kappa, < \delta) \to j_{\vec{U}}(\mathbb{P})$$

such that ι is the identity on $\mathbb{P}$.[6] *Let $G * H$ be generic for $\mathbb{P} * \dot{\mathbb{C}}_{\mathrm{Silv}}(\kappa, < \delta)$ over V. Then in $V[G * H]$ there is a normal ideal $\mathcal{I}(j_{\vec{U}})$ on κ such that $\wp(\kappa)/\mathcal{I}(j_{\vec{U}})$ is forcing-equivalent to $j_{\vec{U}}(\mathbb{P})/\iota[G * H]$.*

We use one more fact about normal ideals.

Fact 27 (see [9]) If $\mathcal{I}$ is a normal, saturated ideal on κ, then $\wp(\kappa)/\mathcal{I}$ is a complete Boolean algebra.

3 Layered Posets

3.1 Basic theory of layered posets A layered poset is, roughly, a poset which has many small regular suborders, where "many" is typically taken to mean club or stationarily many. There are two common definitions of stationarity for subsets of $P_\kappa(H)$, where H is any set of cardinality at least κ:

- strong stationarity, as in Jech [14, Definition 8.21]: a set $S \subseteq P_\kappa(H)$ is *strongly stationary in $P_\kappa(H)$* if and only if $S \cap C \neq \emptyset$ for all $C \subseteq P_\kappa(H)$ which are cofinal in $(P_\kappa(H), \subset)$ and closed under increasing unions of length $< \kappa$;
- Shelah's weak stationarity or generalized stationarity, for example, as in Larson [19] and Foreman [9]: a set $S \subseteq P_\kappa(H)$ is *weakly stationary in $P_\kappa(H)$* if and only if whenever $F : [H]^{<\omega} \to H$ there is some $X \in S$ which is closed under F. The set of all $X \subseteq H$ which are closed under F is denoted by C_F.

For $\kappa = \omega_1$ the notions "strongly stationary in $P_\kappa(H)$" and "weakly stationary in $P_\kappa(H)$" are equivalent, but for $\kappa \geq \omega_2$ they may differ, due to the presence or absence of Chang's conjecture; this is discussed in detail in Foreman [9].

Convention 28 Unless stated otherwise, in this article "stationary" will refer to Jech's strong stationarity.[7]

For example, in the proof of Lemma 4, it is essential that the stationary layering occurs on models M such that $M \cap \kappa \in \kappa$; this happens automatically in any strongly stationary set, but if $\kappa > \omega_1$, then it might fail for a weakly stationary set.

The following is the main definition.

Definition 29 Let $\mathbb{P}$ be a poset, let κ be a regular uncountable cardinal with $\kappa \leq |\mathbb{P}|$, and let

$$\mathrm{Reg}_\kappa(\mathbb{P}) := \left\{X \in P_\kappa(\mathbb{P}) \;\middle|\; X \text{ is a regular suborder of } \mathbb{P}\right\}.$$

1. $\mathbb{P}$ is *κ-stationarily layered* if and only if $\mathrm{Reg}_\kappa(\mathbb{P})$ is stationary in $P_\kappa(\mathbb{P})$.
2. $\mathbb{P}$ is *κ-club-layered* if and only if $\mathrm{Reg}_\kappa(\mathbb{P})$ contains a club (i.e., $P_\kappa(\mathbb{P}) - \mathrm{Reg}_\kappa(\mathbb{P})$ is not stationary).
3. $\mathbb{P}$ is *κ-cofinally layered* if and only if $\mathrm{Reg}_\kappa(\mathbb{P})$ is cofinal in $(P_\kappa(\mathbb{P}), \subseteq)$.

Recall Convention 28; we are referring to Jech's strong stationarity here.

Remark 30 By standard lifting and projecting of stationary sets (see [14, Theorem 8.27]), the following are equivalent:

1. $\mathbb{P}$ is κ-stationarily layered.
2. The set
$$\left\{X \in P_\kappa(H) \;\middle|\; X \cap \mathbb{P} \text{ is a regular suborder of } \mathbb{P}\right\}$$
is stationary in $P_\kappa(H)$, where H is any set such that $\mathbb{P} \subset H$.

The analogous equivalence also holds if we replace "stationary" by "contains a club" (and if we replace "stationary" by "weakly stationary").

Definition 31 Assume that $\mathbb{P} \subset H$ and that $S \subseteq P_\kappa(H)$ is stationary. We say that $\mathbb{P}$ is layered almost everywhere on S if and only if $\{M \cap \mathbb{P} \mid M \in S\} - \mathrm{Reg}_\kappa(\mathbb{P})$ is nonstationary. Equivalently, $M \cap \mathbb{P}$ is regular in $\mathbb{P}$ for all but nonstationarily many $M \in S$.

A special case of the following lemma that appears in the literature is Shelah's work on *layered ideals*.

Lemma 32 (Shelah; see [9]) *If $\mathbb{P}$ is κ-weakly-stationarily layered—that is, if the set of regular suborders of $\mathbb{P}$ is weakly stationary in $P_\kappa(\mathbb{P})$—then $\mathbb{P}$ is κ-cc.*

Proof Let A be a maximal antichain of $\mathbb{P}$. Let θ be a sufficiently large regular cardinal so that $A, \mathbb{P} \in H_\theta$. By Remark 30 (the "weakly stationary" version), there is an $X \prec (H_\theta, \in, \{\mathbb{P}, A\})$ (note that $A \in X$) with $|X| < \kappa$ such that $X \cap \mathbb{P}$ is a regular suborder of $\mathbb{P}$. Because $X \prec (H_\theta, \in, \{\mathbb{P}, A\})$, we have that $A \cap X$ is a maximal antichain in $X \cap \mathbb{P}$. Because $X \cap \mathbb{P}$ is regular in $\mathbb{P}$, we have that $A \cap X$ is a maximal antichain in $\mathbb{P}$. It follows that $A \cap X = A$; that is, that A is a subset of the $< \kappa$-sized set X. So $|A| < \kappa$. $\square$

For posets of size κ, it is often more natural to work with *filtrations* of the poset to determine if the poset is layered. A filtration of a set X is a $\subset$-increasing, $\subset$-continuous sequence $\langle X_\xi \mid \xi < \mathrm{cf}(|X|)\rangle$ such that $|X_\xi| < |X|$ for all $\xi < |X|$, and $X = \bigcup_{\xi < \mathrm{cf}(|X|)} X_\xi$. If $\mathrm{cf}(|X|)$ is uncountable, then any two filtrations agree on a club subset of $\mathrm{cf}(|X|)$. The proof of the following is standard so we omit it.

Lemma 33 *Suppose that $\mathbb{P}$ is a poset of size κ, where κ is regular and uncountable. The following are equivalent:*

1. *$\mathbb{P}$ is κ-stationarily layered.*
2. *For some filtration $\langle \mathbb{Q}_\alpha \mid \alpha < \kappa\rangle$ of $\mathbb{P}$, there are stationarily many $\alpha < \kappa$ such that $\mathbb{Q}_\alpha$ is a regular suborder of $\mathbb{P}$.*
3. *For every filtration $\langle \mathbb{Q}_\alpha \mid \alpha < \kappa\rangle$ of $\mathbb{P}$, there are stationarily many $\alpha < \kappa$ such that $\mathbb{Q}_\alpha$ is a regular suborder of $\mathbb{P}$.*

If $M \prec (H_\theta, \in, \mathbb{P})$ and $p \in \mathbb{P}$, then p is called an *$(M, \mathbb{P})$-master condition* if and only if $p \Vdash \check{M}[\dot{G}] \cap \mathrm{ORD} = \check{M} \cap \mathrm{ORD}$. The following correspondence between κ-cc and master conditions is due to Mekler.

Lemma 34 (Mekler [21]) *Given a poset $\mathbb{P}$ and a regular uncountable cardinal κ, the following are equivalent:*

1. *$\mathbb{P}$ is κ-cc.*
2. *There are stationarily many $M \in P_\kappa(H_\theta)$ such that $1_{\mathbb{P}}$ is an $(M, \mathbb{P})$-master condition.*

If $\kappa = \omega_1$, then these are also equivalent to the following: there are club many $M \in P_\kappa(H_\theta)$ such that $1_{\mathbb{P}}$ is an $(M, \mathbb{P})$-master condition.

Just as Lemma 34 characterizes κ-cc in terms of master conditions, κ-stationary layering can be expressed in terms of *strong* master conditions, as defined by Mitchell [22]. If M is a model, $\mathbb{P}$ is a poset, and $p \in \mathbb{P}$, Mitchell [23] defined p to be an $(M, \mathbb{P})$-strong master condition if and only if every $p' \leq p$ has a reduction in $M \cap \mathbb{P}$. Note that if $\mathbb{P} \subset H_\theta$ and $X \prec (H_\theta, \in, \mathbb{P})$, then by elementarity $X \cap \mathbb{P}$ is a suborder of $\mathbb{P}$ (i.e., we have order and $\perp$ preservation in both directions). The following observation is straightforward, using the "reduct" characterization of regularity in Lemma 11.

Observation 35 If $X \prec (H_\theta, \in \mathbb{P})$, then the following are equivalent:

- $X \cap \mathbb{P}$ is a regular suborder of $\mathbb{P}$.
- $1_{\mathbb{P}}$ is an $(X, \mathbb{P})$-strong master condition (in the sense of Mitchell [23]).

Corollary 36 *Given a poset $\mathbb{P}$ and a regular uncountable cardinal κ, the following are equivalent:*

1. *$\mathbb{P}$ is κ-stationarily layered.*
2. *There are stationarily many $M \in P_\kappa(H_\theta)$ such that $1_{\mathbb{P}}$ is an $(M, \mathbb{P})$*-strong *master condition.*

Proof This follows immediately from Observation 35. □

3.2 Examples of layered posets Many commonly used posets are stationarily layered under mild cardinal arithmetic assumptions; in fact, they are often layered almost everywhere on some natural stationary set S (i.e. "club-layered relative to S"). Here are a few common examples of κ-stationarily layered posets, all under the assumption that κ is (strongly) inaccessible. Much less than inaccessibility is needed for most of these, given the right cardinal arithmetic assumptions.

1. Trivially, any poset of density $< \kappa$ is κ-club-layered.
2. Any poset as in the hypothesis of Corollary 3 is κ-stationarily layered.
3. If κ is inaccessible, then the Levy collapse $\mathrm{Col}(\mu, < \kappa)$ is layered almost everywhere on the stationary set of $< \mu$-closed models in $P_\kappa(H_\kappa)$. (Recall that we are using strong stationarity here, so almost every such model has transitive intersection with κ.) If $\mu = \omega$, then it is club-layered on $P_{\omega_1}(V_\kappa)$.
4. If κ is inaccessible, then the Silver collapse $\mathbb{C}_{\mathrm{Silv}}(\mu, < \kappa)$ that turns κ into μ^+ is layered almost everywhere on the stationary set of μ-closed elements of $P_\kappa(H_\kappa)$. The poset $\mathbb{C}_{\mathrm{Silv}}(\mu, < \kappa)$ is the μ-support product of $\{\mathrm{Col}(\mu, \eta) \mid \eta < \kappa\}$ with bounded domain. That is, conditions are partial functions $p : \mu \times \kappa \to \kappa$ such that $|p| \leq \mu$, $p(\alpha, \eta) < \eta$ whenever $(\alpha, \eta) \in \mathrm{dom}(p)$, and there is some $\xi_p < \mu$ such that, whenever $(\alpha, \eta) \in \mathrm{dom}(p)$, we have that $\alpha < \xi$.

5. If κ is Mahlo, then the Easton collapse $\mathbb{E}(\mu, <\kappa)$ is layered almost everywhere on the stationary set of $W \in P_\kappa(V_\kappa)$ such that $|W| = W \cap \kappa \in \kappa$, $\mu \in W$, and ${}^{<|W|}W \subset W$. Conditions are partial functions $p : \mu \times \kappa \to \kappa$ such that $p(\alpha, \eta) < \eta$ whenever $(\alpha, \eta) \in \mathrm{dom}(p)$, there is some $\xi < \mu$ such that $\alpha < \xi$ whenever $(\alpha, \eta) \in \mathrm{dom}(p)$, and the support of p—that is, the set $\{\eta < \kappa \mid \exists\alpha\ (\alpha, \eta) \in \mathrm{dom}(p)\}$—is *Easton above* μ. A set $X \subset \kappa$ is Easton above μ if and only if $|X \cap \gamma| < \gamma$ whenever $\gamma \in (\mu, \kappa)$ is regular.
6. The poset $\mathbb{B}(\mu, \alpha, \kappa)$ from Foreman and Komjath [10] is layered almost everywhere on the stationary set of $<\alpha$-closed elements of $P_\kappa(V_\kappa)$. Here $\mu < \alpha < \kappa$ are regular cardinals and conditions are elements of the $<\alpha$-support product of κ-many copies of $\mathrm{Col}(\mu, \alpha)$ which have a bounded domain. More precisely, let $\mathbb{C}_i := \mathrm{Col}(\mu, \alpha)$ for each $i < \kappa$ (so these are κ-many copies of $\mathrm{Col}(\mu, \alpha)$). A condition is a $<\kappa$-supported function $p \in \prod_{i<\kappa} \mathbb{C}_i$ such that there is some $\xi_p < \mu$ such that $\mathrm{dom}(p(i)) \subseteq \xi_p$ for all $i \in \mathrm{supp}(p)$.
7. Any κ-cc poset—and any μ-support product of κ-cc posets where $\mu < \kappa$—is κ-stationarily layered, if κ is weakly compact, by [3].

We prove that the Easton collapse $\mathbb{E}(\mu, <\kappa)$ is layered on the stationary set indicated above; the proofs that the other posets are layered is similar (with the exception of the weakly compact example, which is proved in [3]). Pick any $W \prec (H_{\kappa^+}, \in)$ such that $\gamma_W := |W| = W \cap \kappa \in \kappa$ and ${}^{<|W|}W \subset W$. Let $p \in \mathbb{E}(\mu, <\kappa)$, and let S be the support of p, which is Easton above μ (as defined in example 5 above). Because S is Easton, γ_W is regular (inaccessible), and W is closed under $<\gamma_W$ sequences, we have that $p \upharpoonright W$ is an element of W. Note that, because $W \cap \kappa \in \kappa$, $p \upharpoonright W$ is the same as $p \upharpoonright (\mu \times \gamma_W)$. We need to prove that it is a reduct. Suppose that $p' \in W$ is a condition extending $p \upharpoonright W$. Then clearly $p' \cup p$ is a function, because p', being in W, does not make any commitments for those $\eta \notin W$. Let $\xi_{p'}$ be the uniform bound for p''s domains, and let ξ_p be the uniform bound for p's domains. Then $\max(\xi_{p'}, \xi_p)$ is a uniform bound for $p \cup p'$. So $p \cup p'$ is a condition.

Remark 37 Naturally defined layered posets as in the examples listed above typically have κ-stationarily layered products, simply because the stationary sets witnessing layering have stationary overlap. For example, Foreman and Komjath [10] consider products of the form

$$\mathbb{S}(\alpha, <\kappa) \times \mathbb{B}(\mu, \alpha, \kappa),$$

where κ is inaccessible. As remarked above, $\mathbb{S}(\alpha, <\kappa)$ is layered almost everywhere on the set of α-closed models, and $\mathbb{B}(\mu, \alpha, \kappa)$ is layered almost everywhere on the set of $<\alpha$-closed models. Because the intersection of these stationary sets is the set of α-closed models, which is stationary, it follows that their product is layered almost everywhere on the set of α-closed models; this and other issues surrounding products are addressed in detail in Cox and Lücke [3]. It also follows from the more general fact that if p_i is a strong master condition for $(M, \mathbb{P}_i)$ for $i \in \{0, 1\}$, then (p_0, p_1) is a strong master condition for $(M, \mathbb{P}_0 \times \mathbb{P}_1)$ (see Mitchell [23]).

3.3 Relation to Knaster property A poset is called κ-*Knaster* if every κ-sized collection of conditions can be refined to a κ-sized collection of pairwise compatible conditions. Clearly κ-Knaster implies κ-cc. The main feature of the Knaster property used in the literature is the following.

Fact 38 (see [5]) If $\mathbb{P}$ is κ-Knaster and $\mathbb{Q}$ is κ-cc, then $\mathbb{P} \times \mathbb{Q}$ is κ-cc.

We now prove Lemma 4, that being κ-stationary layered implies being κ-Knaster. The proof is modeled after the proof of Soukup [26, Theorem 4.1].

Let $\mathbb{P}$ be κ-stationarily layered, and let X be a κ-sized collection of conditions in $\mathbb{P}$. By the layering assumption of $\mathbb{P}$, there is an $M \prec (H_\theta, \in, \mathbb{P})$ such that $X \in M$, $|M| < \kappa$, $M \cap \kappa \in \kappa$, and $M \cap \mathbb{P}$ is a regular suborder of $\mathbb{P}$. Because $|M| < \kappa = |X|$, there is some $p \in X - M$. Let $p|M$ be any reduct of p into $M \cap \mathbb{P}$. Let Z be a maximal subset of X with the property that Z consists of pairwise compatible conditions, and every member of Z is compatible with $p|M$. Since X and $p|M$ are members of M, we can without loss of generality take $Z \in M$. We claim that $|Z| = \kappa$. If not, then because $Z \in M$ and $|Z| < \kappa$, it follows that $|Z| \in M \cap \kappa$. And because $M \cap \kappa$ is transitive,[8]

$$Z \subset M. \tag{4}$$

It follows that $Z \subsetneq Z \cup \{p\}$. Now p is clearly compatible with $p|M$, and $Z \cup \{p\}$ is a subset of X; thus we will obtain a contradiction to maximality of Z if we can prove that $Z \cup \{p\}$ is pairwise compatible. Let $r \in Z$; then (4) implies $r \in M \cap \mathbb{P}$ and, by the requirements on Z, we have $r \parallel_{\mathbb{P}} p|M$. Since r and $p|M$ are in M, this compatibility is witnessed by some $r' \leq r, p|M$ such that $r' \in M$. Because $r' \leq p|M$ and $r' \in M$, it follows that $r' \parallel_{\mathbb{P}} p$, which implies that $r \parallel_{\mathbb{P}} p$. This proves that $Z \cup \{p\}$ is pairwise compatible, completing the proof.

Remark 39 It is natural to wonder if the implication of Lemma 4 can be reversed. If κ is weakly compact, then the answer is yes; and for nonweakly compact cardinals, the answer is, in general, no (see Cox and Lücke [3]).

3.4 Layering and elementary embeddings

The following simple but useful lemma will be used in the proof of Theorem 6 (and also in [3]).

Lemma 40 *Suppose that*

- $j : H \to N$ *is a* Σ_0*-elementary embedding of* ZFC$^-$ *models;*
- $\kappa = \mathrm{crit}(j)$*;*
- $j[H] \in N$ *(equivalently* $j \in N$*);*
- $\mathbb{Q} \in H$ *is a poset;*
- $N \models \mathbb{Q}$ *is* κ*-cc;*
- $([\mathbb{Q}]^{<\kappa})^H = ([\mathbb{Q}]^{<\kappa})^N$.

Then $j[H] \cap j(\mathbb{Q})$ *is a regular suborder of* $j(\mathbb{Q})$.

Proof Because $j[H] \in N$ and "X is a regular suborder of Y" is a Σ_0-property (using the reduct characterization), it suffices to prove that N believes $j[H] \cap j(\mathbb{Q})$ is regular in $j(\mathbb{Q})$. Clearly, $j \restriction \mathbb{Q} : \mathbb{Q} \to j(\mathbb{Q})$ is order- and incompatibility-preserving, so we only need to check that, whenever $A \in N$ is a maximal antichain in $\mathbb{Q}$, $j[A]$ is maximal in $j(\mathbb{Q})$. So let $A \in N$ be maximal antichain in $\mathbb{Q}$. Because $([\mathbb{Q}]^{<\kappa})^H = ([\mathbb{Q}]^{<\kappa})^N$ and $N \models$ "$\mathbb{Q}$ is κ-cc" we have that $A \in H$ and it has size $< \kappa$ in H. Since $\mathrm{crit}(j) = \kappa$ this implies that $j(A) = j[A]$. But by elementarity, N believes that $j(A)$ is maximal in $j(\mathbb{Q})$, completing the proof. $\square$

4 Coherent, Conservative Systems of Reduction Operations

The iteration we describe in the main theorem is not presented in the usual way, so we will make the following ad hoc definition; it is really a definition about how a poset is presented.

Definition 41 A sequence of posets $\langle \mathbb{L}_\alpha \mid \alpha < \theta \rangle$ will be called a *generalized iteration* if and only if

- each $\mathbb{L}_\alpha$ consists of partial functions on α;
- the restriction map π^β_α is a forcing projection from $\mathbb{L}_\beta \to \mathbb{L}_\alpha$ for all $\beta \geq \alpha$; that is, π^β_α is order-preserving, and whenever $q \leq \pi^\beta_\alpha(p)$, there is some $p' \leq p$ such that $\pi^\beta_\alpha(p') \leq q$;
- the identity map is a regular embedding from $\mathbb{L}_\alpha \to \mathbb{L}_\beta$ for all $\alpha \leq \beta$.

The following definition is very important for the main theorem. If $\mathbb{L}$ is a suborder of $\mathbb{R}$, a map $\pi : \mathbb{R} \to \mathbb{L}$ is called a *reduction operation* if and only if $\pi(r)$ is a reduct of r into $\mathbb{L}$ for all $r \in \mathbb{R}$;[9] that is, whenever $\ell \leq \pi(r)$ and $\ell \in \mathbb{L}$, ℓ is compatible with r.[10]

Definition 42 Assume that $\vec{\mathbb{L}} = \langle \mathbb{L}_\alpha \mid \alpha < \theta \rangle$ and $\vec{\mathbb{R}} = \langle \mathbb{R}_\alpha \mid \alpha < \theta \rangle$ are two generalized iterations of the same length θ, and assume that $\mathbb{L}_\alpha$ is a regular suborder of $\mathbb{R}_\alpha$ for all $\alpha < \theta$. A system of reduction maps $\langle \pi_\alpha : \mathbb{R}_\alpha \to \mathbb{L}_\alpha \mid \alpha < \theta \rangle$ will be called

- *coherent* if and only if $\pi_\alpha(r \restriction \alpha) = \pi_\beta(r) \restriction \alpha$ for all $\alpha \leq \beta < \theta$ and all $r \in \mathbb{R}_\beta$;
- *conservative* if and only if whenever
  - $\alpha \leq \beta$;
  - $\ell_\beta \leq \pi_\beta(r_\beta)$;
  - r'_α is some condition in $\mathbb{R}_\alpha$ witnessing the compatibility of $\ell_\beta \restriction \alpha$ with $r_\beta \restriction \alpha$;

  then there is some $r'_\beta \in \mathbb{R}_\beta$ such that $r'_\beta \restriction \alpha = r'_\alpha$, and r'_β witnesses the compatibility of ℓ_β with r_β.

The conservativity requirement is used to pass inverse limit stages of inductively defined reduction mappings. Roughly, it says that if you are checking compatibility (in a proof in which some condition is a reduct), you can do it without strengthening what you did earlier.

Lemma 43 *Assume that $\vec{\mathbb{L}}$ and $\vec{\mathbb{R}}$ are generalized iterations of limit length θ, and assume that $\vec{\pi} = \langle \pi_\alpha : \mathbb{R}_\alpha \to \mathbb{L}_\alpha \mid \alpha < \theta \rangle$ is a coherent, conservative system of reduction maps. Assume that $\vec{\mathbb{L}}$ and $\vec{\mathbb{R}}$ use some mix of inverse and direct limits, and assume that their limit scheme is the same. More precisely, for each limit ordinal α, either*

1. *$\mathbb{L}_\alpha$ and $\mathbb{R}_\alpha$ are both direct limits or*
2. *$\mathbb{L}_\alpha$ and $\mathbb{R}_\alpha$ are both inverse limits.*

Let $\mathbb{L}_\theta$ and $\mathbb{R}_\theta$ be limits of $\vec{\mathbb{L}}$ and $\vec{\mathbb{R}}$, respectively, where either both are direct limits or both are inverse limits. Then there is a reduction map $\pi_\theta : \mathbb{R}_\theta \to \mathbb{L}_\theta$ such that $\vec{\pi} \cup \{(\theta, \pi_\theta)\}$ is a coherent, conservative system of reduction maps.

Proof We prove the lemma assuming that both $\mathbb{L}_\theta$ and $\mathbb{R}_\theta$ are inverse limits, as this is the hardest case. Note that the coherency of $\vec{\pi}$ ensures that

$$\forall \gamma \in \theta \cap \mathrm{Lim} \ \ \forall r_\gamma \in \mathbb{R}_\gamma \ \ \pi_\gamma(r_\gamma) = \bigcup_{\alpha<\gamma} \pi_\alpha(r_\gamma \restriction \alpha). \tag{5}$$

Also, the coherency requirement that we desire of π_θ compels us to define

$$\pi_\theta(r) := \bigcup_{\alpha<\theta} \pi_\alpha(r \restriction \alpha).$$

That $\pi_\theta(r)$ is a condition in $\mathbb{L}_\theta$ follows from the facts that

- the coherency of $\vec{\pi} \restriction \theta$ ensures that $\pi_\beta(r \restriction \beta)$ end-extends $\pi_\alpha(r \restriction \alpha)$ whenever $\alpha \le \beta$;
- $\mathbb{L}_\theta$ and $\mathbb{R}_\theta$ use the same limit type.

We must prove that conservativity is preserved. (The proof will also show that π_θ is a reduction mapping.) So assume that $\alpha < \theta$, and assume that

- $\ell_\theta \le \pi_\theta(r_\theta)$;
- r'_α is some condition in $\mathbb{R}_\alpha$ witnessing the compatibility of $\ell_\theta \restriction \alpha$ with $r_\theta \restriction \alpha$.

We need to find some $r'_\theta \in \mathbb{R}_\theta$ such that $r'_\theta \restriction \alpha = r'_\alpha$ and r'_θ witnesses the compatibility of ℓ_θ with r_θ. We construct such an r'_θ using the assumption that $\vec{\pi}$ is conservative, together with (5). Namely, recursively define a sequence $\langle r'_\beta \mid \alpha \le \beta \le \theta \rangle$ as follows:

- If r'_β is defined, let $r'_{\beta+1}$ be some condition in $\mathbb{R}_{\beta+1}$ given by the conservativity between β and $\beta+1$. That is,
  - $r'_{\beta+1} \restriction \beta = r'_\beta$;
  - $r'_{\beta+1}$ witnesses the compatibility of $\ell_\theta \restriction (\beta+1)$ with $r_\theta \restriction (\beta+1)$. If index θ is a trivial coordinate for both ℓ_θ and r_θ—that is, if $\ell_\theta \restriction (\beta+1) = \ell_\theta \restriction \beta$ and $r_\theta \restriction (\beta+1) = r_\theta \restriction \beta$—then we require that the θth coordinate of $r'_{\beta+1}$ is also trivial; more precisely that $r'_{\beta+1} = r'_\beta$.
- If γ is limit, let $r'_\gamma = \bigcup_{\beta\in[\alpha,\gamma)} r'_\beta$.

The triviality requirement at successor steps of the construction, together with the assumption that $\vec{\mathbb{L}}$ and $\vec{\mathbb{R}}$ use the same limit scheme, ensures that each r'_β is a condition in $\mathbb{R}_\beta$. It is clear that $r'_\beta = r'_{\beta'} \restriction \beta$ whenever $\beta \le \beta'$ and that r'_θ is as required. □

Though Lemma 43 assumes that the iterations use some mix of inverse and direct limits, we suspect a similar fact holds for many common iteration schemes (e.g., revised countable support iterations).

5 Universal Kunen Iterations

The universal Kunen iteration was introduced by Kunen [17] in the first consistency proof of a saturated ideal on ω_1 and is a useful tool when huge embeddings are involved. Roughly, one has a large cardinal κ and defines some sort of iteration of length κ, where successor steps are handled as follows: given that $0 < \alpha < \kappa$ and $\mathbb{P}_\alpha$ has been defined, if $V_\alpha \cap \mathbb{P}_\alpha$ is a regular suborder of $\mathbb{P}_\alpha$, set $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}_\alpha$ is some poset *from the inner model* $V[\dot{G}_\alpha \cap V_\alpha]$; otherwise, set $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \{1\}$.

In the applications below it will be helpful to describe the Kunen iteration a bit differently from typical iterations (but these will still be iterations in the general sense of iteration, that is, a commutative system of forcing projections). The scheme below was roughly described in Foreman [9].

Definition 44 Let κ be an uncountable regular cardinal. A sequence $\langle \mathbb{P}_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ will be called a *universal Kunen iteration* if and only if we have the following:

1. $\langle \mathbb{P}_\alpha \mid \alpha \leq \kappa \rangle$ is a generalized iteration as in Definition 41.
2. For each $\alpha < \kappa$, the poset $\mathbb{P}_{\alpha+1}$ has the following form. If $V_\alpha \cap \mathbb{P}_\alpha$ is *not* a regular suborder of $\mathbb{P}_\alpha$, then we will call α a *passive stage* and set $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha$. Otherwise—that is, if $V_\alpha \cap \mathbb{P}_\alpha$ is a regular suborder of $\mathbb{P}_\alpha$—we will call α an *active stage* and require that $\dot{Q}_\alpha$ is a $V_\alpha \cap \mathbb{P}_\alpha$-name for a poset,[11] and $\mathbb{P}_{\alpha+1}$ consists of all partial functions f on $\alpha + 1$ with the following properties:
   - $f \restriction \alpha \in \mathbb{P}_\alpha$;
   - if $\alpha \in \mathrm{dom}(f)$, then $f(\alpha) \in V^{V_\alpha \cap \mathbb{P}_\alpha}$ (note that this is a $V_\alpha \cap \mathbb{P}_\alpha$-name, not merely a $\mathbb{P}_\alpha$-name) and $\Vdash_{V_\alpha \cap \mathbb{P}_\alpha} f(\alpha) \in \dot{Q}_\alpha$,

   where the partial ordering is given by

   $$\begin{aligned} f_1 \leq_{\mathbb{P}_{\alpha+1}} f_0 \;:\equiv\; & f_1 \restriction \alpha \leq_{\mathbb{P}_\alpha} f_0 \restriction \alpha \text{ and if } \alpha \in \mathrm{dom}(f_0)\text{, then} \\ & \alpha \in \mathrm{dom}(f_1) \text{ and } f_1 \restriction \alpha \Vdash^{\text{Convention}}_{\mathbb{P}_\alpha} f_1(\alpha) \leq_{\dot{Q}_\alpha} f_0(\alpha), \end{aligned}$$

   where we are using Convention 18, together with the assumption that $V_\alpha \cap \mathbb{P}_\alpha$ is a regular suborder of $\mathbb{P}_\alpha$, to view the statement "$f_1(\alpha) \leq_{\dot{Q}_\alpha} f_0(\alpha)$" as a statement in the language of $\mathbb{P}_\alpha$.

As usual, we can arrange that $\mathbb{P}$ is a set (rather than a proper class) by choosing a single representative of the various names in the definition (modulo equivalence of names). We will insist that such a representative is always chosen of minimal rank. The following is a standard fact.

Fact 45 Suppose that γ is regular and that $\mathbb{R} \in H_\gamma$ is a poset. If τ is a $\mathbb{R}$-name which is forced to be in $H_\gamma[\dot{G}_\mathbb{R}]$, then there is a $\mathbb{R}$-name $\tau' \in H_\gamma$ which is $\mathbb{R}$-equivalent to τ (i.e., $\Vdash_\mathbb{R} \tau = \tau'$).

Lemma 46 *Suppose that $\langle \mathbb{P}_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ is a universal Kunen iteration as in Definition 44 which uses some mix of inverse and direct limits, and suppose that, for every active $\alpha < \kappa$,*

$$\Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \dot{Q}_\alpha \subseteq V_\kappa[\dot{g}_\alpha], \tag{6}$$

where $\dot{g}_\alpha$ is the canonical $V_\alpha \cap \mathbb{P}_\alpha$-name for its generic object. Also assume that $\mathbb{P}_\kappa$ is a direct limit. Then we have the following:

1. $\mathbb{P}_\kappa \subset V_\kappa$.
2. *If $j : V \to N$ is an elementary embedding with critical point κ, then*
   - $j[\mathbb{P}_\kappa] = \mathbb{P}_\kappa = j(\vec{\mathbb{P}})_\kappa \cap V_\kappa$;
   - *if $\mathbb{P}_\kappa$ is κ-cc, then κ is an active stage of $j(\vec{\mathbb{P}})$ (and thus by reflection there are many active stages of $\vec{\mathbb{P}}$).*

Proof For item (1), because $\mathbb{P}_\kappa$ is a direct limit, it suffices to prove that each $\mathbb{P}_\alpha \subset V_\kappa$ (for $\alpha < \kappa$). Note that, given the setup of Definition 44, a direct limit is simply a union.

We prove by induction that $\mathbb{P}_\alpha \subseteq V_\kappa$ for every $\alpha < \kappa$.[12] For limit stage α this follows simply from the induction hypothesis and inaccessibility of κ. (The latter is needed if $\mathbb{P}_\alpha$ is, say, an inverse limit.) Now suppose that $\mathbb{P}_\alpha \subseteq V_\kappa$, and let $f \in \mathbb{P}_{\alpha+1}$. We need to prove that $f \in V_\kappa$; because $f \restriction \alpha \in \mathbb{P}_\alpha \subset V_\kappa$ by the induction assumption, it only remains to show that $f(\alpha) \in V_\kappa$. This follows from assumption (6) and Fact 45 (together with our convention that a representative for a name is always chosen of minimal rank). The proof that $V_\kappa \cap j(\vec{\mathbb{P}})_\kappa \subset \mathbb{P}_\kappa$ is proved similarly.

For item (2), by item (1) $j \restriction \mathbb{P}_\kappa$ is just the identity map on $\mathbb{P}_\kappa$. If $f \in \mathbb{P}_\kappa$, then because $\mathbb{P}_\kappa$ is a direct limit, $f \in \mathbb{P}_\alpha$ for some $\alpha < \kappa$. So $j(f) = f$. Pick any $\beta < \alpha$ in the domain of f. Then $f \restriction \beta \in \mathbb{P}_\beta$ and $V_\beta \cap \mathbb{P}_\beta$ forces that $f(\beta) \in \dot{Q}_\beta$. By the elementarity of j, N believes that $j(f) \restriction j(\beta) = f \restriction \beta$ is an element of the $j(\beta) = \beta$th member of $j(\vec{\mathbb{P}})$ and that $j(f(\beta)) = f(\beta)$ is forced by $j(V_\beta \cap \mathbb{P}_\beta) = V_\beta \cap j(\vec{\mathbb{P}})_{j(\beta)} = V_\beta \cap j(\vec{\mathbb{P}})_\beta$ to be a member of $j(\dot{Q}_\beta)$. This proves that $j(f) = f$ is a member of $j(\vec{\mathbb{P}})_\alpha$, so, in particular, it is a member of $j(\vec{\mathbb{P}})_\kappa$.

Finally, if we know that $\mathbb{P}_\kappa$ is κ-cc, then because $\mathrm{crit}(j) = \kappa$ it follows that $j \restriction \mathbb{P}_\kappa = \mathrm{id}_{\mathbb{P}_\kappa}$ is a regular embedding from $\mathbb{P}_\kappa \to j(\mathbb{P}_\kappa) = j(\vec{\mathbb{P}})_{j(\kappa)}$; that is,

$$\mathbb{R}_0 := j[\mathbb{P}_\kappa] \text{ is a regular suborder of } \mathbb{R}_2 := j(\vec{\mathbb{P}})_{j(\kappa)}.$$

But because we now know that $\mathbb{R}_0 = j[\mathbb{P}_\kappa]$ is actually a suborder of $\mathbb{R}_1 := j(\vec{\mathbb{P}})_\kappa$ (by the previous part of the proof) and because $\mathbb{R}_1 = j(\vec{\mathbb{P}})_\kappa$ is regular in $\mathbb{R}_2 = j(\vec{\mathbb{P}})_{j(\kappa)}$ (because $j(\vec{\mathbb{P}})$ is a generalized iteration), it abstractly follows that $\mathbb{R}_0 = j[\mathbb{P}_\kappa]$ is a regular suborder of $\mathbb{R}_1 = j(\vec{\mathbb{P}})_\kappa$.[13] Since we showed earlier that $j[\mathbb{P}_\kappa] = V_\kappa \cap j(\vec{\mathbb{P}})_\kappa$, it follows that κ is an active stage of $j(\vec{\mathbb{P}})$. □

The following lemma ensures the preservation of closure properties.

Lemma 47 *Let $\mu < \kappa$ be regular cardinals. Suppose that $\langle \mathbb{P}_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ is a universal Kunen iteration with the following properties:*

- *For all (active) $\alpha < \kappa$,*

$$\Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \dot{Q}_\alpha \text{ is } < \mu\text{-closed.}$$

- *The iteration uses $< \mu$-supports; that is, for all limits $\alpha \leq \kappa$, $\mathbb{P}_\alpha$ consists of those partial functions f such that $\mathrm{dom}(f)$ is a $< \mu$-sized subset of α and $f \restriction \bar{\alpha} \in \mathbb{P}_{\bar{\alpha}}$ for all $\bar{\alpha} \in \mathrm{dom}(f)$.*

Then $\mathbb{P}_\kappa$ is $< \mu$-closed.

Proof This is basically the standard proof that $< \mu$-support iterations of $< \mu$-closed posets are $< \mu$-closed (as in [1, Section 2]), with a variation at successor steps due to the requirement of Definition 44 that we only use $V_\alpha \cap \mathbb{P}_\alpha$-names, rather than $\mathbb{P}_\alpha$-names.

Fix $\lambda < \mu$, and suppose that $\langle f_i \mid i < \lambda \rangle$ is a descending sequence in $\mathbb{P}_\kappa$. Note that $\langle \mathrm{dom}(f_i) \mid i < \lambda \rangle$ is a $\subseteq$-increasing sequence of $< \mu$-sized sets, and by the regularity of μ it follows that

$$d := \bigcup_{i<\lambda} \mathrm{dom}(f_i)$$

has size $< \mu$. We recursively define a function f with domain d and verify that f is a condition in $\mathbb{P}_\kappa$ and is a lower bound for the f_i's. Suppose that $\alpha \leq \kappa$ and $f(\bar{\alpha})$ is defined for all $\bar{\alpha} < \alpha$, and suppose that the following induction hypothesis $\mathrm{IH}_{\bar{\alpha}}$ holds for each $\bar{\alpha} < \alpha$:

$$\begin{aligned}\mathrm{IH}_{\bar{\alpha}}:\ & f \restriction (\bar{\alpha}+1) \text{ is a lower bound in } \mathbb{P}_{\bar{\alpha}+1} \text{ for the sequence} \\ & \big\langle f_i \restriction (\bar{\alpha}+1) \bigm| i < \lambda \big\rangle \text{ and } \operatorname{dom}\big(f \restriction (\bar{\alpha}+1)\big) \subseteq d.\end{aligned}$$

First observe that the fact that $\mathrm{IH}_{\bar{\alpha}}$ holds for all $\bar{\alpha} < \alpha$ immediately implies that

$$f \restriction \alpha \text{ is a lower bound in } \mathbb{P}_\alpha \text{ for } \langle f_i \restriction \alpha \mid i < \lambda \rangle. \tag{7}$$

If $\alpha = \kappa$, then we are done; otherwise, we need to define $f(\alpha)$ and verify that IH_α holds. If $\alpha \notin d$, then $f(\alpha)$ is not defined; that is, $\alpha \notin \operatorname{dom}(f)$. Then IH_α is immediate. So from now on assume $\alpha \in d$; then $\alpha \in \operatorname{dom}(f_i)$ for all sufficiently large $i < \lambda$, say, for all $i \geq i_\alpha$. Note that, by the definition of the ordering for $\mathbb{P}_{\alpha+1}$, together with the assumption that the f_i's are descending, it follows that

$$i_\alpha \leq i < j < \lambda \implies f_j \restriction \alpha \Vdash^{\text{Convention}}_{V_\alpha \cap \mathbb{P}_\alpha} f_j(\alpha) \leq_{\dot{Q}_\alpha} f_i(\alpha). \tag{8}$$

Then (7) and (8) together imply that

$$f \restriction \alpha \Vdash^{\text{Convention}}_{V_\alpha \cap \mathbb{P}_\alpha} \big\langle f_i(\alpha) \bigm| i_\alpha \leq i < \lambda \big\rangle \text{ is descending in } \dot{Q}_\alpha.$$

Because $V_\alpha \cap \mathbb{P}_\alpha$ forces (by assumption) that $\dot{Q}_\alpha$ is $< \mu$-closed, by the maximality principle there is a $V_\alpha \cap \mathbb{P}_\alpha$-name $\dot{q}$ such that

$$f \restriction \alpha \Vdash^{\text{Convention}}_{V_\alpha \cap \mathbb{P}_\alpha} \dot{q} \text{ is a lower bound for } \big\langle f_i(\alpha) \bigm| i_\alpha \leq i < \lambda \big\rangle.$$

Then set $f(\alpha) := \dot{q}$. Then IH_α clearly holds. □

Here is how such universal iterations are typically used; we concentrate on Kunen's original construction for concreteness. Suppose that $j : V \to N$ is a huge embedding with critical point κ, and let $\delta = j(\kappa)$. Kunen defined a finite support iteration $\langle \mathbb{P}_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ as in Definition 44, where one starts by forcing with $\mathbb{P}_1 = \mathrm{Col}(\omega, <\kappa)$ and then at each active α one forces with $\mathbb{S}^{V^{V_\alpha \cap \mathbb{P}_\alpha}}(\alpha, <\kappa)$. Kunen's proof that $\mathbb{P}_\kappa$ is κ-cc goes by induction; limit steps are taken care of by Theorem 1, and successor steps $\alpha \to \alpha + 1$ are taken care of by the induction hypothesis that $\mathbb{P}_\alpha$ is κ-cc, together with the fact that $\mathbb{S}^{V^{V_\alpha \cap \mathbb{P}_\alpha}}(\alpha, <\kappa)$ is κ-Knaster; so the product of $\mathbb{S}^{V^{V_\alpha \cap \mathbb{P}_\alpha}}(\alpha, <\kappa)$ with the κ-cc quotient $\mathbb{P}_\alpha/(V_\alpha \cap \mathbb{P}_\alpha)$ is κ-cc by Fact 38. Then by Lemma 46, κ is an active stage of $j(\vec{\mathbb{P}})$, and so the identity regular embedding from $\mathbb{P}_\kappa \to j(\mathbb{P}_\kappa)$ lifts to a regular embedding from $\mathbb{P}_\kappa * \mathbb{S}^{V^{\mathbb{P}_\kappa}}(\kappa, <\delta) \to j(\mathbb{P}_\kappa)$. Thus, if G' is generic over N for $j(\mathbb{P}_\kappa)$, then in $N[G']$ there is a $G * H$ which is generic over V for $\mathbb{P}_\kappa * \mathbb{S}^{V^{\mathbb{P}_\kappa}}(\kappa, <\delta)$; then j lifts to $j_{G'} : V[G] \to N[G']$ and $N[G']$ sees that $m := \bigcup j_{G'}[H]$ is a condition in its Silver collapse $\mathbb{S}^{N[G']}(\delta, < j(\delta))$. Forcing below m yields a lifting of j to a map from $V[G * H] \to N[G' * H']$, but one can approximate such a lifting in the model $V[G']$ (without actually forcing with H'). Because this occurs in the δ-cc extension $V[G']$, the resulting derived ideal in $V[G * H]$ will be $\delta = \kappa^{+V[G*H]}$-cc, that is, saturated. We refer the reader to Foreman [9] for more details.

Table 1 includes some examples of such iterations in the literature on saturated ideals; the columns of the table correspond to the scheme described in Definition 44 above. Kunen's original paper [17] dealt only with ideals on ω_1 and used finite support iteration; Laver showed how to do $< \mu$-support iterations for arbitrary regular μ in certain situations. In all cases, $\mu < \kappa$ are regular cardinals and $\mathbb{P}_0$ is the Levy

Table 1 Some examples of universal Kunen iterations.

Paper	$\dot{\mathbb{Q}}_\alpha$ (when $V_\alpha \cap \mathbb{P}_\alpha \lessdot \mathbb{P}_\alpha$)	Supports
Kunen [17]	$\mathbb{S}^{V[\dot{g}_\alpha]}(\alpha, <\kappa)$	finite (i.e., $<\mu = \omega$)
Magidor (see [9])	$\mathrm{Col}^{V[\dot{g}_\alpha]}(\alpha, <\kappa)$	$<\mu$
Laver [20]	$\mathbb{E}^{V[\dot{g}_\alpha]}(\alpha, <\kappa)$	$<\mu$
Foreman and Komjath [10]	$\mathbb{S}^{V[\dot{g}_\alpha]}(\alpha, <\kappa) \times \mathbb{B}^{V[\dot{g}_\alpha]}(\mu, \alpha^{<\alpha}, \kappa)$	$<\mu$
Foreman [8] and Foreman and Laver [11]	A certain product of Silver collapses	$<\mu$

collapse $\mathrm{Col}(\mu, <\kappa)$ or the Silver collapse $\mathbb{S}(\mu, <\kappa)$ (see Section 3.2 for the definitions of these posets). In Table 1, $\dot{g}_\alpha$ denotes the canonical $V_\alpha \cap \mathbb{P}_\alpha$-name for its generic object.

The following is the key technical theorem of this section.

Theorem 48 *Let κ be Mahlo, and let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \kappa,\ \beta < \kappa \rangle$ be a universal Kunen iteration as in Definition 44, which uses some mix of inverse and direct limits. Assume $W \prec (H_{\kappa^+}, \in, \vec{\mathbb{P}})$ is such that $|W| = W \cap \kappa =: \gamma_W \in \kappa$ and ${}^{<|W|}W \subset W$. Assume that*

1. *$\mathbb{P}_\kappa$ is a direct limit;*
2. *$\mathbb{P}_{\gamma_W}$ is a direct limit;*
3. *for every $\alpha < \gamma_W$*
$$\Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \dot{Q}_\alpha \subseteq V_\kappa[\dot{g}_\alpha],$$
where $\dot{g}_\alpha$ is the canonical $V_\alpha \cap \mathbb{P}_\alpha$-name for its generic;
4. *for every $\alpha < \gamma_W$,*
$$\Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \check{W}[\dot{g}_\alpha] \cap \dot{Q}_\alpha \text{ is a regular suborder of } \dot{Q}_\alpha.$$
(Note that this is trivially true for nonactive α.)

Then $W \cap \mathbb{P}_\kappa$ is a regular suborder of $\mathbb{P}_\kappa$.

Proof Note that Lemma 46 implies that $\mathbb{P}_\kappa \subset V_\kappa$. Fix any $W \prec (H_{\kappa^+}, \in, \mathbb{P}_\kappa)$ as in the hypotheses of the theorem, and set $\gamma_W := W \cap \kappa$. We will recursively construct a coherent, conservative system of reduction operations
$$\langle \pi^W_\alpha : \mathbb{P}_\alpha \to W \cap \mathbb{P}_\alpha \mid \alpha < \gamma_W \rangle. \tag{9}$$
At the end of the proof we will use the system from (9) to get a reduction operation from $\mathbb{P}_\kappa \to W \cap \mathbb{P}_\kappa$.

Let $\beta < \gamma_W$, and assume that the system (9) has been constructed below β; we need to define π^W_β. Assume first that β is a successor ordinal, say, $\beta = \alpha + 1$. So $\pi^W_\alpha : \mathbb{P}_\alpha \to W \cap \mathbb{P}_\alpha$ is a reduction operation. If α is passive, then $\mathbb{P}_\beta = \mathbb{P}_\alpha$ and $\pi^W_\beta := \pi^W_\alpha$ works. Otherwise, α is active, and assumption (4) and the maximality principle imply that there is some $V_\alpha \cap \mathbb{P}_\alpha$-name $\dot{\rho}$ such that
$$\Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \dot{\rho} : \dot{Q}_\alpha \to \check{W}[\dot{g}_\alpha] \cap \dot{Q}_\alpha \text{ is a reduction operation.}$$
Given $f \in \mathbb{P}_{\alpha+1}$, define $\pi^W_{\alpha+1}(f)$ as follows. The first α coordinates are given by $\pi^W_\alpha(f \restriction \alpha)$. Now consider the $V_\alpha \cap \mathbb{P}_\alpha$-name $\dot{\rho}(f(\alpha))$; assumption (3) implies that $\dot{\rho}(f(\alpha))$ is an element of $V_\kappa[\dot{g}_\alpha] \cap \check{W}[\dot{g}_\alpha]$. Because $\alpha \in W \in \Gamma$,

$V_\alpha \cap \mathbb{P}_\alpha \subset V_{\gamma_W} = W \cap V_\kappa$; so $V_\kappa[\dot{g}_\alpha] \cap \check{W}[\dot{g}_\alpha]$ is forced to be the same as $(W \cap V_\kappa)[\dot{g}_\alpha] = V_{\gamma_W}[\dot{g}_\alpha]$. By Fact 45, there is some $V_\alpha \cap \mathbb{P}_\alpha$-name $\dot{y}^W_{f(\alpha)} \in V_{\gamma_W} \subset W$ such that

$$\Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \dot{y}^W_{f(\alpha)} = \dot{\rho}\big(f(\alpha)\big). \tag{10}$$

We use this $\dot{y}^W_{f(\alpha)}$ as the αth coordinate; that is,

$$\pi^W_{\alpha+1}(f) := \pi^W_\alpha(f \upharpoonright \alpha) \cup \big\{(\alpha, \dot{y}^W_{f(\alpha)})\big\}.$$

Note that, because $\dot{y}^W_{f(\alpha)} \in W$ and π^W_α maps into $W \cap \mathbb{P}_\alpha$ by the induction assumption, $\pi^W_{\alpha+1}$ maps into $W \cap \mathbb{P}_{\alpha+1}$. We must verify that $\vec{\pi}^W \upharpoonright (\alpha+1) \cup \{(\alpha+1, \pi^W_{\alpha+1})\}$ is conservative and coherent.

The coherence of $\vec{\pi}^W \upharpoonright (\alpha+1) \cup \{(\alpha+1, \pi^W_{\alpha+1})\}$ follows immediately from the inductively assumed coherency of $\vec{\pi}^W \upharpoonright (\alpha+1)$ together with the definition of $\pi^W_{\alpha+1}$. Finally, we check the conservativity of $\vec{\pi}^W \upharpoonright (\alpha+1) \cup \{(\alpha+1, \pi^W_{\alpha+1})\}$ (which will also show that $\pi^W_{\alpha+1}$ is a reduction operation, as defined immediately before Definition 42). Assume that

- $f \in \mathbb{P}_{\alpha+1}$;
- $h \leq_{\mathbb{P}_{\alpha+1}} \pi^W_{\alpha+1}(f)$, where $h \in W$;
- $\eta \leq \alpha$, $f'_\eta \leq_{\mathbb{P}_\eta} f \upharpoonright \eta$, and $f'_\eta \leq_{\mathbb{P}_\eta} h \upharpoonright \eta$.

We need to find some $f' \in \mathbb{P}_{\alpha+1}$ such that $f' \leq_{\mathbb{P}_{\alpha+1}} f$, $f' \upharpoonright \eta = f'_\eta$, and $f' \leq_{\mathbb{P}_{\alpha+1}} h$. Our inductive hypothesis that $\vec{\pi}^W \upharpoonright (\alpha+1)$ is conservative ensures that there is an $f'_\alpha \in \mathbb{P}_\alpha$ such that $f'_\alpha \leq_{\mathbb{P}_\alpha} f \upharpoonright \alpha$, $f'_\alpha \upharpoonright \eta = f'_\eta$, and $f'_\alpha \leq_{\mathbb{P}_\alpha} h \upharpoonright \alpha$.

Now consider the $V_\alpha \cap \mathbb{P}_\alpha$-names $f(\alpha)$ and $h(\alpha)$. Now $h \upharpoonright \alpha \leq_{\mathbb{P}_\alpha} \pi^W_\alpha(f \upharpoonright \alpha)$ and

$$h \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha} h(\alpha) \leq_{\dot{Q}_\alpha} \pi^W_{\alpha+1}(f)(\alpha) = \dot{y}^W_{f(\alpha)}, \tag{11}$$

where again we are using Convention 18. Let $\text{WeakestReduct}_{\text{ro}(V_\alpha \cap \mathbb{P}_\alpha)}(h \upharpoonright \alpha)$ be the maximum reduct of $h \upharpoonright \alpha$ into $\text{ro}(V_\alpha \cap \mathbb{P}_\alpha)$ as in Definition 17. Then (11) and Lemma 19 imply that

$$\begin{aligned}\text{WeakestReduct}_{\text{ro}(V_\alpha \cap \mathbb{P}_\alpha)}(h \upharpoonright \alpha) \Vdash_{\text{ro}(V_\alpha \cap \mathbb{P}_\alpha)} h(\alpha) &\leq_{\dot{Q}_\alpha} \pi^W_{\alpha+1}(f)(\alpha)\\ &= \dot{y}^W_{f(\alpha)} = \dot{\rho}\big(f(\alpha)\big).\end{aligned} \tag{12}$$

Because $h \in W$ and $\alpha \in W$, it follows that $h(\alpha) \in W$, which in turn implies that

$$\text{WeakestReduct}_{\text{ro}(V_\alpha \cap \mathbb{P}_\alpha)} \Vdash_{\text{ro}(V_\alpha \cap \mathbb{P}_\alpha)} h(\alpha) \in \dot{Q}_\alpha \cap \check{W}[\dot{g}_\alpha]. \tag{13}$$

Because $\dot{\rho}$ is forced by $V_\alpha \cap \mathbb{P}_\alpha$ to be a reduction from $\dot{Q}_\alpha \to \dot{Q}_\alpha \cap W[\dot{g}_\alpha]$, (12) and (13) imply that

$$\begin{aligned}&\text{WeakestReduct}_{\text{ro}(V_\alpha \cap \mathbb{P}_\alpha)}(h \upharpoonright \alpha) \Vdash_{\text{ro}(V_\alpha \cap \mathbb{P}_\alpha)} \exists q \leq_{\dot{Q}_\alpha} f(\alpha)\\ &\quad \text{such that } q \leq_{\dot{Q}_\alpha} h(\alpha).\end{aligned} \tag{14}$$

By the maximality principle there is a $V_\alpha \cap \mathbb{P}_\alpha$-name $\dot{q}_{h(\alpha), f(\alpha)}$ such that

$$\begin{aligned}\text{WeakestReduct}_{\text{ro}(V_\alpha \cap \mathbb{P}_\alpha)}(h \upharpoonright \alpha) \Vdash_{\text{ro}(V_\alpha \cap \mathbb{P}_\alpha)} \dot{q}_{h(\alpha), f(\alpha)} &\leq_{\dot{Q}_\alpha} f(\alpha) \qquad \text{and}\\ \dot{q}_{h(\alpha), f(\alpha)} &\leq_{\dot{Q}_\alpha} h(\alpha),\end{aligned} \tag{15}$$

and by the standard "definition of names by cases" we can assume without loss of generality that $\emptyset \Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \dot{q}_{h(\alpha), f(\alpha)} \in \dot{Q}_\alpha$ and that $\emptyset \Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \dot{q}_{h(\alpha), f(\alpha)} \leq_{\dot{Q}_\alpha} f(\alpha)$.

So

$$f' := f'_\alpha \cup \left\{(\alpha, \dot{q}_{h(\alpha), f(\alpha)})\right\}$$

is a condition in $\mathbb{P}_{\alpha+1}$, $f' \leq_{\mathbb{P}_{\alpha+1}} f$, and $f' \restriction \eta = f'_\alpha \restriction \eta = f'_\eta$. We only have left to check that

$$f' \leq_{\mathbb{P}_{\alpha+1}} h. \tag{16}$$

Now

$$f' \restriction \alpha = f'_\alpha \leq_{\mathbb{P}_\alpha} h \restriction \alpha \tag{17}$$

and $f'(\alpha) = \dot{q}_{h(\alpha), f(\alpha)}$; so we only have left to prove that

$$f'_\alpha \Vdash^{\text{Convention}}_{\mathbb{P}_\alpha} \dot{q}_{h(\alpha), f(\alpha)} \leq_{\dot{Q}_\alpha} h(\alpha). \tag{18}$$

Moreover, by (17) it in turn suffices to prove

$$h \restriction \alpha \Vdash^{\text{Convention}}_{\mathbb{P}_\alpha} \dot{q}_{h(\alpha), f(\alpha)} \leq_{\dot{Q}_\alpha} h(\alpha). \tag{19}$$

But this follows from (15) and Lemma 19. This completes the proof that $\vec{\pi}^W \restriction (\alpha + 1) \cup \{(\alpha+1, \pi^W_{\alpha+1})\}$ is a coherent, conservative system of reduction operations. This completes the case in which β is a successor ordinal.

Now assume that β is a limit ordinal (still assuming $\beta < \gamma_W$). Because stage β of $\mathbb{P}_\beta$ is either an inverse or a direct limit and because ${}^{<\gamma_W} W \subset W$ and $\beta \in W$, it follows that $\mathbb{P}_\beta$ and $\mathbb{P}_\beta \cap W$ use the same limit scheme (with respect to $\langle \mathbb{P}_\alpha \mid \alpha < \beta \rangle$ and $\langle \mathbb{P}_\alpha \cap W \mid \alpha < \beta \rangle$, respectively). So Lemma 43 applies and ensures that the natural limit of $\langle \pi^W_\alpha \mid \alpha < \beta \rangle$ (depending on the limit scheme used at β) is a reduction, and thus, we obtain a coherent, conservative system of reductions $\langle \pi^W_\alpha \mid \alpha \leq \beta \rangle$.

We have constructed a coherent, conservative system $\langle \pi^W_\alpha \mid \alpha < \gamma_W \rangle$. Now by the assumptions that ${}^{<\gamma_W} W \subset W$ and that $\mathbb{P}_{\gamma_W}$ is a direct limit, $\mathbb{P}_{\gamma_W}$ and $W \cap \mathbb{P}_{\gamma_W}$ are direct limits of $\langle \mathbb{P}_\alpha \mid \alpha < \gamma_W \rangle$ and $\langle W \cap \mathbb{P}_\alpha \mid \alpha < \gamma_W \rangle$, respectively. So we may again invoke Lemma 43 to extend this to a coherent, conservative system $\langle \pi^W_\alpha \mid \alpha \leq \gamma_W \rangle$. Although $\mathbb{P}_{\gamma_W} \notin W$, because $\mathbb{P}_{\gamma_W}$ is a direct limit, we still have that $\text{id} : W \cap \mathbb{P}_{\gamma_W} \to \mathbb{P}_{\gamma_W}$ is order- and incompatibility-preserving. This, together with the existence of the reduction operation $\pi^W_{\gamma_W} : \mathbb{P}_{\gamma_W} \to W \cap \mathbb{P}_{\gamma_W}$, implies that

$$W \cap \mathbb{P}_{\gamma_W} \text{ is a regular suborder of } \mathbb{P}_{\gamma_W}.$$

Because $\mathbb{P}_{\gamma_W}$ is a regular suborder of $\mathbb{P}_\kappa$ it follows that

$$W \cap \mathbb{P}_{\gamma_W} \text{ is a regular suborder of } \mathbb{P}_\kappa. \tag{20}$$

Finally, observe that, because both $\mathbb{P}_{\gamma_W} = \mathbb{P}_{W \cap \kappa}$ and $\mathbb{P}_\kappa$ are direct limits,

$$W \cap \mathbb{P}_\kappa = W \cap \mathbb{P}_{\gamma_W}. \tag{21}$$

Note that this is a literal equality, because our conditions are partial functions (i.e., direct limits are simply unions).[14] So (20) and (21) together imply that $W \cap \mathbb{P}_\kappa$ is a regular suborder of $\mathbb{P}_\kappa$. □

The following theorem is almost an immediate corollary of Theorem 48; however, it is typically less useful than Theorem 50, because typically κ is a very large cardinal in the context of universal Kunen iterations.

Theorem 49 *Assume that κ is Mahlo, that $\langle \mathbb{P}_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ is a universal Kunen iteration as in Definition 44 which uses some mix of inverse and direct limits, and that $\mathbb{P}_\gamma$ is a direct limit for all inaccessible $\gamma \leq \kappa$. Let $\Phi(W, \kappa)$ denote the* *statement that* $W \prec H_{\kappa^+}$, $|W| = W \cap \kappa \in \kappa$, *and* ${}^{<|W|}W \subset W$. *Assume that, for each* $\alpha < \kappa$,

$$\Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \dot{Q}_\alpha \text{ is layered almost everywhere on } \{M \mid \Phi^{V[\dot{g}_\alpha]}(M, \kappa)\}$$

and

$$\Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \dot{Q}_\alpha \subset V_\kappa[\dot{g}_\alpha],$$

where $\dot{g}_\alpha$ *is the canonical* $V_\alpha \cap \mathbb{P}_\alpha$*-name for its generic. Then* $\mathbb{P}_\kappa$ *is layered almost everywhere on the stationary set*

$$\Gamma := \{W \mid \Phi^V(W, \kappa)\}.$$

That is, for all but nonstationarily many $W \in \Gamma$, $W \cap \mathbb{P}_\kappa$ *is a regular suborder of* $\mathbb{P}_\kappa$. *In particular,* $\mathbb{P}_\kappa$ *is* κ*-Knaster by Lemma 4.*

Proof Note that Γ is stationary because κ is Mahlo, and the assumptions guarantee (by Lemma 46) that $\mathbb{P}_\kappa \subset V_\kappa$. For each active $\alpha < \kappa$ let $\dot{\mathfrak{A}}_\alpha$ be a $V_\alpha \cap \mathbb{P}_\alpha$-name for a first-order structure on $H_{\kappa^+}[\dot{g}_\alpha]$ witnessing the "almost everywhere" part of the assumption about $\dot{Q}_\alpha$; that is, so that

$$\begin{aligned} &\Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \forall M \;\; M \prec \dot{\mathfrak{A}}_\alpha \wedge \Phi(M, \kappa) \\ &\qquad \Longrightarrow \dot{Q}_\alpha \cap M \text{ is a regular suborder of } \dot{Q}_\alpha. \end{aligned} \tag{22}$$

Now let $W \in \Gamma$ such that $W \prec (H_{\kappa^+}, \in, \vec{\mathbb{P}}, \vec{\dot{\mathfrak{A}}})$. Let $\alpha \in W \cap \kappa =: \gamma_W$. Because $|V_\alpha \cap \mathbb{P}_\alpha| < \gamma_W$, in particular, $V_\alpha \cap \mathbb{P}_\alpha$ has the γ_W-cc, so together with the $< \gamma_W$ closure of W and the fact that $\dot{\mathfrak{A}}_\alpha \in W$ we have

$$\Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \Phi\big(W[\dot{g}_\alpha], \kappa\big) \qquad \text{and} \qquad W[\dot{g}_\alpha] \prec \dot{\mathfrak{A}}_\alpha.$$

Thus, by (22) we obtain

$$\forall \alpha < \gamma_W \quad \Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \dot{Q}_\alpha \cap W[\dot{g}_\alpha] \text{ is a regular suborder of } \dot{Q}_\alpha. \tag{23}$$

Then (23), together with our assumption that $\mathbb{P}_\gamma$ is a direct limit for all inaccessible $\gamma \leq \kappa$, implies by Theorem 48 that $W \cap \mathbb{P}_\kappa$ is a regular suborder of $\mathbb{P}_\kappa$. □

We finally state and prove the precise version of Theorem 6: it tells us that if κ is weakly compact and direct limits are used at all inaccessibles, then *any* universal Kunen iteration of κ-cc posets will be κ-stationarily layered (and thus κ-Knaster). We emphasize that Theorem 50(3) is only assuming that $V_\alpha \cap \mathbb{P}_\alpha$ forces $\dot{Q}_\alpha$ to be κ-cc; it is *not* assuming that $\dot{Q}_\alpha$ remains κ-cc in $V^{\mathbb{P}_\alpha}$ (though this will be the case automatically in the end).

Theorem 50 *Suppose that* κ *is weakly compact, and suppose that* $\langle \mathbb{P}_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ *is a universal Kunen iteration as in Definition 44 which uses some mix of inverse and direct limits. Suppose that*

1. *direct limits are taken at all inaccessible* $\gamma \leq \kappa$;
2. *for every active* $\alpha < \kappa$, $\Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \dot{Q}_\alpha \subset V_\kappa[\dot{g}_\alpha]$;
3. *each* $\dot{Q}_\alpha$ *is forced by* $V_\alpha \cap \mathbb{P}_\alpha$ *to be* κ*-cc.*[15]

Then $\mathbb{P}_\kappa \subset V_\kappa$ *and is layered on some stationary subset of*

$$\Gamma := \{W \in P_\kappa(V_\kappa) \mid W = V_\gamma \text{ for some inaccessible } \gamma < \kappa\}.$$

In particular, $\mathbb{P}_\kappa$ *is* κ*-Knaster by Lemma 4.*

Proof First note that Γ is stationary, because κ is Mahlo. Also Lemma 46 implies that $\mathbb{P}_\kappa \subset V_\kappa$. Suppose toward a contradiction that there is some algebra $\mathfrak{A} = (V_\kappa, \in, \dots)$ such that

$$\forall W \in \Gamma \ W \prec \mathfrak{A} \implies W \cap \mathbb{P}_\kappa \text{ is } \textit{not} \text{ a regular suborder of } \mathbb{P}_\kappa. \tag{24}$$

Fix some transitive κ-sized, $<\kappa$-closed models H and H' such that

- $\kappa \subset H \in H'$;
- $H \prec H' \prec (H_{\kappa^+}, \in, \{\mathfrak{A}, \vec{\mathbb{P}}\})$.

Because κ is weakly compact, there are some transitive N' and an elementary $j : H' \to N'$ with critical point κ. Because $H \in H'$, $\text{crit}(j) = \kappa$, and $H' \models |H| = \kappa$, we have

$$j[H] \in N'. \tag{25}$$

Note also that $P_\kappa(H) \in H'$ and that $P_\kappa(H) \subset H'$ because H' is $<\kappa$-closed and $H \subset H'$. Define

$$U := \{X \in H' \cap P\big(P_\kappa(H)\big) \mid j[H] \in j(X)\}. \tag{26}$$

Then U is an ultrafilter on $H' \cap P(P_\kappa(H))$ which is $<\kappa$-closed in V and normal with respect to κ-sequences from H'; that is, if $F \in H'$ is a regressive function on some $Z \in U$, then there is a $Z' \in U$ on which F is constant.[16]

Claim 50.1 *Let Z be the set of $W \in H' \cap P_\kappa(H)$ which satisfies Theorem 48(4). Then $Z \in U$.*

Proof of Claim 50.1 First note that Z is an element of H', because $\vec{\mathbb{P}} \in H'$. Suppose for a contradiction that $Z \notin U$. Then because $Z \in H'$ and U is an ultrafilter on $H' \cap P(P_\kappa(H))$, there are U-many $W \in \Gamma$ such that Theorem 48(4) fails for some $\alpha < W \cap \kappa$; that is, there are U-many W such that

$$\exists \alpha_W < \gamma_W \ \exists p_W \in V_{\alpha_W} \cap \mathbb{P}_{\alpha_W},$$
$$p_W \Vdash_{V_{\alpha_W} \cap \mathbb{P}_{\alpha_W}} \dot{Q}_{\alpha_W} \cap W[\dot{g}_{\alpha_W}] \text{ is } \textit{not} \text{ a regular suborder of } \dot{Q}_{\alpha_W}.$$

Note that the map $W \mapsto (\alpha_W, p_W)$ is regressive and also an element of H'. So by the normality of U with respect to regressive functions from H', there is some fixed α^* and a fixed $p^* \in V_{\alpha^*} \cap \mathbb{P}_{\alpha^*}$ such that, for U-many $W \in \Gamma$,

$$p^* \Vdash_{V_{\alpha^*} \cap \mathbb{P}_{\alpha^*}} \dot{Q}_{\alpha^*} \cap W[\dot{g}_{\alpha^*}] \text{ is } \textit{not} \text{ a regular suborder of } \dot{Q}_{\alpha^*}.$$

Let A denote the set of such W (so $A \in U$, i.e., $j[H] \in j(A)$). Note that $A \in H'$. (It is definable from the regressive function and (α^*, p^*).)

Let g^* be generic over V for $V_{\alpha^*} \cap \mathbb{P}_{\alpha^*}$ with $p^* \in g^*$. Because $p^* \in g^*$ it follows that

$$\forall W \in A \ \ \mathbb{Q}_{\alpha^*} \cap W[g^*] \text{ is } \textit{not} \text{ a regular suborder of } \mathbb{Q}_{\alpha^*}, \tag{27}$$

where $\mathbb{Q}_{\alpha^*}$ is the evaluation of $\dot{Q}_{\alpha^*}$ by g^*.

Because $V_{\alpha^*} \cap \mathbb{P}_{\alpha^*} \in V_\kappa$, we have that j lifts in $V[g^*]$ to an elementary

$$\tilde{j} : H'[g^*] \to N'[g^*] = N'\big[\tilde{j}(g^*)\big].$$

Because H is $<\kappa$-closed in V and g^* is generic for a κ-cc poset (in fact, a poset of size $<\kappa$), we have that $H[g^*]$ is $<\kappa$-closed in $V[g^*]$ and, in particular, in $N'[g^*]$. Since $\mathbb{Q}_{\alpha^*}$ is κ-cc in $V[g^*]$ and thus in $N'[g^*]$, we have

$$N'[g^*] \text{ and } H[g^*] \text{ have the same maximal antichains of } \mathbb{Q}_{\alpha^*}. \tag{28}$$

Then $H[g^*]$, $\tilde{j} \restriction H[g^*]$, $N'[g^*]$, and $\mathbb{Q}_{\alpha^*}$ satisfy the assumptions of Lemma 40; note we are using here that $j[H] \in N'$ to conclude that $j[H][g^*] = \tilde{j}(H[g^*]) \in N'[g^*]$. So by Lemma 40,

$$\tilde{j}[H][g^*] \cap \tilde{j}(\mathbb{Q}_{\alpha^*}) \text{ is a regular suborder of } \tilde{j}(\mathbb{Q}_{\alpha^*}). \tag{29}$$

Because $\tilde{j}[H] = j[H] \in j(A) = \tilde{j}(A)$ and $\tilde{j}(g^*) = g^*$, by (29) we get

$$N[g^*] \models \exists X \in \tilde{j}(A) \;\; X\big[\tilde{j}(g^*)\big] \cap \tilde{j}(\mathbb{Q}_{\alpha^*}) \text{ is a regular suborder of } \tilde{j}(\mathbb{Q}_{\alpha^*}).$$

By the elementarity of $\tilde{j}$, we have

$$H'[g^*] \models \exists X \in A \;\; X[g^*] \cap \mathbb{Q}_{\alpha^*} \text{ is a regular suborder of } \mathbb{Q}_{\alpha^*},$$

which contradicts (27) and completes the proof of Claim 50.1. □

Let B be the U-set given by Claim 50.1. Because $B \in U$, we have that $H' \models$ "B is stationary in $P_\kappa(H)$," and because $H' \prec H_{\kappa^+}$, we have that B really is stationary in $P_\kappa(H)$. By Theorem 48, $\mathbb{P}_\kappa$ is layered almost everywhere on B. Thus, $\mathbb{P}$ is κ-stationarily layered. □

Remark 51 If the κ in the assumptions of Theorem 50 is measurable and U is any normal ultrafilter on κ, then essentially the same proof shows that $\mathbb{P}_\kappa$ is layered on a U-measure one set; that is, there are U-many $\alpha < \kappa$ such that $V_\alpha \cap \mathbb{P}_\kappa$ is a regular suborder of $\mathbb{P}_\kappa$ (i.e., U-many active α's).

6 Applications

In this section we use our main iteration result (Theorem 50) to prove Theorems 7 and 9. A few remarks are in order first:

1. The special case where $\mu = \omega$ in the conclusion of Theorem 7 follows easily from Kunen's original argument (see [17]), without having to use our new Theorem 50. The reason that the case $\mu = \omega$ is significantly easier is because of the Solovay–Tennenbaum Theorem 1 about finite support iterations.
2. Similarly, the special case $\mu = \omega$ of Theorem 9 could have easily been proven using Kunen's original construction, together with Kanamori's Theorem 57 below, without having to use our new Theorem 50. The reason, again, is that in this particular case (i.e., when κ becomes ω_1) finite support iterations are used and so the Solovay–Tennenbaum Theorem 1 is applicable.
3. Certain specific instances of Theorem 7 were already known for arbitrary μ, for example, when the poset $\mathbb{S}^{H_{\kappa^{++}}}_{r,\phi}$ from the statement of Theorem 7 is the poset $\mathrm{Col}(\mu, <\kappa^+)$[17] and for certain other collapsing-type posets. The examples listed in Corollary 8 are all new, however, and the template provided by Theorem 7 for absorbing posets into quotients of saturated ideals is very general.

6.1 Proof of Theorem 7 Fix $\phi(-,-)$ as in the statement of the theorem. Assume that $j : V \to N$ is a huge embedding with critical point κ. Let $\delta = j(\kappa)$. Assume that μ is a regular cardinal and that $\mu < \kappa$. Define a $<\mu$-support universal Kunen iteration (as in Definition 44)

$$\langle \mathbb{P}_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$$

as follows: $\mathbb{P}_0$ is $\mathrm{Col}(\mu, <\kappa)$. For active $\alpha < \kappa$—that is, if $V_\alpha \cap \mathbb{P}_\alpha$ is a regular suborder of $\mathbb{P}_\alpha$—the $V_\alpha \cap \mathbb{P}_\alpha$-name that we force with is the following poset, assuming that $\alpha = \mu^+$ in $V^{V_\alpha \cap \mathbb{P}_\alpha}$:

$$\dot{Q}_\alpha := \left(\mathbb{C}_{\mathrm{Silv}}(\alpha, <\kappa) \ * \ \dot{\mathbb{S}}^{H_{\alpha^{++}}}_{r_\alpha, \phi}\right)^{V[\dot{g}_\alpha]},$$

where $\dot{g}_\alpha$ is the $V_\alpha \cap \mathbb{P}_\alpha$-name for its generic object, $\mathbb{C}_{\mathrm{Silv}}(\alpha, <\kappa)$ is the Silver collapse (see Section 3.2) that turns κ into α^+, and r_α is some $(V_\alpha \cap \mathbb{P}_\alpha) * \dot{\mathbb{C}}_{\mathrm{Silv}}(\alpha, <\kappa)$-name for a parameter which witnesses the analogue of (1) from the statement of Theorem 7. Note that, from the point of view of the model $V[g_\alpha]^{\mathbb{C}_{\mathrm{Silv}}(\alpha, <\kappa)}$, the poset $\dot{\mathbb{S}}^{H_{\alpha^{++}}}_{r_\alpha, \phi}$ satisfies the assumptions of Theorem 7.

It is routine to verify that

$$\Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \ \dot{Q}_\alpha \text{ is } \kappa\text{-cc, is } <\mu\text{-closed, and has size } \kappa.$$

Then by Theorem 50 it follows that

$$\mathbb{P} := \mathbb{P}_\kappa \text{ is } \kappa\text{-cc,} \tag{30}$$

and Lemma 47 ensures that $\mathbb{P}$ is $<\mu$-closed. Because $\mathbb{P}_0$ collapses all cardinals between μ and κ, these facts together imply that

$$\Vdash_{\mathbb{P}} \ \kappa = \mu^+. \tag{31}$$

For reasons that will be discussed in Section 6.1.3, we will primarily work with the *almost huge embedding* $j_{\vec{U}} : V \to N_{\vec{U}}$ *derived from* j, as in Fact 25, rather than the huge embedding j (though j will play a key role). Let $k : N_{\vec{U}} \to N$ be the map from Fact 25, and recall from Fact 25 that $\mathrm{crit}(k) = (\delta^+)^{N_{\vec{U}}}$. (Recall that $\delta = j(\kappa)$.)

By (30) and Lemma 46 it follows that κ is an active stage of $j_{\vec{U}}(\vec{\mathbb{P}})$. Let G be $(V, \mathbb{P})$-generic. Then by the definition of the iteration, combined with (31), stage κ of the $j_{\vec{U}}(\vec{\mathbb{P}})$ iteration forces with the following poset:

$$\left(\mathbb{C}_{\mathrm{Silv}}(\kappa, <\delta) \ * \ \dot{\mathbb{S}}^{H_{\kappa^{++}}}_{j_{\vec{U}}(\vec{r})(\kappa), \phi}\right)^{N_{\vec{U}}[G]}. \tag{32}$$

Because $N_{\vec{U}}$ is $<\delta$-closed in V and G is generic for a δ-cc poset, we have that $N_{\vec{U}}[G]$ is $<\delta$-closed in $V[G]$. In particular, $V[G]$ and $N_{\vec{U}}[G]$ have the same κ sequences, and hence $\mathbb{C}_{\mathrm{Silv}}(\kappa, <\delta)$ is computed the same in $V[G]$ and $N_{\vec{U}}[G]$. Let H be $(V[G], \mathbb{C}_{\mathrm{Silv}}(\kappa, <\delta))$-generic, and consider the evaluation of the second step of (32); that is, the poset

$$(\mathbb{S}^{H_{\kappa^{++}}}_{r, \phi})^{N_{\vec{U}}[G*H]},$$

where r is the evaluation of $j_{\vec{U}}(\vec{r})(\kappa)$ by $G * H$; so r is a parameter in $(H_{\kappa^{++}})^{N_{\vec{U}}[G*H]} = (H_{\delta^+})^{N_{\vec{U}}[G*H]}$. Now we recall the assumptions about the parameter r, namely,

$$\begin{aligned} N_{\vec{U}}[G * H] \models \mathbb{S}^{H_{\kappa^{++}}}_{r, \phi} &= \left\{ z \in H_{\kappa^{++}} \;\middle|\; (H_{\kappa^{++}}, \in) \models \phi(z, r) \right\} \\ &\text{is a } <\mu\text{-closed, } \kappa^+\text{-cc poset of size at most } \kappa^+. \end{aligned} \tag{33}$$

This implies that $(\mathbb{S}^{H_{\kappa^{++}}}_{r, \phi})^{N_{\vec{U}}[G*H]}$ is actually an element of $(H_{\kappa^{++}})^{N_{\vec{U}}[G*H]}$ and that

$$\begin{aligned} (H_{\kappa^{++}})^{N_{\vec{U}}[G*H]} \models \ &(\mathbb{S}^{H_{\kappa^{++}}}_{r, \phi})^{N_{\vec{U}}[G*H]} \\ &\text{is a } <\mu\text{-closed, } \kappa^+\text{-cc poset of size at most } \kappa^+. \end{aligned} \tag{34}$$

Let $\mathbb{S} := (\mathbb{S}^{H_{\kappa^{++}}}_{r,\phi})^{N_{\vec{U}}[G*H]}$. By the definition of $j_{\vec{U}}(\vec{\mathbb{P}})$, there is an $\iota \in N_{\vec{U}}$ such that

$$\iota : \mathbb{P} * \dot{\mathbb{C}}_{\text{Silv}}(\kappa, <\delta) * \dot{\mathbb{S}} \to j_{\vec{U}}(\mathbb{P}) \text{ is a regular embedding and is the identity on } \mathbb{P}.$$

By Theorem 26, in $V[G * H]$ there is a normal ideal $\mathcal{I}(j_{\vec{U}})$ on κ such that

$$\wp(\kappa)/\mathcal{I}(j_{\vec{U}}) \text{ is forcing-equivalent to } j_{\vec{U}}(\mathbb{P})/\iota[G * H]. \tag{35}$$

By standard arguments involving quotient forcings, the map ι can be used to show that the poset $\mathbb{S} = \dot{\mathbb{S}}^{V[G*H]}$ is absorbed as a regular suborder of $j_{\vec{U}}(\mathbb{P})/\iota[G * H]$. Combining this with (35) yields

$$V[G * H] \models \text{ there is a regular embedding } e : \mathbb{S} \to \text{ro}\big(\wp(\kappa)/\mathcal{I}(j_{\vec{U}})\big), \tag{36}$$

where ro$(-)$ denotes the Boolean completion.

Remark 52 Regarding the "ro" appearing in (36): below we will show that in fact $\mathcal{I}(j_{\vec{U}})$ is saturated, which implies by Fact 27 that $\wp(\kappa)/\mathcal{I}(j_{\vec{U}})$ is a complete Boolean algebra, that is, that $\wp(\kappa)/\mathcal{I}(j_{\vec{U}})$ is isomorphic to (not just densely embeddable into) the complete Boolean algebra ro$(\wp(\kappa)/\mathcal{I}(j_{\vec{U}}))$. So the "ro" in (36) will ultimately be redundant.

Remark 53 By using the duality theorem from Foreman [9], it can be arranged that the regular embedding e from (36) has the following additional property: whenever K is $(V[G*H], \wp(\kappa)/\mathcal{I}(j_{\vec{U}}))$-generic and $j_K : V[G*H] \to \text{ult}(V[G*H], K)$ is the generic ultrapower map, we have that $e^{-1}[K]$ is an element of ult$(V[G*H], K)$ (although K will never be an element of ult$(V[G * H], K)$). However, we choose not to include this additional feature in the theorem, in order to avoid lengthy detours into duality theory.

Now it remains to prove the following two claims.

Claim 54 *The ideal $\mathcal{I}(j_{\vec{U}})$ is saturated; that is,*

$$V[G * H] \models \wp(\kappa)/\mathcal{I}(j_{\vec{U}}) \textit{ is } \delta = \kappa^+\textit{-cc.}$$

Claim 55 *The poset $\mathbb{S}$—which, recall, is the poset $(\mathbb{S}^{H_{\kappa^{++}}}_{r,\phi})^{N_{\vec{U}}[G*H]}$ which was defined in $N_{\vec{U}}[G * H]$—has the required definability properties from the point of view of $V[G * H]$. In other words, it remains to prove that the following holds from the point of view of $V[G * H]$:*

$$\mathbb{S} = \big\{z \in (H_{\kappa^{++}})^{V[G*H]} \mid \big((H_{\kappa^{++}})^{V[G*H]}, \in\big) \models \phi(r, z)\big\} \qquad \text{and}$$
$$\mathbb{S} \textit{ is a } <\mu\textit{-closed, } \kappa^+\textit{-cc poset of size at most } \kappa^+.$$

Remark 56 The equality

$$\mathbb{S} = \big\{z \in (H_{\kappa^{++}})^{V[G*H]} \mid \big((H_{\kappa^{++}})^{V[G*H]}, \in\big) \models \phi(r, z)\big\}$$

from Claim 55 does *not* follow automatically from (33), because

$$(\kappa^{++})^{V[G*H]} = \delta^{+V} > \delta^{+N_{\vec{U}}} = (\kappa^{++})^{N_{\vec{U}}[G*H]},$$

where the inequality is by Fact 25. We seem to need some additional assumption, such as the huge embedding used in the argument below.

We will use the huge embedding j to help prove Claims 54 and 55. Let $k : N_{\vec{U}} \to N$ be the map from Fact 25. Recall that

$$\tau := \mathrm{crit}(k) = (\delta^+)^{N_{\vec{U}}} = (\kappa^{++})^{N_{\vec{U}}[G*H]}. \tag{37}$$

6.1.1 Proof of Claim 54 By (35), proving Claim 54 is equivalent to showing that $V[G*H] \models$ "the poset $j_{\vec{U}}(\mathbb{P})/\iota[G*H]$ is δ-cc." By Fact 21, in turn it suffices to prove that $V \models$ "the poset $j_{\vec{U}}(\mathbb{P})$ is δ-cc." But because $j_{\vec{U}}(\mathbb{P}) \subset j_{\vec{U}}(V_\kappa) = (V_\delta)^{N_{\vec{U}}}$, we have by (37) that the map $k : N_{\vec{U}} \to N$ fixes $j_{\vec{U}}(\mathbb{P})$. Also because $\mathbb{P}$ is κ-cc in V, we have that $j_{\vec{U}}(\mathbb{P})$ is δ-cc in $N_{\vec{U}}$. Then by the elementarity of k, $k(j_{\vec{U}}(\mathbb{P})) = j_{\vec{U}}(\mathbb{P})$ is $k(\delta) = \delta$-cc from the point of view of N. Because N is closed under δ-sequences in V, it follows that $j_{\vec{U}}(\mathbb{P})$ is really δ-cc in V as well, which completes the proof.

6.1.2 Proof of Claim 55 First note that k lifts to an elementary $k : N_{\vec{U}}[G*H] \to N[G*H]$, because $G*H$ is generic for a poset which is a subset of $V_\delta^{N_{\vec{U}}}$ and $\delta < \mathrm{crit}(k)$. Because $\mathbb{S} \in (H_\tau)^{N_{\vec{U}}[G*H]}$ and $\mathrm{crit}(k) = \tau$, we have $k(\mathbb{S}) = \mathbb{S}$; similarly because $r \in (H_\tau)^{N_{\vec{U}}[G*H]}$, we have $k(r) = r$. Then by (33) and the elementarity of k it follows that

$$N[G*H] \models k(\mathbb{S}) = \mathbb{S} = \left\{ z \in (H_{\kappa^{++}})^{N[G*H]} \mid \left((H_{\kappa^{++}})^{N[G*H]}, \in\right) \models \phi(z,r)\right\}$$
$$\text{and this poset is } <\mu\text{-closed}, \delta\text{-cc, and of size} \leq \delta. \tag{38}$$

Because the $\models$ relation is absolute, (38) implies

$$V[G*H] \models \mathbb{S} = \left\{ z \in (H_{\kappa^{++}})^{N[G*H]} \mid \left((H_{\kappa^{++}})^{N[G*H]}, \in\right) \models \phi(z,r)\right\}. \tag{39}$$

Because N is closed under δ-sequences and $G*H$ is generic for a δ-cc poset, $N[G*H]$ is closed under δ-sequences in $V[G*H]$; in particular, $\delta^{+V[G*H]} = \delta^{+N[G*H]}$ and

$$(H_{\kappa^{++}})^{V[G*H]} = (H_{\delta^+})^{V[G*H]} = (H_{\delta^+})^{N[G*H]} = (H_{\kappa^{++}})^{N[G*H]}. \tag{40}$$

Thus, by substituting into (39) we obtain

$$V[G*H] \models \mathbb{S} = \left\{ z \in (H_{\kappa^{++}})^{V[G*H]} \mid \left((H_{\kappa^{++}})^{V[G*H]}, \in\right) \models \phi(z,r)\right\}. \tag{41}$$

This shows that $\mathbb{S}$ indeed has the correct definition from the point of view of $V[G*H]$. Finally, because $N[G*H]$ is δ-closed in $V[G*H]$, the fact that $N[G*H]$ believes $\mathbb{S}$ is $<\mu$-closed, δ-cc, and of size $\leq \delta$ is upward absolute to $V[G*H]$.

6.1.3 A remark about the use of the huge embedding A remark is in order regarding why we insisted on using the almost huge embedding $j_{\vec{U}}$ rather than the huge embedding j to construct the saturated ideal $\mathcal{I}(j_{\vec{U}})$ in the proof of Theorem 7. It certainly is possible to use liftings of the huge embedding j, together with a "pseudogeneric tower" construction, to define a saturated ideal $\mathcal{I}(j)$ on κ; this was Kunen's original argument. However, that argument relied on the coarse fact that there exists an incompatibility-preserving embedding i from $\wp(\kappa)/\mathcal{I}(j)$ into the $\delta = \kappa^+$-cc poset

$$\mathbb{P}^* := j(\mathbb{P})/\iota[G*H] = j_{\vec{U}}(\mathbb{P})/\iota[G*H],$$

which guarantees that $\wp(\kappa)/\mathcal{I}(j)$ is κ^+-cc, that is, that $\mathcal{I}(j)$ is saturated. However, it is not clear if the map i is a dense embedding or even a regular embedding. So, in particular, it is not clear if $\mathbb{S}$—which is a regular suborder of $\mathbb{P}^*$—can be regularly embedded into $\wp(\kappa)/\mathcal{I}(j)$. On the other hand, $\wp(\kappa)/\mathcal{I}(j_{\vec{U}})$ *is* forcing-equivalent to $\mathbb{P}^*$ and thus absorbs $\mathbb{S}$ as a regular suborder, which is why we worked with $\mathcal{I}(j_{\vec{U}})$

instead of $\mathcal{I}(j)$. Nonetheless, the map j played an important background role in the proofs of Claims 54 and 55.

6.2 Proof of Theorem 9 Now we proceed to the proof of Theorem 9. For regular cardinals $\kappa < \delta$, let $\mathrm{Sacks}(\kappa, \delta)$ denote the $\leq \kappa$-support iteration of length δ which uses $\mathrm{Sacks}(\kappa)$ at each step (see Definition 22 for the definition of $\mathrm{Sacks}(\kappa)$). We use the following theorem of Kanamori, which generalizes a theorem from Baumgartner and Laver (see [2, Theorem 6.4]).

Theorem 57 (Kanamori [15, Theorem 4.2]) *Suppose that $\kappa < \delta$ are regular uncountable, that δ is weakly compact, and that $\diamondsuit_\kappa$ holds. Then the $\leq \kappa$-support iteration of length δ of* $\mathrm{Sacks}(\kappa)$ *is δ-cc and forces the following:*

- *$\delta = \kappa^{++}$;*
- *the tree property holds at $\delta = \kappa^{++}$.*

For regular $\kappa < \delta$ let $\mathrm{Sacks}_\diamondsuit(\kappa, \delta)$ denote the $\leq \kappa$-support iteration

$$\langle \mathbb{R}_\gamma, \dot{\mathbb{Q}}_\beta \mid \gamma \leq \delta, \beta < \delta \rangle,$$

where $\mathbb{Q}_0 = \mathrm{Add}(\kappa)$ and $\dot{\mathbb{Q}}_\gamma = \mathrm{Sacks}^{\mathbb{R}_\gamma}(\kappa)$ for all $\gamma \in (0, \delta)$. It is a standard fact that $\mathrm{Add}(\kappa)$ forces $\diamondsuit(\kappa)$; so by Theorem 57 we have the following result.

Corollary 58 *If $\kappa < \delta$ are regular uncountable and δ is weakly compact, then* $\mathrm{Sacks}_\diamondsuit(\kappa, \delta)$ *is δ-cc and forces that $\delta = \kappa^{++}$ and the tree property holds at κ^{++}.*

Assume that $j : V \to N$ is a huge embedding with $\mathrm{crit}(j) = \kappa$ and $j(\kappa) = \delta$. Fix a regular $\mu < \kappa$. Define a $< \mu$-support universal Kunen iteration (as in Definition 44)

$$\langle \mathbb{P}_\alpha, \dot{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$$

as follows: $\mathbb{P}_0$ is $\mathrm{Col}(\mu, < \kappa)$. For active $\alpha < \kappa$—that is, if $V_\alpha \cap \mathbb{P}_\alpha$ is a regular suborder of $\mathbb{P}_\alpha$—the $V_\alpha \cap \mathbb{P}_\alpha$-name that we force with is the following poset:

$$\dot{Q}_\alpha := \big(\mathrm{Sacks}_\diamondsuit(\alpha, \kappa)\big)^{V[\dot{g}_\alpha]},$$

where $\dot{g}_\alpha$ is the $V_\alpha \cap \mathbb{P}_\alpha$-name for its generic object. Note that if $\alpha < \kappa$ and g_α is generic for $V_\alpha \cap \mathbb{P}_\alpha$, then κ is still weakly compact in $V[g_\alpha]$ (because g_α is generic for a $< \kappa$-sized poset), and so by Corollary 58 it follows that

$$\Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \dot{Q}_\alpha \text{ is } \kappa\text{-cc and forces } \kappa = \alpha^{++} \ \wedge\ \mathrm{TP}(\kappa).$$

By Theorem 50, $\mathbb{P} := \mathbb{P}_\kappa$ is κ-cc; then by Lemma 46 it follows that κ is an active stage of $j(\vec{\mathbb{P}})$. Moreover, because $\mathbb{P}_0$ collapses all cardinals in the interval (μ, κ) and each $\dot{Q}_\alpha$ is forced to be $< \mu$-closed, we have that $\kappa = \mu^+$ in $V^{\mathbb{P}}$.

Let G be $(V, \mathbb{P})$-generic. Then stage κ of the $j(\vec{\mathbb{P}})$ iteration forces with the poset

$$\big(\mathrm{Sacks}_\diamondsuit(\kappa, \delta)\big)^{N[G]}. \tag{42}$$

So there is an $e \in N[G]$ such that

$$e : \big(\mathrm{Sacks}_\diamondsuit(\kappa, \delta)\big)^{N[G]} \to j(\mathbb{P})/G \text{ is a regular embedding.}$$

Because N is closed under δ sequences in V (really all we need for this part is that it is closed under $< \delta$ sequences), it follows that

$$\mathbb{Q} := \big(\mathrm{Sacks}_\diamondsuit(\kappa, \delta)\big)^{V[G]} = \big(\mathrm{Sacks}_\diamondsuit(\kappa, \delta)\big)^{N[G]}. \tag{43}$$

Note that, from the point of view of $V[G]$, the poset $\mathbb{Q}$ is a δ-length, $\leq \kappa$-support iteration of $< \kappa$-directed closed posets, where the poset at each step has size $< \delta$. That each step is $< \kappa$-directed closed follows from Lemma 24.

Let K be generic over $V[G]$ for the poset $\mathbb{Q}$. Then by Corollary 58, in $V[G * K]$ the tree property holds at κ^{++}. Also $\kappa = (\mu^+)^{V[G*K]}$, because $\mathbb{P}$ forces $\kappa = \mu^+$ and $\mathbb{Q}$ is $< \kappa$ (directed) closed. Also μ is still a cardinal by Lemma 47. It remains to prove that

$$V[G * K] \models (\kappa^{++}, \kappa) \twoheadrightarrow (\kappa, \mu).$$

Let G' be generic over $V[G * K]$ for

$$\big(j(\mathbb{P})/G\big)/e[K].$$

Then in $V[G']$, the map j lifts to

$$j : V[G] \to N[G'].$$

We want to further lift j to the domain $V[G * K]$ in some forcing extension. Note, in particular, that $K \in N[G']$, because $e \in N[G]$ and $K = e^{-1}[G']$. Since N is closed under δ sequences and $j(\mathbb{P})$ is δ-cc, it follows that $N[G']$ is closed under δ sequences from $V[G']$; in particular, $j \restriction A \in N[G']$ for every $A \in V[G]$ of size δ. So the map $j : V[G] \to N[G']$ and the iteration $\mathbb{Q}$ satisfy the requirements of the following lemma, with $V[G]$ playing the role of W and $N[G']$ playing the role of W'.

Lemma 59 (implicit in Kunen [17]) *Suppose that $j : W \to W'$ is a (possibly external) elementary embedding with critical point κ; let $\delta := j(\kappa)$, and assume that δ is inaccessible in W. Suppose that $\vec{\mathbb{Q}} = \langle \mathbb{Q}_\gamma, \dot{R}_\beta \mid \gamma \leq \delta, \beta < \delta \rangle$ is a $\leq \kappa$-support iteration in W such that*

$$\forall \gamma < \delta \ \Vdash^W_{\mathbb{Q}_\gamma} \ \dot{R}_\gamma \textit{ is } < \kappa\textit{-directed closed and of size } < \delta. \tag{44}$$

Assume that the iteration $\vec{\mathbb{Q}}$ is also an element of W', that

$$\forall A \in \wp^W(V_\delta) \quad j \restriction A \in W', \tag{45}$$

and that there exists some $K \in W'$ which is $(W, \mathbb{Q}_\delta)$-generic.

Then we have the following:

- *there is a condition $m_K \in j(\mathbb{Q}_\delta)$ such that if K' is $(W', j(\mathbb{Q}_\delta))$-generic and $m_K \in K'$, then $j[K] \subset K'$ and thus j can be lifted to $\tilde{j} : W[K] \to W'[K']$;*
- *$W[K] \models (\delta, \kappa) \twoheadrightarrow (\kappa, < \kappa)$.*

Proof The master condition m_K will essentially be the condition with support $j[\delta]$ such that, for each $\gamma < \delta$, the $j(\gamma)$th component of m_K is a lower bound for the pointwise image of the γth component of K; this is possible because each component of the iteration is of size $< \delta$ and the image of the components of the iteration are $< \delta$-directed closed.

More precisely, we will recursively define a sequence

$$\vec{m} = \langle m_\gamma \mid \gamma \leq \delta \rangle$$

and inductively verify that

1. $m_\gamma \in j(\mathbb{Q}_\gamma)$ and has support exactly $j[\gamma]$;
2. whenever $\gamma < \gamma'$, we have that $m_\gamma = m_{\gamma'} \restriction j(\gamma)$;

3. if $K'_{j(\gamma)}$ is generic for $j(\mathbb{Q}_\gamma)$ over W' and $m_\gamma \in K'_{j(\gamma)}$, then $j[K|\gamma] \subset K'_{j(\gamma)}$ and thus j lifts to an embedding $j : W[K|\gamma] \to W'[K'_{j(\gamma)}]$.

The recursion goes as follows. If γ is a limit ordinal, then m_γ is simply the union of the m_β's for $\beta < \gamma$. If γ is a successor ordinal, say, $\gamma = \bar{\gamma} + 1$, then we define $m_{\bar{\gamma}+1}$ as $m_{\bar{\gamma}} \cup \{(j(\bar{\gamma}), \dot{k}_{\bar{\gamma}})\}$, where $\dot{k}_{\bar{\gamma}}$ is the $j(\mathbb{Q}_{\bar{\gamma}})$-name defined conditionally as follows (here $\dot{K}'_{j(\bar{\gamma})}$ denotes the $j(\mathbb{Q}_{\bar{\gamma}})$-name for its generic object):

- If $m_{\bar{\gamma}} \notin \dot{K}'_{j(\bar{\gamma})}$, then $\dot{k}_{\bar{\gamma}}$ is the trivial condition of $j(\dot{R}_{\bar{\gamma}})$.
- If $m_{\bar{\gamma}} \in \dot{K}'_{j(\bar{\gamma})}$, then let

$$j_{\bar{\gamma}} : W[K|\bar{\gamma}] \to W'[\dot{K}'_{j(\bar{\gamma})}]$$

denote the lifting which is guaranteed by the induction hypothesis. Let $H_{\bar{\gamma}}$ be the $\bar{\gamma}$th component of K, that is,

$$H_{\bar{\gamma}} := \left\{ \left(r(\bar{\gamma})\right)_{K|\bar{\gamma}} \;\middle|\; r \in K\middle|(\bar{\gamma}+1) \right\} \subset (\dot{R}_{\bar{\gamma}})_{K|\bar{\gamma}}.$$

Note that $H_{\bar{\gamma}} \in W'$ because $K \in W'$. Because $(\dot{R}_{\bar{\gamma}})_{K|\bar{\gamma}}$ has size $< \delta$ and $j_{\bar{\gamma}}$ is elementary, we have that $j_{\bar{\gamma}}[H_{\bar{\gamma}}]$ is a directed set of conditions of size $< \delta$ in the model $W'[\dot{K}'_{j(\bar{\gamma})}]$. And because $(\dot{R}_{\bar{\gamma}})_{K|\bar{\gamma}}$ is $< \delta$-directed closed in $W'[\dot{K}'_{j(\bar{\gamma})}]$, we have that $j_{\bar{\gamma}}[H_{\bar{\gamma}}]$ has a lower bound. Let $\dot{k}_{\bar{\gamma}}$ be any such lower bound for $j_{\bar{\gamma}}[H_{\bar{\gamma}}]$.

It is now routine to verify the inductive assumptions. Note that m_δ has support $j[\delta]$, which is acceptable because $j(\vec{\mathbb{Q}})$ is a $\leq \delta$ support iteration and $|j[\delta]| = \delta$.

Finally, we prove that $W[K] \models (\delta, \kappa) \twoheadrightarrow (\kappa, < \kappa)$. In $W[K]$ fix a structure $\mathfrak{A} = (\delta, \dots)$ in a countable language. Let K' be $(W', j(\mathbb{Q}_\delta))$-generic with $m_K \in K'$, and let $j_\delta : W[K] \to W'[K']$ be the lifting of j. The elementarity of j_δ ensures that $j_\delta[\delta] = j[\delta] \prec j_\delta(\mathfrak{A})$. Moreover, $j[\delta] \in W'$ and $W'[K']$ believes that $j[\delta] \cap j(\kappa) = \kappa \in j(\kappa)$ and that $|j[\delta]| = \delta = j(\kappa)$; so

$$W'[K'] \models \exists X \prec j_\delta(\mathfrak{A}) \quad X \cap j(\kappa) \in j(\kappa) \text{ and } |X| = j(\kappa).$$

So by the elementarity of j_δ,

$$W[K] \models \exists X \prec \mathfrak{A} \quad X \cap \kappa \in \kappa \text{ and } |X| = \kappa,$$

which completes the proof. □

Remark 60 The model just constructed actually has a κ^{++}-saturated ideal on

$$\left\{ z \in [\kappa^{++}]^\kappa \;\middle|\; z \cap \kappa \in \kappa \right\}.$$

This can be proved using the "pseudogeneric tower" argument from Kunen [17], which is described in detail in Foreman [9].

7 Questions

Question 61 Suppose that we alter the assumptions of Theorem 50 by

- weakening the large cardinal assumption (e.g., only assume that κ is Mahlo) and
- strengthening the assumptions on $\dot{Q}_\alpha$ by requiring them to be κ-Knaster instead of merely κ-cc. More precisely, assume for all active $\alpha < \kappa$ that $\Vdash_{V_\alpha \cap \mathbb{P}_\alpha} \dot{Q}_\alpha$ is κ-Knaster.

Can we conclude that $\mathbb{P}_\kappa$ has the κ-cc? Note that if $\vec{\mathbb{P}}$ is a finite support iteration, then the answer is yes.

Remark 56 in the proof of Theorem 7 raises the following question.

Question 62 Is Theorem 7 still true if κ is merely assumed to be *almost* huge in V?

Notes

1. Namely, $\dot{Q}_\alpha$ is the Silver collapse that turns κ into α^+, as defined in the model $V^{V_\alpha \cap \mathbb{P}_\alpha}$ in the case in which $V_\alpha \cap \mathbb{P}_\alpha$ is a regular suborder of $\mathbb{P}_\alpha$, and is trivial otherwise. See Section 5 for a discussion of such iterations.

2. For example, see [10, Lemma 4.8] and Foreman [7].

3. Note that this is an apparently weaker assumption than requiring that $\dot{Q}_\beta$ is forced by $\mathbb{P}_\beta$ to be κ-cc, though in the end they are equivalent if κ is weakly compact.

4. Note that κ is definable in $(H_{\kappa^{++}}, \in)$, as "the second largest cardinal." So, for example, if one has in mind a definable κ^+-cc forcing which only has the desired properties when, say, $\kappa = \omega_2$, then one can simply insert "and the second largest cardinal is ω_2" into the formula $\phi(-,-)$. This will have the effect of making $\mathbb{S}^{H_{\kappa^{++}}}_{r,\phi}$ into the trivial poset when $\kappa \neq \omega_2$. Similarly, if one has in mind a definable κ^+-cc forcing which only makes sense when $2^\kappa = \kappa^+$, then simply insert "and the second largest cardinal satisfies GCH" into the formula ϕ; this would have the effect of making $\mathbb{S}^{H_{\kappa^{++}}}_{r,\phi}$ into the trivial poset when $2^\kappa > \kappa^+$.

5. See Theorem 7 for precisely what is meant here.

6. That is, $\iota(p, \dot{1}) = p$ for every $p \in \mathbb{P}$.

7. We could alternatively stick with weak stationarity, but require that our stationary subsets of $P_\kappa(H)$ concentrate on those $M \in P_\kappa(H)$ such that $M \cap \kappa \in \kappa$.

8. Note the essential use of transitivity of $M \cap \kappa$ in this part of the argument.

9. See Section 2 for a discussion of reducts.

10. Note that this is *not* the same as a forcing projection; a forcing projection would require instead that there is some $r' \leq r$ such that $\pi(r') \leq \ell$.

11. We emphasize that $\dot{Q}_\alpha$ is a $V_\alpha \cap \mathbb{P}_\alpha$-name, not merely a $\mathbb{P}_\alpha$-name.

12. Recall that we use the convention that all names for conditions are taken to be of minimal rank.

13. In general, if $\mathbb{R}_0$ is a suborder of $\mathbb{R}_1$, $\mathbb{R}_1$ is a suborder of $\mathbb{R}_2$, and $\mathbb{R}_0$ is a *regular* suborder of $\mathbb{R}_2$, it follows easily that $\mathbb{R}_0$ is a regular suborder of $\mathbb{R}_1$.

14. If $f \in W \cap \mathbb{P}_\kappa$, then W believes there is some $\beta < \kappa$ such that $f \in \mathbb{P}_\beta$; so this β must be strictly below $W \cap \kappa = \gamma_W$. Thus, $f \in W \cap \mathbb{P}_{\gamma_W}$.

15. Note again that this is a weaker assumption than requiring $\mathbb{P}_\alpha$ to force that $\dot{Q}_\alpha$ is κ-cc, though in the end that will be true as well.

16. It is not really necessary to work with both H and H' here. We could really use V_κ in place of the H, work with the ultrafilter on $H' \cap P(\kappa)$ derived from j (rather than the ultrafilter on $H' \cap P(P_\kappa(H))$ derived from j), and then make an argument at the end involving Skolem hulls of $\gamma < \kappa$. We use H merely so that the U-measure one sets concentrate directly on the kinds of models described in Theorem 48, so we can directly apply that theorem after the following claim.

17. Equivalently, the $< \mu$-support product of κ^+-many copies of $\mathrm{Col}(\mu, \kappa)$.

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Department of Mathematics and Applied Mathematics
Virginia Commonwealth University
1015 Floyd Avenue
Richmond, VA 23284
USA
scox9@vcu.edu