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Semigroups in Stable Structures

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Abstract Assume that G is a definable group in a stable structure M. Newelski showed that the semigroup $S_G(M)$ of complete types concentrated on G is an inverse limit of the ∞-definable (in M^{eq}) semigroups $S_{G,\Delta}(M)$. He also showed that it is strongly π-regular: for every $p \in S_{G,\Delta}(M)$, there exists $n \in \mathbb{N}$ such that p^n is in a subgroup of $S_{G,\Delta}(M)$. We show that $S_{G,\Delta}(M)$ is in fact an intersection of definable semigroups, so $S_G(M)$ is an inverse limit of definable semigroups, and that the latter property is enjoyed by all ∞-definable semigroups in stable structures.

1 Introduction

A *semigroup* is a set together with an associative binary operation. Although the study of semigroups originated at the beginning of the 20th century, not much attention has been given to semigroups in stable structures. One of the few facts known about them is the following.

Proposition (Hrushovski [6]) *A stable semigroup with left and right cancellation, or with left cancellation and right identity, is a group.*

Recently, ∞-definable semigroups in stable structures made an appearance in a work by Newelski [13]. Let G be a definable group inside a stable structure M. Define $S_G(M)$ to be all the types of $S(M)$ which are concentrated on G. We can give $S_G(M)$ a structure of a semigroup by defining for $p, q \in S_G(M)$,

$$p \cdot q = tp(a \cdot b/M),$$

where $a \models p$, $b \models q$ and $a \mathop{\smile\!\!\!\!\!\!\;|}_M b$.

Newelski gives an interpretation of $S_{G,\Delta}(M)$ (where Δ is a finite set of invariant formulas) as an ∞-definable set in M^{eq}, and thus $S_G(M)$ may be interpreted as an inverse limit of ∞-definable semigroups in M^{eq}.

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As a result, he shows that for every local type $p \in S_{G,\Delta}(M)$, there exists $n \in \mathbb{N}$ such that p^n is in a subgroup of $S_{G,\Delta}(M)$. In fact, he shows that p^n is equal to a translate of a Δ-generic of a Δ-definable connected subgroup of $G(M)$.

Definition A semigroup S is called *strongly π-regular* or an *epigroup* if for all $a \in S$, there exists $n \in \mathbb{N}$ such that a^n is in a subgroup of S.

Question Is this property enjoyed by all ∞-definable semigroups in stable structures?

Since we are dealing with ∞-definable semigroups, remembering that every ∞-definable group in a stable structure is an intersection of definable groups, an analogous question arises.

Question Is every ∞-definable semigroup in a stable structure an intersection of definable ones? Is $S_{G,\Delta}(M)$ an intersection of definable semigroups?

We answer these questions in this article.

It is a classical result about affine algebraic semigroups that they are strongly π-regular. Recently, Brion and Renner [1] proved that this is true for all algebraic semigroups. In fact, we will show the following.

Proposition *Let S be an ∞-definable semigroup inside a stable structure. Then S is strongly π-regular.*

At least in the definable case, this is a direct consequence of stability; the general case is not harder but a bit more technical.

One can ask if what happens in $S_{G,\Delta}(M)$ is true in general ∞-definable semigroups. That is, is every element a power away from a translation of an idempotent? However, this already is not true in $M_2(\mathbb{C})$.

As for the second question, in Section 4 we show that $S_{G,\Delta}(M)$ is an intersection of definable semigroups. In fact, we show the following.

Theorem *$S_G(M)$ is an inverse limit of definable semigroups in M^{eq}.*

Unfortunately, not all ∞-definable semigroups are an intersection of definable ones.

Milliet [11] showed that every ∞-definable semigroup inside a small structure is an intersection of definable semigroups. Hence, it is also true for ω-stable structures and, for instance, in algebraically closed fields (ACFs). This is not true for any stable theory (or even superstable theory). See Example 3.2.1 for a counterexample.

However, there are some classes of semigroups for which this does hold. We recall some basic definitions from semigroup theory that we will need. (See Section 2.2 for more information.)

Definition

1. An element $e \in S$ in a semigroup S is an *idempotent* if $e^2 = e$.
2. A semigroup S is called an *inverse semigroup* if for every $a \in S$ there exists a unique $a^{-1} \in S$ such that
$$aa^{-1}a = a, \qquad a^{-1}aa^{-1} = a^{-1}.$$
3. A *Clifford semigroup* is an inverse semigroup in which the idempotents are central. A *surjective Clifford monoid* is a Clifford monoid in which for every $a \in S$ there exist $g \in G$ and idempotent e such that $a = ge$, where G is the unit group of S.

These kinds of semigroups do arise in the context of $S_G(M)$. It is probably folklore, but one may show (see Section 4.2) that if G is 1-based, then $S_G(M)$ is an inverse monoid. In Section 4.2 we give a condition on G for $S_G(M)$ to be Clifford.

Theorem *Let S be an ∞-definable surjective Clifford monoid in a stable structure. Then S is contained in a definable monoid, extending the multiplication on S. This monoid is also a surjective Clifford monoid.*

As a result of the proof, every such monoid is an intersection of definable ones.

In the process of proving the above theorem, we show two results which might be interesting in their own right.

Since ∞-definable semigroups in stable structures are sπr, one may define a partial order on them given by

$$a \leq b \Leftrightarrow a = be = fb \quad \text{for some } e, f \in E(S^1),$$

where S^1 is $S \cup \{1\}$ and where we define 1 to be the identity element. If for every $a, b \in S$, $a \cdot b \leq a, b$, one may show that there exists $n \in \mathbb{N}$ such that every product of $n + 1$ elements is already a product of n of them (see Proposition 3.3.4). As a result, any such semigroup is an intersection of definable ones. In particular, we have the following.

Proposition *Let E be an ∞-definable commutative idempotent semigroup inside a stable structure. Then E is contained in a definable commutative idempotent semigroup. Furthermore, it is an intersection of definable ones.*

2 Preliminaries

2.1 Notation We fix some notation. We will usually not distinguish between singletons and sequences unless otherwise necessary to avoid confusion. Thus, we may write $a \in M$ and actually mean $a = (a_1, \ldots, a_n) \in M^n$. We will denote $A, B, C, \ldots$ as parameter sets and denote $M, N, \ldots$ as models. When talking specifically about semigroups, monoids, and groups (either definable, ∞-definable, or models) we will denote them by S, M, and G, respectively. We use juxtaposition ab for concatenation of sequences, or AB for $A \cup B$ if dealing with sets. That being said, since we will be dealing with semigroups, when there is a chance of confusion we will try to differentiate between the concatenation ab and the semigroup multiplication ab by denoting the latter by $a \cdot b$.

2.2 Semigroups The work of Clifford and Preston (see [2], [3]) is still a very good reference for the theory of semigroups, but Higgins [4] and Howie [5] are much more recent sources.

A set S with an associative binary operation is called a *semigroup*. An element $e \in S$ is an *idempotent* if $e^2 = e$. We will denote by $E(S)$ the subset of all idempotents of S. By a subgroup of S, we mean a subsemigroup $G \subseteq S$ such that there exists an idempotent $e \in G$ such that $(G, \cdot)$ is a group with neutral element e. Moreover, S is *strongly π-regular* (sπr) if for each $a \in S$, there exits $n > 0$ such that a^n lies in a subgroup of S.

Remark These types of semigroups are also known as *epigroups*, and their elements are known as *group-bound*.

A semigroup with an identity element is called a *monoid*. Observe that any semigroup can be extended to a monoid by artificially adding an identity element. We will denote it by S^1. If S is a monoid, we will denote by $G(S)$ its subgroup of invertible elements.

Given two semigroups S, S', a *homomorphism of semigroups* is a map

$$\varphi : S \to S'$$

such that $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in S$. If S, S' are monoids, then we say that φ is a *homomorphism of monoids* if in addition $\varphi(1_S) = 1_{S'}$.

Definition 2.2.1 The *natural partial order* on $E(S)$ is defined by

$$e \leq f \Leftrightarrow ef = fe = e.$$

Proposition 2.2.2 ([2, Section 1.7]) *For every $e, f \in E(S)$, we have the following:*

1. *eSe is a subsemigroup of S; in fact, it is a monoid with identity element e;*
2. *$eSe \subseteq fSf \Leftrightarrow e \leq f$;*
3. *every maximal subgroup of S is of the form $G(eSe)$ (the unit group of eSe) for $e \in E(S)$;*
4. *if $e \neq f$, then $G(eSe) \cap G(fSf) = \emptyset$.*

There are various ways to extend the partial order on the idempotents to a partial order on the entire semigroup. (See [4, Section 1.4] for a discussion about them.) We will use the *natural partial order* on S. While it has various equivalent definitions, we present the one given in [4, Proposition 1.4.3].

Definition 2.2.3 The relation

$$a \leq b \Leftrightarrow a = xb = by, xa = a \quad \text{for some } x, y \in S^1$$

is called the *natural partial order* on S.

Notice that this extends the partial order on $E(S)$. If S is sπr, this partial order takes a more elegant form as follows.

Proposition 2.2.4 ([4, Corollary 1.4.6]) *On sπr semigroups there is a natural partial order extending the order on $E(S)$:*

$$a \leq b \Leftrightarrow a = be = fb \quad \textit{for some } e, f \in E(S^1).$$

2.2.1 Clifford and inverse semigroups

Definition 2.2.5 A semigroup S is called *regular* if for every $a \in S$, there exists at least one element $b \in S$ such that

$$aba = a, \qquad bab = b.$$

Such an element b is a called a *pseudo-inverse* of a.

Definition 2.2.6 A semigroup S is called an *inverse semigroup* if for every $a \in S$, there exists a unique $a^{-1} \in S$ such that

$$aa^{-1}a = a, \qquad a^{-1}aa^{-1} = a^{-1}.$$

Some basic facts about inverse semigroups include the following.

Proposition 2.2.7 ([5, Section V.1, Theorem 1.2, Proposition 1.4]) *Let S be an inverse semigroup.*

1. *For every* $a, b \in S$, $(a^{-1})^{-1} = a$ *and* $(ab)^{-1} = b^{-1}a^{-1}$.
2. *For every* $a \in S$, aa^{-1} *and* $a^{-1}a$ *are idempotents.*
3. *The idempotents commute. Thus* $E(S)$ *is a commutative subsemigroup, and hence a semilattice.*

The basic example for inverse semigroups is the set $\mathcal{I}(X)$ of partial one-to-one mappings for a set X, which means that the domain is a (possibly empty) subset of X. The composition of two "incompatible" mappings will be the empty mapping. The first surprising fact is that this is an *inverse* semigroup, but one can say even more (a generalization of Cayley's theorem for groups).

Theorem 2.2.8 (Wagner–Preston representation theorem [5, Section V.1, Theorem 1.10]) *If* S *is an inverse semigroup, then there exist a set* X *and a monomorphism* $\phi : S \to \mathcal{I}(X)$.

If S is an inverse semigroup, then the partial order on S gets the form $a \leq b$ if there exists $e \in E(S)$ such that $a = eb$.

Proposition 2.2.9

1. $\leq$ *is a partial order relation.*
2. *If* $a, b, c \in S$ *such that* $a \leq b$, *then* $ac \leq bc$ *and* $ca \leq cb$. *Furthermore,* $a^{-1} \leq b^{-1}$.

Definition 2.2.10 A *Clifford semigroup* is an inverse semigroup in which the idempotents are central.

Remark Different sources give different, but equivalent, definitions of a Clifford semigroup. For instance, Howie defines a Clifford semigroup to be a regular semigroup S in which the idempotents are central (see [5, Section IV.2]). One may show that S is an inverse semigroup if and only if it is regular and the idempotents commute (see [5, Section V.1, Theorem 1.2]), so the definitions coincide.

The following is well known, but we will add a proof instead of adding another source.

Proposition 2.2.11 S *is a Clifford semigroup if and only if it is an inverse semigroup and* $aa^{-1} = a^{-1}a$ *for all* $a \in S$.

Proof Assume that S is a Clifford semigroup, and let $a \in S$. Since aa^{-1} and $a^{-1}a$ are idempotents and central,

$$aa^{-1} = a(a^{-1}a)a^{-1} = (a^{-1}a)aa^{-1} = a^{-1}a(aa^{-1}) = a^{-1}(aa^{-1})a = a^{-1}a.$$

Conversely, we must show that the idempotents are central. For $a \in S$ and $e \in E(S)$, we will show that $ea = (ea)(a^{-1}e)(ea) = (ea)(ea^{-1})(ea)$ and thus, by the uniqueness of the pseudoinverses, $ea = ae$. By our assumption,

$$(ea)(ea^{-1})(ea) = eae(a^{-1}e)(ea) = eae(ea)(a^{-1}e) = eaeaa^{-1}e.$$

Again, $(ea)(a^{-1}e) = (a^{-1}e)(ea)$ so

$$= eaa^{-1}eea = eaa^{-1}ea,$$

and by the commutativity of the idempotents (e and aa^{-1}),

$$= eaa^{-1}a = ea. \qquad \square$$

Definition 2.2.12 ([5, Chapter IV]) A semigroup S is said to be a *strong semilattice of semigroups* if there exist a semilattice Y, disjoint subsemigroups $\{S_\alpha : \alpha \in Y\}$, and homomorphisms $\{\phi_{\alpha,\beta} : S_\alpha \to S_\beta : \alpha, \beta \in Y, \alpha \geq \beta\}$ such that

1. $S = \bigcup_\alpha S_\alpha$,
2. $\phi_{\alpha,\alpha}$ is the identity,
3. for every $\alpha \geq \beta \geq \gamma$ in Y, $\phi_{\beta,\gamma}\phi_{\alpha,\beta} = \phi_{\alpha,\gamma}$.

Theorem 2.2.13 ([5, Section IV.2, Theorem 2.1]) *S is a Clifford semigroup if and only if it is a strong semilattice of groups. The semilattice is $E(S)$, the disjoint groups are*

$$\{G_e = G(eSe) : e \in E(S)\},$$

the maximal subgroups of S, and the homomorphism $\phi_{e,ef}$ is given by multiplication by f.

3 ∞-Definable Semigroups and Monoids

Let S be an ∞-definable semigroup in a stable structure. Assume that S is defined by

$$\bigwedge_i \varphi_i(x).$$

Remark We assume that S is defined over $\emptyset$ just for notational convenience. Moreover, we assume that the φ_i's are closed under finite conjunctions.

3.1 Strongly π-regular Our goal is to prove that an ∞-definable semigroup inside a stable structure is sπr. To better understand what is going on, we start with an easier case.

Definition 3.1.1 A *stable semigroup* is a stable structure S such that there is a definable binary function $\cdot$ which makes $(S, \cdot)$ into a semigroup.

The following was already noted by Losey and Schneider in [9] for semigroups with chain conditions, but we give it in a "stable semigroup" setting.

Proposition 3.1.2 *Any stable semigroup has an idempotent.*

Proof Let $a \in S$, and let

$$\theta(x, y) = \exists u(u \cdot a = a \cdot u \wedge u \cdot x = y).$$

Obviously, $S \models \theta(a^{3^m}, a^{3^n})$ for $m < n$. Moreover, S is stable, and hence θ does not have the order property. Thus there exists $m < n$ such that $S \models \theta(a^{3^n}, a^{3^m})$. Let $C \in S$ be such that $C \cdot a^{3^n} = a^{3^m}$ and commutes with a.

Since $3^n > 2 \cdot 3^m$, then multiplying by $a^{3^n - 2\cdot 3^m}$ yields $Ca^{2(3^n - 3^m)} = a^{3^n - 3^m}$. Notice that since C commutes with a, $Ca^{3^n - 3^m}$ is an idempotent. □

Proposition 3.1.3 *Any stable semigroup is $s\pi r$.*

Proof Let $a \in S$. From the proof of Proposition 3.1.2, there exists $C \in S$ that commutes with a and $n > 0$ such that $Ca^{2n} = a^n$. Set $e := Ca^n$. Indeed, $a^n = e \cdot a^n \cdot e$ and $a^n \cdot eCe = e$. □

Remark Given $a \in S$, there exists a unique idempotent $e = e_a \in S$ such that a^n belongs to the unit group of eSe for some $n > 0$. Indeed, for two idempotents $e \neq f$ the unit groups of eSe and fSf are disjoint (see Proposition 2.2.2).

Furthermore, we have the following.

Lemma 3.1.4 (Munn [12]) *Let S be a semigroup, and let $x \in S$. If for some n, x^n lies in a subgroup of S with identity e, then x^m lies in the unit group of eSe for all $m \geq n$.*

Corollary 3.1.5 *There exists $n > 0$ (depending only on S) such that for all $a \in S$, a^n belongs to the unit group of $e_a S e_a$.*

Proof Let $\phi_i(x)$ be the formula "$x^i \in$ the unit group of $e_x S e_x$," and let $\bigcup_i [\phi(x)] = S_1(S)$, since every elementary extension of S is also stable and hence $s\pi r$. By compactness, there exist $n_1, \ldots, n_k > 0$ such that $S_1(S) = [\phi_{n_1} \vee \cdots \vee \phi_{n_k}]$. Our desired integer is $n = n_1 \cdots n_k$. □

We return to the general case of S being an ∞-definable semigroup inside a stable structure. The following is an easy consequence of stability.

Proposition 3.1.6 *Every chain of idempotents in S, with respect to the partial order on them, is finite and uniformly bounded.*

Our goal is to show that for every $a \in S$, there exists an idempotent $e \in S$ and $n \in \mathbb{N}$ such that a^n is in the unit group of eSe.

We will want to assume that S is a conjunction of countably many formulas. For that we will need to make some observations. The following is well known but we add a proof for completion,

Lemma 3.1.7 *Let S be an ∞-definable semigroup. Then there exist ∞-definable semigroups H_i such that each H_i is defined by at most a countable set of formulas, and $S = \bigcap H_i$.*

Proof Let $S = \bigwedge_{i \in I} \varphi_i$, and assume that the φ_i's are closed under finite conjunctions. By compactness, we may assume that for all i and x, y, z,

$$\varphi_i(x) \wedge \varphi_i(y) \wedge \varphi_i(z) \rightarrow (xy)z = x(yz).$$

Let $i^0 \in I$. By compactness, there exists $i_1^0 \in I$ such that for all x, y,

$$\varphi_{i_1^0}(x) \wedge \varphi_{i_1^0}(y) \rightarrow \varphi_{i^0}(xy).$$

Thus, construct a sequence $i^0, i_1^0, i_2^0, \ldots$, and define

$$H_{i^0} = \bigwedge_j \varphi_{i_j^0}.$$

This is indeed a semigroup, and

$$S = \bigcap_{i \in I} H_i.$$

□

The following is also well known,

Proposition 3.1.8 ([6]) *An ∞-definable semigroup in a stable structure with left and right cancellation, or with left cancellation and right identity, is a group.*

As a consequence, we have the following.

Lemma 3.1.9 *Let S be an ∞-definable semigroup, and let $G_e \subseteq S$ be a maximal subgroup (with idempotent $e \in E(S)$), where G_e is relatively definable in S.*

Proof Let $S = \bigwedge_i \varphi_i(x)$. By compactness, there exists a definable set $S \subseteq S_0$ such that for all $x, y, z \in S_0$,

$$x(yz) = (xy)z.$$

Let $G_e(x)$ be

$$\bigwedge_i \varphi_i(x) \wedge (xe = ex = x) \wedge \bigwedge_i (\exists y \in S_0)\big(\varphi_i(y) \wedge ye = ey = y \wedge yx = xy = e\big).$$

This ∞-formula defines the maximal subgroup G_e.

Let $G_e \subseteq G_0$ be a definable group containing G_e (see [6]). We have that $G_0 \cap S$ is an ∞-definable subsemigroup of S with cancellation, and hence a subgroup. It is thus contained in the maximal subgroup G_e and so equal to it. $\square$

Lemma 3.1.10 *Let S be an ∞-definable semigroup, and let $S \subseteq S_1$ be an ∞-definable semigroup containing it. If S_1 is $s\pi r$, then so is S.*

Proof Let $a \in S$, and let $a^n \in G_e \subseteq S_1$, where G_e is a maximal subgroup of S_1. Thus $a^n \in G_e \cap S$. Since $G_e \cap S$ is an ∞-definable subsemigroup of S with cancellation, it is a subgroup. $\square$

We may, thus, assume that S is the conjunction of countably many formulas.

Furthermore, we may, and will, assume that S is commutative. Indeed, let $a \in S$. By compactness, we may find a definable set $S \subseteq S_0$ such that for all $x, y, z \in S_0$,

$$x(yz) = (xy)z.$$

Define $D_1 = \{x \in S_0 : xa = ax\}$ and then

$$D_2 = \big\{x \in S : (\forall c \in D_1)\, xc = cx\big\},$$

where D_2 is an ∞-definable commutative subsemigroup of S with $a \in D_2$.

Lemma 3.1.11 *There exist definable sets S_i such that $S = \bigcap S_i$, the multiplication on S_i is commutative, and that for all $1 < i$ there exists $C_i \in S_i$ and $n_i, m_i \in \mathbb{N}$ such that*

1. $n_i > 2m_i$;
2. $e_i := C_i a^{n_i - m_i}$ *is an idempotent;*

and furthermore, for all $1 < j \leq i$:

3. $n_j - m_j \leq n_i - m_i$;
4. $e_j e_i = e_i$;
5. $e_i a^{n_i - m_i} = a^{n_i - m_i}$.

Proof By compactness, we may assume that $S = \bigcap S_i$, where

$$S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$$

are definable sets such that for all $i > 1$, we are allowed to multiply associatively and commutatively at most 20 elements of S_i and get an element of S_{i-1}.

Let $i > 1$, and let $\theta(x, y)$ be

$$\exists u \in S_i \quad ux = y.$$

Obviously, $\models \theta(a^{3^k}, a^{3^l})$ for $k < l$. By stability, θ does not have the order property. Thus there exist $k < l$ such that $\models \theta(a^{3^l}, a^{3^k})$. Let $C_i \in S_i$ be such that $C_i a^{3^l} = a^{3^k}$.

Since $l > k$, we have $3^l > 2 \cdot 3^k$ (this gives (1)). Let $n_i = 3^l$ and $m_i = 3^k$. Then $e_i := C_i a^{n_i - m_i} \in S_{i-1}$ is an idempotent (this gives (2)). For this, first note that

$$C_i a^{2n_i - 2m_i} = C_i a^{n_i} a^{n_i - 2m_i} = a^{m_i} a^{n_i - 2m_i} = a^{n_i - m_i}.$$

Hence,

$$(C_i a^{n_i - m_i})(C_i a^{n_i - m_i}) = C_i^2 a^{2n_i - 2m_i} = C_i a^{n_i - m_i}.$$

We may take $n_i - m_i$ to be minimal, but then since $S_i \subseteq S_j$ for $j < i$, we have $n_j - m_j \leq n_i - m_i$ (this gives (3)).

As for (4), if $1 < j < i$, then

$$e_i e_j = C_i a^{n_i - m_i} C_j a^{n_j - m_j} = C_i a^{n_i - m_i + m_j} C_j a^{n_j - 2m_j},$$

but $n_i - m_i + m_j \geq n_j$, so

$$C_i a^{n_i - m_i + m_j - n_j} C_j a^{2n_j - 2m_j} = C_i a^{n_i - m_i + m_j - n_j} a^{n_j - m_j} = e_i.$$

Statement (5) follows quite similarly to what we have done. □

Proposition 3.1.12 *Let S be an ∞-definable semigroup inside a stable structure. Then S is strongly π-regular.*

Proof Let $a \in S$. For all $i > 1$, let S_i, C_i, n_i, and m_i be as in Lemma 3.1.11. Set $k_i = n_i - m_i$, and let

$$e_{i-1} = C_i a^{k_i} \qquad \text{and} \qquad \beta_{i-1} = e_i C_i e_i.$$

Notice that these are both elements of S_{i-1} (explaining the subindex).

By Lemma 3.1.11(4), we get a descending sequence of idempotents

$$e_1 \geq e_2 \geq \cdots$$

with respect to the partial order on the idempotents. By stability it must stabilize. Thus we may assume that $e := e_1 = e_2 = \cdots$ and that it is an element of S.

Moreover, for all $i > 1$,

$$\beta_1 = \beta_1 \cdot e = \beta_1 a^{k_{i+1}} \cdot \beta_i = e \cdot a^{k_{i+1} - k_2} \beta_i.$$

So

$$\beta_1 = e \cdot a^{k_{i+1} - k_2} e C_{i+1} e,$$

which is a product of ≤ 20 elements of S_{i+1} and thus $\in S_i$. Also $\beta_1 \in S$. In conclusion, by setting $k := k_2$ and $\beta := \beta_1$,

$$a^k e = e a^k = a^k, \qquad a^k \beta = \beta a^k = e, \qquad \text{and} \qquad \beta e = e \beta = \beta.$$

So a^k is in the unit group of eSe. □

Corollary 3.1.13 *There exists $n \in \mathbb{N}$ such that for all $a \in S$, a^n is an element of a subgroup of S.*

Proof This is by compactness. □

Corollary 3.1.14 *A semigroup S has an idempotent.*

Remark In the notation of Section 4, Newelski [13] showed that $S_{G,\Delta}(M)$ is an ∞-definable semigroup in M^{eq} and that it is sπr. Proposition 3.1.12, thus, gives another proof.

3.2 A counterexample It is known that every ∞-definable group inside a stable structure is an intersection of definable ones. It would be even better if every such semigroup were an intersection of definable semigroups. Milliet [11] showed that every ∞-definable semigroup inside a small structure is an intersection of definable semigroups. This is particularly true for ω-stable structures. So, for instance, any ∞-definable subsemigroup of $M_n(k)$ for $k \models ACF$ is an intersection of definable semigroups. Unfortunately, this is not true already in the superstable case, as the following example will show.

Example 3.2.1 Pillay and Poizat [15] give an example of an ∞-definable equivalence relation which is not an intersection of definable ones. This will give us our desired semigroup structure.

Consider the theory of a model which consists of universe $\mathbb{Q}$ (the rationals) with the unary predicates

$$U_a = \{x \in \mathbb{Q} : x \leq a\}$$

for $a \in \mathbb{Q}$. The equivalence relation E is defined by

$$\bigwedge_{a<b} \big(\big(U_a(x) \to U_b(y)\big) \wedge \big(U_a(y) \to U_b(x)\big)\big).$$

It is an equivalence relation and, in particular, a preorder (reflexive and transitive). Notice that it also follows that E cannot be an intersection of definable preorders. For if $E = \bigwedge R_i$ (for preorders R_i), then we also have

$$E = \bigwedge (R_i \wedge \overline{R}_i),$$

where $x\overline{R}_i y = yR_i x$ (since E is symmetric). But $R_i \wedge \overline{R}_i$ is a definable equivalence relation (the symmetric closure) and hence trivial. So the R_i's are trivial.

Milliet [11] showed that in an arbitrary structure, every ∞-definable semigroup is an intersection of definable semigroups if and only if this is true for all ∞-definable preorders. As a consequence, in the above structure we can define an ∞-definable semigroup which will serve as a counterexample. Specifically, it will be the following semigroup.

If the preorder is on a set X, add a new element 0 and add $0R0$ to the preorder. Define a semigroup multiplication on R:

$$(a,b)\cdot(c,d) = \begin{cases} (a,d) & \text{if } b = c, \\ (0,0) & \text{otherwise.} \end{cases}$$

Remark This example also shows that even "presumably well behaved" ∞-definable semigroups need not be an intersection of definable ones. In the example at hand, the maximal subgroups are uniformly definable (each of them is finite) and the idempotents form a commutative semigroup.

3.3 Semigroups with negative partial order We showed in Proposition 3.1.12 that every ∞-definable semigroup in a stable structure is strongly π-regular, and hence the natural partial order on it has the following form:

For any $a, b \in S$, $a \leq b$ if there exists $f, e \in E(S^1)$ such that $a = be = fb$.

Remark Note that this order generalizes the order on the idempotents.

In a similar manner to what was done with the order of the idempotents, we have the following.

Proposition 3.3.1

1. *Every chain of elements with regard to the natural partial order is finite.*
2. *By compactness, the length of the chains is uniformly bounded.*

Definition 3.3.2 We will say that a semigroup S is *negatively ordered* with respect to the partial order if

$$a \cdot b \leq a, b$$

for all $a, b \in S$.

Example 3.3.3 A commutative idempotent semigroup (an (inf)-semilattice) is negatively ordered.

Negatively ordered semigroups were studied by Maia and Mitsch [10]. We will only need the definition.

Proposition 3.3.4 *Let S be a negatively ordered semigroup. Assume that the length of chains is bounded by n. Then any product of $n + 1$ elements is a product of n of them.*

Proof Let $a_1 \cdot \ldots \cdot a_{n+1} \in S$. Since S is negatively ordered,

$$a_1 \cdot \ldots \cdot a_{n+1} \leq a_1 \cdot \ldots \cdot a_n \leq \cdots \leq a_1 \cdot a_2 \leq a_1.$$

Since n bounds the length of chains, we must have

$$a_1 \cdot \ldots \cdot a_i = a_1 \cdot \ldots \cdot a_{i+1}$$

for a certain $1 \leq i \leq n$. □

This property is enough for an ∞-definable semigroup to be contained inside a definable one.

Proposition 3.3.5 *Let S be an ∞-definable semigroup (in any structure). If every product of $n + 1$ elements in S is a product of n of them, then S is contained inside a definable semigroup. Moreover, S is an intersection of definable semigroups.*

Proof Let $S \subseteq S_0$ be a definable set where the multiplication is defined. By compactness, there exists a definable subset $S \subseteq S_1 \subseteq S_0$ such that

(i) any product of $\leq 3n$ elements of S_1 is an element of S_0;
(ii) associativity holds for products of $\leq 3n$ elements of S_1;
(iii)] any product of $n + 1$ elements of S_1 is already a product of n of them.

Let

$$S_1 \subseteq S_2 = \left\{x \in S_0 : \exists y_1, \ldots, y_n \in S_1 \bigvee_{i=1}^{n} x = y_1 \cdot \ldots \cdot y_i\right\}.$$

We claim that if $a \in S_1$ and $b \in S_2$, then $ab \in S_2$; indeed, this follows from the properties of S_1. Define

$$S_3 = \{x \in S_2 : xS_2 \subseteq S_2\}.$$

Our desired definable semigroup is S_3. □

As a consequence of these two propositions, we have the following.

Proposition 3.3.6 *Every ∞-definable negatively ordered semigroup inside a stable structure is contained inside a definable semigroup. Furthermore, it is an intersection of definable semigroups.*

Since every commutative idempotent semigroup is negatively ordered, we have the following corollary.

Corollary 3.3.7 *Let E be an ∞-definable commutative idempotent semigroup inside a stable structure. Then E is contained in a definable commutative idempotent semigroup. Furthermore, it is an intersection of definable ones.*

Proof We only need to show that the definable semigroup containing E can be made commutative idempotent. For that to happen, we need to demand that all the elements of S_1 (in the proof of Proposition 3.3.5) be idempotents and that they commute, but that can be satisfied by compactness. □

3.4 Clifford monoids We assume that S is an ∞-definable Clifford semigroup (see Section 2.2.1) inside a stable structure.

The simplest case of Clifford semigroups, commutative idempotent semigroups (semilattices), were considered in Section 3.3.

Understanding the maximal subgroups of a semigroup is one of the first steps when one wishes to understand the semigroup itself. Lemma 3.1.9 is useful and will be used implicitly.

Recall that every Clifford semigroup is a strong semilattice of groups. Between each two maximal subgroups G_e and G_{ef} there exists a homomorphism $\phi_{e,ef}$ given by multiplication by f.

Definition 3.4.1 By a *surjective Clifford monoid* we mean a Clifford monoid M such that for every $a \in M$, there exist $g \in G(M)$ and $e \in E(M)$ such that $a = ge$.

Surjectivity refers to the fact that these types of Clifford monoids are exactly the ones with $\phi_{e,ef}$ surjective.

We restrict ourselves to ∞-definable surjective Clifford monoids.

Theorem 3.4.2 *Let M be an ∞-definable surjective Clifford monoid in a stable structure. Then M is contained in a definable monoid, extending the multiplication on M. This monoid is also a surjective Clifford monoid. Furthermore, every such monoid is an intersection of definable surjective Clifford monoids.*

Proof Let $M \subseteq M_0$ be a definable set where the multiplication is defined. By compactness, there exists a definable subset $M \subseteq M_1 \subseteq M_0$ such that

(i) associativity holds for ≤ 6 elements of M_1;
(ii) any product of ≤ 6 elements of M_1 is in M_0;
(iii) 1 is a neutral element of M_1;
(iv) if x and y are elements of M_1 with y an idempotent, then $xy = yx$.

By the standard argument for stable groups, there exists a definable group

$$G_1 \subseteq G \subseteq M_1,$$

where $G_1 \subseteq M$ is the maximal subgroup of M associated with the idempotent 1. By Proposition 3.3.7, there exists a definable commutative idempotent semigroup $E(M) \subseteq E \subseteq M_1$. Notice that for every $g \in G$ and $e \in E$,

$$ge = eg.$$

Define

$$M_2 = \{m \in M_0 : \exists g \in G, e \in E\ m = ge\}.$$

The desired monoid is M_2.

This, furthermore, is a standard corollary of the above proof. □

We do not have an argument for Clifford monoids which are not necessarily surjective. But we do have a proof for a certain kind of inverse monoids. We will need this result in Section 4.

Theorem 3.4.3 *Let M be an ∞-definable monoid in a stable structure such that*

1. *its unit group G is definable,*
2. *$E(M)$ is commutative, and*
3. *for every $a \in M$, there exist $g \in G$ and $e \in E(M)$ such that*

$$a = ge.$$

Then M is contained in a definable monoid, extending the multiplication on M. This monoid also has these properties.

Remark Incidentally, M is an inverse monoid (recall the definition from Section 2.2.1). It is obviously regular and the pseudo-inverse is unique since the idempotents commute (see the preliminaries). Also, as before, every such monoid is an intersection of definable ones.

Proof Let $M \subseteq M_0$ be a definable set where the multiplication is defined and associative. By compactness, there exists a definable subset $M \subseteq M_1 \subseteq M_0$ such that

(i) associativity holds for ≤ 6 elements of M_1;
(ii) any product of ≤ 6 elements of M_1 is in M_0;
(iii) 1 is a neutral element of M_1;
(iv) if x and y are idempotents of M_1, then $xy = yx$.

By Proposition 3.3.7, there exists a definable commutative idempotent semigroup $E(M) \subseteq E \subseteq M_1$.

Let

$$E_1 = \{e \in E : \forall g \in G \; g^{-1}eg \in E\}.$$

E_1 is still a definable commutative idempotent semigroup that contains $E(M)$. Moreover, for every $e \in E_1$ and $g \in G$,

$$g^{-1}eg \in E_1.$$

Define

$$M_2 = \{m \in M_0 : \exists g \in G, e \in E_1 \; m = ge\}.$$

The desired monoid is M_2. Indeed, if $g, h \in G$ and $e, f \in E_1$, then there exist $h' \in G$ and $e' \in E_1$ such that

$$eh = h'e',$$

thus

$$ge \cdot hf = gh' \cdot e'f.$$ □

4 The Space of Types $S_G(M)$ on a Definable Group

Let G be a definable group inside a stable structure M. Assume that G is definable by a formula $G(x)$. Define $S_G(M)$ to be all the types of $S(M)$ which are on G.

Definition 4.0.1 Let $p, q \in S_G(M)$, and define

$$p \cdot q = tp(a \cdot b/M),$$

where $a \models p$, $b \models q$ and $a \downarrow_M b$.

Notice that the above definition may also be stated in the following form:

$$U \in p \cdot q \Leftrightarrow d_q(U) \in p,$$

where U is a formula and $d_q(U) := \{g \in G(M) : g^{-1}U \in q\}$ (see [13]). Thus, if Δ is a finite family of formulas, in order to restrict the multiplication to $S_{G,\Delta}(M)$, the set of Δ-types on G, we will need to consider invariant families of formulas.

Definition 4.0.2 Let $\Delta \subseteq L$ be a finite set of formulas. We will say that Δ is *$(G$-$)$invariant* if the family of subsets of G definable by instances of formulas from Δ is invariant under left and right translation in G.

From now on, unless stated otherwise, we will assume that Δ is a finite set of invariant formulas. For $\Delta_1 \subseteq \Delta_2$, let

$$r^{\Delta_2}_{\Delta_1} : S_{G,\Delta_2}(M) \to S_{G,\Delta_1}(M)$$

be the restriction map. These are semigroup homomorphisms. Thus

$$S_G(M) = \varprojlim_{\Delta} S_{G,\Delta}(M).$$

Newelski [13] showed that $S_{G,\Delta}(M)$ may be interpreted in M^{eq} as an ∞-definable semigroup. Our aim is to show that these ∞-definable semigroups are in fact an intersection of definable ones and, as a consequence, that $S_G(M)$ is an inverse limit of definable semigroups of M^{eq}.

4.1 $S_{G,\Delta}(M)$ is an intersection of definable semigroups Let $\varphi(x, y)$ be a G-invariant formula. The proof that $S_{G,\varphi}(M)$ is interpretable as an ∞-definable semigroup in M^{eq} is given in [13]. We will show that it may be given as an intersection of definable semigroups.

Proposition 4.1.1 (Pillay [14]) *There exists $n \in \mathbb{N}$ and a formula $d_\varphi(y, u)$ such that, for every $p \in S_{G,\varphi}(M)$, there exists a tuple $c_p \subseteq G$ such that*

$$d_\varphi(y, c_p) = (d_p x)\varphi(x, y).$$

Moreover, d_φ may be chosen to be a positive Boolean combination of φ-formulas.

Let E_{d_φ} be the equivalence relation defined by

$$c_1 E_{d_\varphi} c_2 \iff \forall y(d_\varphi(y, c_1) \leftrightarrow d_\varphi(y, c_2)).$$

Set $Z_{d_\varphi} := \frac{M}{E_{d_\varphi}}$, which is the sort of canonical parameters for a potential φ-definition.

Remark We may assume that c_p is the canonical parameter for $d_\varphi(M, c_p)$, namely, that it lies in Z_{d_φ}. Just replace the formula $d_\varphi(y, u)$ with the formula

$$\psi(y, v) = \forall u\big((\pi(u) = v) \to d_\varphi(y, u)\big),$$

where v lies in the sort $\frac{M}{E_{d_\varphi}}$ and $\pi : M \to \frac{M}{E_{d_\varphi}}$.

Each element $c \in Z_{d_\varphi}$ corresponds to a complete (but not necessarily consistent) set of φ-formulas:

$$\begin{aligned} p_c^0 := & \big\{\varphi(x,a) : a \in M \text{ and } \models d_\varphi(a,c)\big\} \\ & \cup \big\{\neg\varphi(x,a) : a \in M \text{ and } \not\models d_\varphi(a,c)\big\}. \end{aligned}$$

Remark Notice that p_c^0 may not be closed under equivalence of formulas, but the set of canonical parameters $c \in Z_{d_\varphi}$ such that p_c^0 is closed under equivalence of formulas is the definable set

$$\big\{c \in Z_{d_\varphi} : \forall t_1 \forall t_2 \big(\varphi(x,t_1) \equiv \varphi(x,t_2) \to \big(d_\varphi(t_1,c) \leftrightarrow d_\varphi(t_2,c)\big)\big)\big\}.$$

Thus we may assume that we only deal with sets p_c^0 which are closed under equivalence of formulas.

The set of $c \in Z_{d_\varphi}$ such that p_c^0 is k-consistent is definable:

$$Z_{d_\varphi}^k = \{c \in Z_{d_\varphi} : p_c^0 \text{ is } k\text{-consistent}\}.$$

Define

$$Z = \bigcap_{k<\omega} Z_{d_\varphi}^k.$$

There is a bijection $(p \mapsto c_p)$ between $S_{G,\varphi}(M)$ and Z.

The following is a trivial consequence of Proposition 4.1.1.

Lemma 4.1.2 *There exists a formula $\Phi(u,v,y)$ with u,v in the sort Z_{d_φ} such that*

$$\Phi(c_p,c_q,a) \Leftrightarrow \varphi(x,a) \in p \cdot q.$$

Moreover, Φ is a positive Boolean combination of d_φ-formulas (and so of φ-formulas as well).

Proof Since φ is G-invariant, for simplicity we will assume that $\varphi(x,y)$ is in fact of the form $\varphi(l \cdot x \cdot r, y)$. Let $c_p, c_q \subseteq G$ be tuples whose images in Z_{d_φ} correspond to the φ-types $p,q \in S_{G,\varphi}(M)$, respectively.

Remembering that $u = (u_{ij})_{1\le i,j\le n}$ is a tuple of variables, we may write

$$d_\varphi(l,r,y,u) = \bigvee_{i<n} \bigwedge_{j<n} \varphi(l \cdot u_{ij} \cdot r, y).$$

Since

$$\begin{aligned} d_q\big(\varphi(b \cdot x \cdot c, a)\big) &= \big\{g \in G(M) : \varphi\big((b \cdot g) \cdot x \cdot c, a\big) \in q\big\} \\ &= \big\{g \in G(M) : \models d_\varphi(b \cdot g, c, a, c_q)\big\} \end{aligned}$$

and

$$d_\varphi(b \cdot g, c, a, c_q) = \bigvee_{i<n} \bigwedge_{j<n} \varphi\big(b \cdot g \cdot \big((c_q)_{ij} \cdot c\big), a\big),$$

we get that

$$\varphi(b \cdot x \cdot c, a) \in p \cdot q \Longleftrightarrow \models \bigvee_{i<n} \bigwedge_{j<n} d_\varphi\big(b, \big((c_q)_{ij} \cdot c\big), a, c_p\big).$$

□

We use this to define a partial binary operation on Z_{d_φ}.

Definition 4.1.3 For $c_1, c_2, d \in Z_{d_\varphi}$, we will say that $c_1 \cdot c_2 = d$ if d is the unique element of Z_{d_φ} that satisfies

$$\models d_\varphi(a, d) \Longleftrightarrow \models \Phi(c_1, c_2, a)$$

for all $a \in M$.

By compactness, there exists $k \in \mathbb{N}$ such that for all $c_1, c_2 \in Z^k_{d_\varphi}$, there exists a unique $d \in Z_{d_\varphi}$ such that $c_1 \cdot c_2 = d$. For simplicity, we will assume that this happens for $Z^1_{d_\varphi}$.

Theorem 4.1.4 *Z is contained in a definable semigroup extending the multiplication on Z.*

Proof By compactness, there exists $k \in \mathbb{N}$ such that the multiplication is associative on $Z^k_{d_\varphi}$ and the product of two elements of $Z^k_{d_\varphi}$ is in $Z^1_{d_\varphi}$. For simplicity, let us assume that this happens for $Z^2_{d_\varphi}$.

Claim *If $c_p \in Z$ and $c \in Z^2_{d_\varphi}$, then $c_p \cdot c \in Z^2_{d_\varphi}$.*

Let $U_1, U_2 \in p_{c_p \cdot c}$. Hence

$$\{g \in G(M) : g^{-1}U_1 \in p^0_c\}, \{g \in G(M) : g^{-1}U_2 \in p^0_c\} \in p.$$

Since p is consistent, there exists $g \in G(M)$ such that $g^{-1}U_1, g^{-1}U_2 \in p^0_c$. Since $c \in Z^2_{d_\varphi}$, p^0_c is 2-consistent. Thus, the claim follows.

Define

$$\widehat{Z^2_{d_\varphi}} = \{c \in Z^2_{d_\varphi} : c \cdot Z^2_{d_\varphi} \subseteq Z^2_{d_\varphi}\}.$$

$\widehat{Z^2_{d_\varphi}}$ is the desired definable semigroup. □

Corollary 4.1.5 *$Z = S_{G,\varphi}(M)$ is an intersection of definable semigroups.*

Looking even closer at the above proof, we may show that $S_G(M)$ is an inverse limit of definable semigroups.

Assume that $\Delta_2 = \{\varphi_1, \varphi_2\}$ and $\Delta_1 = \{\varphi_1\}$. In the above notation,

$$Z_{\Delta_2} = Z_{d_{\varphi_1}} \times Z_{d_{\varphi_2}}.$$

For $c = \langle c_1, c_2\rangle \in Z_{\Delta_2}$, define

$$p^0_c = p^0_{c_1} \cup p^0_{c_2}$$

and then

$$Z(\Delta_2) = \bigcap Z^k_{\Delta_2}$$

similarly.

For $c, c', d \in Z(\Delta_2)$, we will say that $c \cdot c' = d$ if d is the unique element $d \in Z_(\Delta_2)$ that satisfies

$$c_1 \cdot c'_1 = d_1 \qquad \text{and} \qquad c_2 \cdot c'_2 = d_2.$$

As before, we assume that such a unique element already exists for any pair of elements in $Z^1_{\Delta_2} = Z^1_{\varphi_1} \times Z^1_{\varphi_2}$. The restriction maps $r^{\Delta_2}_{\Delta_1} : Z^1_{\Delta_2} \to Z^1_{\Delta_1}$ are definable homomorphisms. Generally, for every $\Delta = \{\varphi_1, \dots, \varphi_n\}$ and $i < \omega$,

$$Z^i_\Delta = Z^i_{\varphi_1} \times \cdots \times Z^i_{\varphi_n},$$

where the multiplication is coordinate-wise. So the restriction commutes with the inclusion. As a result, we have the following.

Theorem 4.1.6 *We have that $S_G(M)$ is an inverse limit of definable semigroups:*

$$\varprojlim_{\Delta,i} Z_i^{\Delta} = \varprojlim_{\Delta} S_{G,\Delta}(M).$$

4.2 The case where $S_G(M)$ is an inverse monoid We would like to use the theorems we proved in Section 3 to improve the result in the situation where $S_G(M)$ is an inverse monoid. We will first see that this situation might occur. Notice that the inverse operation $^{-1}$ on $S_G(M)$ is an involution.

Proposition 4.2.1 (Lawson [8]) *Let S be a compact semitopological $*$-semigroup (a semigroup with involution) with a dense unit group G. Then the following are equivalent for any element $p \in S$:*

1. *$p = pp^* p$;*
2. *p has a unique quasi-inverse;*
3. *p has a quasi-inverse.*

Remark In the situation of $G(M) \hookrightarrow S_G(M)$, the above proposition can be proved directly using model theory and stabilizers.

Translating the above result to our situation and using results in 1-based groups (see [14]), we have the following.

Corollary 4.2.2 *The following are equivalent:*

1. *for every $p \in S_G(M)$, p is the generic of a right coset of a connected M-∞-definable subgroup of G;*
2. *for every $p \in S_G(M)$, $p \cdot p^{-1} \cdot p = p$;*
3. *$S_G(M)$ is an inverse monoid;*
4. *$S_G(M)$ is a regular monoid.*

Thus if G is 1-based, then $S_G(M)$ is an inverse monoid.

Proof Statements (2), (3), and (4) are equivalent by Proposition 4.2.1, and (1) is equivalent to (2) by Kowalski [7, Lemma 1.2]. □

With a little more work, one may characterize when $S_G(M)$ is a Clifford monoid.

Definition 4.2.3 A right-and-left coset of a subgroup H is a right coset Ha such that $aH = Ha$.

Proposition 4.2.4 ([7]) *$p \in S_G(M)$ is a generic of a right-and-left coset of an M-∞-definable connected subgroup of G if and only if $p \cdot p \cdot p^{-1} = p$.*

We have the following easy lemma.

Lemma 4.2.5 *Assuming that $pp^{-1}p = p$, we have*

$$pp^{-1} = p^{-1}p \Leftrightarrow ppp^{-1} = p.$$

We use the previous lemma to get the following.

Proposition 4.2.6 *The following are equivalent:*

1. *every $p \in S_G(M)$ is the generic of a right-and-left coset of a connected M-∞-definable subgroup of G;*
2. *$S_G(M)$ is a Clifford monoid.*

As a result, it may happen that $S_G(M)$ is an inverse (or Clifford) monoid. One may wonder if in these cases we may strengthen the result.

Lemma 4.2.7 *If $S_G(M)$ is a Clifford monoid, then so is $S_{G,\Delta}(M)$. The same goes for inverse monoids.*

Proof In order to show that $S_{G,\Delta}(M)$ is a Clifford monoid, we must show that it is regular and that the idempotents are central.

Indeed, this follows from the fact that the restriction maps are surjective homomorphisms and that if $q|_\Delta$ is an idempotent, there exists an idempotent $p \in S_G(M)$ such that $p|_\Delta = q|_\Delta$ (see [13]). □

Assume that Δ is a finite invariant set of formulas. We will show that if $S_G(M)$ is an inverse monoid, then $S_{G,\Delta}(M)$ is an intersection of definable inverse monoids.

We recall some definition from [13]. Since Δ is invariant for $p \in S_{G,\Delta}(M)$, we have a map

$$d_p : \mathrm{Def}_{G,\Delta}(M) \to \mathrm{Def}_{G,\Delta}(M)$$

defined by

$$U \mapsto \left\{g \in G(M) : g^{-1}U \in p\right\}.$$

Here $\mathrm{Def}_{G,\Delta}(M)$ are the Δ-M-definable subsets of $G(M)$.

Furthermore, for $p \in S_{G,\Delta}(M)$, define

$$\mathrm{Ker}(d_p) = \left\{U \in \mathrm{Def}_{G,\Delta}(M) : d_p(U) = \emptyset\right\}.$$

Lemma 4.2.8 ([13, Proposition 4.12]) *Let Δ be a finite invariant set of formulas, and let $p \in S_{G,\Delta}(M)$ be an idempotent. Then*

$$\left\{q \in S_{G,\Delta}(M) : \mathrm{Ker}(d_q) = \mathrm{Ker}(d_p)\right\} = \left\{g \cdot p : g \in G(M)\right\} = G(M)p.$$

In particular, it is definable (in M^{eq}).

Corollary 4.2.9 *If $S_{G,\Delta}(M)$ is a regular semigroup, then*

$$S_{G,\Delta}(M) = \bigcup_{p \text{ idempotent}} G(M)p.$$

Proof Let $q \in S_{G,\Delta}(M)$. By regularity, there exists $\tilde{q}$ such that

$$q = q\tilde{q}q \qquad \text{and} \qquad \tilde{q} = \tilde{q}q\tilde{q},$$

where $\tilde{q}q$ is the desired idempotent and $\mathrm{Ker}(q) = \mathrm{Ker}(\tilde{q}q)$. □

Recall Theorem 3.4.3. Note that a semigroup S fulfilling the requirements of the theorem is an inverse monoid. It is obviously regular and the pseudoinverse is unique since the idempotents commute. We get the following.

Corollary 4.2.10 *If $S_G(M)$ is an inverse semigroup, then $S_{G,\Delta}(M)$ is an intersection of definable inverse semigroups.*

Proof Since $S_G(M)$ is inverse, so are the $S_{G,\Delta}(M)$. By [13] the unit group of $S_{G,\Delta}(M)$ is definable, and by the previous corollary, for every $p \in S_{G,\Delta}(M)$, there exist an idempotent e and $g \in G(M)$ such that

$$p = ge.$$

□

References

[1] Brion, M., and L. E. Renner, "Algebraic semigroups are strongly π-regular," pp. 55–59 in *Algebraic Monoids, Group Embeddings, and Algebraic Combinatorics*, vol. 71 of *Fields Institute Communications*, Springer, New York, 2014. MR 3308315. DOI 10.1007/978-1-4939-0938-4_2. 418

[2] Clifford, A. H., and G. B. Preston, *The Algebraic Theory of Semigroups, Vol. I*, vol. 7 of *Mathematical Surveys*, American Mathematical Society, Providence, 1961. Zbl 0111.03403. MR 0132791. 419, 420

[3] Clifford, A. H., and G. B. Preston, *The Algebraic Theory of Semigroups, Vol. II*, vol. 7 of *Mathematical Surveys*, American Mathematical Society, Providence, 1967. Zbl 0178.01203. MR 0218472. 419

[4] Higgins, P. M., *Techniques of Semigroup Theory*, Oxford University Press, New York, 1992. Zbl 0744.20046. MR 1167445. 419, 420

[5] Howie, J. M., *Fundamentals of Semigroup Theory*, vol. 12 of *London Mathematical Society Monographs Series*, Oxford University Press, New York, 1995. Zbl 0835.20077. MR 1455373. 419, 420, 421, 422

[6] Hrushovski, E., "Unidimensional theories are superstable," *Annals of Pure and Applied Logic*, vol. 50 (1990), pp. 117–38. Zbl 0713.03015. MR 1081816. DOI 10.1016/0168-0072(90)90046-5. 417, 423, 424

[7] Kowalski, P., "Stable groups and algebraic groups," *Journal of the London Mathematical Society, Second Series*, vol. 61 (2000), pp. 51–57. Zbl 1026.20025. MR 1745401. DOI 10.1112/S0024610799008261. 433

[8] Lawson, J. D., "Points of continuity for semigroup actions," *Transactions of the American Mathematical Society*, vol. 284 (1984), pp. 183–202. Zbl 0516.54031. MR 0742420. DOI 10.2307/1999282. 433

[9] Losey, G., and H. Schneider, "Group membership in rings and semigroups," *Pacific Journal of Mathematics*, vol. 11 (1961), pp. 1089–98. Zbl 0103.01103. MR 0136672. DOI 10.2140/pjm.1961.11.1089. 422

[10] Maia, A. F., and H. Mitsch, "Semigroups with negative natural partial order," *Pure Mathematics and Applications*, vol. 14 (2003), pp. 289–303. Zbl 1066.20054. MR 2153944. 427

[11] Milliet, C., "On enveloping type-definable structures," *Journal of Symbolic Logic*, vol. 76 (2011), pp. 1023–34. Zbl 1244.03112. MR 2849257. DOI 10.2178/jsl/1309952532. 418, 426

[12] Munn, W. D., "Pseudo-inverses in semigroups," *Proceedings of the Cambridge Philosophical Society*, vol. 57 (1961), pp. 247–50. Zbl 0228.20057. MR 0121410. 423

[13] Newelski, L., "Topological dynamics of stable groups," *Journal of Symbolic Logic*, vol. 79 (2014), pp. 1199–223. Zbl 1353.03022. MR 3343536. DOI 10.1017/jsl.2014.25. 417, 425, 430, 434

[14] Pillay, A., *Geometric Stability Theory*, vol. 32 of *Oxford Logic Guides*, Oxford University Press, New York, 1996. Zbl 0871.03023. MR 1429864. 430, 433

[15] Pillay, A., and B. Poizat, "Pas d'imaginaires dans l'infini!" *Journal of Symbolic Logic*, vol. 52 (1987), pp. 400–403. Zbl 0631.03014. MR 0890448. DOI 10.2307/2274390. 426

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