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Definable Open Sets As Finite Unions of Definable Open Cells

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Abstract We introduce CE-*cell decomposition*, a modified version of the usual o-minimal cell decomposition. We show that if an o-minimal structure $\mathcal{R}$ admits CE-cell decomposition then any definable open set in $\mathcal{R}$ may be expressed as a finite union of definable open cells. The dense linear ordering and linear o-minimal expansions of ordered abelian groups are examples of such structures.

Fix an o-minimal structure $\mathcal{R} = (R, <, \ldots)$. It is true of any o-minimal structure that definable, open sets in one variable are expressible as finite unions of definable open cells. Whether this is true of any definable open set $X \subseteq R^m$ has not yet been decided for all o-minimal structures. If any definable open set in $\mathcal{R}$ can be expressed as a finite union of definable open cells, then $\mathcal{R}$ is said to admit the *open cell property* (OCP).

It has been shown that definable, bounded, open sets are equal to finite unions of definable open cells in expansions of the real closed field (see Wilkie [4]) and non-linear o-minimal expansions of ordered abelian groups (see Edmundo [2]). Wilkie mentions in [4] that boundedness is necessary. Gareth Jones provided me with the example that

$$\left\{(x, y) \in \mathbb{R}^2 : x \neq 0 \wedge y = |\frac{1}{x}|\right\} \cup (0 \times (0, +\infty))$$

is definable in $(\mathbb{R}, +, -, \cdot, 0, 1)$ and is not expressible as a finite union of definable open cells.

Notation 1 "Definable" abbreviates "definable in $\mathcal{R}$ with parameters from $\mathcal{R}$." We write $\Gamma(f)_C = \{(x, y) \in R^{m+1} : x \in C \wedge y = f(x)\}$ for the graph of a function $f : C \to R$ and $(f, g)_C = \{(x, y) \in R^{m+1} : x \in C \wedge f(x) < y < g(x)\}$ for the interval between functions $f, g : C \to R$. We write $\pi : R^m \to R^{m-1}$ for the

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projection map onto the first $m-1$ coordinates and $\pi^* : R^m \to R$ is projection onto the last coordinate. We write $\partial X := \operatorname{cl} X \setminus X$ for the *frontier* of a set X. For the purposes of this paper, a *stratification* $\mathcal{S}$ of a set $X \subseteq R^m$ is a finite partition of X into cells so that, for all $A, B \in \mathcal{S}$, $B \subseteq \operatorname{cl} A$ whenever $B \cap \operatorname{cl} A \neq \varnothing$. A function $f : C \to R$ *defines a cell*, or is a *cell-defining function*, if $\Gamma(f)_C$ is a cell or there is a definable function $g : C \to R$ so that $(f, g)_C$ or $(g, f)_C$ is a cell.

CE-cell decomposition A definable continuous function $f : C \subseteq D \to R$ has a *continuous extension* to D if there is a definable continuous function $F : D \to R$ such that $F \restriction_C = f$. We define *continuous extension cells*, or CE-*cells*, inductively. In R the CE-cells are exactly the points and definable open intervals. $\Gamma(f)_C, (g, h)_C \subseteq R^{m+1}$ are CE-cells if C is a CE-cell and there exist continuous extensions of f, g, and h to $\operatorname{cl} C$. We say that an o-minimal structure admits CE-*cell decomposition* if any cell decomposition admits a refinement by CE-cells.

The main result of this paper is the following theorem.

Theorem 2 *If $\mathcal{R}$ admits* CE-*cell decomposition, then $\mathcal{R}$ admits* OCP.

Example 3 $(R, <)$ admits CE-cell decomposition. By §1.6.3 of [1], cell decomposition in $(R, <)$ is equivalent to partitioning by constant functions and projection maps. Such functions are globally definable and continuous.

Example 4 Linear o-minimal expansions of $(R, <, +)$ admit CE-cell decomposition. By §1.7.4 of [1], cell decomposition in linear expansions of $(R, <, +)$ yields cells whose defining functions are restrictions of affine (linear) functions. Such functions are globally definable and continuous.

The Trichotomy Theorem and the above examples show that the o-minimal structures for which OCP remains to be decided are the trivial structures, excluding $(R, <)$. Defined in [3], $\mathcal{R}$ is trivial if it admits no locally definable group structure.

This work began as an attempt to extend the results of Wilkie and Edmundo to trivial and linear o-minimal structures. However, it was suggested to me through correspondence with Pantelis Eleftheriou that, in these cases, boundedness may not be necessary. It turns out that my original proof is amenable to the unbounded case; this is due to the following lemma.

Lemma 5 *If C is a* CE-*cell, then* $\operatorname{cl} \pi C = \pi \operatorname{cl} C$.

Proof $\pi \operatorname{cl} C \subseteq \operatorname{cl} \pi C$ is always true for cells. For the other direction, suppose $C = \Gamma(f)_{\pi C}$ and let F be the continuous extension of f to $\operatorname{cl} \pi C$. If $x \in \operatorname{cl} \pi C$, then $(x, F(x)) \in \operatorname{cl} C$ by continuity. Hence $x \in \pi \operatorname{cl} C$. The same argument works for the case $C = (f, g)_{\pi C}$. □

Lemma 6 *If $C \subseteq R^m$ is a definable* CE-*cell, then there exists a global retraction $H : R^m \to \operatorname{cl} C$.*

Proof We proceed by induction on m. For $m = 1$:

1. $C = \{a\}$. Define $H(x) := a$.
2. $C = (a, b)$. Define

$$H(x) := \begin{cases} a, & \text{if } x \leq a; \\ x, & \text{if } x \in C; \\ b, & \text{if } b \leq x. \end{cases}$$

Assume $m > 1$ and that the lemma holds for all lower values of m. Let $H_{\text{ind}} : R^{m-1} \to \operatorname{cl} \pi C = \pi \operatorname{cl} C$ be the retraction from the inductive assumption. We write $(x', x^m) \in R^{m-1} \times R$. For $f, g : \pi C \to R$ defining CE-cells we write F and G for their continuous extensions to $\operatorname{cl} \pi C$.

1. $C = \Gamma(f)_{\pi C}$. Define $H(x', x^m) := (H_{\text{ind}}(x'), F(x'))$.
2. $C = (f, g)_{\pi C}$. Define

$$H(x', x^m) := \begin{cases} (H_{\text{ind}}(x'), F \circ H_{\text{ind}}(x')), & x^m \leq F \circ H_{\text{ind}}(x'); \\ (H_{\text{ind}}(x'), x^m), & F \circ H_{\text{ind}}(x') < x^m < G \circ H_{\text{ind}}(x'); \\ (H_{\text{ind}}(x'), G \circ H_{\text{ind}}(x')), & G \circ H_{\text{ind}}(x') \leq x^m. \end{cases}$$

H partitions R^m into the following cells:

$$\begin{aligned} D_1 &:= (-\infty, F \circ H_{\text{ind}})_{R^{m-1}}, \\ D_2 &:= \Gamma(F \circ H_{\text{ind}})_{R^{m-1}}, \\ D_3 &:= (F \circ H_{\text{ind}}, G \circ H_{\text{ind}})_{R^{m-1}}, \\ D_4 &:= \Gamma(G \circ H_{\text{ind}})_{R^{m-1}}, \\ D_5 &:= (G \circ H_{\text{ind}}, +\infty)_{R^{m-1}}. \end{aligned}$$

Since H_{ind}, F and G are continuous functions, H is continuous on each of these sets. To see that H is continuous, suppose $(x_i', x_i^m)_{i \in \mathbb{N}} \to (y', y^m) \in R^m$. If $(y', y^m) \in D_k$ ($k = 1, 3$, or 5) then $(x_n', x_n^m) \in D_k$ for all sufficiently large n by convergence, openness of D_1, D_3, and D_5 and the preceding remarks. Since H is continuous on these sets the result follows. For the final cases we assume, without loss of generality, that $y^m = F \circ H_{\text{ind}}(y')$. Since H_{ind} is continuous we have $H_{\text{ind}}(x_i') \to H_{\text{ind}}(y')$, so it suffices to consider the sequence $\pi^*(H(x_i', x_i^m))_{i \in \mathbb{N}}$. If $(x_n', x_n^m) \in D_k$ ($k = 1, 2$, or 3) for all sufficiently large n then, because $F \circ H_{\text{ind}}(x_i') \to y^m$ and $x_i^m \to y^m$, we have the desired result. If $G \circ H_{\text{ind}}(x_i') \leq x^m$ then we have the following inequality over $\operatorname{cl} \pi C$: $F \circ H_{\text{ind}}(x_i') < \pi^*(H(x_i', x_i^m)) = G \circ H_{\text{ind}}(x_i') \leq x_i^m$. Since $F \circ H_{\text{ind}}(x_i') \to y^m$ and $x_i^m \to y^m$, the result follows. □

Corollary 7 *If f defines a* CE-*cell, then there is a global definable continuous extension of f.*

Proof If $f : C \subseteq R^m \to R$ defines a CE-cell and F is its continuous extension to $\operatorname{cl} C$ then, letting $H : R^m \to \operatorname{cl} C$ be given by Lemma 5, take $F \circ H$. □

Fix a definable open set $X \subseteq R^m$.

Proof of Theorem 2 Induction on m. The result holds for $m = 1$ by o-minimality. Assume $m > 1$ and that the theorem holds for all lower values of m. Let $\mathcal{E}'$ be a CE-cell decomposition of R^m compatible with X and let $\mathcal{F}$ denote the set containing the global continuous extensions of all defining functions of the cells in $\mathcal{E}'$, as well as $+\infty$ and $-\infty$. Refine $\pi \mathcal{E}'$—the set of cells in R^{m-1} which are the images of the cells in $\mathcal{E}'$ under the projection π—to $\mathcal{D}$, a CE-cell decomposition of R^{m-1} that is a stratification and compatible with $\mathcal{F}$. That is, if $D \in \mathcal{D}$ and $g, h \in \mathcal{F}$, then, when g and h are restricted to D, exactly one of $g < h$, $g = h$, or $g > h$ is true. Now refine $\mathcal{E}'$ to $\mathcal{E}$ so that $\pi \mathcal{E} = \mathcal{D}$.

Fix a nonopen cell $N \in \mathcal{E}$ that is also a subset of X. We claim that N is contained in a finite union of definable open cells, all of which are contained in X; this claim proves the theorem.

Case 1 πN is open.

Then $N = \Gamma(f)_{\pi N}$ for a definable continuous function $f : \pi N \to R$. Since X is open, there are cells $(g, f)_{\pi N}, (f, h)_{\pi N} \in \mathcal{E}$ that are also subsets of X. Taking the open cell $(g, h)_{\pi N}$ gives the desired result.

Case 2 πN is nonopen and $N = (f, g)_{\pi N}$ for definable continuous functions f, g on $\pi N \subseteq R^{m-1}$.

Note that, since X is open and $\mathcal{E}$ is compatible with X, there are functions f', g' such that $\Gamma(f'), \Gamma(g') \in \mathcal{E}$, $\Gamma(f' \restriction_{\pi N}), \Gamma(g' \restriction_{\pi N}) \subseteq \partial X$, and $N = (f, g)_{\pi N} \subseteq (f', g')_{\pi N} = N'$. Replacing N by N' if necessary, we may assume that $\Gamma(f \restriction_{\pi N})$ and $\Gamma(g \restriction_{\pi N})$ are contained in ∂X. We proceed by constructing a neighborhood U of πN and continuous extensions $F, G : U \to R$ of f and g such that $F < G$ on U and $(F, G)_U \subseteq X$. Define

$$\mathcal{U} := \{C \in \mathcal{D} : \operatorname{cl} C \cap \pi N \neq \varnothing\} \text{ and } U := \bigcup \mathcal{U}.$$

Note that U is open, since D is a stratification, and connected, by the definition of $\mathcal{U}$.

Remark 8 For each $C \in \mathcal{U}$, the set $(C \times R) \cap X$ consists of finitely many connected components. Label these components $(f_1, g_1)_C, \ldots, (f_{n(C)}, g_{n(C)})_C$ such that $(g_k, f_{k+1})_C \subseteq X^c$ for any $1 \leq k < n(C) - 1$ (either of $f_1 = -\infty$, $g_{n(C)} = +\infty$ is possible). Since X is open, $N \subseteq \operatorname{cl}(f_{j(C)}, g_{j(C)})_C$ for exactly one $j(C) \in \{1, \ldots, n(C)\}$. For each $C \in \mathcal{U}$ we will write F_C and G_C for the global continuous extensions of $f_{j(C)}$ and $g_{j(C)}$. We denote by $\bar{f}$ and $\bar{g}$ the global continuous extensions of f and g. Note that all of these global functions belong to $\mathcal{F}$.

Proposition 9 *For any $C \in \mathcal{U}$ there exist definable functions $P, Q \in \mathcal{F}$ satisfying $P < Q$ on C, $P \restriction_{\pi N} = \bar{f}$, $Q \restriction_{\pi N} = \bar{g}$, and $(P, Q)_C \subseteq X$.*

Proof Fix $C \in \mathcal{U}$. Using the notation in Remark 8, we must have $\bar{g} \leq G_C$ on πN. Otherwise, choose $(x, y) \in (G_C, \bar{g})_{\pi N}$. By continuity, for any open set $y \in V \subseteq R$ there is an open set $x \in U_V \subseteq R^{m-1}$ so that $G_C(x) < y$ for all $x \in U_V$. Since $\pi N \subseteq \operatorname{cl} C$ we have $(U_V \times V) \cap (G_C, \bar{g})_C \neq \varnothing$. However, by definition of G_C, we then have $(U_V \times V) \cap (G_C, \bar{g})_C \cap X^c \neq \varnothing$. Since X is open, this is a contradiction. A similar argument shows that $F_C \leq \bar{f}$ on πN. Since $(F_C, G_C)_C \subseteq X$ and all functions are globally continuous, at least one of the following four combinations satisfies the last three conditions: $P = \bar{f}$ or F_C and $Q = \bar{g}$ or G_C. $P < Q$ follows from continuity and the fact that $\pi N \subseteq \operatorname{cl} C$. □

Now on $C \in \mathcal{U}$ define cells $(P_C, Q_C)_C$ where $Q_C := \min\{T \in \mathcal{F} : T \restriction_{\pi N} = \bar{g}\}$, $P_C := \max\left\{S \in \mathcal{F} : S \restriction_{\pi N} = \bar{f}\right\}$ and the maximum and minimum are taken with respect to the linear ordering of the functions in $\mathcal{F}$ over C. By Proposition 9 we have that $(P_C, Q_C)_C \subseteq X$ for each $C \in \mathcal{U}$. Now define $F, G : U \to R$ by $F(x) := P_C(x)$ and $G(x) := Q_C(x)$ for $x \in C \in \mathcal{U}$.

Proposition 10 *F and G are continuous on U.*

Proof We prove the claim for G, the proof for F being similar. Take $C_1, C_2 \in U$ with $C_2 \subseteq \operatorname{cl} C_1$. Suppose G is not continuous on $C_1 \cup C_2$. Then, on C_1, either $Q_{C_1} < Q_{C_2}$ or $Q_{C_2} < Q_{C_1}$. Since $Q_{C_2} \restriction_{\pi N} = G = Q_{C_1} \restriction_{\pi N}$, the first case contradicts the definition of Q_{C_2} and the second case contradicts the definition of Q_{C_1}. □

Finally, recall that U is open and connected and F and G are continuous on U. From Proposition 9 and the definitions of F and G we have $F < G$ on U, from which it follows that $(F, G)_U$ is connected and open and a subset of X. By the inductive assumption, U may be written as a finite union of open cells $\mathcal{O}_1, \ldots, \mathcal{O}_n$. Case 2 is completed by writing $N \subseteq (F, G)_U = \bigcup_{i=1}^{n} (F, G)_{\mathcal{O}_i}$.

Case 3 πN is nonopen and $N = \Gamma(f)$ for a definable function on $\pi N \subseteq R^{m-1}$.

The argument from Case 1 gives a cell $(g, h)_{\pi N} \subseteq X$ with $N \subset (g, h)_{\pi N} \subseteq X$. We have reduced to Case 2. □

References

[1] van den Dries, L., *Tame Topology and O-Minimal Structures*, vol. 248 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, Cambridge, 1998. Zbl 0953.03045. MR 1633348. 248

[2] Edmundo, M., "Coverings by open cells in nonlinear o-minimal expansions of groups," Preprint, 2008. 247

[3] Peterzil, Y., and S. Starchenko, "A trichotomy theorem for o-minimal structures," *Proceedings of the London Mathematical Society. Third Series*, vol. 77 (1998), pp. 481–523. Zbl 0904.03021. MR 1643405. 248

[4] Wilkie, A., "Covering definable open sets by open cells," in *Proceedings of the RAAG Summer School Lisbon 2003: O-minimal Structures*, edited by M. Edmundo, D. Richardson, and A. J. Wilkie, RAAG, Lisbon. 2005. 247

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