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Past Probabilities

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Abstract The probability that a fair coin tossed yesterday landed heads is either 0 or 1, but the probability that it *would* land heads was 0.5. In order to account for the latter type of probabilities, past probabilities, a temporal restriction operator is introduced and axiomatically characterized. It is used to construct a representation of conditional past probabilities. The logic of past probabilities turns out to be strictly weaker than the logic of standard probabilities.

1 Introduction

Yesterday I tossed a coin that I believe for good reasons to be fair. Consider the following two statements about that occurrence:

(1) The probability that the coin landed heads is 1.
(2) The probability that the coin would land heads was 0.5.

We have no difficulty in accepting both these statements as true. The first refers to the probability that a particular event *happened*. It is either 0 or 1. In contrast, the second refers to the probability that a particular event *would happen*. As was observed by Blackburn, these are two distinctly different types of probabilities. "We can say that the probability of an event was high at some time previous to its occurrence or failure to occur, and this is not to say that it is now probable that it did happen" (Blackburn [1], p. 102). I will use the term *past probabilities* for probabilities of the type exemplified in (2) and *present probabilities* for those exemplified in (1). (Both statements refer to an event in the past, but only in the second statement does the probability belong to the past.) The distinction is closely parallel to that between the subjunctive and indicative moods, as used in the philosophical analysis of conditional sentences.

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According to some authors, notably Toulmin [17] and Swinburne [16], what I have called past probabilities are past only in the sense that the probability assignment was made on (the basis of the information available on) some previous occasion, not in the sense of referring to what obtained in the world at that particular point in time. However, such a reductive account cannot do justice to all the references that we make to probabilities in the past. We often speak of past probabilities that are independent of any belief states at the point in time in question. In discussions about prehuman evolution, probabilistic statements are made about events, such as mutations and the development of a new species, that took place before anyone was present who could have beliefs about these probabilities. More mundanely, as noted by Blackburn, if we learn that a racehorse had the flu then this may change our view about the probability that it would win a race. “It is his catching flu that ruins his chances, not our discovery of it” (Blackburn [1], p. 104). That discovery corrects our estimate of the chances, rather than changing the chances.

Furthermore, the reductive account makes it difficult to account for the common understanding, expressed already by Aristotle (*Rhetoric*, 1402a, 10–15), that improbable events sometimes do happen. It makes sense to tell someone who won the first prize in the national lottery that this was a highly improbable occurrence. The retort: “No, the probability of her winning was clearly 1, since it happened” would scarcely be taken seriously. Admittedly, such an answer would be appropriate on the assumption that we live in a completely predetermined (“Laplacean”) world (Cooper [3], p. 230). However, such a world view is neither supported by science nor by everyday intuitions. Therefore, although the exploration of its philosophical implications may be useful, it should not be given preeminence in philosophical accounts of the notion of probability. In an account of probabilities that connects adequately with the probability assignments that we actually make, probabilities other than 0 and 1 must be assignable to physical events themselves, as distinct from our beliefs about these events.

However, although some probability assignments refer to the physical events themselves, it does not follow that this is true of all probability assignments. As Keynes pointed out very clearly, our ordinary concept of probability includes both probabilities that reflect our own ignorance and probabilities that reflect tendencies inherent in mind-independent matter (Keynes [11], p. 281). In the terminology of Lewis [12], our notion of probability covers both chance and credence. (Cf. also Carnap [2].) Usually, we do not distinguish between the two types of probabilities, and this for good reason: since we do not have direct access to physical chance, the probability statements that refer to it always express our subjective estimates of it. In this way, objective chance is embedded in our subjective probability assignments (Hansson [7]). Ordinary probability statements, as they are made both in science and in other walks of life, are therefore *unified* in the sense that they include, in one and the same representation, both subjective and objective probabilities.

In spite of the subjective qualities of our unified probabilities, we tend to modify them to make them reflect objective chances as closely as possible. This is primarily done by comparing them to actual frequencies. This process is applied also to single-event probabilities. For our subjective estimates of single-event probabilities to be calibrated with actual frequencies, about 5% of the events assigned the probability 0.05 should actually occur, and similarly for events with other probability assignments. (This notion of calibration has been developed in psychology for other purposes; see Lichtenstein et al. [13].) Therefore, contrary to what is often believed (Eagle [4], p. 401), our single-event probabilities are underminable in the sense that some future histories would be incompatible with them.

Another important consequence of the embeddedness of objective probabilities in subjective probabilities is that our estimates of past probabilities are influenceable by information about what happened after the events in question. My beliefs about the chance of a six when a particular die was tossed yesterday morning may legitimately be changed when I learn that a thousand tosses with that same die were made yesterday evening, showing beyond reasonable doubt that the die is biased. Humphreys ([10], p. 670) was right in pointing out that future events leave propensities of prior events unchanged, but the propensities are inaccessible to us. All we have are our estimates of them, and these estimates can be legitimately influenced by information about what happened later (Hansson [9]).

It is the purpose of the present contribution to provide a formal explication of past probabilities. In Section 2, an operator for temporal restriction of information will be introduced in order to account for the distinction between past and present probabilities. In Section 3, the temporal restriction operator is used in the construction of a representation of conditional past probabilities. All proofs of formal results are deferred to an Appendix.

2 The Temporal Restriction Operator

An important clue to the formal representation of past probabilities can be taken from David Lewis. When discussing the probability that a certain coin *would* yield heads when tossed on a particular previous occasion, we do not take into account all the information that we have today about that coin. In particular, we do not directly use the information (if we have it) about the outcome of that particular toss. However, we may use all the information that we have about the characteristics and behavior of that coin before the toss, as well as other information about particular facts before that point in time. We may also use the information we have about (probabilistic and other) laws. The information we may use is called "admissible" by Lewis ([12], p. 272). (Cf. Hall [5], p. 506 and [6], p. 99.) In order to operationalize this notion, it will be useful to introduce a *temporal restriction operator*. But first of all, a (fairly standard) logical framework is needed for the formal developments.

Definition 2.1 $\mathcal{L}$ is a language that is closed under truth-functional operations. $\top$ is an arbitrary tautology and $\perp$ an arbitrary contradiction. Cn is a consequence operator operating on $\mathcal{L}$. Cn satisfies the standard properties: inclusion ($A \subseteq \mathrm{Cn}(A)$), monotony (if $A \subseteq B$, then $\mathrm{Cn}(A) \subseteq \mathrm{Cn}(B)$), and iteration ($\mathrm{Cn}(A) = \mathrm{Cn}(\mathrm{Cn}(A))$). Furthermore, Cn is supraclassical (if p follows from A by classical truth-functional logic, then $p \in \mathrm{Cn}(A)$), and satisfies the deduction property ($q \in \mathrm{Cn}(A \cup \{p\})$ if and only if $(p \rightarrow q) \in \mathrm{Cn}(A)$). $A \vdash p$ is an alternative notation for $p \in \mathrm{Cn}(A)$. $\mathrm{Cn}(\varnothing)$ is the set of tautologies. For any set A such that $\mathrm{Cn}(A) = \mathrm{Cn}(A')$ for some finite A', $\&A$ is a sentence such that $\mathrm{Cn}(A) = \mathrm{Cn}(\{\&A\})$. The set $\mathfrak{W}$ (the possible worlds) is the set of maximal Cn-consistent subsets of $\mathcal{L}$. For any $X \subseteq \mathcal{L}$, $[X] = \{w \in \mathfrak{W} | X \subseteq \mathfrak{W}\}$. For any $x \in \mathcal{L}$, $[x]$ is an abbreviation of $[\{x\}]$. Points in time are represented by real numbers.

Sentences in $\mathcal{L}$ may contain information about what obtains at one or several points in time. They may also refer to general (timeless) properties of the world.

Definition 2.2 A *temporal restriction operator* is a function h from points in time and sentences to sentences. $h_t(p)$ is an alternative notation for $h(t, p)$.

The intended interpretation of the temporal restriction operator is as follows: Points in time are represented by (some set of) real numbers such that a larger number represents a later point in time. For every proposition-representing sentence p and every point in time t, $h_t(p)$ is a sentence representing the maximal proposition implied by p that satisfies Lewis's criterion of admissibility at t. In other words, $h_t(p)$ contains all the information that p carries about particular facts before t and about (probabilistic and other) laws. Hence, $h_t(p)$ will contain information about what happened and obtained before t that can be deduced from information in p about what happened at t or later. However, it contains no direct information about events at t or later. For an example, let t be the very first time when a certain coin was tossed. Suppose that beginning at t we have tossed the coin 1000 times, and it yielded heads only 98 of these times. Let p be a sentence that includes this information. Then $h_t(p)$ will include the information that the probability that the first toss would yield heads was about 0.1, but it will not include the information that the first toss nevertheless yielded heads.

The following seven postulates are offered as plausible properties of the temporal restriction operator.

(1) $\vdash p \rightarrow h_t(p)$. *(implication)*

Implication is a direct consequence of the intended property of h_t that it restricts the contents of a proposition, that is, removes parts of it but does not add anything.

(2) If $\vdash p \rightarrow q$, then $\vdash h_t(p) \rightarrow h_t(q)$. *(inheritance)*

According to *inheritance*, if we lose information (go from p to the logically weaker q), then this does not lead to any addition to the t-restricted part of the information.

(3) There is no infinite series $p_1, p_2, \ldots$ such that for each $k \geq 1$, $\vdash h_t(p_k) \rightarrow h_t(p_{k+1})$ and $\nvdash h_t(p_k + 1) \rightarrow h_t(p_k)$. *(groundedness).*

Groundedness ensures that our beliefs about the past are not infinitely fine-grained, that is, divisible infinitely many times into smaller and smaller parts. This condition is closely related to the property of finite-basedness in belief revision (Hansson [8]). It is included partly for technical reasons. (See the proof of Theorem 2.10 below.)

(4) $\vdash h_t(p \vee q) \leftrightarrow h_t(p) \vee h_t(q)$. *(disjunctive distribution)*

The right-to-left direction of *disjunctive distribution* follows from *inheritance*. The left-to-right direction is perhaps best understood in the equivalent form

$$\text{For all } x,\ (h_t(p) \rightarrow x)\ \&\ (h_t(q) \rightarrow x) \rightarrow (h_t(p \vee q) \rightarrow x),$$

which says that whatever follows from the temporal restriction of each of two sentences also follows from the temporal restriction of their disjunction. For an example, let p denote that a fossilized skeleton of a Kentriodon has been found in a Miocene stratum in a particular location, and q that a fossilized skeleton of an Hadrodelphis has been found in the same place. (These are two groups of prehistoric whales.) Let x denote that the place in question was a sea in the Miocene period, and let t denote the end of the Miocene era. Then we have $h_t(p) \to x$ and $h_t(q) \to x$. If the identification of the skeleton is uncertain, we may know that $p \vee q$ without knowing either p or q, but x can nevertheless be inferred.

(5) For each p there is some $\bar{p}$ such that $\vdash h_t(\bar{p}) \leftrightarrow \neg h_t(p)$. *(negatability)*

Negatability is a condition on the richness of the language. It follows immediately from the plausible condition

$$\vdash h_t(\neg h_t(p)) \leftrightarrow \neg h_t(p),$$

which says essentially that the negation of a t-reduced sentence is also t-reduced.

(6) If $t_1 \leq t_2$, then $\vdash h_{t_1}(h_{t_2}(p)) \leftrightarrow h_{t_1}(p)$. *(postreduction)*

According to *postreduction*, if we perform temporal reduction twice, first to a later and then to an earlier point in time, then this has the same effect as performing the latter reduction directly.

(7) If $t_1 \leq t_2$, then $\vdash h_{t_2}(p) \to h_{t_1}(p)$. *(successive specification)*

Finally, *successive specification* says that a restriction to a certain point in time removes all the information that is removed by a restriction to some later point in time. The information that Russell's autobiography contains about the first two decades of his life is a subset of the information that it contains about his first three decades.

Observation 2.3 Let h_t satisfy *implication*, *postreduction*, and *successive specification*. Then it satisfies

If $t_1 \leq t_2$, then $\vdash h_{t_2}(h_{t_1}(p)) \leftrightarrow h_{t_1}(p)$. *(prereduction)*

It follows from *pre-* and *postreduction* that any series of temporal restrictions such as $h_{t_a}(h_{t_b}(h_{t_c}(p)))$ is equivalent to a single restriction, namely, to that which has the earliest cutoff time.

The seven postulates are proposed as plausible properties for a temporal restriction operator that corresponds to the notion of admissibility outlined above. These postulates may also hold for other types of temporal restriction operators (such as an operator that cuts off all information obtained after a specified point in time t), but such interpretations will not be pursued here.

A semantics for the temporal restriction operator can be constructed in a possible worlds framework. The crucial constructive element that we need in addition to possible worlds is a representation of branching development. For this purpose, we can introduce, for each point in time t, an equivalence relation E_t over the set of possible worlds. It represents indistinguishability before t. The most obvious interpretation of E_t is that it reflects actual indeterminacy in physical reality. Hence, if a quantum-mechanical randomizing device will determine at time t whether a certain lamp will be lit, then there are worlds w and w' such that the lamp is lit in w but not in w', and wE_tw'. The branching structure of the world is obtained with a series of equivalence classes, as follows.

Definition 2.4 Let E and E' be two equivalence relations over the same set. Then E' is a *refinement* of E if and only if (1) for all x and y, $xE'y \to xEy$, and (2) there are at least two elements x and y such that xEy and not $xE'y$.

Definition 2.5 A *time-setter* is a function E from points in time to equivalence classes in $\mathfrak{W}$. E_t is an alternative notation for $E(t)$. A time-setter E is *weakly branching* if and only if it satisfies

$$\text{If } t_1 < t_2 \text{ then } E_{t_2} \text{ is either equal to or a refinement of } E_{t_1}.$$

It is *strictly branching* if and only if it satisfies

$$\text{If } t_1 < t_2 \text{ then } E_{t_2} \text{ is a refinement of } E_{t_1}.$$

Furthermore, a time-setter E is *finitely* weakly (strictly) branching if and only if it is weakly (strictly) branching and each E_t has a finite number of equivalence classes.

Clearly, a time-setter can only be finitely strictly branching if it operates on a finite number of points in time. To simplify the formal treatment it is useful to focus on models that have sentential representations, in the following sense.

Definition 2.6 Let $\mathfrak{A} \subseteq \mathfrak{W}$. Then a set X of sentences is a *sentential representation* of $\mathfrak{A}$ if and only if it holds for all $W \in \mathfrak{W}$ that

$$W \in \mathfrak{A} \text{ if and only if } X \subseteq W.$$

Furthermore, a sentence x is a *monosentential representation* of $\mathfrak{A}$ if and only if it holds for all $W \in \mathfrak{W}$ that

$$W \in \mathfrak{A} \text{ if and only if } x \in W.$$

A set $\mathfrak{A} \subseteq \mathfrak{W}$ is *sententially representable* if and only if it has a sentential representation. It is *monosententially representable* if and only if it has a monosentential representation.

Definition 2.7 The time-setter E is *monosententially representable* if and only if it holds for all sentences p and all points in time t that $\{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})(WE_tW' \;\&\; p \in W')\}$ is monosententially representable.

As the following observation shows, these conditions are substantial restrictions in the sense that if $\mathfrak{W}$ is infinite, then it has subsets that are not sententially representable.

Observation 2.8 Let $\mathcal{L}$ have an infinite set of logically independent sentences. Then there are subsets of $\mathfrak{W}$ that do not have a sentential representation.

If we assume monosentential representability, then we can connect the above semantics with the temporal restriction operator in a simple and straightforward way. For that purpose, the following notation will be useful.

Definition 2.9 Let E be a time-setter for $\mathfrak{W}$. Then

$$[p]_{E_t} = \{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})(WE_tW' \;\&\; p \in W')\}. \tag{1}$$

Furthermore, if E is monosententially representable, then

$$\lceil p \rceil_{E_t} = \& \bigcap [p]_{E_t}. \tag{2}$$

The simplified notation $[p]_t$ and $\lceil p \rceil_t$ is used whenever 'E' can be omitted without creating ambiguity.

$[p]_t$ is the union of the equivalence classes in E_t in which p has at least one member. Intuitively, for any sentence p and point in time t, $[p]_t$ is the set of worlds such that $W \in [p]_t$ if and only if the information contained in p concerning the status of the world up to the time t is compatible with W.

For an example, let p denote that I won £50 yesterday on a lottery ticket that I bought a week ago. Let t be the point in time when the winning ticket was drawn (presumably by a truly randomizing method). Then $[p]_t$ contains the worlds that are compatible with the information contained in p concerning the status of the world at points in time before t. This includes the information that I bought a lottery ticket, but not the information that I won. Hence, $[p]_t$ consists of worlds in which I bought the lottery ticket in question. Among these worlds there are both worlds in which I won and worlds in which I did not.

$\lceil p \rceil_t$ is a monosentential representation of $[p]_t$. It contains all the information that p contains about the status of the world up to the time t. As should be evident, $\lceil p \rceil_t$ has been constructed to be the semantical counterpart of the temporal restriction operator. The following theorem confirms the fit of the construction.

Theorem 2.10 *The following two conditions on a temporal restriction operator h are equivalent:*

(1) *h satisfies* implication, inheritance, groundednesss, disjunctive distribution, negatability, postreduction, *and* successive specification.
(2) *There is a monosententially representable and finitely weakly branching time-setter E such that $h_t(p) = \lceil p \rceil_t$ holds for all sentences p and points in time t.*

3 The Logic of Past Probabilities

We can now turn to the task of representing past probabilities in formal language. This can be done with conditional probabilities. Let $P(\ )$ and $P(\ |\ )$ be the standard monadic and dyadic probability functions.

Definition 3.1 $P(\ )$ is a probability function over $\mathcal{L}$ that satisfies the standard properties. $P(\ |\ )$ is the standard dyadic probability function based on P; that is,

$$P(p \mid q) = \frac{P(p \,\&\, q)}{P(q)}.$$

The function that provides us with conditional past probabilities will be denoted $Q(\ |\ )$. The difference can be illustrated with the coin example referred to at the beginning of Section 1. Let p be the statement that the coin I tossed yesterday yielded heads, and q the statement that the coin in question is fair. Then $P(p \mid q)$ is the probability that the coin actually landed heads (and is thus equal to either 0 or 1), whereas $Q(p \mid q)$ is the probability that the coin would land heads on the occasion in question (and is thus equal to 0.5).

In order to define Q, we need the temporal restriction operator as defined above and, in addition, a function that assigns to each event an appropriate time index for that operator.

Definition 3.2 τ is a function from $\mathcal{L}$ to the set of real numbers.

If p represents a single event that takes place at one moment, then $\tau(p)$ is the real number representing that moment. This applies for instance if p represents a possible outcome of tossing a coin on one particular occasion. If p instead represents a sequence of events, then $\tau(p)$ will represent the time when that sequence begins. Hence, if p denotes that three consecutive tosses of a fair coin all yield heads, then $\tau(p)$ will represent the time of the first of these tosses.

We can now define Q as follows.

Definition 3.3 $Q(p \mid q) = P(p \mid h_{\tau(p)}(q))$.

Before examining the properties of Q, we need to introduce an essential property of the probability function P.

Definition 3.4 A probability function P satisfies *strict coherence* if and only if

$$\text{If } \nvdash p \text{ and } \nvdash \neg p, \text{ then } 0 < P(p) < 1.$$

According to strict coherence, only a priori falsehoods can have the initial probability 0 and only a priori truths can have the initial probability 1 (Stalnaker [14]). Assuming that probabilities are updated in the standard Bayesian way when new information arrives (i.e., when we learn that r is true, then we replace $P(p)$ by $P(p \mid r)$ for all p), probabilities between 0 and 1 will tend to be gradually adjusted in the direction of actual physical frequencies, but probabilities equal to 0 or 1 will never change. Therefore, strict coherence is what we need to ensure that subjective probability tends to move in the direction of experienced frequencies in the physical world.

The following properties have been obtained for Q.

Observation 3.5 Let P satisfy *strict coherence* and let h satisfy *implication*. Let Q be based on P, τ, and h according to Definition 3.3. Then

(1) $0 \leq Q(p \mid q) \leq 1$,
(2) $Q(p \mid p) > 0$ if $p \nvdash \perp$,
(3) it does not hold in general that $Q(p \mid p) = 1$,
(4) if $p \nvdash \perp$, then $Q(\top \mid p) = 1$,
(5) if $\tau(p_1) = \tau(p_2) = \tau(p_1 \,\&\, p_2) = \tau(p_1 \vee p_2)$, then

$$Q(p_1 \vee p_2 \mid q) = Q(p_1 \mid q) + Q(p_2 \mid q) - Q(p_1 \,\&\, p_2 \mid q),$$

(6) if p_1 and p_2 are mutually exclusive, and $\tau(p_1) = \tau(p_2) = \tau(p_1 \,\&\, p_2) = \tau(p_1 \vee p_2)$, then

$$Q(p_1 \vee p_2 \mid q) = Q(p_1 \mid q) + Q(p_2 \mid q).$$

In the (unrealistic) limiting case when $h_t(p) = p$ for all p and t, P and Q will coincide. Therefore, Q is a weaker version of P; that is, the theorems that apply to Q are a subset of those that apply to the standard conditional probability function P.

Appendix: Proofs

Definition 3.6 Let h be a temporal restriction operator and t a point in time. An h_t*-atom* of $\mathfrak{W}$ is a nonempty set $\mathfrak{A} \subseteq \mathfrak{W}$ such that

(i) $\mathfrak{A} = [h_t(p)]$ for some p, and
(ii) there is no q such that $\varnothing \subset [h_t(q)] \subset \mathfrak{A}$.

Lemma 3.7 *Let $\mathfrak{A} \subseteq \mathfrak{W}$. If $\mathfrak{A}$ has a monosentential representation, then* (1) $\bigcap \mathfrak{A}$ *is finite-based and* (2) $\& \bigcap \mathfrak{A}$ *is a monosentential representation of* $\mathfrak{A}$.

Proof of Lemma 3.7

Part 1 Suppose to the contrary that $\mathfrak{A}$ has a monosentential representation and $\bigcap \mathfrak{A}$ is not finite-based. Let r be a monosentential representation of $\mathfrak{A}$. Then clearly $r \in \bigcap \mathfrak{A}$ but since $\bigcap \mathfrak{A}$ is not finite-based we have $\mathrm{Cn}(\{r\}) \subset \bigcap \mathfrak{A}$. There must then be some sentence $s \in \bigcap \mathfrak{A}$ such that $r \nvdash s$. Since $r \,\&\, \neg s$ is consistent, there must be some $W \in \mathfrak{W}$ such that $r \,\&\, \neg s \in W$. It follows from $\neg s \in W$ and $s \in \bigcap \mathfrak{A}$ that $W \notin \mathfrak{A}$. Since r is a monosentential representation of $\mathfrak{A}$ it follows from $r \in W$ that $W \in \mathfrak{A}$. This contradiction completes this part of the proof.

Part 2 Let $\mathfrak{A} \subseteq \mathfrak{W}$ have a monosentential representation. We know from Part 1 that $\& \bigcap \mathfrak{A}$ is well defined. Now suppose for reductio that $\& \bigcap \mathfrak{A}$ is not a monosentential representation of $\mathfrak{A}$. Then there must be some W such that either (i) $W \in \mathfrak{A}$ and $\& \bigcap \mathfrak{A} \notin W$ or (ii) $W \notin \mathfrak{A}$ and $\& \bigcap \mathfrak{A} \in W$. It is obvious that (i) does not hold.

For (ii), let r be a monosentential representation of $\mathfrak{A}$. Then clearly $r \in \bigcap \mathfrak{A}$, thus $\vdash \& \bigcap \mathfrak{A} \to r$. Since $\& \bigcap \mathfrak{A} \in W$ it follows that $r \in W$. However, since r is a monosentential representation of $\mathfrak{A}$ it follows from $W \notin \mathfrak{A}$ that $r \notin W$. This contradiction concludes the proof. □

Lemma 3.8 *Let $\mathfrak{A}_1, \dots \mathfrak{A}_n$ be monosententially representable subsets of $\mathfrak{W}$. Then*

$$\vdash \& \bigcap(\mathfrak{A}_1 \cup \dots \cup \mathfrak{A}_n) \leftrightarrow \left(\& \bigcap \mathfrak{A}_1\right) \vee \dots \vee \left(\& \bigcap \mathfrak{A}_n\right).$$

Proof of Lemma 3.8

$$\begin{aligned}
&\mathrm{Cn}\left(\{\& \bigcap(\mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \dots \cup \mathfrak{A}_n)\}\right) \\
&\quad = \mathrm{Cn}\left(\bigcap(\mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \dots \cup \mathfrak{A}_n)\right) \\
&\quad = \mathrm{Cn}\left(\left(\bigcap \mathfrak{A}_1\right) \cap \left(\bigcap \mathfrak{A}_2\right) \cap \dots \cap \left(\bigcap \mathfrak{A}_n\right)\right) \\
&\quad = \mathrm{Cn}\left(\& \bigcap \mathfrak{A}_1\right) \cap \mathrm{Cn}\left(\& \bigcap \mathfrak{A}_2\right) \cap \dots \cap \mathrm{Cn}\left(\& \bigcap \mathfrak{A}_n\right) \\
&\qquad \text{(monosentential representability and Lemma 3.7)} \\
&\quad = \mathrm{Cn}\left(\left(\& \bigcap \mathfrak{A}_1\right) \vee \left(\& \bigcap \mathfrak{A}_2\right) \vee \dots \vee \left(\& \bigcap \mathfrak{A}_n\right)\right).
\end{aligned}$$

□

Lemma 3.9 *Let E_1 and E_2 be equivalence relations with the same domain such that E_2 is either equal to or a refinement of E_1. Then it holds for all elements W_1 and W_2 of their range of arguments that*

$$W_1 E_1 W_2 \textit{ if and only if } (\exists V)(W_1 E_1 V E_2 W_2).$$

Proof of Lemma 3.9 (Left to right) Since E_2 is reflexive, $W_1 E_1 W_2$ yields $W_1 E_1 W_2 E_2 W_2$.

(Right to left)	$W_1 E_1 V E_2 W_2$	
	$W_1 E_1 V E_1 W_2$	(since E_2 is equal to or a refinement of E_1)
	$W_1 E_1 W_2$	(transitivity of E_1).

□

Lemma 3.10 *Let h_t satisfy* implication, inheritance, *and* postreduction. *If* $\varnothing \subset [h_t(p)] \cap [h_t(q)] \subset [h_t(p)]$, *then* $\varnothing \subset [h_t(h_t(p) \,\&\, h_t(q))] \subset [h_t(p)]$.

Proof of Lemma 3.10 Let $\varnothing \subset [h_t(p)] \cap [h_t(q)] \subset [h_t(p)]$. It follows that there is some $W \in [h_t(p)] \cap [h_t(q)]$ and also some $W' \in [h_t(p)] \setminus [h_t(q)]$.

1 It follows from $W \in [h_t(p)]$ that $h_t(p) \in W$ and similarly from $W \in [h_t(q)]$ that $h_t(q) \in W$. Thus $h_t(p) \& h_t(q) \in W$. It follows from *implication* that $h_t(h_t(p) \& h_t(q)) \in W$. Thus, $W \in [h_t(h_t(p) \& h_t(q))]$ and consequently $\varnothing \subset [h_t(h_t(p) \& h_t(q))]$.

2 It follows from *inheritance* that $\vdash h_t(h_t(p) \& h_t(q)) \rightarrow h_t(h_t(p))$ and from *postreduction* that $\vdash h_t(h_t(p)) \rightarrow h_t(p)$. It follows from this that $\vdash h_t(h_t(p) \& h_t(q)) \rightarrow h_t(p)$. Thus, $[h_t(h_t(p) \& h_t(q))] \subseteq [h_t(p)]$.

3 Next, suppose that $[h_t(p)] \subseteq [h_t(h_t(p) \& h_t(q))]$. We have $W' \in [h_t(p)]$, and it would then follow that $W' \in [h_t(h_t(p) \& h_t(q))]$.

It follows from *inheritance* that $\vdash h_t(h_t(p) \& h_t(q)) \rightarrow h_t(h_t(q))$ and from *postreduction* that $\vdash h_t(h_t(q)) \rightarrow h_t(q)$. Hence, $\vdash h_t(h_t(p) \& h_t(q)) \rightarrow h_t(q)$; thus $[h_t(h_t(p) \& h_t(q))] \subseteq [h_t(q)]$. Combining this with $W' \in [h_t(h_t(p) \& h_t(q))]$ we obtain $W' \in [h_t(q)]$, contrary to the defining assumption of W'. It follows from this contradiction that $[h_t(p)] \nsubseteq [h_t(h_t(p) \& h_t(q))]$.

4 We now have $\varnothing \subset [h_t(h_t(p) \& h_t(q))]$, $[h_t(h_t(p) \& h_t(q))] \subseteq [h_t(p)]$, and $[h_t(p)] \nsubseteq [h_t(h_t(p) \& h_t(q))]$. It follows that $\varnothing \subset [h_t(h_t(p) \& h_t(q))] \subset [h_t(p)]$, as desired. □

Lemma 3.11 *Let h_t satisfy* implication, inheritance, postreduction, *and* negatability. *If $W \in [h_t(p)]$ and there is some q such that $\varnothing \subset [h_t(q)] \subset [h_t(p)]$, then there is some r such that $W \in [h_t(r)] \subset [h_t(p)]$.*

Proof of Lemma 3.11 The case when $W \in [h_t(q)]$ is trivial. When $W \notin [h_t(q)]$, it follows from *negatability* that there is some $\bar{q}$ such that $W \in [h_t(\bar{q})]$ and $[h_t(q)] \cap [h_t(\bar{q})] = \varnothing$. It follows that $\varnothing \subset [h_t(\bar{q})] \cap [h_t(p)] \subset [h_t(p)]$, and then from Lemma 3.10 that $[h_t(h_t(p)) \& h_t(\bar{q}))] \subset [h_t(p)]$.

It follows from *implication* that $\vdash h_t(p) \& h_t(\bar{q}) \rightarrow h_t(h_t(p) \& h_t(\bar{q}))$. Thus, $[h_t(p) \& h_t(\bar{q})] \subseteq [h_t(h_t(p) \& h_t(\bar{q}))]$. We also have $W \in [h_t(p)]$ and $W \in [h_t(\bar{q})]$. Thus, $W \in [h_t(p) \& h_t(\bar{q})]$. Hence, $W \in [h_t(h_t(p) \& h_t(\bar{q}))]$. Combining these results we obtain $W \in [h_t(h_t(p) \& h_t(\bar{q}))] \subset [h_t(p)]$, as desired. □

Lemma 3.12 *Let h_t satisfy* inheritance *and* postreduction. *If $[h_t(r)] \subset [h_t(p)]$, then there is some q such that $\vdash q \rightarrow p$ and $[h_t(q)] \subset [h_t(p)]$.*

Proof of Lemma 3.12 We are going to let $r \& h_t(p)$ fill the function of q in the lemma. From *inheritance* we have $\vdash h_t(r \& h_t(p)) \rightarrow h_t(h_t(p))$ and then from *postreduction*, $\vdash h_t(r \& h_t(p)) \rightarrow h_t(p)$, from which it follows that $[h_t(r \& h_t(p))] \subseteq [h_t(p)]$.

Next, suppose that $[h_t(p)] \subseteq [h_t(r \& h_t(p))]$. Due to *inheritance* we have $\vdash h_t(r \& h_t(p)) \rightarrow h_t(r)$ and thus $[h_t(r \& h_t(p))] \subseteq [h_t(r)]$ so that we obtain $[h_t(p)] \subseteq [h_t(r)]$, contrary to the conditions. We may conclude that $[h_t(p)] \nsubseteq [h_t(r \& h_t(p))]$. Thus we have $[h_t(r \& h_t(p))] \subset [h_t(p)]$, as desired. □

Lemma 3.13 *Let h_t satisfy* inheritance, postreduction, *and* negatability, *and let $[h_t(q)]$ contain no p-world. Then $[h_t(q)] \cap [h_t(p)] = \varnothing$.*

Proof of Lemma 3.13

$[h_t(q)] \cap [p] = \varnothing$
$\vdash p \rightarrow \neg h_t(q)$
$\vdash p \rightarrow h_t(\bar{q})$ (*negatability*)
$\vdash h_t(p) \rightarrow h_t(h_t(\bar{q}))$ (*inheritance*)
$\vdash h_t(p) \rightarrow h_t(\bar{q})$ (*postreduction*)
$\vdash h_t(p) \rightarrow \neg h_t(q)$
$[h_t(q)] \cap [h_t(p)] = \varnothing$

□

Lemma 3.14 *Let h_t satisfy* implication*,* inheritance*, and* postreduction*, and let $[h_t(q)]$ be an h_t-atom that contains at least one p-world. Then $[h_t(q)] \subseteq [h_t(p)]$.*

Proof of Lemma 3.14 Let $[h_t(q)]$ be an h_t-atom and let $p \in W \in [h_t(q)]$. It follows from *implication* that $h_t(p) \in W$. Thus, $W \in [h_t(p)]$. Hence, $[h_t(q)] \cap [h_t(p)] \neq \varnothing$. Now suppose that $[h_t(q)] \cap [h_t(p)] \subset [h_t(q)]$. It then follows from Lemma 3.10 that $[h_t(q)]$ is not an atom, contrary to the assumptions. Hence, $[h_t(q)] \cap [h_t(p)] = [h_t(q)]$; that is, $[h_t(q)] \subseteq [h_t(p)]$. □

Proof of Observation 2.3 Let $t_1 \leq t_2$. $\vdash h_{t_1}(p) \rightarrow h_{t_2}(h_{t_1}(p))$ follows from *implication*. For the other direction, let $h_{t_2}(h_{t_1}(p))$. It follows from *successive specification* that $h_{t_1}(h_{t_1}(p))$ and then from *postreduction* that $h_{t_1}(p)$. □

Proof of Observation 2.8 Suppose to the contrary that all subsets of $\mathfrak{W}$ have a sentential representation. Then it follows that for each $W \in \mathfrak{W}$, the set $\mathfrak{W} \setminus W$ has a sentential representation. Thus, for each $W \in \mathfrak{W}$ there is some set X_W such that $X_W \subseteq \cap(\mathfrak{W} \setminus W)$ and $X_W \not\subseteq W$. It follows that there is some sentence $x_W \in X_W$ such that $x_W \in \cap(\mathfrak{W} \setminus W)$ and $x_W \notin W$. Hence, for each $W \in \mathfrak{W}$ there is a sentence $\neg x_W$ that is an element of W but not of any other element of $\mathfrak{W}$. But this is impossible due to cardinality considerations. □

Proof of Theorem 2.10

From Construction to Postulates

Implication Due to the reflexivity of E_t, all p-worlds are in $[p]_t$, that is, $[p] \subseteq [p]_t$. Due to the monosentential representability of E_t and Lemma 3.7, $\lceil p \rceil_t = \& \bigcap [p]_t$ is a monosentential representation of $[p]_t$, that is, $[p]_t = [\lceil p \rceil_t]$. Thus $[p] \subseteq [\lceil p \rceil_t]$, from which it follows that $\vdash p \rightarrow \lceil p \rceil_t$.

Inheritance

$\vdash p \rightarrow q$
$\{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})(W E_t W' \;\&\; p \in W'\}$
$\quad \subseteq \{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})(W E_t W' \;\&\; q \in W'\}$
$[p]_t \subseteq [q]_t$
$\bigcap [q]_t \subseteq \bigcap [p]_t$
$\vdash \& \bigcap [p]_t \rightarrow \& \bigcap [q]_t$ (sentential representability and Lemma 3.7)
$\vdash \lceil p \rceil_t \rightarrow \lceil q \rceil_t.$

Groundedness Since E_t has a finite number of equivalence classes, it follows from the definition of $[\ ]_t$ that there is only a finite number of sets expressible as $[p]_t$ for some p. Consequently, there is only a finite number of nonequivalent sentences expressible as $\lceil p \rceil_t$ for some p. The rest is obvious.

Disjunctive distribution We have

$$\begin{aligned}\lceil p \vee q \rceil_t &= \& \bigcap [p \vee q]_t \text{ (monosentential representability and Lemma 3.7.)}\\ &= \& \bigcap \{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})((W E_t W') \;\&\; (p \vee q \in W'))\}\\ &= \& \bigcap \{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})((W E_t W') \;\&\; (p \in W' \vee q \in W'))\}\\ &= \& \bigcap \{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})((W E_t W') \;\&\; (p \in W'))\}\\ &\quad \cup \{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})((W E_t W') \;\&\; (q \in W'))\}.\end{aligned}$$

Since the equivalence classes of E_t are finite in number and all monosententially representable, there are sets of sentences $\{a_1, \dots, a_k\}$ and $\{b_1, \dots, b_m\}$ such that

$$\{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})((W E_t W') \;\&\; (p \in W'))\} = [a_1] \cup \dots [a_k] \text{ and}$$
$$\{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})((W E_t W') \;\&\; (q \in W'))\} = [b_1] \cup \dots [b_m].$$

We can therefore continue,

$$\begin{aligned}&\& \bigcap (\{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})((W E_t W') \;\&\; (p \in W'))\} \cup\\ &\quad \{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})((W E_t W') \;\&\; (q \in W'))\})\\ &= \& \bigcap ([a_1] \cup \cdots [a_k] \cup [b_1] \cup \cdots [b_m])\\ &= \& \bigcap ([a_1 \vee \cdots a_k \vee b_1 \vee \cdots b_m])\\ &= \& \bigcap ([a_1 \vee \cdots a_k]) \vee \& \bigcap ([b_1 \vee \cdots b_m])\\ &= \& \bigcap ([a_1] \cup \cdots [a_k]) \vee \& \bigcap ([b_1] \cup \cdots [b_m])\\ &= \& \bigcap \{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})((W E_t W') \;\&\; (p \in W'))\} \vee\\ &\quad \& \bigcap \{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})((W E_t W') \;\&\; (q \in W'))\}\\ &= \& \bigcap [p]_t \vee \& \bigcap [q]_t\\ &= \lceil p \rceil_t \vee \lceil q \rceil_t.\end{aligned}$$

Negatability Let $\mathfrak{A}_1, \dots \mathfrak{A}_m$ be the equivalence classes by E_t that contain at least one p-world, and $\mathfrak{B}_1, \dots \mathfrak{B}_n$ those that contain no p-world. Let $\bar{p} = (\& \bigcap \mathfrak{B}_1) \vee \dots \vee (\& \bigcap \mathfrak{B}_n))$. Then

$$\begin{aligned}&\neg h_t(p) \leftrightarrow \neg \lceil p \rceil_t\\ &\leftrightarrow \neg \& \bigcap \{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})(W E_t W' \;\&\; p \in W')\}\\ &\leftrightarrow \neg \& \bigcap (\mathfrak{A}_1 \cup \cdots \cup \mathfrak{A}_m)\\ &\leftrightarrow \& \bigcap (\mathfrak{B}_1 \cup \cdots \cup \mathfrak{B}_n).\end{aligned}$$

(Monosentential representability. Due to Lemma 3.7, if $\mathfrak{C} \subseteq \mathfrak{W}$ has a monosentential representation, then $\& \bigcap \mathfrak{C}$ is a monosentential representation of $\mathfrak{C}$, and thus $\neg\& \bigcap \mathfrak{C}$ a monosentential representation of $\mathfrak{W} \setminus \mathfrak{C}$.)

$$\leftrightarrow (\& \bigcap \mathfrak{B}_1) \vee \cdots \vee (\& \bigcap \mathfrak{B}_n) \quad \text{(Lemma 3.8)}$$
$$\leftrightarrow \& \bigcap(\{W \in \mathfrak{W} \mid \& \bigcap \mathfrak{B}_1 \in W\}) \vee \cdots \vee \& \bigcap(\{W \in \mathfrak{W} \mid \& \bigcap \mathfrak{B}_n \in W\})$$
$$\leftrightarrow \& \bigcap(\{W \in \mathfrak{W} \mid W \in \mathfrak{B}_1\}) \vee \cdots \vee \& \bigcap(\{W \in \mathfrak{W} \mid W \in \mathfrak{B}_n \in W\})$$
$$\leftrightarrow \& \bigcap(\{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})(W E_t W' \ \& \ W' \in \mathfrak{B}_1)\}) \vee \cdots \vee$$
$$\& \bigcap(\{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})(W E_t W' \ \& \ W' \in \mathfrak{B}_n)\})$$
(Since $\mathfrak{B}_1, \ldots, \mathfrak{B}_n$ are equivalence classes by E_t.)
$$\leftrightarrow \& \bigcap(\{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})(W E_t W' \ \& \ \& \bigcap \mathfrak{B}_1 \in W')\}) \vee \cdots \vee$$
$$\& \bigcap(\{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})(W E_t W' \ \& \ \& \bigcap \mathfrak{B}_n \in W')\})$$
$$\leftrightarrow \& \bigcap\{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})(W E_t W' \& ((\& \bigcap \mathfrak{B}_1) \vee \cdots \vee (\& \bigcap \mathfrak{B}_n) \in W'))\}$$
$$\leftrightarrow \& \bigcap\{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})(W E_t W' \ \& \ \bar{p} \in W')\}$$
$$\leftrightarrow \lceil \bar{p} \rceil_t$$
$$\leftrightarrow h_t(\bar{p}).$$

Postreduction

$$\lceil \lceil p \rceil_{t_2} \rceil_{t_1}$$
$$\leftrightarrow \& \bigcap\{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})(W E_{t_1} W' \ \& \ \lceil p \rceil_{t_2} \in W')\}$$
$$\leftrightarrow \& \bigcap\{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})(W E_{t_1} W' \ \&$$
$$\& \bigcap\{V \in \mathfrak{W} \mid (\exists V' \in \mathfrak{W})(V E_{t_2} V' \ \& \ p \in V')\} \in W')\}$$
$$\leftrightarrow \& \bigcap\{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})(W E_{t_1} W' \ \&$$
$$W' \in \{V \in \mathfrak{W} \mid (\exists V' \in \mathfrak{W})(V E_{t_2} V' \ \& \ p \in V')\}\} \quad \text{(Lemma 3.7.)}$$
$$\leftrightarrow \& \bigcap\{W \in \mathfrak{W} \mid (\exists W', V' \in \mathfrak{W})(W E_{t_1} W' E_{t_2} V' \ \& \ p \in V')\}$$
$$\leftrightarrow \& \bigcap\{W \in \mathfrak{W} \mid (\exists V' \in \mathfrak{W})(W E_{t_1} V' \ \& \ p \in V')\} \quad \text{(Lemma 3.9)}$$
$$\leftrightarrow \& \bigcap[p]_{t_1}$$
$$\leftrightarrow \lceil p \rceil_{t_1}.$$

Successive specification

$$t_1 \leq t_2$$
$$[p]_{t_2} \subseteq [p]_{t_1} \quad \text{(weak branching)}$$
$$\bigcap[p]_{t_1} \subseteq \bigcap[p]_{t_2}$$
$$\vdash \& \bigcap[p]_{t_2} \to \& \bigcap[p]_{t_1} \quad \text{(monosentential representability and Lemma 3.7)}$$
$$\vdash \lceil p \rceil_{t_2} \to \lceil p \rceil_{t_1}.$$

From Postulates to Construction Let E be the time-setter so constructed that for each t, E_t is the relation such that for all $W, W' \in \mathfrak{W}$,

$W E_t W'$ if and only if W and W' are elements of the same h_t-atom.

We need to show that for each t, E_t is an equivalence relation with the atoms as equivalence classes. According to a standard result (Stoll [15], p. 34), this is done by showing that

(1) all elements of $\mathfrak{W}$ are elements of one of its atoms under h_t, and

(2) no element of $\mathfrak{W}$ belongs to more than one atom under h_t.

Furthermore, we have to show the following:

(3) E_t has a finite number of equivalence classes,
(4) E_t is monosententially representable,
(5) E_t is weakly branching,
(6) $\vdash h_t(p) \leftrightarrow \lceil p \rceil_t$ for all p and t.

Part 1 Let $W \in \mathfrak{W}$. For each $p \in W$ we have $h_t(p) \in W$ due to *implication*. It follows that $W \in [h_t(p)]$. In order to show that there is some $p \in W$ for which there is no q with $\varnothing \subset [h_t(q)] \subset [h_t(p)]$, suppose to the contrary that this is not the case. Then it follows from Lemma 3.11 that there is an infinite series $q_1, q_2, \ldots$ such that $W \in [h_t(q_k)]$ for each q_k and that $[h_t(p)] \supset [h_t(q_1)] \supset [h_t(q_2)] \ldots$. It follows from $[h_t(q_1)] \subset [h_t(p)]$ that $\vdash h_t(q_1) \to h_t(p)$ and $\nvdash h_t(p) \to h_t(q_1)$, and similarly for the rest of the series $p, q_1, q_2 \ldots$. This is impossible due to *groundedness*.

We can conclude from this contradiction that for each $W \in \mathfrak{W}$ there is some $p \in W$ such that $W \in [h_t(p)]$ and that there is no q with $\varnothing \subset [h_t(q)] \subset [h_t(p)]$; that is, W is an element of one of the atoms.

Part 2 Let $W \in [h_t(p)] \cap [h_t(q)]$. It follows from Lemma 3.10 that $[h_t(p)]$ is not an atom.

Part 3 Suppose to the contrary that for some t, E_t has an infinite number of atoms (equivalence classes). It follows from Part 2 that there is then an infinite set of mutually exclusive atoms, $[h_t(p_1)], [h_t(p_2)] \ldots$. Due to *disjunctive distribution* there is then an infinite series $[h_t(p_1)] \subset [h_t(p_1 \vee p_2)] \ldots \subset [h_t(p_1 \vee p_2 \cdots \vee p_k)] \ldots$, contrary to *groundedness*.

Part 4 It follows directly from the construction for this part of the proof that each atom, that is, each equivalence class by E_t, has a monosentential representation. Since the number of such equivalence classes if finite (due to Part 3) it follows from *disjunctive distribution* that all subsets of $\mathfrak{W}$ that can be formed as a union of such equivalence classes have a monosentential representation. According to Definition 2.7 this is what it takes for E to be monosententially representable.

Part 5 Let $t_1 \leq t_2$. We have to show that if $W E_{t_2} W'$ then $W E_{t_1} W'$. Let $W E_{t_2} W'$. Then there is some p such that $W, W' \in [h_{t_2}(p)]$ and that $[h_{t_2}(q)] \not\subset [h_{t_2}(p)]$ for all q. It follows from *successive specification* that $[h_{t_2}(p)] \subseteq [h_{t_1}(p)]$. Thus $W, W' \in [h_{t_1}(p)]$. It remains to be shown that $[h_{t_1}(p)]$ is an atom. For that purpose, suppose to the contrary that there is some r such that $[h_{t_1}(r)] \subset [h_{t_1}(p)]$.

It follows from $[h_{t_1}(r)] \subseteq [h_{t_1}(p)]$ and Lemma 3.12 that there is a sentence q such that $\vdash q \to p$ and $[h_{t_1}(q)] \subset [h_{t_1}(p)]$. It follows from $\vdash q \to p$ and *inheritance* that $\vdash h_{t_2}(q) \to h_{t_2}(p)$. Suppose that also $\vdash h_{t_2}(p) \to h_{t_2}(q)$. Then we have $\vdash h_{t_2}(p) \leftrightarrow h_{t_2}(q)$, and due to *inheritance* we then have $\vdash h_{t_1}(h_{t_2}(p)) \leftrightarrow h_{t_1}(h_{t_2}(q))$. *Postreduction* yields $\vdash h_{t_1}(p) \leftrightarrow h_{t_1}(q)$ so that $[h_{t_1}(p)] = [h_{t_1}(q)]$, contrary to what we have just shown. Hence, $\nvdash h_{t_2}(p) \to h_{t_2}(q)$. From this and $\vdash h_{t_2}(q) \to h_{t_2}(p)$ it follows that $[h_{t_2}(q)] \subset [h_{t_2}(p)]$, contrary to the conditions. With this we have disproved the supposition that there is some r such that $[h_{t_1}(r)] \subset [h_{t_1}(p)]$. Thus $[h_{t_1}(p)]$ is an atom, as desired.

Part 6 Let $[h_t(p_1)], \ldots [h_t(p_n)]$ be the h_t-atoms that contain at least one p-world. We then have

$$\begin{aligned}
&\lceil p \rceil_t \leftrightarrow \& \bigcap [p]_t \\
&\leftrightarrow \& \bigcap \{W \in \mathfrak{W} \mid (\exists W' \in \mathfrak{W})(W E_t W' \;\&\; p \in W'\} \\
&\leftrightarrow \& \bigcap ([h_t(p_1)] \cup \cdots \cup [h_t(p_n)]) \\
&\leftrightarrow (\& \bigcap [h_t(p_1)]) \vee \cdots \vee (\& \bigcap [h_t(p_n)]) \text{ (Lemma 3.8)} \\
&\leftrightarrow h_t(p_1) \vee \cdots \vee h_t(p_n).
\end{aligned}$$

We now need to show that $\vdash h_t(p) \leftrightarrow h_t(p_1) \vee \cdots \vee h_t(p_n)$. It follows from Part 1 of the present direction of the proof that all elements of $\mathfrak{W}$ are elements of an h_t-atom. From this and Lemmas 3.13 and 3.14 it follows that $[h_t(p)] = [h_t(p_1)] \cup \cdots \cup [h_t(p_n)]$, thus $[h_t(p)] = [h_t(p_1) \vee \cdots \vee h_t(p_n)]$, thus $\vdash h_t(p) \leftrightarrow h_t(p_1) \vee \cdots \vee h_t(p_n)$. □

Proof of Observation 3.5

Part 1 Directly from the properties of the probability function P.

Part 2 We have

$$Q(p \mid p) = P(p \mid h_{\tau(p)}(p)) = \frac{P(p \;\&\; h_{\tau(p)}(p))}{P(h_{\tau(p)}(p))}.$$

It follows from *implication* that $\vdash p \rightarrow h_{\tau(p)}(p)$. From this and $p \nvdash \perp$ we can conclude that $h_{\tau(p)}(p) \nvdash \perp$. *Strict coherence* yields $P(h_{\tau(p)}(p)) \neq 0$. Furthermore, *implication* yields $\vdash p \leftrightarrow p \;\&\; h_{\tau(p)}(p)$, thus $p \;\&\; h_{\tau(p)}(p) \nvdash \perp$. Thus due to *strict coherence* $P(p \;\&\; h_{\tau(p)}(p)) \neq 0$. The rest is obvious.

Part 3 Let p be such that $P(p) < P(h_{\tau(p)}(p))$. Then due to *implication* we have $P(p \;\&\; h_{\tau(p)}(p)) < P(h_{\tau(p)}(p))$. Since

$$Q(p \mid p) = P(p \mid h_{\tau(p)}(p)) = \frac{P(p \;\&\; h_{\tau(p)}(p))}{P(h_{\tau(p)}(p))}$$

it follows that $Q(p \mid p) < 1$.

Part 4 We have

$$Q(\top \mid p) = P(\top \mid h_{\tau(\top)}(p)) = \frac{P(\top \;\&\; h_{\tau(\top)}(p))}{P(h_{\tau(\top)}(p))} = \frac{P(h_{\tau(\top)}(p))}{P(h_{\tau(\top)}(p))}.$$

It follows from *implication* that $\vdash p \rightarrow h_{\tau(\top)}(p)$. From this and $p \nvdash \perp$ it follows that $h_{\tau(\top)}(p) \nvdash \perp$. *Strict coherence* yields $P(h_{\tau(\top)}(p)) \neq 0$. Hence,

$$\frac{P(h_{\tau(\top)}(p))}{P(h_{\tau(\top)}(p))} = 1.$$

Part 5

$$\begin{aligned}
Q(p_1 \vee p_2 \mid q) &= P(p_1 \vee p_2 \mid h_{\tau(p_1)}(q)) \\
&= \frac{P((p_1 \vee p_2) \,\&\, h_{\tau(p_1)}(q))}{P(h_{\tau(p_1)}(q))} \\
&= \frac{P((p_1 \,\&\, h_{\tau(p_1)}(q)) \vee (p_2 \,\&\, h_{\tau(p_1)}(q)))}{P(h_{\tau(p_1)}(q))} \\
&= \frac{P((p_1 \,\&\, h_{\tau(p_1)}(q)) + P(p_2 \,\&\, h_{\tau(p_1)}(q)) - P(p_1 \,\&\, p_2 \,\&\, h_{\tau(p_1)}(q))}{P(h_{\tau(p_1)}(q))} \\
&= P(p_1 \mid h_{\tau(p_1)}(q)) + P(p_2 \mid h_{\tau(p_1)}(q)) - P(p_1 \,\&\, p_2 \mid h_{\tau(p_1)}(q)) \\
&= Q(p_1 \mid q) + Q(p_2 \mid q) - Q(p_1 \,\&\, p_2 \mid q).
\end{aligned}$$

Part 6 Since p_1 and p_2 are mutually exclusive, so are $p_1 \,\&\, h_{\tau(p_1)}(q))$ and $p_2 \,\&\, h_{\tau(p_1)}(q))$. Hence, $Q(p_1 \,\&\, p_2 \mid q) = P(p_1 \,\&\, p_2 \mid h_{\tau(p_1)}(q)) = 0$, which we can insert into the result from Part 5. □

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