Notre Dame Journal of Formal Logic
Volume 51, Number 2, 2010

The Consistency Strength of $\mathbf{MP}_{\mathrm{CCC}}(\mathbb{R})$

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Abstract The Maximality Principle $\mathbf{MP}_{\mathrm{CCC}}$ is a scheme which states that if a sentence of the language of **ZFC** is true in some CCC-forcing extension $V^{\mathbb{P}}$, and remains true in any further CCC-forcing extension of $V^{\mathbb{P}}$, then it is true in all CCC-forcing extensions of V, including V itself. A parameterized form of this principle, $\mathbf{MP}_{\mathrm{CCC}}(\mathbb{R})$, makes this assertion for formulas taking real parameters. In this paper, we show that $\mathbf{MP}_{\mathrm{CCC}}(\mathbb{R})$ has the same consistency strength as **ZFC**, solving an open problem of Hamkins. We extend this result further to parameter sets larger than $\mathbb{R}$.

1 Introduction

The *Maximality Principle*, **MP**, is a scheme over all sentences of **ZFC**. It states that if a sentence of **ZFC** is true in some forcing extension $V^{\mathbb{P}}$ of V, and remains true in any subsequent forcing extension of $V^{\mathbb{P}}$, then it is true in any forcing extension of V. Equivalently (see [2]), the principle states that if a sentence of **ZFC** is true in some forcing extension $V^{\mathbb{P}}$ of V, and remains true in any subsequent forcing extension of $V^{\mathbb{P}}$, then it is true in V itself. This principle and modified versions of it were discussed by Hamkins in [2]. If we let Γ be any class of posets, then $\mathbf{MP}_{\Gamma}$ makes a similar assertion, but only allowing forcing with posets in Γ. Thus, the class of CCC forcing notions produces the corresponding maximality principle $\mathbf{MP}_{\mathrm{CCC}}$. The principle **MP** and its variations $\mathbf{MP}_{\Gamma}$ can be further modified to include formulas which take parameters from some set X; these schemes are denoted as $\mathbf{MP}(X)$ and $\mathbf{MP}_{\Gamma}(X)$ respectively. So the class of CCC forcing notions, regarding formulas which take real parameters, produces the corresponding maximality principle $\mathbf{MP}_{\mathrm{CCC}}(\mathbb{R})$.

Allowing parameters may add strength to such a principle since the allowable forcing notions may produce models which enlarge the parameter set. Indeed, in [2] Hamkins showed that the maximality principle **MP** is equiconsistent with **ZFC**, while $\mathbf{MP}(\mathbb{R})$, the maximality principle with real parameters, has consistency

Received June 21, 2009; accepted September 6, 2009; printed May 20, 2010
2010 Mathematics Subject Classification: Primary, 03E35, 03E40
Keywords: forcing, forcing axioms, modal logic
 10.1215/00294527-2010-011

strength strictly greater than ZFC alone. Since he also showed that $\mathrm{MP_{CCC}}$ is also equiconsistent with ZFC, a natural expectation is that $\mathrm{MP_{CCC}}(\mathbb{R})$ will have a greater consistency strength than ZFC alone. The parameter set is certainly enlarged, since CCC-forcing adds new reals. In this paper, we show that this intuition is false and that $\mathrm{MP_{CCC}}(\mathbb{R})$ has the same consistency strength as ZFC alone, solving an open problem from [2].

We will use terminology from the language of modal logic by regarding models of ZFC as possible worlds in a Kripke model in which world B is accessible from world A if B is a forcing extension of A. This leads to natural set-theoretic interpretations of the modal concepts of possibility and necessity. A statement φ is *possible* or *forceable* ($\Diamond\varphi$ in the notation of modal logic) if it is true in some forcing extension and *necessary* (denoted by $\Box\varphi$) if it is true in every forcing extension. A sentence is *possibly necessary*, or *forceably necessary*, if it is true in some forcing extension and remains true in any subsequent forcing extension. MP becomes a scheme, the collection of all instances of the statement "$\Diamond\Box\varphi$ implies $\Box\varphi$" (or its **S4** equivalent, "$\Diamond\Box\varphi$ implies φ")[1] where φ ranges over all sentences in the language of ZFC.

If Γ is a class of forcing notions, an accessibility relation can be expressed between models M_1 and M_2 by saying that M_2 is a Γ*-forcing extension* of M_1 (that is, that $M_2 = M_1[G]$ where G is M_1-generic over some $\mathbb{P}$ where $\mathbb{P} \in \Gamma$). A formula φ is Γ*-necessary* (denoted by $\Box_\Gamma\varphi$) when it is true in all Γ-forcing extensions. So a formula is CCC*-necessary* if it is true in all CCC-forcing extensions. A formula φ is Γ*-forceable* (denoted by $\Diamond_\Gamma\varphi$) when it is true in some Γ-forcing extension. (Note that, as in other interpretations of possibility and necessity, that $\Diamond$ and $\Box$ are dual to each other—$\Diamond\varphi$ can be defined as $\neg\Box\neg\varphi$.) A formula φ is Γ*-possibly necessary* or Γ*-forceably necessary* (denoted by $\Diamond_\Gamma\Box_\Gamma\varphi$) if it is Γ-forceable that φ is Γ-necessary. The determination as to whether a forcing notion $\mathbb{P}$ in a Γ-forcing extension is itself in Γ is made *de dicto*—the formula defining the class is interpreted in the model of ZFC in which $\mathbb{P}$ will be forced. In terms of the symbols just introduced, the principle MP_Γ can be expressed as the scheme $\Diamond_\Gamma\Box_\Gamma\varphi \implies \Box_\Gamma\varphi$ where φ can be any statement in the language of ZFC.

The rest of this paper is organized as follows. Section 2 addresses the technical requirement of providing elementary submodels of V in which a uniform definition of the forcing relation can exist. Section 3 contains the main results of the paper.[2]

2 Elementary Submodels of V

The main results of this paper require definitions of forcing iterations. The successor stages force sentences which are particular instances of a maximality principle for which we seek a model. The full principle is then satisfied in the iterated extension, hence consistent, because each instance has been handled at some stage. But such a partial order at a successor stage uses the forcing relation in its definition. Unless the ground model at this stage is a set, there is no uniform definition of the forcing relation $p \Vdash \varphi$, as this would provide a definable truth predicate.

To provide set models of ZFC at successor stages, we employ the strategy of [2] of using an initial segment of the universe as an elementary submodel of it.[3] In particular, we generalize Lemma 2.5 of [2]. We first add a constant symbol δ to the language of ZFC, intended to stand for some ordinal. The following arguments will take place in this expanded language, with a corresponding structure to interpret this constant.

Let '$V_\delta \prec V$' stand for the scheme that asserts, of any formula φ with a parameter x, that

$$\text{for every } x \in V_\delta,\ V_\delta \models \varphi[x] \text{ if and only if } \varphi(x).$$

Lemma 2.1 *If T is any theory containing* ZFC *as a subtheory, then* $\mathrm{Con}(T)$ *if and only if* $\mathrm{Con}(T + V_\delta \prec V)$.

Proof To prove the nontrivial direction, let M be a model for T. I will show that, with a suitable interpretation of δ, M is a model for any finite collection of formulas in $T + V_\delta \prec V$, which is therefore consistent. The conclusion then follows by compactness.

Let Ψ^* be any finite collection of instances of the scheme $V_\delta \prec V$. Let Ψ be the collection of formulas $\psi(x)$, in the language of T, for which there is an instance in Ψ^* of the form '$\forall x \in V_\delta\ V_\delta \models \psi[x]$ if and only if $\psi(x)$'. We can write $\Psi^* = \{$"$\forall x \in V_\delta\ V_\delta \models \psi[x]$ if and only if $\psi(x)$"$| \psi \in \Psi\}$.

Ψ is finite, so by Lévy reflection there is an initial segment M_γ of M such that for all $\psi \in \Psi$, and for all $x \in M_\gamma$, if $M_\gamma \models \psi[x]$, then $\psi(x)$. So, interpreting δ as γ, $M \models \Psi^*$ (so Ψ^* is consistent). As Ψ^* was an arbitrary finite collection of formulas from $V_\delta \prec V$, and since M also satisfies any finite fragment of T (being a model of T), the entire theory $T + V_\delta \prec V$ is therefore consistent. □

Applying Lemma 2.1 where T is ZFC, we can find a model M' for ZFC $+ V_\delta \prec V$ given the existence of a model M for ZFC alone. In fact, a sharper observation can be made which relates these models, namely, it can be arranged that $M \prec M'$, by "reproving" the compactness theorem, that is, by building M' as an ultraproduct.

We first introduce a notation for an idea that will recur throughout the section. Let T be any theory containing ZFC as a subtheory. Let $I = \{\Phi \subseteq T | \Phi \text{ is finite}\}$ be the set of finite collections of formulas in the theory T. For each Φ in I, let $V_\delta \prec_\Phi V$ denote the collection of statements $\{$"$\forall x \in V_\delta (V_\delta \models \varphi[x]$ if and only if $\varphi(x))$"$| \varphi \in \Phi\}$, that is, those instances of $V_\delta \prec V$ only mentioning those formulas φ in Φ. Notice that any finite subcollection of the scheme $V_\delta \prec V$ can be so represented.

Lemma 2.2 *Let T be any theory containing* ZFC *as a subtheory. Suppose $M \models T$. Then there is $M' \models T + V_\delta \prec V$ such that $M \prec M'$.*

Proof Let M be a model for theory T, and let I be the set of finite collections of formulas as defined above. By the Lévy Reflection Theorem, for each Φ in I, there is a $\delta = \delta_\Phi$ for which $V_\delta \prec_\Phi V$ holds in M. So the expansion of the model M to $\langle M, \delta_\Phi \rangle$ is a model of $T + V_\delta \prec_\Phi V$. Denote this expanded model by M_Φ. (Notice that if $\Phi \subseteq \Psi$, then $M_\Psi \models V_\delta \prec_\Phi V$.) Now construct an ultrafilter on I as follows: For each $\Phi \in I$, define $d_\Phi = \{\Psi \in I | \Phi \subseteq \Psi\}$, the set of finite collections of formulas containing Φ as a subcollection. Then $D_I = \{d_\Phi | \Phi \in I\}$ is easily seen to be a filter on I. And by Zorn's Lemma, there is an ultrafilter $U_I \supseteq D_I$ on I. We can now take the ultraproduct $\langle M', \delta \rangle = \prod M_\Phi / U_I$. Then $\langle M', \delta \rangle$ is a model of $V_\delta \prec_\Phi V$, for any Φ in I, by Łos's Theorem applied to the set $\{\Psi \in I | M_\Psi \models V_\delta \prec_\Phi V\} \supseteq d_\Phi \in U_I$.

Since $\langle M', \delta \rangle$ is a model of any finite subcollection of $V_\delta \prec V$ (where δ is the element of the ultraproduct which represents the equivalence class of the mapping I to M via $\Phi \mapsto \delta_\Phi$), it must satisfy the entire scheme $V_\delta \prec V$. Finally, the reduct M' of $\langle M', \delta \rangle$ is simply the ultrapower of the model M over the ultrafilter U_I, so $M' \models T$ and $M \prec M'$. □

The following lemma says that $V_\delta \prec V$ persists over forcing extensions when forcing with "small" forcing notions, that is, those contained in V_δ. (The condition $\mathbb{P} \in V_\delta$ precludes, say, collapsing δ to ω, which would destroy the scheme $V_\delta \prec V$.) In the following results, $V_\delta[G] \prec V[G]$ will mean the obvious thing, namely, that $V[G] \models V_\delta \prec V$, where V_δ is interpreted as $V_\delta[G]$ in $V[G]$. The expression $V_\delta[G]$ is unambiguous, since in our usage, G is always generic over small forcing, in which case $(V_\delta)[G] = (V[G])_\delta$.

Lemma 2.3 *Let $V_\delta \prec V$, let $\mathbb{P} \in V_\delta$ be a notion of forcing, and let G be V-generic over $\mathbb{P}$. Then $V_\delta[G] \prec V[G]$.*

Proof Suppose $x \in V_\delta[G]$ such that $V_\delta[G] \models \varphi(x)$. Then there is a condition $p \in G \subseteq \mathbb{P}$ such that $V_\delta \models p \Vdash \varphi(\dot{x})$, so by elementarity $V \models p \Vdash \varphi(\dot{x})$; hence $V[G] \models \varphi(x)$. □

This is the way an initial segment V_δ, an elementary submodel of the universe, is used in a forcing iteration to obtain a model in which a desired maximality principle holds. Once a forcing notion has been found at each stage to force the necessity of a particular forceably necessary formula in the elementary submodel, a generic G can then be taken to produce an actual extension which is also an elementary submodel in which the next iteration is definable.

Another direction in which Lemma 2.1 can be generalized is to show there is a closed unbounded (club) class of cardinals δ for which $V_\delta \prec V$. This uses the following strong form of the Reflection Principle.

Lemma 2.4 (Lévy) *For any finite list of formulas Φ in the language of* ZFC *the following is a theorem of* ZFC*:*

There is a club class of cardinals C such that for each δ in C, $V_\delta \prec_\Phi V$.

Proof (This follows the usual proof of the Lévy Reflection Principle.) Without loss of generality, suppose the list $\Phi = \{\varphi_1 \ldots \varphi_n\}$ is closed under subformulas. Define a class function $f : \mathrm{ORD} \to \mathrm{ORD}$ as follows: For α in ORD, let $f(\alpha)$ be the least ordinal γ such that, for all $\vec{x}$ in V_α, and for all $i = 1, \ldots, n$, there exists y such that $\varphi_i(\vec{x}, y)$ and y is in V_γ. If no such y exists, let $f(\alpha) = 0$. Let D be the set of closure points of f. (An ordinal α is a *closure point* of f if $f``\alpha \subseteq \alpha$.) It is easy to see that D is a club class. For any δ in D, absoluteness for $\varphi_1, \ldots, \varphi_n$ between V_δ and V can be proven by induction on the complexity of each φ_i: Absoluteness will be preserved under Boolean connectives, and the same is true under existential quantification (the Tarski-Vaught criterion is satisfied since δ is a closure point of the function f). Since the cardinals also form a club class, the intersection of them with D will be the desired club C. □

This leads to the next theorem, another variation of Lemma 2.1. As in Lemma 2.2, the ground model is expanded to interpret a new predicate symbol in the language, which in this case is the symbol for a club class.

Theorem 2.5 *Let T_0 be a theory containing* ZFC*. Then the following are equivalent:*

(1) $\mathrm{Con}(T_0)$,
(2) $\mathrm{Con}(T_0 + T)$,

where T is the theory in the language $\{\in, C\}$ which contains all instances of the Replacement and Comprehension axiom schemes augmented with the relation symbol C, and which asserts

(i) $C = \langle \delta_\alpha | \alpha \in \mathrm{ORD} \rangle$ *is a closed unbounded class of cardinals;*
(ii) *for all δ in C, $V_\delta \prec V$;*
(iii) *for all α in* ORD, $\delta_\alpha < \mathrm{cf}(\delta_{\alpha+1})$.

Proof For the nontrivial direction—(1) implies (2)—we first give a model where theory T only includes assertions (i) and (ii). Let M be a model of theory T_0. It will suffice to show that every finite subtheory of $T_0 + T$ is consistent. So let F be such a finite subtheory. That is, $F \subseteq \mathrm{ZFC} + \{\sigma_1, \ldots, \sigma_n\} \cup \{$"$C$ is club"$\}$, where $\sigma_i =$ "$\forall \delta \in C \forall x \in V_\delta (V_\delta \models \varphi_i[x]$ if and only if $\varphi_i(x))$". We show that M is a model for F: Fix $\{\varphi_1, \ldots, \varphi_n\}$. By Lemma 2.4, there is a definable club class of cardinals C such that for all i in $1, \ldots, n$, $M \models$ "C is a club class" + "$\forall \delta \in C, V_\delta \models \varphi_i[x]$ if and only if $\varphi_i(x)$". But this is σ_i, so $M \models \sigma_i$. And since M models T_0, it is a model for F. And by compactness, M models all of T_0 and T, where T includes assertions (i) and (ii). But from M we can find a model which has a club C which satisfies assertion (iii) as well. Simply define a new club class to be a continuous subsequence of the original one by inductively defining, for each ordinal α, $\delta_{\alpha+1} =$ the δ_α^+th element that follows δ_α in C. Take suprema at limits. Interpreting the symbol C to be this thinned club ensures that $\delta_\alpha < \mathrm{cf}(\delta_{\alpha+1})$. □

One should stress that in all applications of Theorem 2.5 one is using the language of ZFC expanded to have the relation symbol C and that the theory includes theory T as described in the theorem as well as ZFC in this expanded language.

Lemma 2.2 makes it conceptually easier to include in the theory T statements about some element κ of V referring to it by a name added to the language of ZFC. This is done by expanding the model M to interpret the name of κ. Equiconsistency of such statements together with $V_\delta \prec V$ follows from Lemma 2.2 since the name for κ is "rigid"—the model for $V_\delta \prec V$ can be taken to be an elementary extension, so the same κ can be found in both models. The next lemma, illustrating this, is an enhancement of Lemma 2.1 that provides a condition on δ.

Lemma 2.6 *Let T be any theory containing* ZFC *as a subtheory. Let κ be any cardinal in a model M of T expanded to include κ. Then there is an elementary extension M' of M which is a model of $T + V_\delta \prec V + \mathrm{cf}(\delta) > \kappa$.*

Proof We proceed exactly as in Lemma 2.1, performing additional work to address the requirement on δ: By Lévy Reflection Lemma 2.4, for any fixed finite $\Phi \subseteq T$, there is a class $\{\alpha \in \mathrm{ORD} | V_\alpha \prec_\Phi V\}$ which is closed and unbounded. So it has a κ^+th member. We interpret δ as this member, giving $\mathrm{cf}(\delta) = \kappa^+$ and $V_\delta \prec_\Phi V$. The rest of the proof of Lemma 2.1 now gives $M' \models T + V_\delta \prec V$+$\mathrm{cf}(\delta) > \kappa$. □

A typical application of this lemma is to ensure that the cofinality of δ is greater than ω.

3 The Consistency Strength of $\mathrm{MP}_{\mathrm{CCC}}(\mathbb{R})$

In [2] Hamkins showed that the maximality principle with real parameters, $\mathrm{MP}(\mathbb{R})$, has consistency strength strictly greater than ZFC, while MP and $\mathrm{MP}_{\mathrm{CCC}}$ are both equiconsistent with ZFC. That article also asks as to the consistency strength of the latter principle when real parameters are added. It would seem that doing so should increase the consistency strength of $\mathrm{MP}_{\mathrm{CCC}}$ as it did for MP, especially since CCC-forcing will certainly add new reals that are not in the ground model. However, contrary to expectation, we have the following theorem.

Theorem 3.1 *The following are equivalent:*

1. Con(ZFC),
2. Con(ZFC $+$ $\mathrm{MP}_{\mathrm{CCC}}(\mathbb{R})$).

We need some preliminary results in order to proceed. We first quote a well-known result (Lemma 5.14 of Chapter VIII from [4]). In the following series of lemmas, α is any ordinal, $\mathcal{I}$ is an ideal on $\alpha+1$, and $\mathcal{I}$ contains all finite subsets of α. Furthermore, to say that an α-stage iteration has *supports in* $\mathcal{I}$ means that if $\lambda < \alpha$ is a limit ordinal, then $p = \langle p_\mu | \mu < \lambda \rangle \in \mathbb{P}_\lambda$ if and only if for all $\xi < \mu$, $p \upharpoonright \xi \in \mathbb{P}_\xi$ and the support of p is in $\mathcal{I}$.

Lemma 3.2 *Assume that in V, α is a limit ordinal,*

$$\langle \langle \mathbb{P}_\xi | \xi \leq \alpha \rangle, \langle \pi_\xi | \xi < \alpha \rangle \rangle$$

is an α-stage iterated forcing construction with supports in $\mathcal{I}$, and each element of $\mathcal{I}$ is bounded in α. Suppose G is V-generic over $\mathbb{P}_\alpha$, $S \in V$, $X \subseteq S$, $X \in V[G]$, and $(|S| < \mathrm{cf}(\alpha))^{V[G]}$. Then for some $\eta < \alpha$, $X \in V[G_\eta]$.

This lemma says that subsets from the forcing extension $V[G]$) of small sets in V appear in some earlier stage. Another lemma we will use, whose proof is almost identical, says that any small sets in $V[G]$ which are subsets of sets in V appear in some earlier stage.

Lemma 3.3 *Assume that in V, α is a limit ordinal,*

$$\langle \langle \mathbb{P}_\xi | \xi \leq \alpha \rangle, \langle \pi_\xi | \xi < \alpha \rangle \rangle$$

is an α-stage iterated forcing construction with supports in $\mathcal{I}$, and each element of $\mathcal{I}$ is bounded in α. Suppose G is V-generic over $\mathbb{P}_\alpha$, $S \in V$, $X \subseteq S$, $X \in V[G]$, and $(|X| < \mathrm{cf}(\alpha))^{V[G]}$. Then for some $\eta < \alpha$, $X \in V[G_\eta]$.

Proof Let σ be a $\mathbb{P}$-name such that $X = \sigma_G$. Then $s \in X$ if and only if there is $p \in G$ such that $p \Vdash_{\mathbb{P}_\alpha} (\check{s} \in \sigma)$. By our assumption on $\mathcal{I}$, $\mathbb{P}_\alpha = \bigcup_{\xi<\alpha} i''_{\xi\alpha}\mathbb{P}_\xi$ and $G = \bigcup_{\xi<\alpha} i''_{\xi\alpha} G_\xi$ where $G_\xi = i^{-1}_{\xi\alpha} G$. In $V[G]$, for each $s \in X$, fix $\xi_s < \alpha$ such that $\exists p \in G_\xi (i_{\xi\alpha}(p) \Vdash_{\mathbb{P}_\alpha} \check{s} \in \sigma)$. Let $\eta = \sup\{\xi_s : s \in X\}$. Since $|X| < \mathrm{cf}(\alpha)$ in $V[G]$, $\eta < \alpha$. Now $X = \{s \in S : \exists p \in G_\eta (i_{\eta\alpha}(p) \Vdash_{\mathbb{P}_\alpha} \check{s} \in \sigma)\}$. Since $\Vdash_{\mathbb{P}_\alpha}$ is defined in V, $X \in V[G_\eta]$. □

The following lemma is an application of Lemma 3.3 to sets of hereditary cardinality.

Lemma 3.4 *Assume that in V, λ is a limit ordinal,*

$$\langle \langle \mathbb{P}_\xi | \xi \leq \lambda \rangle, \langle \pi_\xi | \xi < \lambda \rangle \rangle$$

is a λ-stage iterated forcing construction with supports in $\mathcal{I}$, and each element of $\mathcal{I}$ is bounded in λ. Suppose G is V-generic over $\mathbb{P}_\lambda$ and $V[G] \models x \in H(\mathrm{cf}(\lambda))$, then $x \in V[G_\eta]$ for some $\eta < \lambda$.

Proof The proof is by induction on the rank of x (up to $\mathrm{cf}(\lambda)$, since $H(\mathrm{cf}(\lambda)) \subset R(\mathrm{cf}(\lambda))$. If x has rank 0, then it is the empty set, found in $V[G_0]$. Now assume x has rank $\alpha > 0$ and that $y \in H(\mathrm{cf}(\lambda))$ implies $y \in V[G_{\eta_y}]$ for some $\eta_y < \lambda$, for all y of rank less than α. In particular, all $y \in x$ satisfy this induction hypothesis. Let γ be the supremum of all the η_y where $y \in x$. Since $|x| < \mathrm{cf}(\lambda)$ we have that $\gamma < \lambda$, and all $y \in x$ occur simultaneously in $V[G_\gamma]$, where they are all members of the set $R(\alpha)$. Considering $V[G] = V[G_\gamma][G_\lambda^{(\gamma)}]$ as a forcing extension of $V[G_\gamma]$, we can apply Lemma 3.3 to see that there is then some $\eta < \lambda$ such that $x \in V[G_\gamma][G_\eta^{(\gamma)}] = V[G_\eta]$. □

Remark 3.5 The referee has pointed out that Lemma 3.4 also follows from the fact that when κ is a cardinal, members of $H(\kappa)$ can be coded by subsets of cardinals less than κ. (Lemma 3.2 can then be applied directly.)

We will say that a class of forcing notions Γ is *closed under iterations of length γ with appropriate support* if there is an ideal $\mathcal{I}$ on $\gamma + 1$ which provides supports (in the sense given for Lemmas 3.2 through 3.4) for any such iteration and produces a forcing notion which is still in Γ. Moreover, we require that if $\mathbb{P}_\gamma$ is a γ-iteration of forcing in Γ, then for every $\alpha < \gamma$, $\mathbb{P}_\gamma = \mathbb{P}_\alpha * \mathbb{P}_{\{\alpha,\gamma\}}$, where $\mathbb{P}_{\{\alpha,\gamma\}}$, an iteration of forcing notions of Γ, is defined in $V^{\mathbb{P}_\alpha}$ with support in the ideal $\mathcal{I}$. An example of such support would be where either direct or inverse limits are always taken; this is Lemma 21.8 in [3].

The following lemma introduces the technique of applying the results of Section 2 in forcing iteration constructions.

Lemma 3.6 *Let X be any set, and let Γ be a class of forcing notions which contains trivial forcing, which is closed under two-step iterations, and is closed under iterations of length $\kappa = |X|$ with appropriate support. Suppose $V_\delta \prec V$ and $\mathrm{cf}(\delta) > \kappa$. Then there is a forcing notion $\mathbb{P}$ in Γ such that $V^{\mathbb{P}} \models \mathrm{MP}_\Gamma(X)$ and $\mathbb{P} \in V_\delta$.*

Remark 3.7 The symbol X occurring in the expression $V^{\mathbb{P}} \models \mathrm{MP}_\Gamma(X)$ is to be interpreted *de re*—it represents the same set in $V^{\mathbb{P}}$ as when interpreted in V. For example, in applying this lemma, the symbol $\mathbb{R}^V$ will replace X. This is a different situation than the one we shall see in this paper's main result, in which the symbol $\mathbb{R}$ represents a set which has a definition whose interpretation is *de dicto*; hence the set varies from model to model.

Proof Let $\kappa = |X|$. Let $\pi : \omega \times \kappa \longrightarrow \kappa$ be a bijective pairing function. Enumerate all formulas with one parameter in the language of set theory as $\langle \varphi_n | n \in \omega \rangle$ and all elements x of X as $\langle x_\mu | \mu \in \kappa \rangle$. Define a κ-iteration $\mathbb{P} = \mathbb{P}_\kappa$ of Γ forcing notions, with appropriate support, as follows. At successor stages, let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where, if $\alpha = \pi(n, \mu)$ and $V_\delta^{\mathbb{P}_\alpha} \models$ "$\varphi_n(x_\mu)$ is Γ-forceably necessary", then $V_\delta^{\mathbb{P}_\alpha} \models$ "$\mathbb{Q}_\alpha$ is a forcing notion in Γ that forces '$\varphi_n(x_\mu)$ is Γ-necessary' "; otherwise, $\mathbb{Q}_\alpha$ is $\{\varnothing\}$, the trivial forcing. Use appropriate support at limit stages. Note that, since $\mathrm{cf}(\delta) > \kappa$, $\mathbb{P}_\alpha$ is in V_δ for all $\alpha < \kappa$.

Let G be V-generic over $\mathbb{P}$. The claim is that $V[G] \models \mathrm{MP}_\Gamma(X)$. To prove the claim, suppose $x \in X$ and $V[G] \models$ "$\varphi(x)$ is Γ-forceably necessary". It will suffice to show that $V[G] \models$ "$\varphi(x)$ is Γ-necessary". Let $\varphi = \varphi_n$ and $x = x_\mu$ for some $\alpha = \pi(n, \mu)$. By factoring, $V[G] = V[G_\alpha][G_{\mathrm{TAIL}}] \models$ "$\varphi(x)$ is Γ-forceably necessary", so $V[G_\alpha] \models$ "$\varphi(x)$ is Γ-forceably necessary" as well. By elementarity,

$V_\delta[G_\alpha] \models$ "$\varphi(x)$ is Γ-forceably necessary". But at stage α, the forcing notion $\mathbb{Q}$ in V_δ has been defined to force $\Box_\Gamma \varphi_n(x_\mu)$. So $V_\delta[G_{\alpha+1}] \models$ "$\varphi(x)$ is Γ-necessary". Again by elementarity, $V[G_{\alpha+1}] \models$ "$\varphi(x)$ is Γ-necessary". And since $V[G]$ is a Γ-forcing extension of $V[G_{\alpha+1}]$, $V[G] \models$ "$\varphi(x)$ is Γ-necessary". This proves the claim. Finally, since the iteration of $\mathbb{P}$ has appropriate support, $\mathbb{P}$ is in Γ. And since $\mathrm{cf}(\delta) > \kappa$, $\mathbb{P}$ is in V_δ. □

To prove the nontrivial implication of Theorem 3.1 it will suffice to prove that if there is a model of ZFC then there is a model of ZFC + $\mathrm{MP}_{\mathrm{CCC}}(\mathbb{R})$. I will give two proofs of this.

First Proof of Theorem 3.1 We prove the consistency of a weak version of the principle stated in the theorem. For any set X, using our notation, $\mathrm{MP}_{\mathrm{CCC}}(X)$ is the modified maximality principle that says any formula with parameters taken from the set X which is CCC-forceably necessary is true. Let $\mathbb{P}$ be a CCC forcing notion. Let us confine ourselves to the model $V^{\mathbb{P}}$ and denote by $\mathbb{R}^V$ the reals of the ground model V. Let the principle $\mathrm{MP}_{\mathrm{CCC}}(\mathbb{R}^V)$ be the form of $\mathrm{MP}_{\mathrm{CCC}}(X)$ interpreted in $V^{\mathbb{P}}$ with parameter set $\mathbb{R}^V$. By Lemma 2.6, it is consistent to assume that $V_\delta \prec V$, and $\mathrm{cf}(\delta) > 2^\omega$. By a direct application of Lemma 3.6, there is a forcing notion $\mathbb{P}$ in CCC such that $V^{\mathbb{P}} \models \mathrm{MP}_{\mathrm{CCC}}(\mathbb{R}^V)$ and $\mathbb{P} \in V_\delta$.

We may now iterate this construction. Suppose $V \models$ ZFC. By Theorem 2.5 we may assume that there is a club class of cardinals C such that for all δ in C, $V_\delta \prec V$. We will construct a forcing extension which is a model of ZFC+$\mathrm{MP}_{\mathrm{CCC}}(\mathbb{R})$. Construct a finite-support ω_1-iteration $\mathbb{P} = \mathbb{P}_{\omega_1}$ such that $V^{\mathbb{P}} \models \mathrm{MP}_{\mathrm{CCC}}(\mathbb{R})$ as follows. Let $\mathbb{P}_0$ be the trivial notion of forcing. At stage α, select δ_α from the club C such that the rank of $\mathbb{P}_\alpha < \delta_\alpha$ (so that $\mathbb{P}_\alpha$ is in V_{δ_α}) and $\mathrm{cf}(\delta_\alpha) > (2^\omega)^{V^{\mathbb{P}_\alpha}}$. Working in $V^{\mathbb{P}_\alpha}$, define $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$, where $\dot{\mathbb{Q}}$ is a $\mathbb{P}_\alpha$-name of a CCC notion of forcing such that $V^{\mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha} \models \mathrm{MP}_{\mathrm{CCC}}(\mathbb{R}^{V^{\mathbb{P}_\alpha}})$. Such a $\mathbb{Q}_\alpha$ is guaranteed to exist by Lemma 3.6, since the conditions $V_{\delta_\alpha} \prec V$ and $\mathrm{cf}(\delta_\alpha) > 2^\omega$ are satisfied by $V^{\mathbb{P}_\alpha}$. This completes the construction of $\mathbb{P} = \mathbb{P}_{\omega_1}$.

Let G be V-generic over $\mathbb{P}$. I claim that $V[G] \models \mathrm{MP}_{\mathrm{CCC}}(\mathbb{R})$. To see this, let $V[G] \models$ "$\varphi(r)$ is CCC-forceably necessary", where $r \in \mathbb{R}$, the reals as interpreted in $V[G]$. It will suffice to show that $V[G] \models$ "$\varphi(r)$ is CCC-necessary". Note that r, as a real, is a subset of ω in $V[G]$, while ω itself is in V. Therefore, since $\mathbb{P}$ is an ω_1 iteration with finite support and $\mathrm{cf}(\omega_1) > |\omega|$, r must be in some $V[G_\alpha]$, where $\alpha < \omega_1$, $\mathbb{P} = \mathbb{P}_\alpha * \dot{\mathbb{P}}^{(\alpha)}_{\omega_1}$ is the factorization of $\mathbb{P}$ at stage α, and G_α is the projection of G to $\mathbb{P}_\alpha$. (This follows from Lemma 3.2.) But the definition of $\mathbb{P}$ required that $\mathbb{Q}_\alpha$ force $\mathrm{MP}_{\mathrm{CCC}}(\mathbb{R}^{V[G_\alpha]})$, which therefore must hold at stage $\alpha + 1$. Indeed, refactoring $\mathbb{P} = \mathbb{P}_{\alpha+1} * \dot{\mathbb{P}}^{(\alpha+1)}_{\omega_1}$ and setting $G_{\alpha+1}$ to be the projection of G to $\mathbb{P}_{\alpha+1}$, we have that $r \in V[G_\alpha]$ and $V[G_{\alpha+1}] \models \mathrm{MP}_{\mathrm{CCC}}(\mathbb{R}^{V[G_\alpha]})$ (as well as $V[G_{\alpha+1}] \models$ "$\varphi(r)$ is CCC-forceably necessary", since $V[G] = V[G_{\alpha+1}][G^{(\alpha+1)}_{\omega_1}]$ is a CCC-forcing extension of $V[G_{\alpha+1}]$). Therefore $V[G_{\alpha+1}] \models$ "$\varphi(r)$ is CCC-necessary". Since $V[G]$ is a CCC-forcing extension of $V[G_{\alpha+1}]$, we have that $V[G] \models$ "$\varphi(r)$ is CCC-necessary" as required. □

Second Proof of Theorem 3.1 This time, we use a bookkeeping function style argument. Suppose $V \models$ ZFC. By Theorem 2.5 we may assume that there is in V a club class of cardinals C such that for all δ in C, $V_\delta \prec V$. Let

$\pi : \mathrm{ORD} \simeq \omega \times \mathrm{ORD} \times \mathrm{ORD}$ be a definable bijective class function $\pi : \alpha \mapsto \langle n, \beta, \mu \rangle$ such that $\beta \leq \alpha$. Using π as a bookkeeping function, we define a sequence of iterated forcing notions $\mathbb{P}_\alpha$, simultaneously with a sequence of cardinals δ_α, by transfinite induction on α in ORD as follows. Let $\mathbb{P}_0$ be trivial forcing. Given $\mathbb{P}_\alpha$, let δ_α be the least cardinal in the club C such that $\mathbb{P}_\alpha$ is in V_{δ_α}. Define $\dot{\mathbb{Q}}_\alpha$ in $V_{\delta_\alpha}^{\mathbb{P}_\alpha}$ as follows: Let $\pi(\alpha) = \langle n, \beta, \mu \rangle$. Consider the statement $\varphi(x) = \varphi_n(x)$, the nth statement in the language of ZFC according to some enumeration, with single parameter $x = x_\mu$, the μth name for a real in the model $V_{\delta_\alpha}^{\mathbb{P}_\beta}$ where $\beta \leq \alpha$, according to some definable well-ordering of the universe. If, in $V_{\delta_\alpha}^{\mathbb{P}_\alpha}$, $\varphi(x)$ is CCC-forceably necessary, let $\dot{\mathbb{Q}}_\alpha$ be the V_{δ_α}-least $\mathbb{P}_\alpha$-name of a forcing notion which performs a forcing that $\varphi(x)$ is CCC-necessary. Otherwise, let $\dot{\mathbb{Q}}_\alpha$ be the $\mathbb{P}_\alpha$-name for trivial forcing. (This will be the case, for example, if μ is not below δ_α, the height of the set model V_{δ_α}.) Now let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. Finally, take finite support at limits. This defines the sequence $\mathbb{P}_\alpha$ for all α in ORD. Note that, for all such α, $\mathbb{P}_\alpha$ is CCC and is contained in V_{δ_α}.

We wish to truncate this sequence at an appropriate length λ to obtain an iterated forcing notion $\mathbb{P}_\lambda$ which forces a model of $\mathrm{MP}_{\mathrm{CCC}}(\mathbb{R})$. This will occur if all reals in $V^{\mathbb{P}_\lambda}$ are introduced at some earlier stage of the iteration and the cofinality of λ is greater than ω. To ensure this, we define λ to be a closure point of the function $f : \mathrm{ORD} \longrightarrow \mathrm{ORD}$ which takes β, the stage at which a real parameter is introduced, to the least stage by which all formulas φ have been applied to all parameters in $V^{\mathbb{P}_\beta}$. These parameters are $V^{\mathbb{P}_\beta}$-names, and by using *nice names*, discussed in detail in [4], of which there are $|\mathbb{P}_\beta|^\omega$ many in $V^{\mathbb{P}_\beta}$, this gives $f(\beta) = \sup_{\mu < |\mathbb{P}_\beta|^\omega}\{\pi(\alpha) = \langle n, \beta, \mu \rangle\}$. Now let λ be the first closure point of $f : \mathrm{ORD} \longrightarrow \mathrm{ORD}$ with cofinality ω_1.

Let $\mathbb{P} = \mathbb{P}_\lambda$, and let G be V-generic over $\mathbb{P}$. By the usual argument, we can now establish that $V[G] \models \mathrm{MP}_{\mathrm{CCC}}(\mathbb{R})$: Suppose $V[G] \models$ "$\varphi(r)$ is CCC-forceably necessary". Then there is $\alpha = \langle n, \mu, \beta \rangle$, where $\varphi = \varphi_n$ and $\dot{r}$ is the μth nice $\mathbb{P}_\beta$-name of a real, for some $\beta \leq \alpha$ (the name $\dot{r}$ appears in the model $V_{\delta_\alpha}^{\mathbb{P}_\beta}$ by construction). Since $V[G]$ is a CCC-forcing extension of $V[G_\alpha]$, where G_α is V-generic over $\mathbb{P}_\alpha$, $V[G_\alpha] \models$ "$\varphi(r)$ is CCC-forceably necessary", whence by elementarity, $V_{\delta_\alpha}[G_\alpha] \models$ "$\varphi(r)$ is CCC-forceably necessary". But by the construction of $\mathbb{P}$, if $G_{\alpha+1}$ is $\mathbb{P}_{\alpha+1}$-generic over $V_{\delta_{\alpha+1}}$, then $V_{\delta_{\alpha+1}}[G_{\alpha+1}] \models$ "$\varphi(r)$ is CCC-necessary", so by elementarity $V[G_{\alpha+1}] \models$ "$\varphi(r)$ is CCC-necessary" and therefore $V[G] \models$ "$\varphi(r)$ is CCC-necessary". □

Remark 3.8 The second proof just given of Theorem 3.1 makes use of the existence of closure points of the defined function $f : \mathrm{ORD} \longrightarrow \mathrm{ORD}$. In order to know such closure points exist one needs to apply the Replacement scheme. Even the first proof makes use of the Replacement Axiom Scheme enhanced with the symbol C in order to construct the iteration $\mathbb{P}_\alpha$. These arguments take place in the language of ZFC expanded with the symbol C interpreted as a club class in an expanded model. This is why we included, in Theorem 2.5, all instances of Replacement and Comprehension that mention the club class C.

One might expect that Theorem 3.1 can be extended to parameter sets which are power sets of sets of cardinality greater than ω, such as ω_1 or $\aleph_{\omega_{17}}$, and in fact this is the case. Let κ be a cardinal. Singling out the second proof strategy above, one can state and prove a generalization of Theorem 3.1 after making the following definitions.

We will call a formula $\psi(x)$ a *definition* of a set A if for all x, $\psi(x)$ if and only if $x \in A$. Let Γ be a class of forcing notions. A formula $\varphi(x)$ is Γ*-absolute* if for any $\mathbb{P}$ in Γ, and for any x in V, $V^{\mathbb{P}} \models \varphi(x)$ if and only if $\varphi(x)$. A set defined by such a formula $\varphi(x)$ in all Γ-forcing extensions as well as in V is said to be Γ*-absolutely definable*.

Theorem 3.9 *Let κ be any* CCC*-absolutely definable cardinal. Then the following are equivalent:*

1. Con(ZFC)*,*
2. Con(ZFC + $\mathrm{MP}_{\mathrm{CCC}}(H(\kappa))$).

Proof Just as in the second proof of Theorem 3.1, the proof in the nontrivial direction consists in building a CCC-forcing extension model for $\mathrm{MP}_{\mathrm{CCC}}(H(\kappa))$ from a model V of ZFC. Without loss of generality, by Theorem 2.5, assume in V a club class of cardinals C such that for all δ in C, $V_\delta \prec V$. We use the same bookkeeping class function. The definition of a sequence of iterated forcing notions $\mathbb{P}_\alpha$, simultaneously with a sequence of cardinals δ_α, again proceeds by transfinite induction on α in ORD. At stage α, in defining $\dot{\mathbb{Q}}_\alpha$ in $V_{\delta_\alpha}^{\mathbb{P}_\alpha}$, α now codes $\langle n, \beta, \mu\rangle$ where μ is now an index for the name for a subset x of κ in the model $V_{\delta_\alpha}^{\mathbb{P}_\beta}$ where $\beta \leq \alpha$, and $\dot{\mathbb{Q}}_\alpha$ forces that $\varphi(x)$ is CCC-necessary if such a forcing notion exists. Again let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ with finite support at limits. This again gives $\mathbb{P}_\alpha$ for all α in ORD, with $\mathbb{P}_\alpha$ being CCC and contained in V_{δ_α}.

Define a function $f : \mathrm{ORD} \longrightarrow \mathrm{ORD}$ which takes β, the stage at which a real parameter is introduced, to the least stage by which all formulas φ have been applied to all parameters in $V^{\mathbb{P}_\beta}$. Since we only need to count nice names, of which there are $(|\mathbb{P}_\beta|^\omega)^{<\kappa} = |\mathbb{P}_\beta|^{<\kappa}$ many in $V^{\mathbb{P}_\beta}$, this gives $f(\beta) = \sup_{\mu < |\mathbb{P}_\beta|^{<\kappa}} \{\alpha = \langle n, \beta, \mu\rangle\}$. The closure points of this function form a club class in ORD. We truncate the class iteration to length λ, the κth element of this sequence.

We claim $\mathbb{P}_\lambda$ produces a forcing extension $V[G]$ which satisfies $\mathrm{MP}_{\mathrm{CCC}}(H(\kappa))$. Since its length has cofinality κ, we know all names of parameters in $V^{\mathbb{P}_\lambda}$ are introduced at earlier stages, by Lemma 3.4. The rest of the argument is identical to the corresponding part of the second proof of Theorem 3.1. □

Finally, we prove what appears to be an optimal equiconsistency with ZFC (in the sense that $\mathrm{MP}_{\mathrm{CCC}}(H(2^\omega))$ is known to require a proper class of inaccessible cardinals—see Theorem 5.6 in [2]). The following proof is a modification of the second proof of Theorem 3.1.

Theorem 3.10 *If there is a club class C such that for all δ in C, $V_\delta \prec V$, then there is a forcing extension which satisfies* ZFC + $\mathrm{MP}_{\mathrm{CCC}}(H(\mathrm{cf}(2^\omega)))$. *Furthermore, in this model, the value of* $\mathrm{cf}(2^\omega)$ *can be as large as any* CCC*-absolutely definable cardinal.*

Proof We use a bookkeeping function to define a sequence of iterated forcing notions $\mathbb{P}_\alpha$, simultaneously with a sequence of cardinals δ_α, proceeding by transfinite induction on α in ORD. Define $\mathbb{P}_0$ to be trivial forcing and $\delta_0 =$ the first element of the sequence C. Let $\pi : \mathrm{ORD} \simeq \omega \times \mathrm{ORD} \times \mathrm{ORD}$ be a definable bijective class function $\pi : \alpha \mapsto \langle n, \beta, \mu \rangle$ such that $\beta \leq \alpha$. At stage α, in defining $\dot{\mathbb{Q}}_\alpha$ in $V_{\delta_\alpha}^{\mathbb{P}_\alpha}$, α codes $\langle n, \beta, \mu \rangle$ where μ is, as before, an index for the name of some set x in the model $V_{\delta_\beta}^{\mathbb{P}_\beta}$ (according to some definable well-ordering of the universe) where $\beta \leq \alpha$, and $\dot{\mathbb{Q}}_\alpha$ forces that $\varphi_n(x)$ is CCC-necessary if such a forcing notion exists. Otherwise, let $\mathbb{Q}_\alpha$ be trivial forcing. Let $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ with finite support at limits, while $\delta_{\alpha+1}$ is chosen from the tail of the class club C after δ_α such that $V_{\delta_{\alpha+1}}$ contains $\mathbb{P}_{\alpha+1}$. Define $\delta_\alpha = \bigcup_{\xi<\alpha} \delta_\xi$ for limit α. This defines $\mathbb{P}_\alpha$ and δ_α for all α in ORD, where $\mathbb{P}_\alpha$ is CCC and is contained in V_{δ_α}.

Let the function $f : \mathrm{ORD} \longrightarrow \mathrm{ORD}$ take any ordinal β to the index of the least stage by which all formulas have been applied to all parameters in $V_{\delta_\beta}^{\mathbb{P}_\beta}$. The set of closure points of f forms a club class in ORD. Let λ be a closure point of f of uncountable cofinality, and truncate the sequence $\langle \mathbb{P}_\alpha | \alpha \in \mathrm{ORD} \rangle$ at λ. By definition of the class club C, V_{δ_λ} contains the entire iteration $\mathbb{P}_\lambda$, and $\delta_\alpha < \mathrm{cf}(\delta_\lambda)$ for all $\alpha < \lambda$.

Let $\mathbb{P} = \mathbb{P}_\lambda$, and let G be V-generic over $\mathbb{P}$. We can now show that $V[G] \models \mathrm{MP}_{\mathrm{CCC}}(H(\mathrm{cf}(2^\omega)))$. We claim that, in $V[G]$, $\mathrm{cf}(\lambda) = \mathrm{cf}(2^\omega)$. First, the continuum is forced to be larger than any cardinal below δ_α for unboundedly many $\alpha < \lambda$ over the partial stages of the iteration $\mathbb{P}_\lambda$, so $V[G]$ satisfies $2^\omega \geq \delta_\lambda$. And second, the forcing notion $\mathbb{P}_\lambda$ is contained in V_{δ_λ}, so $V[G]$ satisfies $2^\omega \leq \delta_\lambda$, whence $2^\omega = \delta_\lambda$ in $V[G]$. Now observe that by continuity, $\mathrm{cf}(\lambda) = \mathrm{cf}(\delta_\lambda)$, proving our claim. We further see that all parameters in $H(\mathrm{cf}(\lambda))$ found in $V[G]$ appear at some previous stage, using Lemma 3.4.

Suppose $V[G] \models$ "$\varphi(x)$ is CCC-forceably necessary and $x \in H(\mathrm{cf}(\lambda))$". (Then $x \in V_{\delta_\lambda}[G] = \bigcup_{\beta<\lambda} V_{\delta_\beta}[G]$, so $x \in V_{\delta_\beta}[G]$ for some $\beta < \lambda$, so it has been handled as a parameter for all forceably necessary statements unboundedly often over this iteration.) Take $\alpha = \langle n, \mu, \beta \rangle$, where $\varphi = \varphi_n$ and $\dot{x}$ is the μth nice $\mathbb{P}_\beta$-name of the parameter x, for some $\beta \leq \alpha$. Since $V[G]$ is a CCC-forcing extension of $V[G_\alpha]$, where G_α is V-generic over $\mathbb{P}_\alpha$, $V[G_\alpha] \models$ "$\varphi(x)$ is CCC-forceably necessary", whence by elementarity, $V_{\delta_\alpha}[G_\alpha] \models$ "$\varphi(x)$ is CCC-forceably necessary". But by the construction of $\mathbb{P}$, if $G_{\alpha+1}$ is $\mathbb{P}_{\alpha+1}$-generic over $V_{\delta_{\alpha+1}}$, then $V_{\delta_{\alpha+1}}[G_{\alpha+1}] \models$ "$\varphi(x)$ is CCC-necessary", so by elementarity $V[G_{\alpha+1}] \models$ "$\varphi(x)$ is CCC-necessary" and therefore $V[G] \models$ "$\varphi(x)$ is CCC-necessary".

Finally, notice that in our choice of λ, we can make $\mathrm{cf}(\lambda)$ any CCC-absolutely definable regular cardinal we like, below λ. (Let the desired cofinality be κ. Now choose λ to be the κth closure point of the function f.) □

Remark 3.11 In the above proof, for example, if $\mathrm{cf}(\lambda)$ is chosen to be ω_{23}, we get a model of ZFC+$\mathrm{MP}_{\mathrm{CCC}}(H(\omega_{23}))$+$\mathrm{cf}(2^\omega) = \omega_{23}$, an improvement over Theorem 3.9.

Remark 3.12 The construction in the proof of this theorem actually provides a stronger parameter set than $H(\mathrm{cf}(2^\omega))$. The sentences which are forced necessary paired with their parameters occur unboundedly often in the forcing iteration. From this fact, we see that for any ordinal η below $\delta_\lambda = 2^\omega$, together with any CCC-forceably necessary formula using it as a parameter, there is a stage α at which it is forced to be necessary. Thus the set of ordinals below 2^ω, that is, the set 2^ω, can be used as a parameter set. So the actual result of the theorem is the following corollary.

Corollary 3.13 *The following theories are equiconsistent:*

1. ZFC,
2. $\mathrm{ZFC} + \mathrm{MP}_{\mathrm{CCC}}(H(\mathrm{cf}(2^{\omega})) \cup 2^{\omega})$.

Furthermore, in the second theory, the value of $\mathrm{cf}(2^{\omega})$ *can be as large as any* CCC-*absolutely definable regular cardinal.*

This answers a question posed in [2], in which Hamkins proves, as part of his Theorem 5.6, the following theorem.

Theorem 3.14 *The following theories are equiconsistent:*

1. $\mathrm{ZFC} + \mathrm{MP}_{\mathrm{CCC}}(2^{\omega})$+*"$2^{\omega}$ is regular"*,
2. $\mathrm{ZFC}+V_{\delta} \prec V$+*"$\delta$ is inaccessible"*.

He goes on to ask whether the regularity of 2^{ω} can be dropped from the first theory while preserving its equiconsistency with the second. But that cannot be done. By Corollary 3.13, a model of ZFC alone implies existence of a model of the first theory of Theorem 3.14 without regularity of 2^{ω}, since $\mathrm{MP}_{\mathrm{CCC}}(H(\mathrm{cf}(2^{\omega})) \cup 2^{\omega})$ implies $\mathrm{MP}_{\mathrm{CCC}}(2^{\omega})$. This would give a model of the second theory of Theorem 3.14 and its inaccessible δ, a contradiction, since inaccessible cardinals have higher consistency strength than ZFC.

Open questions remain, such as whether results similar to those in this paper are possible for other kinds of forcing, say, for the class of proper forcing notions.

Notes

1. The **S4** system of modal logic consists of the schemes (for any formula φ),
   (1) $\Diamond_{\Gamma}\varphi \leftrightarrow \neg\Box_{\Gamma}\neg\varphi$,
   (2) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,
   (3) $\Box\varphi \to \varphi$, and
   (4) $\Box\varphi \to \Box\Box\varphi$.
   The equivalence is left as an easy exercise. See [1] for a good introduction to modal logic systems.

2. While preparing these results, the author noticed that the paper [5] addresses areas that overlap topics in this paper and [2]. It prefigures, by a quarter of a century, the current work on maximality principles. Further, it includes another class of reflection principles, and applies all these ideas to abstract logics. Its results are powerful and interesting. However, in spite of the intuitions behind its forcing arguments which convinced this author of the truth of its claims, [5] is flawed. In several proofs, it makes use of a truth predicate over the universe, which is forbidden by the fundamental result of Tarski. In the proof of Theorem 28 in [5], a stage in a forcing iteration is defined as "If $\mathbb{B}_{\alpha}$ is defined, we let $\mathbb{B}_{\alpha+1}$ be determined as follows:... such that $[\![R(x_1, \ldots, x_n)]\!]^{\mathbb{B}_{\alpha}} = 1 \ldots$" The condition that a Boolean value of a statement is equal to 1 is equivalent to that statement being true in the Boolean-valued extension. This cannot be expressed in the language of ZFC, nor serve as a definition, unless the ground model itself is somehow exempt from Tarski's result. For example, one could ensure that the ground model is a set by assuming $V_{\delta} \prec V$, the tactic used in [2] and in this paper. Other results in [5] which depend on this argument (Theorems 33, 34, and 35) share this flaw.

   In fact, Theorem 32 in [5] is false. It says that for any ground model, there is always a forcing extension which satisfies $\mathrm{MP}_{\mathrm{CCC}}(A)$ for any set A. A counterexample can be found within the proof of Theorem 2.8 in [2], which gives a model of ZFC for which the definable ordinals are unbounded. This is easy to arrange by taking the "definable cut"

W_{DEF} of a model W of **ZFC**–the union of all W_α where α is definable. We also need that W is a model where all such definitions are **CCC**-forcing absolute, but any model which satisfies $V = L$ will suffice. By the Tarski-Vaught criterion, W_{DEF} is an elementary submodel of W and therefore is a model of **ZFC**. Clearly the class of definable ordinals is unbounded in W_{DEF}. Assume there is a forcing extension $W_{\mathrm{DEF}}[G]$ which satisfies $\mathbf{MP}_{\mathrm{CCC}}(A)$. It still has the same ordinals as the ground model; hence the definable ordinals are still unbounded. Working in $W_{\mathrm{DEF}}[G]$, let γ be any definable ordinal. Consider the **CCC**-forcing which enlarges the continuum to some cardinal greater than γ. Since this inequality, $2^\omega > \gamma$, is **CCC**-forceably necessary, it is already true by $\mathbf{MP}_{\mathrm{CCC}}(A)$. So in $W_{\mathrm{DEF}}[G]$, 2^ω is greater than any definable ordinal γ. But the definable ordinals are unbounded in $W_{\mathrm{DEF}}[G]$, a contradiction.

3. The models of **ZFC** in which this occurs will be non-well-founded models. An alternative approach, not taken here, would be to avoid these nonstandard models by constructing a model for a maximality principle that applies to any finite fragment of **ZFC** and then finally applying the compactness theorem to find a model of the full maximality principle. Our approach will be to apply the compactness theorem earlier on and allow iterations to construct models of the full maximality principles.

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Acknowledgments

This work is a revision and extension of part of the author's dissertation written under the supervision of Joel David Hamkins. The author would like to thank the referee for pointing out several places in the draft of this paper which required some further clarification.

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