

A Silver-like Perfect Set Theorem with an Application to Borel Model Theory

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Abstract A number of results have been obtained concerning Borel structures starting with Silver and Friedman followed by Harrington, Shelah, Marker, and Louveau. Friedman also initiated the model theory of Borel (in fact totally Borel) structures. By this we mean the study of the class of Borel models of a given first-order theory. The subject was further investigated by Steinhorn. The present work is meant to go further in this direction. It is based on the assumption that the study of the class of, say, countable models of a theory reduces to analyzing a single ω_1 -saturated model. The question then arises as to when such a model can be totally Borel. We present here a partial answer to this problem when the theory under investigation is superstable.

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1 Introduction

This work is organized as follows. In Section 2, we introduce a notion of dependence, which generalizes that of a pregeometry and then we state the Silver-like theorem, to the effect that if we suppose the dependence notion is coanalytic then every analytic set which contains an uncountable independent set contains a perfect one. Section 3 points out a number of corollaries. Section 4 reviews the basics of the Gandy-Harrington topology. In Section 5, we supply the proof of the theorem.

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It boils down to checking that Harrington’s proof of Silver’s theorem carries over to the relevant situation. In Section 6, we remind the reader of the required facts concerning stable theories. Section 7 extends the facts of Section 6 when instead of models of a stable theory we consider structures of the following form: the quotient of \mathfrak{M}_0 under an equivalence relation E , where (\mathfrak{M}_0, E) is a model of a stable theory; the extension is proved under a strong Borelness assumption on (\mathfrak{M}_0, E) . This is used as a lemma in Section 8, which proves the model theoretic result of this article: an ω_1 -saturated, totally Borel model of a superstable theory is saturated.

2 Statement of the Main Lemma

In this section, we state a generalization of Silver’s theorem on the number of equivalence classes of a coanalytic equivalence relation. The proof is postponed until Section 5.

2.1 Let X be a set and suppose $D = (D_n)_n$ is a family of relations with $D_n \subseteq X^n$, for each integer $n \geq 0$. Given $A \subseteq X$, $[A]_D$ will denote the D -closure of A , that is, the set of $x \in X$ such that $(x, x_1, \dots, x_n) \in D_{n+1}$, for some $x_1, \dots, x_n \in A$. Also, we write $[x_1, \dots, x_n]$ for $[\{x_1, \dots, x_n\}]_D$.

2.2 Let X and $D = (D_n)_n$ as above. D is said to be a *notion of dependence on X* if, for all $A \subseteq X$,

1. $A \subseteq [A]_D$,
2. if $x, y \notin [A]_D$, then

$$x \in [A \cup \{y\}]_D \text{ iff } y \in [A \cup \{x\}]_D.$$

D is a *pregeometry* if it satisfies the additional condition

3. $[A]_D = [[A]_D]_D$.

Let $D = (D_n)_n$ be a notion of dependence on X . $(x_1, \dots, x_n) \in X^n$ is a *D -free sequence* if $x_1 \notin [x_2, \dots, x_n]_D$, \dots , $x_{n-1} \notin [x_n]_D$ and $x_n \notin [\emptyset]_D$. $S \subseteq X$ is a *D -free set* if every finite sequence of distinct members of S is D -free.

A subset S of A *spans* A if $A \subseteq [S]_D$. A is *finitely spanned* if it is spanned by a finite D -free set. Note that a D -free subset of A spans A if and only if it is a maximal D -free subset of A .

As is well known, free sets behave nicely when we restrict ourselves to pregeometries. More precisely, if D is a pregeometry, any two D -free sets spanning the whole space have the same cardinality, which is called the *dimension* of the space. We now introduce a weakening of the concept of pregeometry, where the *dimension* is no longer well defined.

2.3 Let $D = (D_n)_n$ be a notion of dependence; D is said to *satisfy EFSW* (“Every Finitely Spanned set has Weight”) if

for every finitely spanned set $A \subseteq X$, there exists a positive integer N such that every D -free subset of A has cardinality $\leq N$.

The smallest such integer is called the *weight* of A and, when it exists, A is said to *have weight*.

By Subsection 2.2, every pregeometry satisfies EFSW. The converse is false. Some typical counterexamples will be described in Section 3. The full strength of

this condition will be made use of when we investigate the model theory of super-stable theories.

2.4 $D = (D_n)_n$ is said to be a Γ -*notion of dependence on X* if D is a notion of dependence on X and D_n is in Γ , for all n . Here, Γ is a class of point sets such as Δ_1^1 , Σ_1^1 , $\Delta_1^1(x)$, and so on, if $X = \mathcal{N} (= \omega^\omega)$ or Borel, analytic, and so on, if X is an arbitrary Polish space. Here is our Silver-like perfect set theorem.

Lemma 2.1 (Main Lemma) *Let $D = (D_n)_n$ be a Π_1^1 notion of dependence on \mathcal{N} . If D satisfies EFSW, then there exists a set $H \subseteq \mathcal{N}$, depending only on D , such that*

1. H is Π_1^1 ;
2. every D -free subset of H is countable;
3. for every Σ_1^1 subset S of \mathcal{N} , either $S \subseteq H$ or S contains a perfect, D -free subset.

Note that if every free set is countable, $H = \mathcal{N}$ works. Also, if $[A]_D = A$ for every A , $H =$ the set of hyperarithmetical reals does the job.

The proof will easily be seen to relativize, thus yielding the following classical version.

Corollary 2.2 (Main Corollary) *Let $D = (D_n)_n$ be a coanalytic notion of dependence on a Polish space X . If D satisfies EFSW, then every analytic subset of X containing an uncountable D -free set has a perfect D -free subset.*

3 Some Easy Corollaries

The main lemma trivially implies a variety of perfect set theorems. We collect here some typical examples.

Corollary 3.1 (Silver) *Let E be a coanalytic equivalence relation on $\mathcal{N} = \omega^\omega$, equipped with the usual product topology; then, either the quotient set \mathcal{N}/E is countable or there exists a perfect subset of \mathcal{N} , any two elements of which are pairwise inequivalent.*

Proof Define

1. $D_o = D_1 = \emptyset$;
2. for $n \geq 1$, $(x, x_1, \dots, x_n) \in D_{n+1}$ iff xEx_1 or \dots or xEx_n ;
3. $D = (D_n)_n$.

The family D is a coanalytic pregeometry. □

Corollary 3.2 (Kuratowski-Mycielski) *Let X be the set \mathbb{R} of real numbers, construed as a vector space over the field \mathbb{Q} . If $S \subseteq \mathbb{R}$ is an analytic subset of \mathbb{R} , then either S is K -dimensional for some $K \leq \omega$, or S contains a perfect set of linearly independent points over \mathbb{Q} . In particular, there exists a perfect subset of \mathbb{R} , the elements of which are linearly independent.*

Proof This is Exercise (19.2) of [7]; here we can deduce it from the main result by using the remark that follows. Define

1. $D_o = \emptyset$;
2. $D_1 = \emptyset$;
3. for $n \geq 1$, $(x, x_1, \dots, x_n) \in D_{n+1}$ iff $\exists \alpha, \alpha_1, \dots, \alpha_n \in \mathbb{Q}$ s. t. $\alpha \neq 0$ and $\alpha x + \sum \alpha_i x_i = 0$;

$$4. D = (D_n)_n.$$

D is Borel pregeometry. □

The following result was proved in [2].

Corollary 3.3 (van Engelen, Kunen, Miller) *Let $X = \mathbb{R}^2$ and suppose $S \subset X$ is analytic. Then either S is covered by countably many lines or S contains a perfect set, no three points of which are collinear.*

Proof Set

1. $\Delta_0 = \emptyset$,
2. $(z, z_1, \dots, z_n) \in \Delta_{n+1}$ iff $z \in \{z_1, \dots, z_n\}$,

and define

1. if $n \neq 3$, $D_n = \Delta_n$,
2. $(z, z_1, z_2) \in D_3$ iff $z_1 \neq z_2$ and z, z_1, z_2 are collinear,
3. $D = (D_n)_n$.

Clearly, D is a notion of dependence.

We contend that D is Borel. To see this, note that the question as to whether $z = (x, y)$, $z_1 = (x_1, y_1)$, $z_2 = (x_2, y_2)$ are collinear reduces to solving for α and β the following system of linear equations.

$$\left. \begin{aligned} \alpha x + \beta y &= 1 \\ \alpha x_1 + \beta y_1 &= 1 \\ \alpha x_2 + \beta y_2 &= 1 \end{aligned} \right\} (S)$$

Hence,

$$(z, z_1, z_2) \in D_3 \text{ iff } \exists \alpha, \beta \in \mathbb{Q}(x, y, x_1, y_1, x_2, y_2) \text{ such that } (S) \text{ holds true.}$$

This shows that D is Borel.

Moreover, if A is a D -free set of cardinality n , there are $n(n-1)/2$ lines through two points of A . Moreover, if $B \subseteq [A]_D$ is D -free, it cannot have more than 2 points lying on the same line among those listed above. Thus, B contains at most $n(n-1)$ points. This shows that D satisfies EFSW. □

The proof of Corollary 3.3 presented in [2] is not easy to generalize. The one given above is, as shown by the following result.

Corollary 3.4 *Let $X = \mathbb{R}^2$ and suppose S is an analytic subset of X . Then either S is covered by countably many lines or S is covered by countably many conics or S contains a perfect set, no three points of which are collinear and no six points of which lie on the same conic.*

Proof Define

1. $D_n = \Delta_n$, if $n \neq 3, 6$;
2. $(z, z_1, z_2) \in D_3$ iff $z_1 \neq z_2$ and z, z_1, z_2 are collinear;
3. $(z, z_1, \dots, z_5) \in D_6$ iff the z_i s are distinct and there exists a conic through (z, z_1, \dots, z_5) ;
4. $D = (D_n)_n$.

Clearly, D is a notion of dependence.

Testing whether six given points lie on a single conic amounts to solving a system of six linear equations with five unknowns. Arguing as above, we conclude that D is Borel.

Suppose a D -free subset A of X has n members. By imitating again the proof of the preceding corollary, it is easily shown that any other D -free set spanning $[A]_D$ has at most $n(n - 1) + 5 \binom{n}{5}$ members. Hence, D satisfies EFSW. \square

4 Preliminaries on the Gandy-Harrington Topology

In this section, we review the basics of the Choquet Space Theory together with the Gandy-Harrington topology (see [4]). We follow closely [12] and [5] and also state and prove a straightforward generalization of a theorem of Louveau concerning Choquet spaces (3.3).

4.1 Let X be a Hausdorff space. The *Choquet game on X* is a two player game which runs as follows. Player \emptyset chooses a nonempty open set V_0 . Then player $\neg\emptyset$ chooses a nonempty open set $V_1 \subseteq V_0$. Then player \emptyset chooses a nonempty open set $V_2 \subseteq V_1$, and so on. \emptyset wins if $\bigcap\{V_i; i < \omega\} = \emptyset$. Otherwise, $\neg\emptyset$ wins. X is a *Choquet space* if $\neg\emptyset$ has a winning strategy for the Choquet game on X .

4.2 We now present some well-known facts about Choquet spaces that will be needed in the sequel.

1. If V is an open subset of a Choquet space, then V is Choquet.
2. The Baire category theorem holds for Choquet spaces: no open subset of a Choquet space is meager.
3. The product of two Choquet spaces is Choquet.
4. The Kuratowski-Ulam theorem carries over to Choquet spaces; in other words, the following are equivalent, whenever X is Choquet and $R \subset X^2$ has the Baire property:
 - (A) $\{x \in X; \{y \in X; (x, y) \in R\} \text{ is comeager in } X\}$ is comeager in X ;
 - (B) R is comeager in X^2 ;
 - (C) $\{y \in X; \{x \in X; (x, y) \in R\} \text{ is comeager in } X\}$ is comeager in X .
5. Complete metric spaces and compact Hausdorff spaces are Choquet.

4.3 First, we introduce some terminology. Given a set X , we define $(X)^n$ to be the set of length n sequences from X whose components are pairwise distinct. A *regular family* $I = (I_n)_n$ on X is a collection such that $I_n \subseteq (X)^n$. If $I = (I_n)_n$ is a regular family, a subset A of X is *I -free* if $(A)^n \subseteq I_n$, for all n .

S_n is the set of permutations of $\{1, \dots, n\}$; for $R \subseteq (X)^n$, R' denotes the set of $(x_1, \dots, x_n) \in (X)^n$ such that $(x_{\pi(1)}, \dots, x_{\pi(n)}) \in R$, for some $\pi \in S_n$ and \check{R} is defined to be $R - (X^n - R)'$. Call $R \subseteq (X)^n$ *symmetric* if $(x_{\pi(1)}, \dots, x_{\pi(n)}) \in R$ whenever $\pi \in S_n$ and $(x_1, \dots, x_n) \in R$. Clearly,

1. R' and \check{R} are symmetric,
2. $R \subseteq R'$ and $\check{R} \subseteq R$,
3. if R is a closed, nowhere dense subset of X^n , so is R' ,
4. if R is an open, dense subset of X^n , so is \check{R} .

Theorem 4.1 *Let X be a perfect, second countable, Choquet space refining a metric space and let $I = (I_n)_n$ be a regular family on X . If each I_n is a comeager subset of the product space X^n , then there exists a perfect, I -free subset of X .*

Proof This is close to the Kuratowski-Mycielski Theorem, see (19.1) in [7]. Here it is natural to prove it in the following way. Fix the metric on X ; since the I_n s are comeager, for all n , one can find a family $(I_n^k)_k$ of dense, open subsets of X^n such that

$$\bigcap \{I_n^k; k < \omega\} \subseteq I_n.$$

Define $J_m \subseteq (X)^{2^m}$ by $(x_1, \dots, x_{2^m}) \in J_m$ if and only if $\forall n \leq m \forall i_1, \dots, i_n$ such that $1 \leq i_1 < \dots < i_n \leq 2^m \forall k \leq m$ we have $(x_{i_1}, \dots, x_{i_n}) \in I_n^k$. It is easily seen that

1. J_m is dense, open in X^{2^m} ;
2. J_m is symmetric.

Let $(U_n)_n$ be a countable basis for X and suppose σ is a winning strategy for the Choquet game on X . After fixing enumerations $\{s_1, \dots, s_{2^n}\}$ of $\{0, 1\}^n$ for each $n < \omega$ we build a family $V_s, s \in \{0, 1\}^{<\omega}$ of nonempty open sets such that

- (1) $V_{s_1} \times \dots \times V_{s_{2^n}} \subseteq J_n$;
- (2) if $s \in \{0, 1\}^n$, $\text{diam } V_s \leq 1/2^n$;
- (3) if n is odd and $s \in \{0, 1\}^n$, then $V_s \subseteq \sigma(V_{s(0)}, \dots, V_{s(n-1)})$;
- (4) V_{s*0}, V_{s*1} are disjoint for all s .

This is done inductively. Suppose V_s s are already constructed for $s \in \{0, 1\}^n$. We define for $1 \leq i \leq 2^n$, $U_i = \sigma(V_{s_i(0)}, \dots, V_{s_i(n-1)})$. Then pick disjoint $V'_i, V''_i \subset U_i$ of diameter $< 2^{-n}$ such that $V'_1 \times V''_1 \times \dots \times V'_{2^n} \times V''_{2^n} \subset J_{n+1}$. This is possible because X is perfect, J_{n+1} is open, dense and so $J_{n+1} \cap (U_1^2 \times \dots \times U_{2^n}^2)$ is open. \square

4.4 Our main tool will be a specific Choquet space, known as the Gandy-Harrington topology. We now state the basic facts on the subject. (See, e.g., [13] or [5] for a more detailed account of this material, together with references.) As usual, $\mathcal{N} = \omega^\omega$. \mathcal{N}^n is endowed with two distinct topology structures.

1. \mathfrak{T}_n : the open sets are (necessarily countable) unions of Σ_1^1 subsets of \mathcal{N}^n .
2. \mathfrak{T}^n is $\mathfrak{T} \times \dots \times \mathfrak{T}$ n times, where $\mathfrak{T} = \mathfrak{T}_1$.

Both \mathfrak{T}_n and \mathfrak{T}^n define a second countable Choquet space on \mathcal{N}^n .

The following fact has been singled out by Louveau. It unravels the connection between \mathfrak{T}_n and \mathfrak{T}^n .

Theorem 4.2 *If $A \subseteq \mathcal{N}$ is comeager (with respect to \mathfrak{T}) and $R \subseteq \mathcal{N}^n$ is Σ_1^1 and nonempty, then $A^n \cap R \neq \emptyset$.*

Proof Let A be comeager and let $R \subseteq \mathcal{N}^n$ be Σ_1^1 . There exists a decreasing family $(V_n)_{n < \omega}$ of dense, open subsets of \mathcal{N} (with respect to \mathfrak{T}) such that

$$B = \bigcap \{V_n : n < \omega\} \subseteq A.$$

Set

$$A_n^i = \mathcal{N}^{i-1} \times V_i \times \mathcal{N}^{n-i}.$$

Then

1. $\bigcap \{A_n^i : 1 \leq i \leq k, k < \omega\} = B^n \subseteq A^n$;
2. each A_n^i is dense, open with respect to \mathfrak{T}_n ; this is because if $S \subseteq \mathcal{N}^n$ is Σ_1^1 and $\neq \emptyset$, the relation S' defined by $S'(x)$ if and only if $\exists x_1 \dots x_{i-1} x_{i+1} \dots x_n$ such that $S(x_1, \dots, x_{i-1}, x, \dots, x_n)$ is Σ_1^1 and nonempty;
3. R is nonempty and open with respect to \mathfrak{T}_n ; thus, by the Baire Category Theorem for the topology \mathfrak{T}_n , $R \cap B^n \neq \emptyset$ and so, $R \cap A^n \neq \emptyset$. \square

5 Proof of the Main Lemma

5.1 From here until the end of this section, we assume

$D = (D_n)_n$ is a Π_1^1 notion of dependence on \mathcal{N} satisfying EFSW.

Also “free” means D -free, $[\cdot]$ is short for $[\cdot]_D$, and so on.

5.2 We introduce an auxiliary definition. Let $A \subseteq X$ be finitely spanned and let N be the height of A . Then the *weak closure of A* , denoted by $\langle A \rangle$, is the set of $x \in X$ such that $x \in [x_1, \dots, x_N]$, for all free $(x_1, \dots, x_N) \in A^N$.

Fact 5.1 Let A be finitely spanned. Then

- (1) $A \subseteq \langle A \rangle \subseteq [A]$,
- (2) $\langle A \rangle$ is finitely spanned,
- (3) if A is Σ_1^1 , then $\langle A \rangle$ is Π_1^1 .

5.3 Define

1. $\mathcal{A} = \{\langle A \rangle : A \text{ is nonempty, finitely spanned, } \Sigma_1^1 \text{ subset of } \mathcal{N}\}$;
2. $H = \bigcup \mathcal{A}$;
3. $H^c = \mathcal{N} - H$.

Fact 5.2

- (1) H is Π_1^1 ;
- (2) every free subset of H is countable.

Proof of (1) First, we claim that

$$H = \bigcup \{\langle A \rangle : A \text{ is nonempty, finitely spanned, } \Delta_1^1 \text{ subset of } \mathcal{N}\}.$$

To see this, let A be a finitely spanned, Σ_1^1 subset of \mathcal{N} . By Fact 5.1(1), $A \subseteq \langle A \rangle$. Moreover, by Fact 5.1(3), A and $\mathcal{N} - \langle A \rangle$ are both Σ_1^1 . Thus, the effective separation theorem supplies a Δ_1^1 set B such that $A \subseteq B$ and $B \cap (\mathcal{N} - \langle A \rangle) = \emptyset$. This proves the claim.

From the claim, it follows that $x \in H$ if and only if $\exists A \in \Delta_1^1$ such that

- (i) $\exists N \forall y_1, \dots, y_{N+1} \in A [(y_1, \dots, y_{N+1}) \text{ is not free}]$,
- (ii) $x \in \langle A \rangle$.

Upon applying the Spector-Gandy theorem (see p. 245 of [14]), we conclude that H is Π_1^1 . \square

Proof of (2) Let X be a free set contained in H . Then

$$X = \bigcup \{X \cap B : B \in \mathcal{A}\}$$

Moreover, $X \cap B$ is finite for every $B \in \mathcal{A}$ and \mathcal{A} is countable. \square

Lemma 5.3 Let $S \subseteq \mathcal{N}$ be Σ_1^1 and put $A = H^c \cap S$; if $A \neq \emptyset$, then, for all n , the set

$$R_n = \{(x_1, \dots, x_n) \in A^n : (x_1, \dots, x_n) \text{ is free}\}.$$

is comeager with respect to \mathfrak{L}^n .

Proof By Fact 5.2(1), A is Σ_1^1 . Thus, so is R_n ; moreover, \mathfrak{T}^n is a Choquet topology extending the standard Baire topology. As proved in [7] p. 153–54 (where it is expressed in terms of Banach-Mazur and Choquet games) this implies that R_n has the property of Baire with respect to this topology.

Moreover,

$$R_n = \{ (x_1, \dots, x_n) \in A^n : (x_1, \dots, x_{n-1}) \text{ is free and } x_n \notin [x_1, \dots, x_{n-1}] \}.$$

Hence, by the Kuratowski-Ulam Theorem for Choquet spaces, it suffices to show that the set

$$A \cap [x_1, \dots, x_{n-1}]$$

is meager in A whenever $x_1, \dots, x_{n-1} \in A$.

Suppose the contrary and pick $x_1, \dots, x_{n-1} \in A$ such that $[x_1, \dots, x_{n-1}] \subset A$ is not meager in A . $[x_1, \dots, x_{n-1}] \subset A$ is Π_1^1 and A is Σ_1^1 . Arguing as above, we conclude that $A \cap [x_1, \dots, x_{n-1}]$ has the property of Baire with respect to \mathfrak{T} .

It follows from this that an open set B of \mathfrak{T} can be found such that

- (i) $[x_1, \dots, x_{n-1}] \cap A \subseteq B \subseteq A$;
- (ii) $[x_1, \dots, x_{n-1}] \cap A$ is comeager in B .

However, D satisfies EFSW. Thus $[x_1, \dots, x_{n-1}]$ has height. Let N be the height of this set. By Theorem 4.2 in Subsection 4.4, we know that

$$[x_1, \dots, x_{n-1}]^{N+1} \cap R_{N+1} \neq \emptyset.$$

This yields a free subset of $[x_1, \dots, x_{n-1}]$ whose cardinality is greater than N , a contradiction. □

5.4 Proof of the Main Lemma Suppose $S \subseteq \mathcal{N}$ is Σ_1^1 and contains an uncountable free subset. By Fact 5.2(2) $A = H^c \cap S$ is nonempty. Using Lemma 5.3, we derive from this that for all n , $R_n \cap A^n$ is comeager in the open set A^n with respect to \mathfrak{T}^n . Moreover, every $\{x\} \in \Sigma_1^1$ lies in H and so, A contains no isolated point. Thus, the requirements for applying Theorem 4.1 in Section 4 are met. This yields a perfect free subset of S .

6 Preliminaries on Stable Theories

The reader is assumed to have some acquaintance with stability theory. The main reference is, of course, [15]. More friendly approaches can be found in several textbooks. We suggest [11]. [8] gives an updated treatment together with important new results. We proceed to remind the reader of a few basic facts that will be needed in the subsequent part of this article.

6.1 From here on,

1. L is a countable, recursive language,
2. \mathfrak{M} is a structure for L ,
3. $T = \text{Th}(\mathfrak{M})$ is the first-order theory of \mathfrak{M} ,
4. $|\mathfrak{M}|$ is the universe of \mathfrak{M} .

Also, if $A \subseteq |\mathfrak{M}|$, $L_m(A)$ denotes the set of formulas of L with parameters in A and free variables among the list v_1, \dots, v_m . $S_m(A)$ is the set of complete types over A (so, every $p \in S_m(A)$ is a subset of $L_m(A)$). A *partial m -type over A* is a consistent subset of $L_m(A)$. If $x = (x_1, \dots, x_m) \in |\mathfrak{M}|^m$, the type of x over $A \subseteq |\mathfrak{M}|$ denoted by $\text{tp}(x/A)$ is the set of formulas $\varphi = \varphi(v_1, \dots, v_m)$ of $L_m(A)$ such that $\mathfrak{M} \models \varphi(x)$.

We extend these notions to the case where $m = \omega$. We will drop mention of m when it is clear from the context.

6.2 Let $B = \{y_i : i < \omega\} \subseteq |\mathfrak{M}|$. We do not distinguish between the set B and the sequence $(y_i : i < \omega)$. Let $A \subseteq |\mathfrak{M}|$ and $B \in |\mathfrak{M}|^\omega$ and suppose $\Phi = \Phi(v)$ is a partial type over $A \cup B$; Φ divides over A if there exists an elementary extension \mathfrak{M}' of \mathfrak{M} and an infinite set $\{B_i : i < \omega\}$ of A -indiscernibles such that

1. $\text{tp}(B_i/A) = \text{tp}(B/A)$ for all $i < \omega$,
2. $\bigcup\{\text{tp}(B_i/A) : i < \omega\}$ is inconsistent.

The following fact was proved by Kim in his Ph.D dissertation. We will use it as a definition of forking. See [8] for the full statement together with a proof.

Theorem 6.1 *If T is stable, $A \subseteq |\mathfrak{M}|$, $B \in |\mathfrak{M}|^\omega$ and $\Phi = \Phi(v)$ is a partial type over $A \cup B$, then Φ forks over A if and only if Φ divides over A .*

6.3 We shall also need the following basic result, a proof of which can be found in [11].

Theorem 6.2 *Let T be stable and suppose \mathfrak{M} is ω_1 -saturated. For every $A \subseteq |\mathfrak{M}|$ and every $p \in S(A)$, there exists a countable subset A_0 of A such that p is the unique nonforking extension of $p \upharpoonright A_0$ over A . Here $p \upharpoonright A_0$ is the type over A_0 whose members are the formulas of p with parameters in A_0 .*

6.4 Recall that a theory is *superstable* if, whenever \mathfrak{M} is a model of T , A is a subset of $|\mathfrak{M}|$ and p is a complete type over A , there exists a finite subset A_0 of A such that p does not fork over A_0 . Every superstable theory is stable.

Fact 6.3 Suppose T is superstable, let $A \subseteq |\mathfrak{M}|$, and let $x \in |\mathfrak{M}|^n$. Then there exists N such that for every $k < \omega$, if $\{y_1, \dots, y_k\}$ is an A -independent set of pairwise distinct tuples from $|\mathfrak{M}|$ and for every i , $(1 \leq i \leq k)$ $\text{tp}(y_i|A \cup \{x\})$ forks over A then $k \leq N$.

A proof of this important result can be found in any good book on stability theory, for example, [11].

6.5 We now present a specific type of notion of dependence. Assume

- (i) \mathfrak{M} is a structure for a countable language,
- (ii) $T = \text{Th}(\mathfrak{M})$ is the first-order theory of \mathfrak{M} ,
- (iii) $A \subseteq |\mathfrak{M}|$.

Definition 6.4

1. $\mathcal{D}_0^{k,A} = \emptyset$;
2. For $x, x_1, \dots, x_n \in |\mathfrak{M}|^k$,

$$(x, x_1, \dots, x_n) \in \mathcal{D}_{n+1}^{k,A} \text{ iff } \text{tp}(x|B) \text{ forks over } A$$

where B denotes the extension of A by the coordinates of (x_1, \dots, x_n) ;

3. $\mathcal{D}^{k,A} = (\mathcal{D}_n^{k,A})_n$.

Fact 6.5

- (1) If T is stable, then $\mathcal{D}^{k,A}$ is a notion of dependence and the $\mathcal{D}^{k,A}$ -free sets are precisely the A -independent sets of k -tuples in the sense of stability theory.

(2) If T is superstable, $\mathcal{D}^{k,A}$ satisfies EFSW.

Proof (1) is a direct translation of the exchange principle for forking in a model of a stable theory. (2) follows from (1) and from Fact 6.3. \square

Nota Bene: the case $k = 1$ will suffice when we shall use the notion of dependence $\mathcal{D}^{k,A}$ in Section 8.

7 Prestructures and Borel over Borel Structures

7.1 Assume

1. L is a first-order language,
2. \mathfrak{M}_0 is a structure for L ,
3. E is an equivalence relation on $|\mathfrak{M}_0|$.

$\mathfrak{M} = \langle \mathfrak{M}_0, E \rangle$ is a *prestructure for L* if the axioms for equality hold true when the nonlogical symbols of L are interpreted using \mathfrak{M}_0 and the equality symbol is interpreted by E . Suppose $\mathfrak{M} = \langle \mathfrak{M}_0, E \rangle$ is a prestructure for L ; define

1. for $x \in |\mathfrak{M}_0|$, \tilde{x} is the equivalence class of x modulo E ,
2. for $x = (x_1, \dots, x_n) \in |\mathfrak{M}_0|^n$, \tilde{x} is $(\tilde{x}_1, \dots, \tilde{x}_n)$,
3. for $A \subseteq |\mathfrak{M}_0|^n$, \tilde{A} is the image of A under $x \mapsto \tilde{x}$,
4. the *factor structure* $\tilde{\mathfrak{M}}$ is the structure for L whose universe is $|\mathfrak{M}_0|/E$ and where the nonlogical symbols are given the natural interpretation.

Let $\mathfrak{M} = \langle \mathfrak{M}_0, E \rangle$ be a prestructure for L and suppose $\varphi = \varphi(v_1, \dots, v_n)$ is a formula of L with free variables among the list v_1, \dots, v_n . For $x_1, \dots, x_n \in |\mathfrak{M}_0|$, set

$$\mathfrak{M} \models \varphi(x_1, \dots, x_n) \quad \text{iff} \quad \tilde{\mathfrak{M}} \models \varphi(\tilde{x}_1, \dots, \tilde{x}_n).$$

This defines the *satisfaction relation for the prestructure \mathfrak{M}* ; also, let

$$\varphi^{\mathfrak{M}} = \{(x_1, \dots, x_n) : \mathfrak{M} \models \varphi(x_1, \dots, x_n)\};$$

the relation $R = \varphi^{\mathfrak{M}}$ on $|\mathfrak{M}_0|$ is said to be *defined by φ in the prestructure \mathfrak{M}* ; we also refer to such relations as (*parameter-free*) *predefinable relations in \mathfrak{M}* . Predefinable relations with parameters from some set $A \subseteq |\mathfrak{M}_0|$ are defined similarly.

7.2 We now extend the notions of Borel and totally Borel structure, as defined in [3] or [16], to make the best of the theory developed thus far. Recall that \mathfrak{M}_0 is a *Borel structure for L* if

1. $|\mathfrak{M}_0| = \omega^\omega$;
2. every nonlogical symbol is interpreted in \mathfrak{M}_0 by a Borel object.

Suppose further that every L -definable relation on ω^ω is Borel. Then \mathfrak{M}_0 is said to be a *totally Borel structure*.

Let $\mathfrak{M} = \langle \mathfrak{M}_0, E \rangle$ be a prestructure; \mathfrak{M} is a *Borel prestructure* if \mathfrak{M}_0 is a Borel structure and E is a Borel equivalence relation. If, in addition to this, every predefinable relation in \mathfrak{M} is Borel, \mathfrak{M} is said to be a *totally Borel prestructure*. The factor structure $\tilde{\mathfrak{M}}$ is *totally Borel over Borel* if \mathfrak{M} is a totally Borel prestructure.

Fact 7.1 Let $\mathfrak{M} = \langle \mathfrak{M}_0, E \rangle$ be a totally Borel prestructure and suppose $A \subseteq \omega^\omega$ ($= |\mathfrak{M}_0|$) is countable. For $x, x_1, \dots, x_n \in (\omega^\omega)^k$ and if B is the extension of A by the coordinates of x_1, \dots, x_n , the condition

$$\text{tp}(\tilde{x}|\tilde{B}) \text{ forks over } \tilde{A}$$

defines a Borel relation on $(\omega^\omega)^k$. As a result, the family $D^{k,A}$ is Borel, for all countable A .

Proof Let $A = \{y_i; i \in \omega\} \subseteq \omega^\omega$, where the y_i s are pairwise inequivalent modulo E . It follows from Subsection 6.2 that the question as to whether

$$\text{tp}(\tilde{x} | \tilde{A} \cup \{\tilde{x}_1, \dots, \tilde{x}_n\}) \text{ forks over } \tilde{A}$$

reduces to deciding whether a uniquely defined first-order theory, obtained from the theory of $\langle \mathfrak{M}_0, E, x, x_1, \dots, x_n, y_0, y_1, \dots \rangle$ by adding a countable set of individual constants together with a given set of axioms is consistent. Since \mathfrak{M} is totally Borel, this expresses a Borel condition on $x, x_1, \dots, x_n, y_0, y_1, \dots$. \square

7.3 The foregoing facts can be rephrased in terms of notions of dependence on a prestructure. Assume

1. $\mathfrak{M} = \langle \mathfrak{M}_0, E \rangle$ is a totally Borel prestructure;
2. $T = \text{Th}(\mathfrak{M})$;
3. $A \subseteq \omega^\omega$ ($= |\mathfrak{M}_0|$) is countable.

$X \subseteq (\omega^\omega)^k$ is *transversal* if it does not contain two distinct k -tuples $(x_1, \dots, x_k), (y_1, \dots, y_k)$ such that $x_i E y_i$ for each i .

Definition 7.2

1. $D_0^{k,A} = \emptyset$.
2. for $x, x_1, \dots, x_n \in (\omega^\omega)^k, (x, x_1, \dots, x_n) \in D_{n+1}^{k,A}$ iff $(\tilde{x}, \tilde{x}_1, \dots, \tilde{x}_n) \in D_{n+1}^{k,\tilde{A}}$.
3. $D^{k,A} = (D_n^{k,A})_n$.

Fact 7.3

- (1) $D^{k,A}$ is Borel.
- (2) If T is stable, then $D^{k,A}$ is a notion of dependence. Every $D^{k,A}$ -free set is a transversal and its image under $x \mapsto \tilde{x}$ is \tilde{A} -independent in $\tilde{\mathfrak{M}}$ in the sense of stability theory.
- (3) Conversely, if T is stable, X is a transversal and \tilde{X} is \tilde{A} -independent in $\tilde{\mathfrak{M}}$, then X is $D^{k,A}$ -free.
- (4) If T is superstable, then $D^{k,A}$ satisfies EFSW.

Proof

- (1) Follows from Fact 7.1 proved in Subsection 7.2.
- (2) In view of Fact 6.5(1), it is left to show that $D^{k,A}$ -free sets are transversals. To see this, note first that, for D a notion of dependence, $A \subseteq [A]_D$. This implies that any D -free sequence (x_1, \dots, x_n) must have no repetition. In particular,

$$\begin{aligned} (x, x_1, \dots, x_n) \in D^{1,A} &\implies (\tilde{x}, \tilde{x}_1, \dots, \tilde{x}_n) \in \mathcal{D}^{1,\tilde{A}} \\ &\implies \tilde{\mathfrak{M}} \models \neg \tilde{x} = \tilde{x}_i \text{ for all } 1 \leq i \leq n \\ &\implies \tilde{\mathfrak{M}} \models x \notin x_i \text{ for all } 1 \leq i \leq n. \end{aligned}$$

A similar statement holds true when $D^{k,A}$ is substituted for $D^{1,A}$.

- (3) This is clear.

- (4) This follows from Subsection 6.5 and the observation (Remark 5.6.5, p. 283 of [1]) that every finitely generated part of a model of a superstable theory has finite weight. \square

8 Forking in Totally Borel Models of a Superstable Theory

8.1 The chief purpose of this section is to establish a result on the saturated models of superstable theories using the machinery developed in the preceding sections. Some preliminary remarks are in order.

Let T be an ω_1 -categorical theory and let \mathfrak{M} be the unique model of T in power $c = 2^\omega$. \mathfrak{M} is known to be saturated. Moreover, Friedman's completeness theorem (see [3] or [16]) implies that, up to isomorphism, \mathfrak{M} is totally Borel. Thus, there are some theories having a totally Borel, saturated model. A moment's thought will convince the reader that the class of theories having such a model is fairly rich, as classifiable theories in the sense of Shelah [15] tend to be of that kind.

However, there are also some theories having no model of this kind. For example, let T be a theory implying that there exists a total ordering of the universe (e.g., Peano arithmetic or the theory of real closed fields). Then T cannot have a totally Borel over Borel ω_1 -saturated model. This follows from a result of Shelah [6] to the effect that a Borel total ordering has no chain of length ω_1 .

Moreover, other theories lie somewhere between these two simple cases. For example, let T be the theory of atomless Boolean algebras and let $\mathcal{B} = \mathcal{P}(\omega)/\text{fin}$, where $\mathcal{P}(\omega)$ is equipped with the usual Boolean operations and fin is the ideal of finite sets. Clearly, $\mathcal{B} \models T$.

Example 8.1 \mathcal{B} is totally Borel over Borel.

To prove this, identify $\mathcal{P}(\omega)$ with $\{0, 1\}^\omega$ and let E_0 be the equivalence relation defined by $x E_0 y$ if and only if $x(n) = y(n)$ for all sufficiently large n . Set $\mathfrak{M} = \langle \mathcal{P}(\omega), E_0 \rangle$. Then $\mathcal{B} = \mathfrak{M}$, which proves the desired result.

Example 8.2 \mathcal{B} is ω_1 -saturated.

See [9], 5.5.

Example 8.3 \mathcal{B} is not ω_2 -saturated. Thus, if $\omega_1 < c$, \mathcal{B} is not saturated.

This follows from a well-known result of Hausdorff to the effect that \mathcal{B} has an (ω_1, ω_1^*) -gap (see [10], Chap. II, ex. 2.4).

Of course, the theory of atomless Boolean algebras is not stable. The following result shows that a model of a superstable theory cannot exhibit the same pattern as \mathcal{B} .

Theorem 8.4 *Let \mathfrak{M} be a totally Borel over Borel model of a superstable theory. If \mathfrak{M} is ω_1 -saturated, then \mathfrak{M} is saturated.*

This section is devoted to proving this fact. First we need a lemma. (This is where our main lemma in Section 2 comes into the picture).

Lemma 8.5 *Let T be superstable and let $\tilde{\mathfrak{M}}$ be a totally Borel over Borel model of T . Suppose further that \tilde{A} is a countable subset of $|\tilde{\mathfrak{M}}|$ and that \tilde{p} is a type over \tilde{A} . If \tilde{p} is realized by an uncountable, \tilde{A} -independent set, then \tilde{p} is realized by an \tilde{A} -independent set of power c .*

Proof of the Lemma Assume

1. $\tilde{\mathfrak{M}}$ is the factor structure of some totally Borel prestructure $\mathfrak{M} = \langle \mathfrak{M}_0, E \rangle$,
2. \tilde{A} is the image under $x \mapsto \tilde{x}$ of some countable $A \subseteq \omega^\omega = |\mathfrak{M}_0|$,
3. there exists some p which is a type over A in the sense of prestructure theory and such that

$$\tilde{p} = \{\varphi(v, \tilde{x}) : p \vdash \varphi(v, x)\},$$

4. S is the set of realizations of p in \mathfrak{M} .

Suppose further that \tilde{S} contains an uncountable set \tilde{X} which is \tilde{A} -independent. We can assume that X is a transversal. Hence, by Fact 7.3(3), X is $D^{k,A}$ -free, where k is the number of variables of the type p . Moreover, by Fact 7.3(1), $D^{k,A}$ is Borel and by Fact 7.3(2) and 7.3(4), $D^{k,A}$ is a notion of dependence satisfying EFSW. Thus, the main lemma (Lemma 2.1 in Section 2.4) implies that the Borel set S contains a perfect $D^{k,A}$ -free subset P . P has cardinality c . Therefore, by Fact 7.3(2), \tilde{P} has also cardinality c . Also, by the same token, \tilde{P} is $\mathcal{D}^{k,\tilde{A}}$ -free, which implies that it is \tilde{A} -independent. □

We now complete the proof of Theorem 8.4.

Proof of the Theorem Let B be a subset of $\tilde{\mathfrak{M}}$ of cardinality $k \leq c$ and suppose p is a type over B with one variable. We show that p is realized in $\tilde{\mathfrak{M}}$.

First notice that B can be assumed to contain an elementary substructure of $\tilde{\mathfrak{M}}$. This implies that p is stationary (i.e., it has a unique nonforking extension on extensions of B). By stability theory, there exists a countable subset A of B such that $p_0 = p \upharpoonright A$ is stationary.

Since $\tilde{\mathfrak{M}}$ is ω_1 -saturated, there exists an uncountable A -independent subset X of $\tilde{\mathfrak{M}}$ whose elements realize p_0 . By Lemma 8.5, X can be assumed to have cardinality c .

Let $F \subseteq B$ be finite. Define $X_F = \{a \in X; \text{tp}(a/A \cup F) \text{ forks over } A\}$. By stability theory, X_F is countable (in fact, we can use Fact 6.3 in Subsection 6.4 to show that it is finite). Now, set

$$Y = \cup\{X_F; F \text{ is a finite subset of } B\}$$

and pick $a \in X - Y$. This is possible because Y has power $k < c = \text{power of } X$. $\text{tp}(a/B)$ is a nonforking extension of p_0 . Therefore, since $p_0 \subseteq p$ is stationary, $\text{tp}(a/B) = p$. Hence, a realizes p . □

9 Final Remarks

The time has come to give questions and perspectives beyond the paper. The referee asked whether the last theorem holds when superstability is reduced to stability. Velickovic suggested to reexamine the work using the methods introduced by Miller in place of the Gandy-Harrington topology (for an account of Miller’s alternative approach, we suggest his five “Paris Lectures,” available on his [home page](#) at the Westfälische Wilhelms-Universität Münster, Germany). The author Joël Combès had a whole program to pursue the investigation of Borel model theory under assumptions related to stability, but he died without writing down his conclusion. The friends who took care that his work be published want to replace the missing part by a final “adieu.”

Joël Combase, 1947–2010

Joël was accepted as a graduate student at Stanford University, due to his brilliant undergraduate career in Paris University. In 1984 he obtained a Ph.D. under the guidance of J. Barwise and S. Feferman and returned to France where he became a teacher in the philosophy department at La Sorbonne. He taught philosophy with emphasis on mathematical logic to his students all his life except in the long periods where he was too ill to work. His lessons in logic were fascinating... for those who were good enough to follow. One of these students, now professor at the University of Lausanne, is certain that he would have missed his love and vocation for mathematical research if Combase had not been his teacher in philosophy. Joël used to explain to the members of the Equipe de Logique his mathematical ideas, which always bore the distinctive mark of his personal approach to logic. To those who became his friends, he has made an unforgettable impression by his intelligence, melancholy, and kindness.

JP Ressayre

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