Homogeneous principal bundles over the upper half-plane

Indranil Biswas

Abstract Let G be a connected complex reductive linear algebraic group, and let $K \subset G$ be a maximal compact subgroup. The Lie algebra of K is denoted by $\mathfrak{k}$. A holomorphic Hermitian principal G-bundle is a pair of the form (E_G, E_K), where E_G is a holomorphic principal G-bundle and $E_K \subset E_G$ is a C^∞-reduction of structure group to K. Two holomorphic Hermitian principal G-bundles (E_G, E_K) and (E'_G, E'_K) are called holomorphically isometric if there is a holomorphic isomorphism of the principal G-bundle E_G with E'_G which takes E_K to E'_K. We consider all holomorphic Hermitian principal G-bundles (E_G, E_K) over the upper half-plane $\mathbb{H}$ such that the pullback of (E_G, E_K) by each holomorphic automorphism of $\mathbb{H}$ is holomorphically isometric to (E_G, E_K) itself. We prove that the isomorphism classes of such pairs are parameterized by the equivalence classes of pairs of the form (χ, A), where $\chi : \mathbb{R} \longrightarrow K$ is a homomorphism, and $A \in \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$ such that $[A, d\chi(1)] = 2\sqrt{-1}\cdot A$. (Here $d\chi : \mathbb{R} \longrightarrow \mathfrak{k}$ is the homomorphism of Lie algebras associated to χ.) Two such pairs (χ, A) and (χ', A') are called equivalent if there is an element $g_0 \in K$ such that $\chi' = \mathrm{Ad}(g_0) \circ \chi$ and $A' = \mathrm{Ad}(g_0)(A)$.

1. Introduction

Let G be a connected complex reductive linear algebraic group. Fix a maximal compact subgroup $K \subset G$. Let $\mathfrak{g}$ (resp., $\mathfrak{k}$) denote the Lie algebra of G (resp., K). The inclusion of $\mathfrak{k}$ in $\mathfrak{g}$ extends (uniquely) to a $\mathbb{C}$-linear isomorphism of Lie algebras $\mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \mathfrak{g}$. Let E_G be a holomorphic principal G-bundle over the upper half-plane $\mathbb{H} \subset \mathbb{C}$. A Hermitian structure on E_G is defined to be a C^∞-reduction of structure group

$$E_K \subset E_G$$

to the subgroup K. By a holomorphic Hermitian principal G-bundle over $\mathbb{H}$ we mean a pair (E_G, E_K) of the above type. Two holomorphic Hermitian principal G-bundles (E_G, E_K) and (E'_G, E'_K) over $\mathbb{H}$ are called holomorphically isometric if there is a holomorphic isomorphism of principal G-bundles

$$\gamma : E_G \longrightarrow E'_G$$

such that $\gamma(E_K) = E'_K$.

Kyoto Journal of Mathematics, Vol. 50, No. 2 (2010), 325–363
DOI 10.1215/0023608X-2009-016,
Received May 20, 2005. Revised December 1, 2009. Accepted December 3, 2009.
Mathematics Subject Classification: Primary 53B35. Secondary 32L05.

We consider all holomorphic Hermitian principal G-bundles (E_G, E_K) over $\mathbb{H}$ such that for each holomorphic automorphism τ of $\mathbb{H}$, the pulled-back holomorphic Hermitian principal G-bundle $(\tau^* E_G, \tau^* E_K)$ is holomorphically isometric to (E_G, E_K) itself. We call such a pair (E_G, E_K) an invariant holomorphic Hermitian principal G-bundle.

Consider all pairs of the form (χ, A), where

- $\chi : \mathbb{R} \longrightarrow K$ is a homomorphism, and
- A is an element of the Lie algebra $\mathfrak{g}$ of G such that $[A, d\chi(1)] = 2\sqrt{-1} \cdot A$,

where $d\chi : \mathbb{R} \longrightarrow \mathfrak{k}$ is the homomorphism of Lie algebras associated to the above homomorphism χ.

Two such pairs (χ, A) and (χ', A') are called equivalent if there is an element $g_0 \in K$ such that

- $\chi'(t) = g_0 \chi(t) g_0^{-1}$ for all $t \in \mathbb{R}$, and
- $A' = \mathrm{Ad}(g_0)(A)$, where $\mathrm{Ad}(g_0)$ is the automorphism of $\mathfrak{g}$ associated to the inner automorphism of G defined by $g \longmapsto g_0 g g_0^{-1}$.

The following theorem classifies all the invariant holomorphic Hermitian principal G-bundles over $\mathbb{H}$ up to a holomorphic isometry (see Theorem 6.4).

THEOREM 1.1

There is a canonical bijection between all the holomorphic isometry classes of invariant holomorphic Hermitian principal G-bundles over $\mathbb{H}$ and all the equivalence classes of pairs of the form (χ, A).

In [BM], all the isomorphism classes of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal $\mathrm{GL}(n,\mathbb{C})$-bundles were classified for every n. If we set $G = \mathrm{GL}(n,\mathbb{C})$ and $K = \mathrm{U}(n)$, then Theorem 1.1 coincides with the classification done in [BM].

The bijection in Theorem 1.1 is constructed in Section 6.

Let $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ denote the universal cover of the group $\mathrm{PSL}(2,\mathbb{R})$. The group of all holomorphic automorphisms of $\mathbb{H}$ is identified with $\mathrm{PSL}(2,\mathbb{R})$, and hence we have an action

$$\phi : \widetilde{\mathrm{SL}(2,\mathbb{R})} \longrightarrow \mathrm{Aut}(\mathbb{H}).$$

An $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle on $\mathbb{H}$ is defined to be a triple $(E_G, E_K; \rho)$, where

- $f : E_G \longrightarrow \mathbb{H}$ is a holomorphic principal G-bundle,
- $E_K \subset E_G$ is a C^∞-reduction of structure group to K, and
- ρ is a C^∞-action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on the total space of E_G,

$$\rho : \widetilde{\mathrm{SL}(2,\mathbb{R})} \times E_G \longrightarrow E_G,$$

such that the following four conditions hold:

(1) $(f \circ \rho)(g, z) = \phi(g)(f(z))$ for all $(g, z) \in \widetilde{\mathrm{SL}(2, \mathbb{R})} \times E_G$,

(2) the actions of G and $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ on E_G commute,

(3) $\rho(\widetilde{\mathrm{SL}(2, \mathbb{R})} \times E_K) = E_K$, and

(4) for each $g \in \widetilde{\mathrm{SL}(2, \mathbb{R})}$, the diffeomorphism $E_G \longrightarrow E_G$ defined by $z \longmapsto \rho(g, z)$ is holomorphic.

Two $\widetilde{\mathrm{SL}(2, \mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundles $(E_G, E_K; \rho)$ and $(E'_G, E'_K; \rho')$ over $\mathbb{H}$ are called isomorphic if there is a holomorphic isomorphism of principal G-bundles

$$\widetilde{h} : E_G \longrightarrow E'_G,$$

such that $\widetilde{h}(E_K) = E'_K$ and $\widetilde{h} \circ \rho = \rho' \circ (\mathrm{Id}_{\widetilde{\mathrm{SL}(2,\mathbb{R})}} \times \widetilde{h})$.

An $\widetilde{\mathrm{SL}(2, \mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle is clearly invariant.

Here we prove that the two categories, namely, the $\widetilde{\mathrm{SL}(2, \mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundles on $\mathbb{H}$ and the invariant holomorphic Hermitian principal G-bundles on $\mathbb{H}$, coincide (see Proposition 2.5). Hence Theorem 1.1 also classifies the $\widetilde{\mathrm{SL}(2, \mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundles on $\mathbb{H}$ (see Theorem 6.3).

2. Invariant principal bundles

2.1. The upper half-plane

Let

$$\mathbb{H} := \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$$

be the upper half of the complex plane. The group of all holomorphic automorphisms of $\mathbb{H}$, which is denoted by $\mathrm{Aut}(\mathbb{H})$, is identified with $\mathrm{PSL}(2, \mathbb{R})$. We recall that $\mathrm{SL}(2, \mathbb{R})$ has an action on $\mathbb{H}$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) = \frac{az + b}{cz + d}.$$

This action descends to a homomorphism

$$\phi_0 : \mathrm{PSL}(2, \mathbb{R}) \longrightarrow \mathrm{Aut}(\mathbb{H}) \tag{2.1}$$

which is in fact an isomorphism.

It is customary to denote the universal cover of the group $\mathrm{PSL}(2, \mathbb{R})$ by $\widetilde{\mathrm{SL}(2, \mathbb{R})}$. Let

$$p : \widetilde{\mathrm{SL}(2, \mathbb{R})} \longrightarrow \mathrm{PSL}(2, \mathbb{R}) \tag{2.2}$$

be the projection. The group $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ acts on $\mathbb{H}$ using the composition homomorphism

$$\phi := \phi_0 \circ p, \tag{2.3}$$

where ϕ_0 is defined in (2.1).

2.2. Invariant principal bundles

Let G be a connected reductive linear algebraic group defined over the field of complex numbers. We fix a maximal compact subgroup

$$K \subset G. \tag{2.4}$$

It is known that any two maximal compact subgroups of G are conjugate (see [He, Theorem 2.1, p. 256]).

We recall that a holomorphic principal G-bundle over $\mathbb{H}$ is a complex manifold E_G and a surjective holomorphic submersion

$$f : E_G \longrightarrow \mathbb{H} \tag{2.5}$$

such that

- the complex manifold E_G is equipped with a right holomorphic action of the complex Lie group G

$$\varphi : E_G \times G \longrightarrow E_G \tag{2.6}$$

(so φ is a holomorphic map),

- $f \circ \varphi = f \circ p_1$, where p_1 is the projection of $E_G \times G$ to the first factor, and
- the action of G is free and transitive on each fiber of the projection f in equation (2.5).

The last condition is equivalent to the assertion that the diagonal map to the fiber product

$$\varphi \times p_1 : E_G \times G \longrightarrow E_G \times_{\mathbb{H}} E_G$$

is an isomorphism. We recall that $E_G \times_{\mathbb{H}} E_G$ is the submanifold of $E_G \times E_G$ consisting of all points (y_1, y_2) such that $f(y_1) = f(y_2)$.

A *Hermitian structure on* a holomorphic principal G-bundle E_G is a C^∞-reduction of structure group of E_G,

$$E_K \subset E_G,$$

to the subgroup K in (2.4). This means that E_K is a K-invariant C^∞-submanifold of E_G, and for each point $y \in \mathbb{H}$, the action of K on $(E_K)_y := E_K \bigcap f^{-1}(y)$ is transitive.

We note that any C^∞-section

$$\sigma : \mathbb{H} \longrightarrow E_G/K,$$

of the natural projection $E_G/K \longrightarrow \mathbb{H}$ constructed using f, yields a C^∞-reduction of structure group of E_G to K. Indeed,

$$E_K := q^{-1}\big(\sigma(\mathbb{H})\big) \subset E_G,$$

where $q : E_G \longrightarrow E_G/K$ is the quotient map, is a C^∞-reduction of structure group of E_G to K.

DEFINITION 2.1

A *holomorphic Hermitian principal G-bundle on* $\mathbb{H}$ is a holomorphic principal G-bundle E_G on $\mathbb{H}$ together with a Hermitian structure E_K on E_G.

Let (E_G, E_K) and (E'_G, E'_K) be two holomorphic Hermitian principal G-bundles over $\mathbb{H}$. Let

$$h : E_K \longrightarrow E'_K$$

be a C^∞-isomorphism of principal K-bundles. Then h extends uniquely to a C^∞-isomorphism

$$\widetilde{h} : E_G \longrightarrow E'_G \tag{2.7}$$

of principal G-bundles. To construct $\widetilde{h}$, we first note that

$$E_G := E_K \times^K G = (E_K \times G)/K,$$

where the action of any $k \in K$ on $E_K \times G$ sends any $(z, g) \in E_K \times G$ to $(zk, k^{-1}g)$. The diffeomorphism

$$h \times \mathrm{Id}_G : E_K \times G \longrightarrow E'_K \times G$$

descends to the map

$$\widetilde{h} : E_K \times^K G \longrightarrow E'_K \times^K G$$

in (2.7) between the quotient spaces.

DEFINITION 2.2

The isomorphism $h : E_K \longrightarrow E'_K$ is called a *holomorphic isometry* if the map $\widetilde{h}$ in (2.7) is holomorphic.

Two holomorphic Hermitian principal G-bundles are called *holomorphically isometric* if there exists a holomorphic isometry between them.

If h is a holomorphic isometry, then $\widetilde{h}$ is also called a *holomorphic isometry.* Since h is the restriction of $\widetilde{h}$ to E_K, the map h is uniquely determined by $\widetilde{h}$. Therefore, there is no abuse of the terminology *holomorphic isometry.*

REMARK 2.3

We recall that a connection ∇ on a holomorphic principal G-bundle is called a *complex connection* if the Lie algebra valued one-form on the total space of E_G defining ∇ is of Hodge type $(1,0)$. Let (E_G, E_K) be a holomorphic Hermitian principal G-bundle over $\mathbb{H}$. There is a unique complex connection ∇ on the holomorphic principal G-bundle E_G which preserves E_K. Equivalently, the principal K-bundle E_K has a unique C^∞-connection ∇_0 such that the connection on E_G induced by ∇_0 is complex. The connection ∇_0 is simply the restriction of ∇ to E_K (see [AB, p. 220] for the details).

For any holomorphic map

$$\tau : \mathbb{H} \longrightarrow \mathbb{H}$$

and any holomorphic Hermitian principal G-bundle (E_G, E_K) over $\mathbb{H}$, the pullback

$$\tau^*(E_G, E_K) := (\tau^* E_G, \tau^* E_K)$$

is clearly a holomorphic Hermitian principal G-bundle over $\mathbb{H}$. This holomorphic Hermitian principal G-bundle $\tau^*(E_G, E_K)$ is called the *pullback* of (E_G, E_K) by τ.

DEFINITION 2.4

A holomorphic Hermitian principal G-bundle (E_G, E_K) on $\mathbb{H}$ is called *invariant* if for each $\tau \in \mathrm{Aut}(\mathbb{H})$, the pulled-back holomorphic Hermitian principal G-bundle $(\tau^* E_G, \tau^* E_K)$ is holomorphically isometric to (E, h).

An $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-*homogeneous* holomorphic Hermitian principal G-bundle on $\mathbb{H}$ is defined to be a triple $(E_G, E_K; \rho)$, where

- (E_G, E_K) is a holomorphic Hermitian principal G-bundle on $\mathbb{H}$ and
- ρ is a C^∞-action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on the total space of E_G,

$$\rho : \widetilde{\mathrm{SL}(2,\mathbb{R})} \times E_G \longrightarrow E_G, \tag{2.8}$$

such that the following four conditions hold:

(1) $(f \circ \rho)(g, z) = \phi(g)(f(z))$ for all $(g, z) \in \widetilde{\mathrm{SL}(2,\mathbb{R})} \times E_G$, where ϕ is constructed in (2.3) and f is the projection as in (2.5),

(2) the actions of G and $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_G commute,

(3) $\rho(\widetilde{\mathrm{SL}(2,\mathbb{R})} \times E_K) = E_K$, and

(4) for each $g \in \widetilde{\mathrm{SL}(2,\mathbb{R})}$, the diffeomorphism $E_G \longrightarrow E_G$ defined by $z \longmapsto \rho(g, z)$ is holomorphic.

Two $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundles $(E_G, E_K; \rho)$ and $(E'_G, E'_K; \rho')$ are called *isomorphic* if there is a holomorphic isometry

$$\widetilde{h} : E_G \longrightarrow E'_G$$

such that $\widetilde{h} \circ \rho = \rho' \circ (\mathrm{Id}_{\widetilde{\mathrm{SL}(2,\mathbb{R})}} \times \widetilde{h})$.

We note that for any $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E_G, E_K; \rho)$ over $\mathbb{H}$, and any element $g \in \widetilde{\mathrm{SL}(2,\mathbb{R})}$, the map

$$E_G \longrightarrow E_G$$

defined by $z \longmapsto \rho(g, z)$ is a holomorphic isometry of the pulled-back holomorphic Hermitian principal G-bundle $(\phi(g^{-1})^* E_G, \phi(g^{-1})^* E_K)$ with (E_G, E_K), where ϕ is constructed in (2.3). Therefore, any $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle is invariant.

Although, a priori, the condition for a holomorphic Hermitian principal G-bundle to be a $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle is stronger than the condition for it to be invariant, the following proposition shows that the two definitions are in fact equivalent.

PROPOSITION 2.5
Let (E_G, E_K) be an invariant holomorphic Hermitian principal G-bundle over $\mathbb{H}$. Then the principal G-bundle E_G admits a C^∞-action

$$\rho : \widetilde{\mathrm{SL}(2,\mathbb{R})} \times E_G \longrightarrow E_G$$

such that the triple $(E_G, E_K; \rho)$ is an $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle.

Furthermore, there is exactly one action ρ of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_G which makes $(E_G, E_K; \rho)$ an $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle.

Proof
Let U denote the group of all holomorphic isometries of the holomorphic Hermitian principal G-bundle (E_G, E_K) with itself. Therefore, U is the group of all holomorphic automorphisms

$$\widetilde{h} : E_G \longrightarrow E_G$$

such that

- $f \circ \widetilde{h} = \widetilde{h}$, where f as in (2.5) is the projection of E_G to $\mathbb{H}$,
- $\widetilde{h}(E_K) = E_K$, and
- $\widetilde{h}$ commutes with the action of G on E_G.

We show that U is a compact Lie group.

To prove that U is a compact Lie group, fix a point $x_0 \in \mathbb{H}$. Let $\mathrm{Aut}((E_K)_{x_0})$ denote the group of all diffeomorphisms of the fiber $(E_K)_{x_0}$ which commute with the action of K on $(E_K)_{x_0}$. Now, fix a point $z_0 \in (E_K)_{x_0}$. We have an isomorphism of groups

$$h_{z_0} : K \longrightarrow \mathrm{Aut}\big((E_K)_{x_0}\big)$$

which sends any $g \in K$ to the automorphism of $(E_K)_{x_0}$ defined by $z_0 g' \longmapsto z_0 g g'$, where $g' \in K$. We note that this isomorphism h_{z_0} depends on the choice of z_0. However, for any $g_0 \in K$, we have

$$h_{z_0 g_0}(g) = h_{z_0}(g_0 g g_0^{-1})$$

for all $g \in K$, where $h_{z_0 g_0} : K \longrightarrow \mathrm{Aut}((E_K)_{x_0})$ is the isomorphism that sends any $g'' \in K$ to the automorphism of $(E_K)_{x_0}$ defined by $z_0 g_0 g' \longmapsto z_0 g_0 g'' g'$, where $g' \in K$. Therefore, the Lie group $\mathrm{Aut}((E_K)_{x_0})$ is compact and connected.

The group $\mathrm{Aut}((E_K)_{x_0})$ has the following alternative description. Let

$$\mathrm{Ad}(E_K) = E_K \times^K K \longrightarrow \mathbb{H} \tag{2.9}$$

be the adjoint bundle. So $\mathrm{Ad}(E_K)$ is associated to the principal K-bundle E_K for the adjoint action of K on itself. We recall that the adjoint action on K of any $g_0 \in K$ is defined by $g \longmapsto g_0 g g_0^{-1}$. Therefore, two points (z, g) and (z', g') of $E_K \times K$ are identified in the quotient space $\mathrm{Ad}(E_K)$ if and only if there is an element $k \in K$ such that $z' = zk$ and $g' = k^{-1}gk$. The fibers of the natural projection $\mathrm{Ad}(E_K) \longrightarrow \mathbb{H}$ are groups isomorphic to K. The fiber $\mathrm{Ad}(E_K)_{x_0}$ of $\mathrm{Ad}(E_K)$ over x_0 is canonically identified with the group $\mathrm{Aut}((E_K)_{x_0})$ defined earlier. To construct the action of any $y \in \mathrm{Ad}(E_K)_{x_0}$ on $(E_K)_{x_0}$, fix a point $(z_0, g_0) \in (E_K)_{x_0} \times K$ which projects to y. The action of y on $(E_K)_{x_0}$ is defined by $z_0 g \longmapsto z_0 g_0 g$, where $g \in K$.

Let

$$\rho_1 : U \longrightarrow \mathrm{Aut}\big((E_K)_{x_0}\big) \tag{2.10}$$

be the homomorphism that sends any holomorphic isometry $\widetilde{h}$ of (E_G, E_K) to its restriction $\widetilde{h}(x_0)|_{(E_K)_{x_0}}$ to the fiber $(E_K)_{x_0}$. The homomorphism ρ_1 in (2.10) is in fact injective. Indeed, any automorphism $\widetilde{h} \in U$ of E_G leaves invariant the unique complex connection on E_G which preserves the reduction $E_K \subset E_G$ (see Remark 2.3). We note that any automorphism of a principal bundle over a connected base which preserves a given connection is uniquely determined by the restriction of the automorphism over one fixed point of the base. Indeed, over any other point of the base, the automorphism is simply the parallel transport of the automorphism over the fixed point of the base. Consequently, the homomorphism ρ_1 in (2.10) is injective.

The subgroup U of the compact Lie group $\mathrm{Aut}((E_K)_{x_0})$ is clearly closed. Hence it follows that U is a compact Lie group.

Let $\widetilde{U}$ denote the group of all pairs of the form (γ, T), where $\gamma \in \mathrm{Aut}(\mathbb{H})$ and

$$T : (\gamma^{-1})^* E_G \longrightarrow E_G$$

is a holomorphic isometry between the two holomorphic Hermitian principal G-bundles (E_G, E_K) and $((\gamma^{-1})^* E_G, (\gamma^{-1})^* E_K)$. The group operation on $\widetilde{U}$ is defined by

$$(\gamma_1, T_1)(\gamma, T) = (\gamma_1 \circ \gamma, T'),$$

where T' is the composition

$$\big((\gamma_1 \circ \gamma)^{-1}\big)^* E_G = (\gamma_1^{-1})^*(\gamma^{-1})^* E_G \xrightarrow{(\gamma_1^{-1})^* T} (\gamma_1^{-1})^* E_G \xrightarrow{T_1} E_G.$$

It can be shown that $\widetilde{U}$ is a finite-dimensional Lie group. (This also follows from the short exact sequence in (2.11).)

Since (E_G, E_K) is an invariant holomorphic Hermitian principal G-bundle, for each $\gamma \in \mathrm{Aut}(\mathbb{H})$ there is at least one holomorphic isometry T such that $(\gamma, T) \in \widetilde{U}$. Consequently, we have a short exact sequence of Lie groups

$$e \longrightarrow U \xrightarrow{I} \widetilde{U} \xrightarrow{H_0} \mathrm{Aut}(\mathbb{H}) \longrightarrow e, \tag{2.11}$$

where H_0 is the homomorphism defined by $(\gamma, T) \longmapsto \gamma$, and I is the homomorphism defined by $\widetilde{h} \longmapsto (\mathrm{Id}_{\mathbb{H}}, \widetilde{h})$.

The Lie algebra of U (resp., $\widetilde{U}$) is denoted by $\mathfrak{u}$ (resp., $\widetilde{\mathfrak{u}}$). Let

$$0 \longrightarrow \mathfrak{u} \xrightarrow{\iota} \widetilde{\mathfrak{u}} \xrightarrow{h_0} \mathrm{sl}(2, \mathbb{R}) \longrightarrow 0 \tag{2.12}$$

be the short exact sequence of Lie algebras associated to the exact sequence in (2.11); the Lie algebra of $\mathrm{Aut}(\mathbb{H})$ is identified with $\mathrm{sl}(2, \mathbb{R})$ using the isomorphism of Lie algebras associated to the homomorphism ϕ in (2.3).

The Lie algebra $\mathrm{sl}(2, \mathbb{R})$ is simple. Hence the homomorphism h_0 in (2.12) splits. In other words, there is a homomorphism of Lie algebras

$$h_1 : \mathrm{sl}(2, \mathbb{R}) \longrightarrow \widetilde{\mathfrak{u}} \tag{2.13}$$

such that $h_0 \circ h_1 = \mathrm{Id}_{\mathrm{sl}(2,\mathbb{R})}$ (see [Bou, Corollaire 3, p. 91]).

We fix a homomorphism of Lie algebras h_1 as in (2.13) such that $h_0 \circ h_1 = \mathrm{Id}_{\mathrm{sl}(2,\mathbb{R})}$. We show that the image of h_1 commutes with the image of the homomorphism ι in (2.12).

To prove that the images of h_1 and ι commute, we first note that $\mathfrak{u}$ in (2.12) is an ideal of the Lie algebra $\widetilde{\mathfrak{u}}$. Hence, using the Lie algebra operation of $\widetilde{\mathfrak{u}}$, we have a homomorphism from $\widetilde{\mathfrak{u}}$ to the Lie algebra of derivations of $\mathfrak{u}$. We denote by $\mathrm{Der}(\mathfrak{u})$ the Lie algebra of derivations of $\mathfrak{u}$. Let

$$\delta : \mathrm{sl}(2, \mathbb{R}) \longrightarrow \mathrm{Der}(\mathfrak{u}) \tag{2.14}$$

be the homomorphism of Lie algebras which sends any $w \in \mathrm{sl}(2, \mathbb{R})$ to the derivation of $\mathfrak{u}$ given by $h_1(w)$, where h_1 is the homomorphism in (2.13).

Let

$$Z(\mathfrak{u}) \subset \mathfrak{u} \tag{2.15}$$

be the center. We recall that $\mathfrak{u}$ is the Lie algebra of a compact Lie group. So the quotient map $\mathfrak{u} \longrightarrow \mathfrak{u}/Z(\mathfrak{u})$ identifies $[\mathfrak{u}, \mathfrak{u}]$ with $\mathfrak{u}/Z(\mathfrak{u})$. Furthermore,

$$\mathrm{Der}(\mathfrak{u}) = \mathfrak{u}/Z(\mathfrak{u}) = [\mathfrak{u}, \mathfrak{u}].$$

The homomorphism $[\mathfrak{u}, \mathfrak{u}] \longrightarrow \mathrm{Der}(\mathfrak{u})$ is defined by the Lie algebra operation of $\mathfrak{u}$. In particular, all the elements of $\mathrm{Der}(\mathfrak{u})$ are semisimple. On the other hand, the Lie algebra $\mathrm{sl}(2, \mathbb{R})$ has nilpotent elements, and also, it is simple. From the properties of the Jordan decomposition (see [Bor, Section 4.4, Theorem, pp. 83–84]), we know that any homomorphism of Lie algebras

$$\mathrm{sl}(2, \mathbb{R}) \longrightarrow [\mathfrak{u}, \mathfrak{u}]$$

takes nilpotent elements of $\mathrm{sl}(2, \mathbb{R})$ to nilpotent elements of $[\mathfrak{u}, \mathfrak{u}]$. Hence there is no nonzero homomorphism of Lie algebras from $\mathrm{sl}(2, \mathbb{R})$ to $[\mathfrak{u}, \mathfrak{u}]$. In particular, the homomorphism δ in (2.14) vanishes.

Since the homomorphism δ in (2.14) vanishes, it follows immediately that the images of h_1 and ι commute. We next show that the splitting homomorphism h_1 in (2.13) is actually unique.

Take any homomorphism of Lie algebras

$$h_1' : \mathrm{sl}(2, \mathbb{R}) \longrightarrow \widetilde{\mathfrak{u}}$$

such that $h_0 \circ h_1' = \mathrm{Id}_{\mathrm{sl}(2,\mathbb{R})}$. (The homomorphism h_0 is defined in (2.12).) From the above observation that the images of h_1 and ι commute, it follows immediately that there is a homomorphism of Lie algebras

$$\alpha_0 : \mathrm{sl}(2, \mathbb{R}) \longrightarrow \mathfrak{u}$$

such that $h_1' = h_1 + \iota \circ \alpha_0$, where h_1 is the fixed homomorphism in (2.13).

Now, $\mathfrak{u} = [\mathfrak{u}, \mathfrak{u}] \bigoplus Z(\mathfrak{u})$ (see (2.15)). We already noted that there is no nonzero homomorphism from $\mathrm{sl}(2, \mathbb{R})$ to $[\mathfrak{u}, \mathfrak{u}] = \mathrm{Der}(\mathfrak{u})$. On the other hand, there is no nonzero homomorphism from $\mathrm{sl}(2, \mathbb{R})$ to the abelian Lie algebra $Z(\mathfrak{u})$. Consequently, there is no nonzero homomorphism from $\mathrm{sl}(2, \mathbb{R})$ to $\mathfrak{u}$. Hence we conclude that there is exactly one homomorphism of Lie algebras

$$h_1 : \mathrm{sl}(2, \mathbb{R}) \longrightarrow \widetilde{\mathfrak{u}}$$

such that

$$h_0 \circ h_1 = \mathrm{Id}_{\mathrm{sl}(2,\mathbb{R})} .$$

Now we are in a position to complete the proof of the proposition. Since the group $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ is simply connected, the space of all homomorphisms

$$f_1 : \widetilde{\mathrm{SL}(2, \mathbb{R})} \longrightarrow \widetilde{U} \tag{2.16}$$

such that $H_0 \circ f_1 = p$, where H_0 and p are constructed in (2.11) and (2.2), respectively, are parameterized by the space of all homomorphism of Lie algebras

$$h_1 : \mathrm{sl}(2, \mathbb{R}) \longrightarrow \widetilde{\mathfrak{u}}$$

such that $h_0 \circ h_1 = \mathrm{Id}_{\mathrm{sl}(2,\mathbb{R})}$, where h_0 is constructed in (2.12). We proved that there is exactly one such homomorphism h_1 of Lie algebras. Consequently, there is exactly one homomorphism of Lie groups f_1 as in (2.16) such that $H_0 \circ f_1 = p$.

Now we define the map

$$\rho : \widetilde{\mathrm{SL}(2, \mathbb{R})} \times E_G \longrightarrow E_G \tag{2.17}$$

which sends any $(g, z) \in \widetilde{\mathrm{SL}(2, \mathbb{R})} \times E_G$ to $f_1(g)(z)$, where f_1 is the unique homomorphism in (2.16). It is straightforward to check that ρ in (2.17) satisfies all the conditions that are imposed on the map in (2.8). Consequently, $(E_G, E_K; \rho)$ is an $\widetilde{\mathrm{SL}(2, \mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle. This proves the first part of the proposition.

Let

$$\rho' : \widetilde{\mathrm{SL}(2, \mathbb{R})} \times E_G \longrightarrow E_G$$

be another action of $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ on E_G which makes $(E_G, E_K; \rho')$ an $\widetilde{\mathrm{SL}(2, \mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle. Let

$$f_1' : \widetilde{\mathrm{SL}(2, \mathbb{R})} \longrightarrow \widetilde{U}$$

be the homomorphism (the group $\widetilde{U}$ is as in (2.11)) that sends any g to the automorphism of E_G defined by $z \longmapsto \rho'(g,z)$. It is easy to see that $H_0 \circ f_1' = p$, where H_0 and p are constructed in (2.11) and (2.2), respectively. Now from the uniqueness of the homomorphism f_1 in (2.16) it follows immediately that f_1' coincides with f_1. Consequently, ρ' coincides with ρ constructed in (2.17). This completes the proof of the proposition. $\square$

In view of Proposition 2.5, given any invariant holomorphic Hermitian principal G-bundle (E_G, E_K) over $\mathbb{H}$, the unique action ρ of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_G, which makes $(E_G, E_K; \rho)$ a $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle, is automatically assumed.

3. Invariant holomorphic Hermitian $\mathbb{C}^*$-bundles

Invariant holomorphic Hermitian principal $\mathbb{C}^*$-bundles on $\mathbb{H}$ were investigated in [BM, Section 3]. We reexamine them in this section from the present point of view, which is to reconstruct invariant holomorphic Hermitian principal bundles from the principal $\mathbb{R}$-bundle on $\mathbb{H}$ obtained from the action on it of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

Set $x_0 := \sqrt{-1} \in \mathbb{H}$. Let

$$H_{x_0} \subset \widetilde{\mathrm{SL}(2,\mathbb{R})} \tag{3.1}$$

be the isotropy subgroup of x_0 for the action ϕ of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathbb{H}$ constructed in (2.3). We have a surjective homomorphism

$$\alpha : \mathbb{R} \longrightarrow p(H_{x_0}) \subset \mathrm{PSL}(2,\mathbb{R}) \tag{3.2}$$

defined by

$$\theta \longmapsto \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix},$$

where p is the homomorphism in (2.2). Let

$$\widetilde{\alpha} : \mathbb{R} \longrightarrow H_{x_0} \tag{3.3}$$

be the unique homomorphism such that $p \circ \widetilde{\alpha} = \alpha$, where α is defined in (3.2). It is easy to see that $\widetilde{\alpha}$ is an isomorphism. Using $\widetilde{\alpha}$, we have

$$\mathbb{H} = \widetilde{\mathrm{SL}(2,\mathbb{R})}/\mathbb{R}.$$

The corresponding quotient map

$$\widetilde{\mathrm{SL}(2,\mathbb{R})} \longrightarrow \widetilde{\mathrm{SL}(2,\mathbb{R})}/\mathbb{R} = \mathbb{H} \tag{3.4}$$

defines a C^∞-principal $\mathbb{R}$-bundle over $\mathbb{H}$. The group $\mathbb{R} = H_{x_0}$ acts on $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ as right translations. Let

$$q : F_{\mathbb{R}} := \widetilde{\mathrm{SL}(2,\mathbb{R})} \longrightarrow \mathbb{H} \tag{3.5}$$

be the principal $\mathbb{R}$-bundle over $\mathbb{H}$ defined by the quotient map in (3.4).

The tangent bundle $T\widetilde{\mathrm{SL}(2,\mathbb{R})}$ of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is equipped with a nondegenerate symmetric bilinear form defined by the Killing form on the Lie algebra $\mathrm{sl}(2,\mathbb{R})$ of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. The restriction of this bilinear form to the line subbundle

$$\mathrm{kernel}(dq) \subset T\widetilde{\mathrm{SL}(2,\mathbb{R})}$$

is clearly nondegenerate, where $dq : T\widetilde{\mathrm{SL}(2,\mathbb{R})} \longrightarrow q^*T\mathbb{H}$ is the differential of the map q in (3.5). Note that the subbundle kernel(dq) coincides with the one defined by the orbits of the right translation action, on $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, of the subgroup H_{x_0} in (3.1). Consider the orthogonal projection of $T\widetilde{\mathrm{SL}(2,\mathbb{R})}$ to kernel(dq). As the Lie algebra of $H_{x_0} = \mathbb{R}$ (see (3.3)) is identified with the Lie algebra $\mathbb{R}$, this orthogonal projection defines a C^∞ one-form

$$\omega_0 \in C^\infty\big(\widetilde{\mathrm{SL}(2,\mathbb{R})}; \Omega^1_{\widetilde{\mathrm{SL}(2,\mathbb{R})}}\big) \tag{3.6}$$

on $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Using the fact that the Killing form on $\mathrm{sl}(2,\mathbb{R})$ is invariant under the adjoint action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathrm{sl}(2,\mathbb{R})$, it is easy to deduce that ω_0 defines a C^∞-connection on the principal $\mathbb{R}$-bundle $F_{\mathbb{R}}$ in (3.5). This C^∞-connection on $F_{\mathbb{R}}$ is denoted by $\nabla^0_{\mathbb{R}}$.

The curvature of the above connection $\nabla^0_{\mathbb{R}}$ on $F_{\mathbb{R}}$ is denoted by $\mathcal{K}(\nabla^0_{\mathbb{R}})$. Since the group $\mathbb{R}$ is abelian and its Lie algebra is $\mathbb{R}$, the adjoint vector bundle $\mathrm{ad}(F_{\mathbb{R}}) = F_{\mathbb{R}} \times^{\mathbb{R}} \mathbb{R}$ is the trivial line bundle $\mathbb{H}\times\mathbb{R}$ over $\mathbb{H}$. (The construction of the adjoint bundle is recalled in (4.10).) In other words, we have a C^∞-isomorphism

$$\mu : \mathrm{ad}(F_{\mathbb{R}}) \longrightarrow \mathbb{H}\times\mathbb{R} \tag{3.7}$$

of vector bundles. Using this isomorphism, we have

$$\mathcal{K}(\nabla^0_{\mathbb{R}}) \in C^\infty(\mathbb{H}; \Omega^2_{\mathbb{H}}), \tag{3.8}$$

or in other words, the curvature $\mathcal{K}(\nabla^0_{\mathbb{R}})$ is a C^∞ two-form on $\mathbb{H}$.

The left translation action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on itself commutes with the right translation action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on itself. Also, the projection in (3.4) intertwines the left translation action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on itself and the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathbb{H}$ defined by ϕ in (2.3). In other words, the left translation action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on itself makes $F_{\mathbb{R}}$ a smooth $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous principal $\mathbb{R}$-bundle. The connection $\nabla^0_{\mathbb{R}}$ on $F_{\mathbb{R}}$ constructed above is preserved by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $F_{\mathbb{R}}$. Consequently, the curvature form $\mathcal{K}(\nabla^0_{\mathbb{R}})$ in (3.8) is preserved by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathbb{H}$. This immediately implies that

$$\mathcal{K}(\nabla^0_{\mathbb{R}}) = \frac{\lambda_0\sqrt{-1}}{\mathrm{Im}(z)}\, dz \bigwedge d\overline{z} \tag{3.9}$$

for some $\lambda_0 \in \mathbb{R}$, where z is the tautological holomorphic function on $\mathbb{C}$.

From the definition of the curvature of a connection, we have

$$q^*\mathcal{K}(\nabla^0_{\mathbb{R}}) = d\omega_0,$$

where ω_0 is the one-form in (3.6), and q is the projection in (3.5). Using this, it can be shown that the constant λ_0 in (3.9) is -1.

Take any $t \in \mathbb{R}$. Let

$$\rho_t : \mathbb{R} \longrightarrow \mathrm{U}(1) \subset \mathbb{C}^* \tag{3.10}$$

be the homomorphism defined by $\lambda \longmapsto \exp(\sqrt{-1} \cdot t\lambda)$. Let

$$F^t_{\mathbb{C}^*} := F_{\mathbb{R}}(\rho_t) \tag{3.11}$$

be the C^∞-principal $\mathbb{C}^*$-bundle over $\mathbb{H}$ obtained by extending the structure group of the principal $\mathbb{R}$-bundle $F_{\mathbb{R}}$ in (3.5) using the homomorphism ρ_t in (3.10). Since ρ_t factors through the subgroup $\mathrm{U}(1) \subset \mathbb{C}^*$, the principal $\mathbb{C}^*$-bundle $F^t_{\mathbb{C}^*}$ in (3.11) is equipped with a tautological reduction of structure group

$$F^t_U \subset F^t_{\mathbb{C}^*} \tag{3.12}$$

to $\mathrm{U}(1)$. Note that F^t_U is identified with the principal $\mathrm{U}(1)$-bundle over $\mathbb{H}$ obtained by extending the structure group of the principal $\mathbb{R}$-bundle $F_{\mathbb{R}}$ using ρ_t.

The connection $\nabla^0_{\mathbb{R}}$ on $F_{\mathbb{R}}$ induces a connection on the principal $\mathrm{U}(1)$-bundle F^t_U in (3.12). This induced connection on F^t_U is denoted by ∇^t. The connection on the principal $\mathbb{C}^*$-bundle $F^t_{\mathbb{C}^*}$ induced by $\nabla^0_{\mathbb{R}}$ is denoted by $\nabla^t_{\mathbb{C}}$. We note that the connection ∇^t on F^t_U induces the connection $\nabla^t_{\mathbb{C}}$ on $F^t_{\mathbb{C}^*}$ using the reduction of structure group in (3.12).

The Lie algebra $\mathrm{Lie}(\mathrm{U}(1))$ of $\mathrm{U}(1)$ is identified with $\mathbb{R}$ using the homomorphism of Lie algebras associated to the homomorphism ρ_1 (see (3.10)). In terms of this identification, the homomorphism of Lie algebras

$$\mathbb{R} = \mathrm{Lie}(\mathbb{R}) \longrightarrow \mathrm{Lie}\big(\mathrm{U}(1)\big) = \mathbb{R}$$

associated to the homomorphism ρ_t in (3.10) is multiplication by t. Hence from (3.9) it follows that the curvature $\mathcal{K}(\nabla^t)$ of the connection ∇^t satisfies the identity

$$\mathcal{K}(\nabla^t) = \frac{t\lambda_0\sqrt{-1}}{\mathrm{Im}(z)}\, dz \bigwedge \overline{z}. \tag{3.13}$$

The curvature of the connection $\nabla^t_{\mathbb{C}}$ on $F^t_{\mathbb{C}^*}$ clearly coincides with $\mathcal{K}(\nabla^t)$.

As the differential form $\mathcal{K}(\nabla^t)$ is of type $(1,1)$ (any nonzero two-form on a Riemann surface is of Hodge type $(1,1)$), there is a unique holomorphic structure on the principal $\mathbb{C}^*$-bundle $F^t_{\mathbb{C}^*}$ which satisfies the following two conditions:

- the natural projection of $F^t_{\mathbb{C}^*}$ to $\mathbb{H}$ is holomorphic, and
- the connection form $\nabla^t_{\mathbb{C}}$ on $F^t_{\mathbb{C}^*}$, which is a complex differential form on $F^t_{\mathbb{C}^*}$, is of Hodge type $(1,0)$

(see [Ko, Proposition 4.17, p. 12]).

Therefore, the pair $(F^t_{\mathbb{C}^*}, F^t_U)$ is a holomorphic Hermitian principal $\mathbb{C}^*$-bundle over $\mathbb{H}$. The following proposition shows that they classify all the invariant holomorphic Hermitian principal $\mathbb{C}^*$-bundles.

PROPOSITION 3.1

The holomorphic Hermitian principal $\mathbb{C}^*$-bundle $(F^t_{\mathbb{C}^*}, F^t_U)$ constructed above is invariant. Given any invariant holomorphic Hermitian principal $\mathbb{C}^*$-bundle $(E_{\mathbb{C}^*}, E_{\mathrm{U}(1)})$ on $\mathbb{H}$, there is a unique $t \in \mathbb{R}$ such that $(F^t_{\mathbb{C}^*}, F^t_U)$ is holomorphically isometric to $(E_{\mathbb{C}^*}, E_{\mathrm{U}(1)})$.

Proof

We noted earlier that using the identification $F_{\mathbb{R}} := \widetilde{\mathrm{SL}(2,\mathbb{R})}$ in (3.5), the left translation action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on itself makes $F_{\mathbb{R}}$ a smooth $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous principal $\mathbb{R}$-bundle. Therefore, we have an induced action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on any fiber bundle associated to the principal $\mathbb{R}$-bundle $F_{\mathbb{R}}$. More precisely, if $\mathbb{R}$ acts on T, then the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $F_{\mathbb{R}} \times T$ defined by the trivial action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on T and the above action on $F_{\mathbb{R}}$ descends to an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on the associated fiber bundle $F_{\mathbb{R}} \times^{\mathbb{R}} T$. In particular, both $F^t_{\mathbb{C}^*}$ and F^t_U are equipped with an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. It is easy to check that these actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $F^t_{\mathbb{C}^*}$ and F^t_U make $(F^t_{\mathbb{C}^*}, F^t_U)$ an $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal $\mathbb{C}^*$-bundle. In particular, the holomorphic Hermitian principal $\mathbb{C}^*$-bundle $(F^t_{\mathbb{C}^*}, F^t_U)$ is invariant.

Let $(E_{\mathbb{C}^*}, E_{\mathrm{U}(1)})$ be an invariant holomorphic Hermitian principal $\mathbb{C}^*$-bundle over $\mathbb{H}$. Let $\nabla^{\mathbb{C}^*}_1$ be the corresponding complex connection on the principal $\mathbb{C}^*$-bundle $E_{\mathbb{C}^*}$ (see Remark 2.3). The curvature

$$\mathcal{K}(\nabla^{\mathbb{C}^*}_1) \in C^\infty(\mathbb{H}; \Omega^2_{\mathbb{H}})$$

of the above connection $\nabla^{\mathbb{C}^*}_1$ satisfies the condition

$$\mathcal{K}(\nabla^{\mathbb{C}^*}_1) = \frac{t_0\sqrt{-1}}{\mathrm{Im}(z)}\, dz \bigwedge \overline{z} \tag{3.14}$$

for some $t_0 \in \mathbb{R}$. Indeed, this follows immediately from the fact that any $\mathrm{Aut}(\mathbb{H})$-invariant real two-form on $\mathbb{H}$ must be a constant real multiple of $(\sqrt{-1}/\mathrm{Im}(z))\, dz \bigwedge \overline{z}$.

Therefore, from (3.13) it follows that the curvature $\mathcal{K}(\nabla^{t_0/\lambda_0})$ of the connection ∇^{t_0/λ_0} on the principal $\mathbb{C}^*$-bundle $F^{t_0/\lambda_0}_{\mathbb{C}^*}$ coincides with $\mathcal{K}(\nabla^{\mathbb{C}^*}_1)$, where λ_0 is the nonzero constant in (3.9). Now using [BM, Lemma 3.1, p. 6], we conclude that the two holomorphic Hermitian principal $\mathbb{C}^*$-bundles $(F^{t_0/\lambda_0}_{\mathbb{C}^*}, F^{t_0/\lambda_0}_U)$ and $(E_{\mathbb{C}^*}, E_{\mathrm{U}(1)})$ are holomorphically isometric.

Since λ_0 in (3.9) is nonzero, from (3.13) we conclude that $\mathcal{K}(\nabla^t) \neq \mathcal{K}(\nabla^s)$ if $t \neq s$. In particular, $(F^t_{\mathbb{C}^*}, F^t_U)$ and $(F^s_{\mathbb{C}^*}, F^s_U)$ are not holomorphically isometric if $t \neq s$. Therefore, any invariant holomorphic Hermitian principal $\mathbb{C}^*$-bundle over $\mathbb{H}$ is holomorphically isometric to $(F^t_{\mathbb{C}^*}, F^t_U)$ for exactly one $t \in \mathbb{R}$. This completes the proof of the proposition. $\square$

COROLLARY 3.2

The holomorphic isometry classes of invariant holomorphic Hermitian principal

$\mathbb{C}^*$-bundles over $\mathbb{H}$ are parameterized by $\mathbb{R}$. For each $t \in \mathbb{R}$, the corresponding invariant holomorphic Hermitian principal $\mathbb{C}^*$-bundle is $(F^t_{\mathbb{C}^*}, F^t_U)$.

Similarly, the isomorphism classes of all $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal $\mathbb{C}^*$-bundles over $\mathbb{H}$ are parameterized by $\mathbb{R}$, and for each $t \in \mathbb{R}$, the corresponding $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal $\mathbb{C}^*$-bundle is $(F^t_{\mathbb{C}^*}, F^t_U)$.

Proof
The first part of the corollary is a reformulation of Proposition 3.1. Also, in view of Proposition 2.5, the second part of the corollary is deduced using its first part. $\square$

4. The center of G and the action of an isotropy subgroup

Let E_G be a holomorphic principal G-bundle over $\mathbb{H}$. Its adjoint bundle

$$\mathrm{Ad}(E_G) := E_G \times^G G \tag{4.1}$$

is the holomorphic fiber bundle over $\mathbb{H}$ associated to E_G for the adjoint action of G on itself. So $\mathrm{Ad}(E_G)$ is a quotient of $E_G \times G$, and two points (z_1, g_1) and (z_2, g_2) of $E_G \times G$ are identified in $\mathrm{Ad}(E_G)$ if and only if there is an element $g_0 \in G$ such that

$$(z_2, g_2) = \big(z_1 g_0^{-1}, \mathrm{Ad}(g_0)g_1\big),$$

where

$$\mathrm{Ad}(g_0) : G \longrightarrow G \tag{4.2}$$

is the inner automorphism defined by $g \longmapsto g_0 g g_0^{-1}$. Since $\mathrm{Ad}(g_0)$ is an automorphism of the group G for all g_0, the fibers of $\mathrm{Ad}(E_G)$ are groups isomorphic to G. For each point $x \in \mathbb{H}$, the fiber $\mathrm{Ad}(E_G)_x$ is the group of all diffeomorphisms of $(E_G)_x$ which commute with the action of G on $(E_G)_x$. Fix a point $z \in \mathrm{Ad}(E_G)_x$. Using it, we get an isomorphism

$$\psi_z : G \longrightarrow \mathrm{Ad}(E_G)_x \tag{4.3}$$

which sends any $g \in G$ to the image of (z, g) in $\mathrm{Ad}(E_G)_x$.

Let

$$Z(G) \subset G \tag{4.4}$$

be the center of the group G. For any element $g \in Z(G)$, let t_g be the automorphism of E_G defined by $z \longmapsto zg$. Since the actions of $Z(G)$ and G on E_G commute, this map t_g defines a holomorphic section of $\mathrm{Ad}(E_G)$ over $\mathbb{H}$. Therefore, for each point $x \in \mathbb{H}$, we have an injective homomorphism of groups

$$\widetilde{\beta}_x : Z(G) \longrightarrow \mathrm{Ad}(E_G)_x \tag{4.5}$$

which sends any $g \in Z(G)$ to the automorphism of $(E_G)_x$ defined by $z \longmapsto zg$.

Let

$$Z(K) = Z(G) \cap K \tag{4.6}$$

be the center of the maximal compact subgroup K in (2.4). Let

$$\beta_x : Z(K) \longrightarrow \mathrm{Ad}(E_G)_x \tag{4.7}$$

be the restriction of the homomorphism $\widetilde{\beta}_x$ in (4.5).

Let (E_G, E_K) be an invariant holomorphic Hermitian principal G-bundle over $\mathbb{H}$. The corresponding action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_G (see Proposition 2.5) induces an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on the total space of $\mathrm{Ad}(E_G)$ which lifts the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathbb{H}$ defined by ϕ in (2.3). We noted earlier that for any point $x \in \mathbb{H}$, the fiber $\mathrm{Ad}(E_G)_x$ is the group of all diffeomorphisms of $(E_G)_x$ which commute with the action of G on $(E_G)_x$. Hence the action of the isotropy subgroup H_{x_0} (see (3.1)) on $(E_G)_{x_0}$ yields a homomorphism

$$\gamma_{x_0} : H_{x_0} = \mathbb{R} \longrightarrow \mathrm{Ad}(E_G)_{x_0}. \tag{4.8}$$

(As before, H_{x_0} is identified with $\mathbb{R}$ using $\widetilde{\alpha}$ in (3.3).)

REMARK 4.1

The image of the injective homomorphism $\widetilde{\beta}_x$ in (4.5) is the center of the group $\mathrm{Ad}(E_G)_x$. Also, the image of the homomorphism γ_{x_0} in (4.8) lies in the subgroup

$$\mathrm{Ad}(E_K)_{x_0} \subset \mathrm{Ad}(E_G)_{x_0}$$

which preserves $(E_K)_{x_0} \subset (E_G)_{x_0}$ (see (2.9)). Therefore, the following two statements are equivalent.

(1) The image of the homomorphism γ_{x_0} in (4.8) lies inside the center of $\mathrm{Ad}(E_G)_{x_0}$.

(2) The homomorphism γ_{x_0} satisfies the condition that there be a homomorphism $\xi : H_{x_0} \longrightarrow Z(K)$, where $Z(K)$ is defined in (4.6), such that $\gamma_{x_0} = \beta_{x_0} \circ \xi$, where β_{x_0} is constructed in (4.7).

PROPOSITION 4.2

Let $(E_G, E_K; \rho)$ be a $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle over $\mathbb{H}$. Let

$$\xi : H_{x_0} \longrightarrow Z(K) \tag{4.9}$$

be a homomorphism, where $Z(K)$ is defined in (4.6). Assume that

$$\gamma_{x_0} = \beta_{x_0} \circ \xi,$$

where β_{x_0} and γ_{x_0} are constructed in (4.7) and (4.8), respectively. Then the following two hold.

(1) *The underlying C^∞-principal G-bundle E_G does not admit any different holomorphic structure $\widehat{E}_G$ for which $(\widehat{E}_G, E_K; \rho)$ is also a $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle.*

(2) *Let* $(E'_G, E'_K; \rho')$ *be another* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$*-homogeneous holomorphic Hermitian principal* G*-bundle over* $\mathbb{H}$ *such that*

$$\gamma'_{x_0} = \beta'_{x_0} \circ \xi,$$

where ξ *is the homomorphism in (4.9), and the two homomorphisms* β'_{x_0} *and* γ'_{x_0} *are constructed exactly as* β_{x_0} *and* γ_{x_0} *are constructed in (4.7) and (4.8), respectively, after replacing* $(E_G, E_K; \rho)$ *with* $(E'_G, E'_K; \rho)$*. Then the two* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$*-homogeneous holomorphic Hermitian principal* G*-bundles* $(E_G, E_K; \rho)$ *and* $(E'_G, E'_K; \rho')$ *are isomorphic.*

Proof

Let $\mathfrak{g}$ denote the Lie algebra of the group G. Let

$$\mathrm{ad}(E_G) = E_G \times^G \mathfrak{g} \tag{4.10}$$

be the holomorphic vector bundle over $\mathbb{H}$ associated to the principal G-bundle E_G for the adjoint action of G on $\mathfrak{g}$. We note that the total space of $\mathrm{ad}(E_G)$ is a quotient of $E_G \times \mathfrak{g}$, and two points (z_1, v_1) and (z_2, v_2) of $E_G \times \mathfrak{g}$ are identified in $\mathrm{ad}(E_G)$ if there is an element $g_0 \in G$ such that $(z_2, v_2) = (z_1 g_0^{-1}, \mathrm{Ad}(g_0)(v_1))$, where $\mathrm{Ad}(g_0)$ is the automorphism of Lie algebras associated to the inner automorphism of G in (4.2).

The action ρ of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_G induces an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathrm{ad}(E_G)$. Indeed, the diagonal action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $E_G \times \mathfrak{g}$ constructed using the trivial action on $\mathfrak{g}$ descends to an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on the quotient space $\mathrm{ad}(E_G)$.

Assume that the C^∞-principal G-bundle E_G admits a different holomorphic structure $\widehat{E}_G$ for which $(\widehat{E}_G, E_K; \rho)$ is an $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle. We note that any two holomorphic structures on the C^∞-principal G-bundle E_G differ by a smooth $(0,1)$-form on $\mathbb{H}$ with values in $\mathrm{ad}(E_G)$.

Let

$$\theta \in C^\infty\big(\mathbb{H}; \Omega^{0,1}_{\mathbb{H}}(\mathrm{ad}(E_G))\big) \tag{4.11}$$

be the section by which the holomorphic structure of $\widehat{E}_G$ differs from that of E_G. Since the holomorphic structures of both E_G and $\widehat{E}_G$ are preserved by the action ρ of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, it follows immediately that the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ leaves the section θ in (4.11) invariant. In particular, the action of the isotropy subgroup H_{x_0} (see (3.1)) on the fiber $(T^{0,1}_{x_0})^* \otimes \mathrm{ad}(E_G)_{x_0}$ fixes the vector

$$\theta(x_0) \in (T^{0,1}_{x_0})^* \otimes \mathrm{ad}(E_G)_{x_0}. \tag{4.12}$$

Consider the adjoint action of G on $\mathfrak{g}$. Its restriction to $Z(G)$ is the trivial action (see (4.4)). Hence from the given condition

$$\gamma_{x_0} = \beta_{x_0} \circ \xi,$$

we conclude that H_{x_0} acts trivially on the fiber $\mathrm{ad}(E_G)_{x_0}$. On the other hand, H_{x_0} acts nontrivially on the complex line $T^{0,1}_{x_0}$. Consequently, there is no nonzero vector in $(T^{0,1}_{x_0})^* \otimes \mathrm{ad}(E_G)_{x_0}$ which is fixed by the action of H_{x_0}. Therefore, the invariant vector $\theta(x_0)$ in (4.12) must vanish. Since the section θ in (4.11) is left invariant by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ and the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathbb{H}$ is transitive, we now conclude that

$$\theta = 0.$$

In other words, the holomorphic structure of $\widehat{E}_G$ coincides with that of E_G. This completes the proof of the first part of the proposition.

To prove the second part, define

$$\ell : M_G(E,E') \longrightarrow \mathbb{H} \tag{4.13}$$

to be the holomorphic fiber bundle whose fiber $M_G(E,E')_x := \ell^{-1}(x)$ over any point $x \in \mathbb{H}$ is the space of all G-equivariant isomorphisms $(E_G)_x \longrightarrow (E'_G)_x$, where E'_G is as in the statement of the proposition. We note that the group $\mathrm{Ad}(E_G)_x$ acts freely transitively on the right of $M_G(E,E')_x$, and the group $\mathrm{Ad}(E'_G)_x$ acts freely transitively on the left of $M_G(E,E')_x$. The two actions ρ and ρ', of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_G and E'_G, respectively, together define an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $M_G(E,E')$. More precisely, the action of any $A \in \widetilde{\mathrm{SL}(2,\mathbb{R})}$ sends any $\xi \in M_G(E,E')_x$ to the map $(E_G)_x \longrightarrow (E'_G)_x$ defined by $\rho(A,z) \longmapsto \rho'(A,\xi(z))$, where $z \in (E_G)_x$.

Let

$$\begin{array}{ccc} M_K(E,E') & \subset & M_G(E,E') \\ \big\downarrow & & \\ \mathbb{H} & & \end{array} \tag{4.14}$$

be the subbundle whose fiber $M_K(E,E')_x$ over any point $x \in \mathbb{H}$ is the space of all K-equivariant isomorphisms $(E_K)_x \longrightarrow (E'_K)_x$. The action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $M_G(E,E')$ preserves $M_K(E,E')$. Indeed, this follows from the fact that the actions ρ and ρ' preserve E_K and E'_K, respectively.

Recall that $\gamma'_{x_0} = \beta'_{x_0} \circ \xi$ and $\gamma_{x_0} = \beta'_{x_0} \circ \xi$ (see the statement of the proposition). From these it follows immediately that the isotropy subgroup H_{x_0} acts trivially on the fiber $M_G(E,E')_{x_0}$. Using this, we show that the fiber bundle $M_K(E,E')$ (see (4.14)) admits a C^∞-section that is left invariant by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

To prove this, fix any element

$$\tau_0 \in M_K(E,E')_{x_0}.$$

Now, consider the orbit $\mathbf{O}(\tau_0)$ of τ_0 under the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $M_K(E,E')$. Since H_{x_0} acts trivially on $M_G(E,E')_{x_0}$ and $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ acts transitively on $\mathbb{H}$, we conclude that the projection

$$\ell|_{\mathbf{O}(\tau_0)} : \mathbf{O}(\tau_0) \longrightarrow \mathbb{H}$$

is a C^∞-section of the fiber bundle $M_K(E,E')$ in (4.14), where ℓ is the projection in (4.13).

Any $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-invariant C^∞-section of the fiber bundle $M_K(E,E')$ defines a C^∞-isomorphism $E_G \longrightarrow E'_G$ of principal G-bundles which takes E_K to E'_K and also intertwines the actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Now the second part of the proposition follows from the first part. This completes the proof of the proposition. $\square$

Fix any homomorphism

$$\chi : H_{x_0} = \mathbb{R} \longrightarrow K \subset G \tag{4.15}$$

to the maximal compact subgroup. (Recall that H_{x_0} is identified with $\mathbb{R}$ using $\widetilde{\alpha}$ in (3.3).) Let

$$E_G^\chi \longrightarrow \mathbb{H} \quad (\text{resp., } E_K^\chi \longrightarrow \mathbb{H}) \tag{4.16}$$

be the C^∞-principal G-bundle (resp., principal K-bundle) obtained by extending the structure group of the principal $\mathbb{R}$-bundle $F_\mathbb{R}$ (see (3.5)) using the homomorphism χ in (4.15). Let

$$\nabla^G \in C^\infty\left(E_G^\chi; \mathfrak{g} \otimes_\mathbb{C} (T^* E_G^\chi)\right) \tag{4.17}$$

be the connection on the principal G-bundle E_G^χ induced by the connection $\nabla_\mathbb{R}^0$ on $F_\mathbb{R}$ defined by the form ω_0 in (3.6). The curvature of ∇^G on E_G^χ is of Hodge type $(1,1)$. (Any nonzero two-form on $\mathbb{H}$ is of Hodge type $(1,1)$.) Therefore, there is a unique holomorphic structure on E_G^χ such that the natural projection $E_G^\chi \longrightarrow \mathbb{H}$ is holomorphic, and the $\mathfrak{g}$-valued one-form ∇^G in (4.17) is of Hodge type $(1,0)$ (see [Ko, Proposition 4.17, p. 12]).

Since $E_G^\chi = \widetilde{\mathrm{SL}(2,\mathbb{R})} \times^\mathbb{R} G$, using the left translation action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on itself we get an action ρ' of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on the principal G-bundle E_G^χ. This action ρ' of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_G^χ clearly preserves the submanifold

$$E_K^\chi = \widetilde{\mathrm{SL}(2,\mathbb{R})} \times^\mathbb{R} K \subset E_G^\chi.$$

Therefore, $(E_G^\chi, E_K^\chi; \rho')$ is a $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle over $\mathbb{H}$.

Consider the fiber $(E_G^\chi)_{x_0}$ of E_G^χ over the point $x_0 \in \mathbb{H}$ (see (3.1)). We have a natural biholomorphism

$$G \longrightarrow (E_G^\chi)_{x_0} \tag{4.18}$$

which sends any $g \in G$ to the image of

$$(e, g) \in \widetilde{\mathrm{SL}(2,\mathbb{R})} \times G$$

in the quotient space $E_G^\chi = \widetilde{\mathrm{SL}(2,\mathbb{R})} \times^\mathbb{R} G$, where e is the identity element of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. The biholomorphism in (4.18) takes the action of G on the fiber $(E_G^\chi)_{x_0}$ to the right translation action of G on itself. Similarly, the fiber $\mathrm{Ad}(E_G^\chi)_{x_0}$ of the adjoint bundle $\mathrm{Ad}(E_G^\chi)$ is identified with the group G. More precisely, we have an isomorphism of algebraic groups

$$G \longrightarrow \mathrm{Ad}(E_G^\chi)_{x_0} \tag{4.19}$$

which sends any $g \in G$ to the image of

$$(e, g) \in \widetilde{\mathrm{SL}(2,\mathbb{R})} \times G$$

in the quotient space $\mathrm{Ad}(E_G^{\chi})$ of $\widetilde{\mathrm{SL}(2,\mathbb{R})} \times G$ (see (4.1)). The isomorphism in (4.19) clearly takes the subgroup $K \subset G$ isomorphically to the subgroup $\mathrm{Ad}(E_K^{\chi})_{x_0} \subset \mathrm{Ad}(E_G^{\chi})_{x_0}$.

Consider the homomorphism

$$\gamma'_{x_0} : H_{x_0} \longrightarrow \mathrm{Ad}(E_G^{\chi})_{x_0} = G$$

constructed as in (4.8) for the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E_G^{\chi}, E_K^{\chi}; \rho')$. It is now straightforward to check that it satisfies the identity

$$\gamma'_{x_0} = \chi, \tag{4.20}$$

where χ is the homomorphism in (4.15).

If we fix a point of $(E_G^{\chi})_{x_0}$, then we get an isomorphism of G with $\mathrm{Ad}(E_G^{\chi})_{x_0}$ (see (4.3)). Let

$$z_0 \in (E_G^{\chi})_{x_0} \tag{4.21}$$

be the image of $(e, e) \in \widetilde{\mathrm{SL}(2,\mathbb{R})} \times G$, where e denotes the identity element. The isomorphism $G \longrightarrow \mathrm{Ad}(E_G^{\chi})_{x_0}$ corresponding to z_0 clearly coincides with the one in (4.19).

We summarize the above construction in the following lemma.

LEMMA 4.3

Take any homomorphism $\chi : H_{x_0} = \mathbb{R} \longrightarrow K$ as in (4.15). Then there is a natural $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E_G^{\chi}, E_K^{\chi}; \rho')$ and a point $z_0 \in (E_K^{\chi})_{x_0}$ (see (4.21)) such that the corresponding identification of $\mathrm{Ad}(E_G^{\chi})_{x_0}$ with G has the following property.

The homomorphism

$$\gamma'_{x_0} : H_{x_0} \longrightarrow \mathrm{Ad}(E_G^{\chi})_{x_0} = G$$

constructed as in (4.8) coincides with χ.

Note that if $z \in (E_K^{\chi})_{x_0}$, then the isomorphism ψ_z in (4.3) takes K to $\mathrm{Ad}(E_K^{\chi})_{x_0}$.

DEFINITION 4.4

Any given $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E_G, E_K; \rho)$ over $\mathbb{H}$ is called *split* if there is a homomorphism

$$\chi : H_{x_0} \longrightarrow K$$

such that the corresponding $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle in Lemma 4.3 is isomorphic to $(E_G, E_K; \rho)$.

An invariant holomorphic Hermitian principal G-bundle over $\mathbb{H}$ is called *split* if it corresponds, by Proposition 2.5, to a split $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle.

REMARK 4.5

From Corollary 3.2 it follows that all invariant holomorphic Hermitian principal $\mathbb{C}^*$-bundles over $\mathbb{H}$ are split.

PROPOSITION 4.6

Let $h_1 : H_{x_0} \longrightarrow K$ and $h_2 : H_{x_0} \longrightarrow K$ be two homomorphisms. Let $(E_G^1, E_K^1; \rho^1)$ and $(E_G^2, E_K^2; \rho^2)$ be the split $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundles over $\mathbb{H}$ corresponding to h_1 and h_2, respectively (see Lemma 4.3). Then $(E_G^1, E_K^1; \rho^1)$ is isomorphic to $(E_G^2, E_K^2; \rho^2)$ if and only if there is an element $g_0 \in K$ such that

$$h_2(t) = g_0 h_1(t) g_0^{-1}$$

for all $t \in H_{x_0}$.

Proof

We recall that E_G^i (resp., E_K^i) is the extension of structure group of the principal $\mathbb{R}$-bundle $F_{\mathbb{R}}$ using h_i (see (4.16)), where $i = 1, 2$.

First, assume that there is an element $g_0 \in K$ such that

$$h_2(t) = g_0 h_1(t) g_0^{-1}$$

for all $t \in H_{x_0}$. Consider the diffeomorphism

$$\widetilde{g}_0 : \widetilde{\mathrm{SL}(2,\mathbb{R})} \times G \longrightarrow \widetilde{\mathrm{SL}(2,\mathbb{R})} \times G$$

defined by $(z, g) \longmapsto (z, g_0 g)$. This diffeomorphism $\widetilde{g}_0$ evidently descends to a holomorphic map between the quotient spaces

$$\widetilde{g}'_0 : E_G^1 \longrightarrow E_G^2. \tag{4.22}$$

This descended map $\widetilde{g}'_0$ clearly takes E_K^1 to E_K^2. Furthermore, $\widetilde{g}'_0$ intertwines the actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. (Note that this also follows immediately from Proposition 2.5.) Therefore, $(E_G^1, E_K^1; \rho^1)$ is isomorphic to $(E_G^2, E_K^2; \rho^2)$.

To prove the converse, let

$$\zeta : E_G^1 \longrightarrow E_G^2 \tag{4.23}$$

be a holomorphic map such that

- $\zeta(E_K^1) = E_K^2$, and
- ζ intertwines the actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$.

For $i = 1, 2$, let

$$\gamma_{x_0}^i : H_{x_0} \longrightarrow \mathrm{Ad}(E_G^i)_{x_0}$$

be the homomorphism constructed as in (4.8) (see also Lemma 4.3). The isomorphism

$$\widetilde{\zeta}:\ \mathrm{Ad}(E_G^1)_{x_0} \longrightarrow \mathrm{Ad}(E_G^2)_{x_0}$$

induced by ζ in (4.23) clearly satisfies the identity

$$\gamma^2_{x_0} = \widetilde{\zeta} \circ \gamma^1_{x_0}. \tag{4.24}$$

For $i = 1, 2$, let

$$z_i \in (E_K^i)_{x_0} \tag{4.25}$$

be a point that satisfies the condition in Lemma 4.3 which says that the corresponding isomorphism

$$G \longrightarrow \mathrm{Ad}(E_G^i)_{x_0} \tag{4.26}$$

(see (4.3)) takes the homomorphism h_i to the homomorphism $\gamma^i_{x_0}$. Let $g_0 \in K$ be the unique element such that

$$\zeta(z_1) = z_2 g_0, \tag{4.27}$$

where z_1 and z_2 are the points in (4.25).

Since the isomorphism in (4.26) takes $\gamma^i_{x_0}$ to h_i, using (4.24) it follows that

$$h_2(t) = g_0 h_1(t) g_0^{-1}$$

for all $t \in H_{x_0}$, where g_0 is the element in (4.27). This completes the proof of the proposition. □

REMARK 4.7

We use the notation of Proposition 4.6. For $i = 1, 2$, let $\mathrm{ad}(E_K^i)$ be the adjoint vector bundle of E_K^i. So $\mathrm{ad}(E_K^i)$ is a quotient of $\widetilde{\mathrm{SL}(2,\mathbb{R})} \times \mathfrak{k}$, where $\mathfrak{k}$ is the Lie algebra of K. Note that two elements (z, k) and (z', k') of $\widetilde{\mathrm{SL}(2,\mathbb{R})} \times \mathfrak{k}$ are identified in $\mathrm{ad}(E_K^i)$ if and only if there is an element $g_0 \in H_{x_0}$ such that $(z', k') = (zg_0^{-1}, \mathrm{Ad}(h_i(g_0))(k))$ (see (4.10)). Consider the isomorphism of Lie algebras

$$\mathfrak{k} \longrightarrow \mathrm{ad}(E_K^i)_{x_0} \tag{4.28}$$

which sends any $k \in \mathfrak{k}$ to the element in $\mathrm{ad}(E_K^i)_{x_0}$ defined by $(e, k) \in \widetilde{\mathrm{SL}(2,\mathbb{R})} \times \mathfrak{k}$, where e is the identity element of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Let

$$\widehat{g}'_0:\ \mathrm{ad}(E_K^1)_{x_0} \longrightarrow \mathrm{ad}(E_K^2)_{x_0}$$

be the isomorphism constructed using $\widetilde{g}'_0$ defined in (4.22). (Recall that $\widetilde{g}'_0(E_K^1) = E_K^2$.) In terms of the identifications in (4.28), the above isomorphism $\widehat{g}'_0$ coincides with the automorphism $\mathrm{Ad}(g_0)$ of $\mathfrak{k}$.

The set of all isomorphism classes of split $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundles over $\mathbb{H}$ are contained in the set of all isomorphism classes of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundles. In the next section we construct a retraction map to this subset (see Remark 5.13).

5. Splitting of an invariant bundle

Let

$$(E_G, E_K; \rho)$$

be an $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle over $\mathbb{H}$. Consider the homomorphism γ_{x_0} constructed in (4.8). Since the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_G preserves E_K, the image of γ_{x_0} lies inside the subgroup $\mathrm{Ad}(E_K)_{x_0} \subset \mathrm{Ad}(E_G)_{x_0}$. Let

$$d\gamma_{x_0} : \mathbb{R} \longrightarrow \mathrm{ad}(E_K)_{x_0} \tag{5.1}$$

be the homomorphism of Lie algebras associated to γ_{x_0}; here the Lie algebra of H_{x_0} is identified with $\mathbb{R}$ using $\widetilde{\alpha}$ in (3.3). Also, note that the Lie algebra of $\mathrm{Ad}(E_K)_{x_0}$ is $\mathrm{ad}(E_K)_{x_0}$.

Let $\mathfrak{k}$ denote the Lie algebra of the compact Lie group K.

We recall that $\mathrm{ad}(E_K)_{x_0}$ (resp., $\mathrm{ad}(E_G)_{x_0}$) is a quotient of $(E_K)_{x_0} \times \mathfrak{k}$ (resp., $(E_G)_{x_0} \times \mathfrak{g}$). Two points of (z_1, v_1) and (z_2, v_2) of $(E_K)_{x_0} \times \mathfrak{k}$ (resp., $(E_G)_{x_0} \times \mathfrak{g}$) are identified in $\mathrm{ad}(E_K)_{x_0}$ (resp., $\mathrm{ad}(E_G)_{x_0}$) if and only if there is $g \in K$ (resp., $g \in G$) such that $(z_2, v_2) = (z_1 g^{-1}, \mathrm{Ad}(g)(v_1))$ (see (4.10)). Let

$$q_K : (E_K)_{x_0} \times \mathfrak{k} \longrightarrow \mathrm{ad}(E_K)_{x_0} \tag{5.2}$$

be the quotient map. Similarly, let

$$q_G : (E_G)_{x_0} \times \mathfrak{g} \longrightarrow \mathrm{ad}(E_G)_{x_0} \tag{5.3}$$

be the quotient map.

Fix any point

$$z_0 \in (E_K)_{x_0}. \tag{5.4}$$

Let

$$\delta_{z_0} : \mathfrak{k} \longrightarrow \mathrm{ad}(E_K)_{x_0} \tag{5.5}$$

be the isomorphism of Lie algebras defined by $v \longmapsto q_K(z_0, v)$, where q_K is the map in (5.2). Now, define

$$\varpi := \big((\delta_{z_0})^{-1} \circ d\gamma_{x_0}\big)(1) \in \mathfrak{k}, \tag{5.6}$$

where $d\gamma_{x_0}$ and δ_{z_0} are defined in (5.1) and (5.5), respectively.

LEMMA 5.1

The conjugacy class of the element ϖ in (5.6) is independent of the choice of the element z_0 in (5.4).

Take any element $\varpi' \in \mathfrak{k}$ in the conjugacy class of ϖ. Then there is some $z' \in (E_K)_{x_0}$ such that if we replace z_0 by z' in the construction of ϖ, then the resulting element in (5.6) is ϖ'.

Proof

Take any $g_0 \in G$. Let

$$\delta_z : \mathfrak{g} \longrightarrow \mathrm{ad}(E_G)_{x_0}$$

be the isomorphism of Lie algebras defined by $v \longmapsto q_G(z_0 g_0, v)$, where q_G is the map in (5.3) and z_0 is the element in (5.4). Then

$$\delta_z = \delta_{z_0} \circ \mathrm{Ad}(g_0) \tag{5.7}$$

($\mathrm{Ad}(g_0)$ is the inner automorphism of $\mathfrak{g}$ defined before; see (4.10)).

Now set

$$\varpi_z := (\delta_z)^{-1}\big(d\gamma_{x_0}(1)\big) \in \mathfrak{g},$$

where $d\gamma_{x_0}$ is defined in (5.1) and δ_z is defined above. From (5.7) we have

$$\varpi_z = \mathrm{Ad}(g_0^{-1})(\varpi). \tag{5.8}$$

This proves the first part of the lemma.

To prove the second part, take $g_0 \in K$ such that $\varpi' = \mathrm{Ad}(g_0^{-1})(\varphi)$. Then from (5.8) it follows that $z' = z_0 g_0$ satisfies the condition in the lemma. $\square$

Fix an element

$$k_0 \in \mathfrak{k} \tag{5.9}$$

which lies in the conjugacy class, in the Lie algebra $\mathfrak{k}$, of the element ϖ in (5.6). From Lemma 5.1 we know that this conjugacy class depends only on $(E_G, E_K; \rho)$. Let

$$K_0 \subset K \tag{5.10}$$

be the centralizer of the element k_0 in (5.9) for the adjoint action of K on $\mathfrak{k}$. Let

$$G_0 \subset G \tag{5.11}$$

be the centralizer of $k_0 \in \mathfrak{k} \subset \mathfrak{g}$ for the adjoint action of G on $\mathfrak{g}$. This subgroup G_0 coincides with the complex Lie subgroup of G generated by the compact subgroup K_0 in (5.10).

Define

$$\mathcal{S}_K := \big\{ z \in (E_K)_{x_0} \;\big|\; q_K(z, k_0) = d\gamma_{x_0}(1) \big\} \subset (E_K)_{x_0}, \tag{5.12}$$

where $d\gamma_{x_0}$ and q_K are the maps in (5.1) and (5.2), respectively, and k_0 is the element in (5.9). Similarly, define

$$\mathcal{S}_G := \big\{ z \in (E_G)_{x_0} \;\big|\; q_G(z, k_0) = d\gamma_{x_0}(1) \big\} \subset (E_G)_{x_0}, \tag{5.13}$$

where q_G is the map in (5.3).

PROPOSITION 5.2

Consider the action ρ of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_G. The $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-invariant subset

$$E_{K_0} := \widetilde{\mathrm{SL}(2,\mathbb{R})}(\mathcal{S}_K) \subset E_K$$

generated by $\mathcal{S}_K$ *in (5.12), equipped with the natural map* $E_{K_0} \longrightarrow \mathbb{H}$*, is a* C^∞*-principal* K_0*-bundle over* $\mathbb{H}$*, where* K_0 *is the subgroup constructed in (5.10). Furthermore,* $E_{K_0} \cap f^{-1}(x_0) = \mathcal{S}_K$*, where* f *is the projection of* E_G *to* $\mathbb{H}$*.*

The $\widetilde{\mathrm{SL}(2,\mathbb{R})}$*-invariant subset*

$$E_{G_0} := \widetilde{\mathrm{SL}(2,\mathbb{R})}(\mathcal{S}_G)$$

generated by $\mathcal{S}_G$ *in (5.13), equipped with the natural map* $E_{G_0} \longrightarrow \mathbb{H}$*, is a* C^∞*-principal* G_0*-bundle over* $\mathbb{H}$*, where* G_0 *is the subgroup constructed in (5.11). Furthermore,* $E_{G_0} \cap f^{-1}(x_0) = \mathcal{S}_G$*.*

Proof

The actions of G and $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_G commute, and the group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ acts transitively on $\mathbb{H}$. Therefore, to prove that E_{G_0} (resp., E_{K_0}) is a C^∞-principal G_0-bundle (resp., K_0-bundle) over $\mathbb{H}$, it suffices to show that the following four statements are valid.

(1) The subset $\mathcal{S}_G$ (resp., $\mathcal{S}_K$) is nonempty.

(2) The action of the isotropy subgroup H_{x_0} (see (3.1)) on $(E_G)_{x_0}$ preserves $\mathcal{S}_G$ (resp., $\mathcal{S}_K$).

(3) The subset $\mathcal{S}_G$ (resp., $\mathcal{S}_K$) is preserved by the action of G_0 (resp., K_0) on $(E_G)_{x_0}$.

(4) The action of the group G_0 (resp., K_0) on $\mathcal{S}_G$ (resp., $\mathcal{S}_K$) is transitive.

Since k_0 lies in the conjugacy class, in $\mathfrak{k}$, of the element ϖ in (5.6), the first statement follows from the second part of Lemma 5.1. (Note that $\mathcal{S}_K \subset \mathcal{S}_G$.)

Take any $z \in \mathcal{S}_K$. Since $q_K(z,k_0) = d\gamma_{x_0}(1)$, from the construction of the homomorphism γ_{x_0} (see (4.8)) it follows immediately that the orbit of z under the action of the isotropy subgroup H_{x_0} is $\{z\exp(tk_0)\}_{t\in\mathbb{R}}$. The image of $(z\exp(tk_0), k_0) \in (E_K)_{x_0} \times \mathfrak{k}$ in $\mathrm{ad}(E_K)_{x_0}$ coincides with that of $(z, \mathrm{Ad}(\exp(tk_0))(k_0))$ (see (4.10)). Since $\mathrm{Ad}(\exp(tk_0))(k_0) = k_0$, we have

$$q_K\big(z\exp(tk_0), k_0\big) = q_K(z,k_0) = d\gamma_{x_0}(1).$$

Hence $z\exp(tk_0) \in \mathcal{S}_K$ for all $t \in \mathbb{R}$. This proves the second statement.

From (5.7) it follows immediately that

- the subset $\mathcal{S}_G$ (resp., $\mathcal{S}_K$) is preserved by the action of G_0 (resp., K_0) on $(E_G)_{x_0}$, and
- the action of G_0 (resp., K_0) on $\mathcal{S}_G$ (resp., $\mathcal{S}_K$) is transitive.

This completes the proof of the proposition. □

Consider G_0 defined in (5.11) as the centralizer of k_0. We note that an element $g \in G$ lies in G_0 if and only if

$$g^{-1}\exp(ck_0)g = \exp(ck_0)$$

for each $c \in \mathbb{C}$. Since $k_0 \in \mathfrak{k}$, we know that k_0 is semisimple. Therefore, from the above characterization of the subgroup G_0, we conclude that G_0 is a Levi subgroup of some parabolic subgroup of G (see [DM, Proposition 1.22, p. 26]). We recall the definition of a Levi subgroup. Let P be a parabolic subgroup of G, and let $R_u(P) \subset P$ be the unipotent radical of P. So the quotient $P/R_u(P)$ is a connected reductive linear algebraic group defined over $\mathbb{C}$. A *Levi subgroup* of P is a closed connected subgroup $L(P) \subset P$ such that the composition

$$L(P) \hookrightarrow P \longrightarrow P/R_u(P)$$

is an isomorphism (see [Bor, Section 11.22, p. 158] and [Hu, Section 30.2, p. 184]).

Therefore, G_0 is a connected reductive linear algebraic group defined over $\mathbb{C}$. Since K is a maximal compact subgroup of G, it follows that $K_0 = G_0 \cap K$ is also a maximal compact subgroup of G_0.

The action ρ of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_G preserves both E_{G_0} and E_{K_0} (see Proposition 5.2). The restriction of ρ to E_{G_0} is denoted by ρ_0.

Consider the triple

$$(E_{G_0}, E_{K_0}; \rho_0). \tag{5.14}$$

We show that E_{G_0} admits a unique holomorphic structure such that this triple becomes an $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G_0-bundle.

Consider the action ρ_0 of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_{K_0}. Restricting it to the isotropy subgroup H_{x_0} (see (3.1)), we get a homomorphism of groups

$$h_0 : H_{x_0} \longrightarrow \mathrm{Ad}(E_{K_0})_{x_0}, \tag{5.15}$$

where $\mathrm{Ad}(E_{K_0})$ is the adjoint bundle of E_{K_0}.

We note that the canonical inclusion of $\mathrm{Ad}(E_{K_0})_{x_0}$ in $\mathrm{Ad}(E_G)_{x_0}$ takes h_0 to the homomorphism γ_{x_0} constructed in (4.8).

LEMMA 5.3

The principal G_0-bundle E_{G_0} admits a unique holomorphic structure such that the triple $(E_{G_0}, E_{K_0}; \rho_0)$ in (5.14) is an $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G_0-bundle.

Proof

The existence of a holomorphic structure on E_{G_0} such that the triple $(E_{G_0}, E_{K_0}; \rho_0)$ is an $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G_0-bundle follows from the construction in Lemma 4.3. More precisely, in Lemma 4.3, set $G = G_0$ and $K = K_0$. To construct the homomorphism χ in Lemma 4.3, fix a point

$$z_0 \in (E_{K_0})_{x_0}.$$

Let

$$\gamma_{z_0} : K_0 \longrightarrow \mathrm{Ad}(E_{K_0})_{x_0}$$

be the isomorphism of groups which sends any $g \in K_0$ to the image of (z_0, g_0) in $(E_{K_0})_{x_0}$ (see (4.1)). Now, if we set χ in Lemma 4.3 to be $(\gamma_{z_0})^{-1} \circ h_0$, where

h_0 is the homomorphism in (5.15), then the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G_0-bundle given by Lemma 4.3 is isomorphic to the one in (5.14).

To prove the uniqueness of the holomorphic structure on E_{G_0}, first recall that $(E_{K_0})_{x_0}$ coincides with $\mathcal{S}_K$ in (5.12) (see Proposition 5.2). From the construction of $\mathcal{S}_K$, it follows immediately that $\mathrm{Ad}(E_{K_0})_{x_0}$ is the centralizer of $d\gamma_{x_0}(1)$ (see (5.1)) for the adjoint action of the group $\mathrm{Ad}(E_K)_{x_0}$ on its Lie algebra $\mathrm{ad}(E_K)_{x_0}$. Consequently, the image of the homomorphism h_0 in (5.15) lies inside the center of $\mathrm{Ad}(E_{K_0})_{x_0}$.

Therefore, the triple $(E_{G_0}, E_{K_0}; \rho_0)$ satisfy the condition in Proposition 4.2 (see also Remark 4.1). Hence from the first part of Proposition 4.2 we know that E_{G_0} admits at most one holomorphic structure such that $(E_{G_0}, E_{K_0}; \rho_0)$ is an $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G_0-bundle. This completes the proof of the lemma. $\square$

Let $(E'_G, E'_K; \rho')$ denote the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle obtained from the above $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G_0-bundle $(E_{G_0}, E_{K_0}; \rho_0)$ using the inclusion of G_0 in G. Therefore,

$$E'_G = E_{G_0} \times^{G_0} G \tag{5.16}$$

is the holomorphic principal G-bundle obtained by extending the structure group of E_{G_0} using the inclusion map $G_0 \hookrightarrow G$. Similarly, $E'_K = E_{K_0} \times^{K_0} K$ is the C^∞-principal K-bundle obtained by extending the structure group of E_{K_0} using the inclusion map $K_0 \hookrightarrow K$. The action ρ' is the one induced by ρ_0.

LEMMA 5.4

The isomorphism class of the above $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E'_G, E'_K; \rho')$ is independent of the choice of the element k_0 in (5.9).

Proof

Take any $g_0 \in K$. Replace k_0 in (5.9) by $\mathrm{Ad}(g_0^{-1})(k_0)$, where $\mathrm{Ad}(g_0^{-1})$ is the Lie algebra automorphism $\mathfrak{k} \longrightarrow \mathfrak{k}$ associated to the inner automorphism

$$\mathrm{Ad}(g_0^{-1}) : K \longrightarrow K$$

defined by $g \longmapsto g_0^{-1} g g_0$. The centralizer K_0 in (5.10) gets replaced by

$$K'_0 := \mathrm{Ad}(g_0^{-1})(K_0) \subset K, \tag{5.17}$$

where $\mathrm{Ad}(g_0^{-1})$ is defined above. Similarly, the centralizer G_0 in (5.11) gets replaced by

$$G'_0 := \mathrm{Ad}(g_0^{-1})(G_0) \subset G. \tag{5.18}$$

(Here $\mathrm{Ad}(g_0^{-1})$ is the inner automorphism of G defined by g_0^{-1}.)

Let $\mathcal{S}'_K$ (resp., $\mathcal{S}'_G$) be the submanifold of $(E_K)_{x_0}$ (resp., $(E_G)_{x_0}$) constructed as in (5.12) (resp., (5.13)) after replacing k_0 by $\mathrm{Ad}(g_0^{-1})(k_0)$. From the quotient construction of the adjoint vector bundle (see (4.10)), it follows that

$$\mathcal{S}'_K = \mathcal{S}_K g_0 \quad \text{and} \quad \mathcal{S}'_G = \mathcal{S}_G g_0 \subset (E_G)_{x_0}. \tag{5.19}$$

Let $E_{K'_0}$ (resp., $E_{G'_0}$) be the principal K'_0-bundle (resp., principal G'_0-bundle) obtained in place of E_{K_0} (resp., E_{G_0}) after we replace k_0 by $\mathrm{Ad}(g_0^{-1})(k_0)$ (see Proposition 5.2). From (5.19) it follows immediately that

$$E_{K'_0} = E_{K_0} g_0 \quad \text{and} \quad E_{G'_0} = E_{G_0} g_0 \subset E_G. \tag{5.20}$$

Consider the subgroup G'_0 in (5.18). Let

$$A : G_0 \longrightarrow G'_0 \tag{5.21}$$

be the isomorphism defined by $g \longmapsto \mathrm{Ad}(g_0^{-1})(g) := g_0^{-1} g g_0$. The restriction of A to K_0 is denoted by $A|_{K_0}$. Let

$$B : \mathcal{S}_G \longrightarrow \mathcal{S}'_G \tag{5.22}$$

be the C^∞-map defined by $z \longmapsto z g_0$.

Note that

$$B(zg) = B(z)A(g)$$

for all $z \in \mathcal{S}_G$ and all $g \in G_0$, where A and B are constructed in (5.21) and (5.22), respectively. Also, the map B intertwines the actions of H_{x_0} on $\mathcal{S}_G$ and $\mathcal{S}'_G$ because the actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ and G on E_G commute. Therefore, using the actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_{G_0} and $E_{G'_0}$, the map B extends to an $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-equivariant isomorphism

$$\widetilde{B} : E_{G_0} \longrightarrow E_{G'_0} \tag{5.23}$$

of principal bundles with respect to the homomorphism A in (5.21). In other words,

$$\widetilde{B}(zg) = \widetilde{B}(z)A(g)$$

for all $z \in E_{G_0}$ and all $g \in G_0$. Now from the first part of Proposition 4.2 (which asserts uniqueness of the holomorphic structure), it follows immediately that the map $\widetilde{B}$ in (5.23) is holomorphic.

Let

$$\widetilde{E}_G = E_{G'_0} \times^{G'_0} G \tag{5.24}$$

be the principal G-bundle obtained by extending the structure group of the principal G'_0-bundle $E_{G'_0}$ in (5.20) using the inclusion of G'_0 in G.

Consider the map

$$E_{G_0} \times G \longrightarrow E_{G'_0} \times G$$

defined by $(z, g) \longmapsto (zg_0, g_0^{-1}g)$. It is straightforward to check that this map descends to a map between the quotient spaces

$$\Psi : E'_G \longrightarrow \widetilde{E}_G, \tag{5.25}$$

where $\widetilde{E}_G$ is defined in (5.24) and E'_G is defined in (5.16). This map Ψ intertwines the actions of G as well as those of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Furthermore, from the holomorphicity of the map $\widetilde{B}$ in (5.23), it follows immediately that the map Ψ is also holomorphic.

The map $\widetilde{B}$ in (5.23) clearly sends E_{K_0} to $E_{K'_0}$. Hence the map Ψ in (5.25) sends $E'_K \subset E'_G$ to $E_{K'_0} \times^{K'_0} K$; here $E_{K'_0} \times^{K'_0} K$ is the principal K-bundle obtained by extending the structure group of the principal K'_0-bundle $E_{K'_0}$ in (5.20) using the inclusion of K'_0 in K. Thus the isomorphism class of the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E'_G, E'_K; \rho')$ coincides with that of the holomorphic Hermitian principal G-bundle given by the pair $(\widetilde{E}_G, E_{K'_0} \times^{K'_0} K)$ equipped with the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. This completes the proof of the lemma. □

The element k_0 in (5.9) was the only choice made in the construction of $(E'_G, E'_K; \rho')$ from $(E_G, E_K; \rho)$. Therefore, Lemma 5.4 has the following corollary.

COROLLARY 5.5

The isomorphism class of the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$*-homogeneous holomorphic Hermitian principal* G*-bundle* $(E'_G, E'_K; \rho')$ *over* $\mathbb{H}$ *depends only on the isomorphism class of the* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$*-homogeneous holomorphic Hermitian principal* G*-bundle* $(E_G, E_K; \rho)$.

The following corollary shows that there a canonical C^∞-isomorphism of E'_G with E_G.

COROLLARY 5.6

There is a canonical C^∞*-isomorphism of the principal* G*-bundle* E_G *with the principal* G*-bundle* E'_G *constructed in (5.16). This isomorphism takes* $E_K \subset E_G$ *to the reduction* $E'_K \subset E'_G$. *Also, it intertwines the actions of* $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ *on* E_G *and* E'_G.

Proof

Consider $E'_G = E_{G_0} \times^{G_0} G$ constructed in (5.16). Note that $E_{G_0} \subset E_G$ (see Proposition 5.2). Therefore, we have a map

$$E_{G_0} \times G \longrightarrow E_G \tag{5.26}$$

defined by the action of G (see (2.6)).

Two points (z_1, g_1) and (z_2, g_2) of $E_{G_0} \times G$ are identified in the quotient space E'_G if and only if there is an element $g \in G_0$ such that $(z_2, g_2) = (z_1 g, g^{-1} g_1)$.

Hence the map in (5.26) descends to a G-equivariant C^∞-isomorphism

$$\tau' : E'_G \longrightarrow E_G \tag{5.27}$$

from the quotient space E'_G of $E_{G_0} \times G$. This map τ' clearly takes the submanifold

$$E_K \subset E_G$$

to $E'_K \subset E'_G$. It is straightforward to check that τ' intertwines the actions ρ and ρ' of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_G and E'_G, respectively. □

REMARK 5.7

It should be emphasized that the isomorphism τ' in (5.27) need not be holomorphic. The isomorphism τ' has the following property.

Consider $\widetilde{E}_G$ constructed in (5.24). Let

$$\widetilde{\tau} : \widetilde{E}_G \longrightarrow E_G$$

be the C^∞-isomorphism of principal G-bundles constructed as in (5.27). So $\widetilde{\tau}$ is obtained from the map $E_{G'_0} \times G \longrightarrow E_G$ defined by the action of G (see (5.26)). It is easy to see that

$$\widetilde{\tau} \circ \Psi = \tau', \tag{5.28}$$

where Ψ is the holomorphic isomorphism in (5.25).

Examining the construction of $(E'_G, E'_K; \rho')$ from $(E_G, E_K; \rho)$ in Corollary 5.5, it can be seen that $(E'_G, E'_K; \rho')$ is a split $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle (see Definition 4.4). Indeed, as noted in the proof of Lemma 5.3, if we set χ in Lemma 4.3 to be $(\gamma_{z_0})^{-1} \circ h_0$, then the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G_0-bundle given by Lemma 4.3 is isomorphic to the one in (5.14). Therefore, we have the following corollary.

COROLLARY 5.8

The $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E'_G, E'_K; \rho')$ in Corollary 5.5 is split.

We see in Proposition 5.11 that the construction in Corollary 5.5 sends a split $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle to itself.

Take any point $x \in \mathbb{H}$. Let

$$H_x \subset \widetilde{\mathrm{SL}(2,\mathbb{R})} \tag{5.29}$$

be the isotropy subgroup of x for the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathbb{H}$ defined by ϕ in (2.3). In view of the fact that H_{x_0} is abelian, the isomorphism $\widetilde{\alpha}$ in (3.3) also gives an isomorphism of H_x with $\mathbb{R}$. To explain this, take any $g \in \widetilde{\mathrm{SL}(2,\mathbb{R})}$ such that $\phi(g)(x_0) = x$. We now have an isomorphism

$$H_{x_0} \longrightarrow H_x \tag{5.30}$$

defined by $x \longmapsto gxg^{-1}$. Since H_{x_0} is abelian, the isomorphism in (5.30) is independent of the choice of g. Therefore, we have a natural isomorphism

$$H_x \xrightarrow{\sim} \mathbb{R}. \tag{5.31}$$

Let

$$(E_G, E_K; \rho)$$

be an $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle over $\mathbb{H}$. Consider the action, given by ρ, of the isotropy subgroup H_x on the fiber $(E_G)_x$. Since actions of G and $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_G commute, we obtain a homomorphism

$$\gamma_x : H_x = \mathbb{R} \longrightarrow \mathrm{Ad}(E_G)_x \tag{5.32}$$

(see (4.8)). Let

$$d\gamma_x : \mathbb{R} \longrightarrow \mathrm{ad}(E_G)_x \tag{5.33}$$

be the corresponding homomorphism of Lie algebras (as in (5.1)), where $\mathrm{Lie}(H_x)$ is identified with $\mathbb{R}$ using the isomorphism in (5.31).

Let

$$\Theta_\rho : \mathbb{H} \longrightarrow \mathrm{ad}(E_G) \tag{5.34}$$

be the C^∞-section of the adjoint vector bundle defined by

$$x \longmapsto d\gamma_x(1),$$

where $d\gamma_x$ is the homomorphism in (5.33).

Let $f : E_G \longrightarrow \mathbb{H}$ be the natural projection (as in (2.5)). Recall that $\mathrm{ad}(E_G)_{f(z)}$ is a quotient space of $(E_G)_{f(z)} \times \mathfrak{g}$ (see (4.10)).

PROPOSITION 5.9

The submanifold $E_{G_0} \subset E_G$ constructed in Proposition 5.2 coincides with the subset of E_G consisting of all points z such that the point in $\mathrm{ad}(E_G)_{f(z)}$ defined by (z, k_0) coincides with $\Theta_\rho(f(z))$, where Θ_ρ and k_0 are defined in (5.34) and (5.9), respectively.

Proof

The action ρ of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_G induces an action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on the vector bundle $\mathrm{ad}(E_G)$ lifting the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathbb{H}$. From the construction of the section Θ_ρ in (5.34), it follows immediately that Θ_ρ is left invariant by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathrm{ad}(E_G)$.

Let

$$S \subset E_G \tag{5.35}$$

be the subset consisting of all points z such that the point in $\mathrm{ad}(E_G)_{f(z)}$ defined by (z, k_0) coincides with $\Theta_\rho(f(z))$. Since the section Θ_ρ is left invariant by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathrm{ad}(E_G)$, it follows immediately that the subset S is also left invariant by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on E_G. Therefore, to prove that S coincides with E_{G_0} it suffices to show that

$$(E_{G_0})_{x_0} = S \cap f^{-1}(x_0), \tag{5.36}$$

where x_0 is the base point in (3.1).

In Proposition 5.2 we saw that $(E_{G_0})_{x_0} = \mathcal{S}_G$. Comparing the constructions of $\mathcal{S}_G$ and Θ_ρ, it follows immediately that (5.36) holds. This completes the proof of the proposition. □

The following lemma is deduced from Proposition 5.9.

LEMMA 5.10

If the section Θ_ρ in (5.34) is holomorphic, then the map τ' in (5.27) is holomorphic.

Proof

If Θ_ρ is holomorphic, then using Proposition 5.9 it follows that the reduction

$$E_{G_0} \subset E_G$$

constructed in Proposition 5.2 is a complex submanifold. Consequently, the map τ' is holomorphic if the section Θ_ρ is holomorphic. □

PROPOSITION 5.11

Let $(E_G, E_K; \rho)$ be a split $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle over $\mathbb{H}$. Then the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E'_G, E'_K; \rho')$ associated to $(E_G, E_K; \rho)$ (see Corollary 5.5) is isomorphic to it.

Proof

Fix a homomorphism

$$\chi : H_{x_0} = \mathbb{R} \longrightarrow K \tag{5.37}$$

such that the corresponding $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E^\chi_G, E^\chi_K; \rho')$ (see Lemma 4.3) is isomorphic to $(E_G, E_K; \rho)$. We interchange $(E^\chi_G, E^\chi_K; \rho')$ and $(E_G, E_K; \rho)$ without any further explanation.

Let

$$d\chi : \mathbb{R} \longrightarrow \mathfrak{k} \tag{5.38}$$

be the homomorphism of Lie algebras corresponding to χ.

The principal G-bundle E^χ_G is the extension of structure group of the principal $\mathbb{R}$-bundle $F_\mathbb{R}$ (defined in (3.5)) using the homomorphism χ in (5.37) (see (4.16)). Hence we have a homomorphism of Lie algebra bundles

$$\Phi : \mathrm{ad}(F_\mathbb{R}) \longrightarrow \mathrm{ad}(E^\chi_G). \tag{5.39}$$

More precisely, the map $F_{\mathbb{R}} \times \mathbb{R} \longrightarrow F_{\mathbb{R}} \times \mathfrak{g}$ defined by $(z, v) \longmapsto (z, d\chi(v))$, where $d\chi$ is defined in (5.38), descends to a map Φ between the quotient spaces.

Recall that the adjoint vector bundle $\mathrm{ad}(F_{\mathbb{R}})$ is the trivial line bundle $\mathbb{H} \times \mathbb{R}$ (see (3.7)). Let

$$\theta_0 : \mathbb{H} \longrightarrow \mathrm{ad}(F_{\mathbb{R}}) \tag{5.40}$$

be the section given by the constant function 1. Let

$$\Theta' := \Phi \circ \theta_0 \in C^{\infty}\big(\mathbb{H}; \mathrm{ad}(E_G^{\chi})\big) \tag{5.41}$$

be the smooth section, where Φ is the homomorphism of vector bundles in (5.39).

We show that the section Θ' in (5.41) coincides with Θ_ρ constructed in (5.34).

To prove that $\Theta' = \Theta_\rho$, first note that both the sections are left invariant by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\mathrm{ad}(E_G^{\chi})$. Hence it suffices to show that they coincide at x_0. It is straightforward to check that they do coincide at x_0. Therefore, we have $\Theta' = \Theta_\rho$.

A connection on a principal bundle induces a connection on its adjoint bundle. Since $\mathbb{R}$ is abelian, the connection on $\mathrm{ad}(F_{\mathbb{R}})$ induced by a connection on $F_{\mathbb{R}}$ coincides with the trivial connection associated to the trivialization of $\mathrm{ad}(F_{\mathbb{R}})$ in (3.7). In particular, the section θ_0 in (5.40) is flat with respect to the connection on $\mathrm{ad}(F_{\mathbb{R}})$ induced by the connection $\nabla^0_{\mathbb{R}}$ on $F_{\mathbb{R}}$ defined by the one-form ω_0 in (3.6).

Since θ_0 is flat, we know that the section Θ' in (5.41) is flat with respect to the connection on $\mathrm{ad}(E_G^{\chi})$ induced by the connection ∇^G in (4.17). We recall that the holomorphic structure of E_G^{χ} is defined using the connection ∇^G. Consequently, the section Θ' is holomorphic.

We now conclude that the section Θ_ρ is holomorphic because $\Theta' = \Theta_\rho$ and Θ' is holomorphic. This in turn implies that the subset $S \subset E_G$ constructed in (5.35) is a complex submanifold. (It is a C^{∞}-submanifold by Proposition 5.9.) Now, using Proposition 5.9, we conclude that the reduction

$$E_{G_0} \subset E_G$$

constructed in Proposition 5.2 is a complex submanifold. Consequently, the C^{∞}-isomorphism τ' constructed in (5.27) is actually holomorphic.

From Lemma 5.3 we know that there is a unique holomorphic structure on the C^{∞}-principal G-bundle E_{G_0} which makes the triple $(E_{G_0}, E_{K_0}; \rho_0)$ an $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G_0-bundle. On the other hand, the holomorphic structure on the C^{∞}-principal G_0-bundle E_{G_0} induced by the holomorphic structure on E_G (we have shown above that E_{G_0} is a complex submanifold of E_G) makes the triple $(E_{G_0}, E_{K_0}; \rho_0)$ an $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G_0-bundle. Therefore, the map τ' constructed in (5.27) is an isomorphism of the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E_G, E_K; \rho)$ with the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E'_G, E'_K; \rho')$ associated to $(E_G, E_K; \rho)$ by Corollary 5.5. This completes the proof of the proposition. $\square$

The proof of Proposition 5.11 yields the following corollary.

COROLLARY 5.12

Let $(E_G, E_K; \rho)$ be any $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle over $\mathbb{H}$. Let $(E'_G, E'_K; \rho')$ be the split $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle associated to it. Let

$$\widetilde{\tau}' : \mathrm{ad}(E_G) \longrightarrow \mathrm{ad}(E'_G)$$

be the C^∞-isomorphism of adjoint vector bundles obtained from the C^∞-isomorphism $(\tau')^{-1}$ in (5.27). Then the section

$$\widetilde{\tau}'(\Theta_\rho) \in C^\infty\big(\mathbb{H}; \mathrm{ad}(E'_G)\big)$$

is holomorphic, where Θ_ρ is the section constructed in (5.34).

Proof

Note that the section Θ_ρ in (5.34) depends only on the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on the C^∞-principal G-bundle E_G. Therefore, if we replace $(E_G, E_K; \rho)$ with $(E'_G, E'_K; \rho')$ in the construction of Θ_ρ in (5.34), then the resulting section of $\mathrm{ad}(E'_G)$ coincides with the section $\widetilde{\tau}'(\Theta_\rho)$.

Now, set (E^χ_G, E^χ_K) in the proof of Proposition 5.11 to be the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E'_G, E'_K; \rho')$. In the proof of Proposition 5.11 we saw that the section Θ_ρ for (E^χ_G, E^χ_K) is holomorphic. This completes the proof of the corollary. $\square$

REMARK 5.13

Combining Corollary 5.8 and Proposition 5.11, we obtain a retraction map from the set of all isomorphism classes of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle over $\mathbb{H}$ to the set of all isomorphism classes of split $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundles over $\mathbb{H}$.

We also have the following characterization of an $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle.

LEMMA 5.14

Let $(E_G, E_K; \rho)$ be any $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle over $\mathbb{H}$. Then $(E_G, E_K; \rho)$ is split if and only if the section Θ_ρ constructed in (5.34) is holomorphic.

Proof

Assume that $(E_G, E_K; \rho)$ is split. In the proof of Proposition 5.11 we showed that Θ_ρ is holomorphic.

Now, assume that Θ_ρ is holomorphic. Then in Proposition 5.11 it was shown that the map τ' in (5.27) is an isomorphism of the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E_G, E_K; \rho)$ with the one associated to it by Corollary 5.5. From Corollary 5.8 we know that the associated $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle is split. This completes the proof of the lemma. $\square$

6. Classification of invariant holomorphic Hermitian principal bundles

Take any $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle

$$(E_G, E_K; \rho)$$

over $\mathbb{H}$. Consider the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E'_G, E'_K; \rho')$ on $\mathbb{H}$ constructed from $(E_G, E_K; \rho)$ (see Corollary 5.5). There is a canonical C^∞-isomorphism of the principal G-bundle E_G with E'_G (see Corollary 5.6). Let $\overline{\partial}_{E_G}$ (resp., $\overline{\partial}_{E'_G}$) denote the Dolbeault operator defining the holomorphic structure of E_G (resp., E'_G). Holomorphic structures on the C^∞-principal G-bundle E_G form an affine space for the vector space of all smooth $(0,1)$-forms on $\mathbb{H}$ with values in the adjoint bundle $\mathrm{ad}(E_G)$. Therefore,

$$\eta := \overline{\partial}_{E_G} - \overline{\partial}_{E'_G} \in C^\infty\big(\mathbb{H}; \Omega^{0,1}_{\mathbb{H}}; (\mathrm{ad}(E_G))\big). \tag{6.1}$$

The Dolbeault operator $\overline{\partial}_{E'_G}$ is considered as a Dolbeault operator on E_G using the C^∞-isomorphism τ' in (5.27).

Let $\overline{\partial}^0_{E_G}$ denote the Dolbeault operator on the adjoint vector bundle $\mathrm{ad}(E_G)$. So $\overline{\partial}^0_{E_G}$ is induced by the Dolbeault operator $\overline{\partial}_{E_G}$ in (6.1). Consider the smooth section Θ_ρ of $\mathrm{ad}(E_G)$ constructed in (5.34). We have

$$\overline{\partial}^0_{E_G}(\Theta_\rho) \in C^\infty\big(\mathbb{H}; \Omega^{0,1}_{\mathbb{H}}(\mathrm{ad}(E_G))\big). \tag{6.2}$$

LEMMA 6.1

The identity

$$\overline{\partial}^0_{E_G}(\Theta_\rho) = 2\sqrt{-1}\cdot\eta$$

holds, where η and $\overline{\partial}^0_{E_G}(\Theta_\rho)$ are constructed in (6.1) and (6.2), respectively.

Proof

Let $\overline{\partial}^0_{E'_G}$ denote the Dolbeault operator on the adjoint vector bundle $\mathrm{ad}(E'_G) = E'_G \times^G \mathfrak{g}$ of the holomorphic principal G-bundle E'_G. So $\overline{\partial}^0_{E'_G}$ is induced by the Dolbeault operator $\overline{\partial}_{E'_G}$ in (6.1). We know that Θ_ρ is holomorphic with respect to $\overline{\partial}^0_{E'_G}$ (see Corollary 5.12). Therefore,

$$\overline{\partial}^0_{E_G}(\Theta_\rho) = \overline{\partial}^0_{E'_G}(\Theta_\rho) + [\eta, \Theta_\rho] = [\eta, \Theta_\rho], \tag{6.3}$$

where η is constructed in (6.1). There is a unique smooth section

$$A : \mathbb{H} \longrightarrow \mathrm{ad}(E_G)$$

such that

$$\eta = A \otimes d\overline{z}. \tag{6.4}$$

We note that $[\eta, \Theta_\rho] = [A, \Theta_\rho] \otimes d\overline{z}$. Therefore, from (6.3) we have

$$\overline{\partial}^0_{E_G}(\Theta_\rho) = [A, \Theta_\rho] \otimes d\overline{z}. \tag{6.5}$$

Since the holomorphic structures of both E_G and E'_G are preserved by the actions ρ and ρ' of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, and since the isomorphism between E_G and E'_G intertwines the actions of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ (see Corollary 5.6), we know that the section η is left invariant by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on the vector bundle $\Omega^{0,1}_{\mathbb{H}}(\mathrm{ad}(E_G))$. Consider the action of the isotropy subgroup $H_{x_0} = \mathbb{R}$ on the cotangent line $(T^{0,1}_{x_0})^*$. (As before, H_{x_0} is identified with $\mathbb{R}$ using $\widetilde{\alpha}$ in (3.3).) Any $t \in \mathbb{R}$ acts on $(T^{0,1}_{x_0})^*$ as multiplication by $\exp(2\sqrt{-1}t)$. It can now be shown that the action of $t \in \mathbb{R} = H_{x_0}$ on $\mathrm{ad}(E_G)_{x_0}$ sends the vector

$$A(x_0) \in \mathrm{ad}(E_G)_{x_0}$$

(see (6.4)) to $\exp(-2\sqrt{-1}t)A(x_0)$. Indeed, this follows immediately using the fact that the action of H_{x_0} on the fiber $\Omega^{0,1}_{\mathbb{H}}(\mathrm{ad}(E_G))_{x_0}$ fixes $\eta(x_0)$.

Since the action of $t \in \mathbb{R} = H_{x_0}$ on $\mathrm{ad}(E_G)_{x_0}$ sends $A(x_0)$ to $\exp(-2\sqrt{-1}t) \times A(x_0)$, from the construction of Θ_ρ in (5.34) we conclude that

$$[\Theta_\rho(x_0), A(x_0)] = -2\sqrt{-1} \cdot A(x_0).$$

Hence from (6.5) we have

$$\big(\overline{\partial}^0_{E_G}(\Theta_\rho)\big)(x_0) = 2\sqrt{-1} \cdot A(x_0) \otimes d\overline{z} = 2\sqrt{-1} \cdot \eta(x_0). \tag{6.6}$$

We already noted that the section η is left invariant by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\Omega^{0,1}_{\mathbb{H}}(\mathrm{ad}(E_G))$. On the other hand, the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on the vector bundle $\mathrm{ad}(E_G)$ preserves the section Θ_ρ. Hence the section $\overline{\partial}^0_{E_G}(\Theta_\rho)$ is also preserved by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ on $\Omega^{0,1}_{\mathbb{H}}(\mathrm{ad}(E_G))$. Therefore, from (6.6) we conclude that

$$\overline{\partial}^0_{E_G}(\Theta_\rho) = 2\sqrt{-1} \cdot \eta.$$

This completes the proof of the lemma. □

Consider all pairs of the form (χ, A), where

- $\chi : H_{x_0} = \mathbb{R} \longrightarrow K$ is a homomorphism and
- A is an element of the Lie algebra $\mathfrak{g}$ of G such that $[A, d\chi(1)] = 2\sqrt{-1} \cdot A$,

where

$$d\chi : \mathbb{R} \longrightarrow \mathfrak{k}$$

is the homomorphism of Lie algebras associated to the homomorphism χ.

DEFINITION 6.2

Two pairs (χ, A) and (χ', A') of the above type are called *equivalent* if there is an element $g_0 \in K$ such that

- $\chi'(t) = g_0\chi(t)g_0^{-1}$ for all $t \in \mathbb{R}$ and
- $A' = \mathrm{Ad}(g_0)(A)$, where $\mathrm{Ad}(g_0)$ is the automorphism of $\mathfrak{g}$ associated to the inner automorphism of G defined by $g \longmapsto g_0 g g_0^{-1}$.

The following theorem classifies the isomorphism classes of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundles.

THEOREM 6.3

There is a canonical bijection between the isomorphism classes of all $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-*homogeneous holomorphic Hermitian principal* G*-bundles over* $\mathbb{H}$ *and the equivalence classes of all pairs of the form* (χ, A) *(see Definition 6.2).*

Proof

Let $(E_G, E_K; \rho)$ be an $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle over $\mathbb{H}$. The Dolbeault operator defining the holomorphic structure of E_G is denoted by $\overline{\partial}_{E_G}$.

Consider the C^∞-section Θ_ρ of the adjoint vector bundle $\mathrm{ad}(E_G)$ constructed in (5.34). Let $\overline{\partial}'$ be the new Dolbeault operator on the C^∞-principal G-bundle E_G defined as

$$\overline{\partial}' := \overline{\partial}_{E_G} - \frac{\overline{\partial}_{E_G}(\Theta_\rho)}{2\sqrt{-1}}.$$

Let E'_G denote the holomorphic principal G-bundle defined by this Dolbeault operator $\overline{\partial}'$ on the C^∞-principal G-bundle E_G. From Lemma 6.1 we know that the triple $(E'_G, E_K; \rho)$ defines a split $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle, which is associated to $(E_G, E_K; \rho)$ by Corollary 5.5.

Fix a homomorphism

$$\chi : H_{x_0} = \mathbb{R} \longrightarrow K$$

which gives the above-defined split $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E'_G, E_K; \rho)$. Fix an identification of $(E'_G, E_K; \rho)$ with the split $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E^\chi_G, E^\chi_K; \rho')$ given by χ (see Lemma 4.3). Recall that

$$E^\chi_G = \widetilde{\mathrm{SL}(2,\mathbb{R})} \times^{\mathbb{R}} G$$

(see (4.16)). Therefore,

$$\mathrm{ad}(E^\chi_K) = \widetilde{\mathrm{SL}(2,\mathbb{R})} \times^{\mathbb{R}} \mathfrak{k}. \tag{6.7}$$

Note that the C^∞-vector bundle $\mathrm{ad}(E_G)$ is identified with $\mathrm{ad}(E'_G)$ because the C^∞-principal G-bundles underlying E_G and E'_G coincide. Therefore, using the above identification of E'_G with E^{χ}_G, the section $\overline{\partial}_{E_G}(\Theta_\rho)$ of $\Omega^{0,1}_{\mathbb{H}}(\mathrm{ad}(E_G))$ gives a C^∞-section of $\Omega^{0,1}_{\mathbb{H}}(\mathrm{ad}(E^{\chi}_G))$. This section of $\Omega^{0,1}_{\mathbb{H}}(\mathrm{ad}(E^{\chi}_G))$ is denoted by $\widetilde{\Theta}^{\chi}_\rho$. Let

$$A' \in \mathrm{ad}(E^{\chi}_G)_{x_0}$$

be the unique element that satisfies the identity

$$\widetilde{\Theta}^{\chi}_\rho(x_0) = A' \otimes d\overline{z}.$$

Now, let

$$A \in \mathfrak{g}$$

be the unique element such that $(e, A) \in \widetilde{\mathrm{SL}(2,\mathbb{R})} \times \mathfrak{g}$ projects to A' by the quotient map (see (6.7)). Using Lemma 6.1 together with (6.5), we know that $[A, d\chi(1)] = 2\sqrt{-1}\cdot A$. In other words, the above pair (χ, A) is of the type considered in Definition 6.2.

In view of Remark 4.7, from Proposition 4.6 it follows immediately that the equivalence class of the pair (χ, A) is uniquely determined by the isomorphism class of the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E_G, E_K; \rho)$.

For the reverse direction, take any pair (χ, A) as in Definition 6.2. Let $(E^{\chi}_G, E^{\chi}_K; \rho')$ be the split $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle over $\mathbb{H}$ associated to χ (see Lemma 4.3). The action of the isotropy subgroup $H_{x_0} \subset \widetilde{\mathrm{SL}(2,\mathbb{R})}$ on the fiber $\Omega^{0,1}_{\mathbb{H}}(\mathrm{ad}(E^{\chi}_G))_{x_0}$ clearly fixes the element defined by $A \otimes d\overline{z}$. Hence the orbit of this element $A \otimes d\overline{z} \in \Omega^{0,1}_{\mathbb{H}}(\mathrm{ad}(E^{\chi}_G))$ by the action of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is a C^∞-section, over $\mathbb{H}$, of the vector bundle $\Omega^{0,1}_{\mathbb{H}}(\mathrm{ad}(E^{\chi}_G))$. Let

$$\widetilde{A} \in C^\infty\big(\mathbb{H}; \Omega^{0,1}_{\mathbb{H}}(\mathrm{ad}(E^{\chi}_G))\big) \tag{6.8}$$

be the section defined by this orbit.

Let $\overline{\partial}_{E^{\chi}_G}$ be the Dolbeault operator of the holomorphic principal G-bundle E^{χ}_G. Let E_G be the holomorphic principal G-bundle over $\mathbb{H}$ defined by the Dolbeault operator $\overline{\partial}_{E^{\chi}_G} + \widetilde{A}$ on the C^∞-principal G-bundle E^{χ}_G, where $\widetilde{A}$ is constructed in (6.8). The reverse construction associates to (χ, A) the $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-homogeneous holomorphic Hermitian principal G-bundle $(E_G, E^{\chi}_K; \rho')$. This completes the proof of the theorem. $\square$

In view of Proposition 2.5, from Theorem 6.3 we have the following classification of the holomorphic isometry classes of the invariant holomorphic Hermitian principal G-bundles over $\mathbb{H}$.

THEOREM 6.4

There is a canonical bijection between the holomorphic isometry classes of all the *invariant holomorphic Hermitian principal G-bundles over $\mathbb{H}$ and all the equivalence classes of pairs of the form (χ, A), where $\chi : \mathbb{R} \longrightarrow K$ is a homomorphism, and A is an element of the Lie algebra $\mathfrak{g}$ of G such that $[A, d\chi(1)] = 2\sqrt{-1} \cdot A$ (see Definition 6.2).*

References

[AB] B. Anchouche and I. Biswas, *Einstein–Hermitian connections on polystable principal bundles over a compact Kähler manifold*, Amer. J. Math. **123** (2001), 207–228.

[BM] I. Biswas and G. Misra, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-*homogeneous vector bundles*, Internat. J. Math. **19** (2008), 1–19.

[Bor] A. Borel, *Linear Algebraic Groups*, 2nd ed., Grad. Texts in Math. **126**, Springer, New York, 1991.

[Bou] N. Bourbaki, *Éléments de mathématique, fasc. 26: Groupes et algèbres de Lie, Chapitre 1: Algèbres de Lie*, 2nd ed., Actualités Sci. Indust. **1285**, Hermann, Paris, 1971.

[DM] F. Digne and J. Michel, *Representations of Finite Groups of Lie Type*, London Math. Soc. Stud. Texts **21**, Cambridge Univ. Press, Cambridge, 1991.

[He] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Pure and Appl. Math. **80**, Academic Press, New York, 1978.

[Hu] J. E. Humphreys, *Linear Algebraic Groups*, Grad. Texts in Math. **21**, Springer, New York, 1975.

[Ko] S. Kobayashi, *Differential Geometry of Complex Vector Bundles*, Publ. Math. Soc. Japan **15**, Kanô Memorial Lectures **5**, Princeton Univ. Press, Princeton, 1987.

School of Mathematics, Tata Institute of Fundamental Research, Bombay 400005, India; indranil@math.tifr.res.in