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Comment

Dietrich Stoyan

Professor Kendall's paper is an excellent survey on a very important topic and describes many deep and complicated results obtained by himself and his colleagues. It is a pleasure to congratulate him on this success and to wish him further progress. The publication in this journal will help to inform many statisticians of these ideas and methods and so lead to further interesting applications. Because my own work has had until now only weak connections to Professor Kendall's theory of shape (with the nice exception of being a coauthor of a book that contains a chapter on shape theory written by W. S. Kendall), I can give marginal comments only; I take the opportunity to ask some questions.

In my opinion, in some cases the original problem of finding collinearities in point patterns can be solved by means of methods of point process statistics. If the point pattern under study can be interpreted as a sample of a stationary point process, then the orientation analysis of Ohser and Stoyan (1981) can be used to detect orientations and collinearities; see also Stoyan, Kendall and Mecke (1987). More interesting is the case of motion-invariant point processes with "inner orientations"; a nice example is the pattern of self-intersection points of a motion-invariant planar line process. Hanisch and Stoyan (1984) suggested statistical characteristics that are based on third-order moment measures or two-point Palm distributions. An example is the mean number of points in a rhombus with vertices at the members of a "typical" point pair of the point process with distance r (see Figure 1). If the corresponding mean, for which an unbiased estimator was given, is clearly greater than "intensity × area of rhombus" for interesting values of r, then some form of collinearity in the point pattern is detected.

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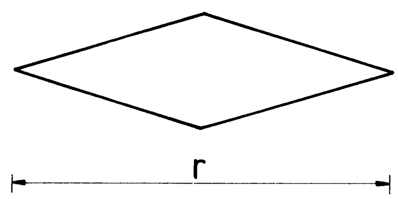

FIG. 1. *A rhombus with vertices of distance r. If the vertices are points of a point process and in the rhombus there are "many" other points of the point process, then this shows some inner orientation.*

Many statisticians and physicists, geographers (see the booklet by Boots, 1987) and others are very much interested in Dirichlet tessellations and the closely related Delaunay tessellations. Therefore the results on the Delaunay tessellation are of great value, both theoretically and practically. In particular, I like the elegant way of simulating "lone" Poisson Delaunay cells.

I think that a promising method for a "shape analysis" of tessellations could be based on the angles at vertices, if all vertices are Y-shaped, with three emanating edges. (This situation very often appears in practical problems, as physicists and materials scientists say.) Then each vertex corresponds to a triangle, which is similar to the Delaunay triangle if the tessellation under study is a Dirichlet tessellation with respect to a point pattern. Most empirical tessellations are not Dirichlet tessellations or, if their generating points are not given, the natural starting points for the shape analysis are the three angles. Therefore it would be helpful to transform shape theory results for triangles into angular coordinates, where, for example, a triangle is described by its maximal and minimal angles.

Perhaps it is of interest to mention a further (additionally to Professor Kendall's findings for PDLY tiles) interesting property of the Dirichlet tessellation, which in future may be better elucidated by the new simulation methods. Together with Dr. H. Hermann, I studied statistically simulated Poisson Dirichlet tessellations, in particular the point process of vertices of cells. Surprisingly we found that the corresponding second-order product density $\rho(r)$ has a striking form: it seems to be true that

$$\lim_{r\to 0} \rho(r) = \infty,$$

or, at least, $\rho(0)$ seems to be very great. Usually, such behavior of a product density is an indicator of a high degree of clustering. By visual inspection of some simulated tessellations we found that clusters of vertices in the usual sense of the word are not typical for these tessellations, but there appear frequently very short edges (of otherwise "normal" cells) or pairs of vertices very close together.

With respect to statistical shape problems related to "landmarks" in the sense of Bookstein (1978, 1986), I should like to ask the following question. Imagine three nonintersecting circles in the plane. Take a random point in each of the circles, for example uniformly or with respect to any distribution. Form the triangle having the three points as their vertices. Is it possible to give the corresponding shape density?

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Rejoinder

David G. Kendall

It is appropriate that Professor Bookstein should open this discussion in view of the importance of his work and the great influence that this has had through his own presentation in *Statistical Science* and his earlier 1978 monograph. I was already deeply involved in shape theory when I first read the latter, but did not at that time foresee how closely our two different and differently motivated approaches would converge. It is all the more valuable, therefore, that he has generously taken the time and trouble to survey their current interactions and differences of emphasis. His remarks will deserve careful study.

Professor Small's contribution is full of wise insights, and novel suggestions are made that I shall think about deeply. "Projection-pursuit" viewing of higher dimensional shape manifolds may well be a reality a few years from now. My current practice, not so technologically ambitious, is to try to understand these spaces as thoroughly as possible, and then to seek dimension-lowering projections that retain the important information and make it visible in a helpful way. One example of such a procedure will be found in my contribution to the discussion on Bookstein's 1986 paper referred to above. Of course I agree with the remarks that he and others have made about the advantages of having a variety of visual displays available. I recall that Kipling wrote a fine poem on a similar topic many years ago.

Professor Mardia's contribution was a shock to me because I did not expect to see so beautiful a solution as that found by Mardia and Dryden to the important problem they have studied. It makes one ask, why is it so beautiful? What has happened to all the horrible noncentral χ^2's? Of course the Gaussian distribution never ceases to spring surprises on us. I discussed Mardia's remarks with Wilfrid Kendall, and it occurred to us that a dynamic approach might at least "explain" what lies behind such a nice formula. So here are a few remarks intended only to illuminate the anatomy of the problem.

To start with it will be necessary to change the notation a little. We identify Mardia's κ with $s_0^2/(4c^2t)$, where c is a diffusion constant, t is the time elapsed during the interval considered and s_0 is a linear measure of the size of the triangle $\Delta_0 = (A_0, B_0, C_0)$ at the beginning of that time interval. The Mardia-Dryden formula then gives the law of distribution of the shape at the end of the time interval when we know what the shape was to start with. Notice that in this formulation it is no longer necessary to exclude $A_0 = B_0 = C_0$ as a possible initial shape, for then $s_0 = 0$, and this makes $\kappa = 0$, and then the Mardia-Dryden formula tells us that the distribution of size at the end of the interval is uniform over the sphere, as it ought to be.

More generally let us write $\zeta(t)$ for the shape of $\Delta_t = (A_t, B_t, C_t)$ at time t, this being undefined at