surprise however is a relative concept and readers of McKean (1973) would not be surprised at all!

No doubt readers will see other ways of addressing these problems using perhaps stochastic calculus without benefit of CA or the theory of Wishart distributions (indeed Mr. James of Leeds University has shown me how to use Wishart matrix theory to establish the Clifford-Green result mentioned above). The main purpose of this work has been to initiate the development of CA as an effective tool in the study of random processes, rather than to develop new results. More recently, and with the same motivation, I have been working on the use of CA to derive the statistics of shape diffusions for k-ads with $k > 3$. Here the technical problem is to find effective ways of dealing with sums involving k summands, when k is not fixed beforehand but must be treated as a symbolic quantity. Some progress has been made, but work is not yet complete.

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Comment

Geoffrey S. Watson

In stochastic geometry as in number theory, it is easy to ask questions that the layman can understand but that the specialist can only answer with difficulty or not at all. Under the older name, geometrical probability, the subject is old, e.g., Buffon's famous problem was invented around the time Buffon was preparing a French version of Newton's "fluxions." I don't know of any ancient and unresolved conjectures like Fermat's but it is easy to give simple-sounding problems that are hard to solve, e.g., the motivating problem of Kendall's theory of shape. How do the shapes of triangles vary when their vertices are independently and uniformly distributed in a fixed rectangle? This problem arises from questions about whether there is too much "collinearity" in sets of points (see Figures 1 and 2). A recent and very readable survey of Kendall's theory has been given by Small (1988).

All but the most mathematically gifted readers will find this paper difficult. Rather more basic details are given in Kendall (1984), but this too is written for mathematicians. I hope the promised book (now in preparation) by Carne, Kendall and Le will make it clear to statisticians, because I'm sure that this is a fascinating area for research and applications. To support this belief I will give a brief summary of my own related efforts, sticking mainly to triangles. This is reasonable because most of the suggested applications use them and they are the simplest case.

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The shape of Δ, a triangle P_1, P_2, P_3, with vertex angles α_1, α_2, α_3, could be defined as the pair (α_1, α_2). But for most problems this is not easy to work with, or to generalize to k labeled points in $\mathscr{R}^m$. There are lots of other ways to define the shape of a triangle. We may think of Δ as a 2×3 matrix $[z_1, z_2, z_3]$, where the column z_i has elements x_i, y_i, and denotes the position of the vertex P_i in the plane. Because we are only interested in the shape of Δ we may translate, dilate and rotate Δ without changing the shape of Δ, so we seek a "canonical" triangle. Kendall's approach is a variant of the following. Change the origin to the centroid of the triangle and consider the singular value decomposition of the new 2×3 matrix, $R\Lambda L'$, where R is a 2×2 rotation and so irrelevant. By scaling we could make $\lambda_1^2 + \lambda_2^2 = 1$. The remaining object defines the shape. See Mannion (1988) for a simple description—it is very similar to the next suggestion—and Small (1988).

I found Kendall's reduction hard to understand and considered (in Watson, 1986) two alternatives, which worked well in the simple planar problem I had posed. Move P_1 to the origin (0, 0), move P_2 to (0, 1), which uses up the available transformations, and denote P_3 by z, which then serves to define the shape of Δ. It is natural to take it as a point in the complex plane. The other alternative came from taking z_1, z_2, z_3 as complex numbers C, and letting them be the elements of a vector Z in C^3. With $\omega = \exp 2\pi i/3$, the vectors $(1, 1, 1)$, $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$, denoted by $\mathbf{1}$, $\mathbf{u}$ and $\bar{\mathbf{u}}$ are an orthogonal basis for C^3, so $Z = c_1\mathbf{1} + c_2\mathbf{u} + c_3\bar{\mathbf{u}}$ for complex numbers c_1, c_2, c_3. $\mathbf{1}$ denotes a degenerate triangle, and $\mathbf{u}$ and $\bar{\mathbf{u}}$ equilateral triangles with centroids at the origin. Clearly $c_1\mathbf{1}$ is of no interest $-$ $\mathbf{1}'Z/3$ is the centroid of Δ and so set $c_1 = 0$. Multiplying Z by a complex number is the same as dilating and rotating Δ. Thus my canonical version of Δ could be written as $u + b\bar{u}$, with b its "shape." I then complicated the problem by going on to ignore the labels on the vertices. Veitch and Watson (1986) used a generalization of this Fourier method for k-labeled points in m dimensions for a generalization of the problem that started me off: Go along side P_1P_2 a distance $S\,|P_1P_2|$ to a point Q_3, go along P_2P_3 a distance $S\,|P_2P_3|$ to a point Q_1, and along P_3P_1 a distance $S\,|P_3P_1|$ to get a point Q_2. Thus one gets a new triangle Q_1, Q_2, Q_3. Let $S_1, S_2, \cdots$ be iid on $[0, 1]$, and repeat this construction successively. Thus we get a random sequence of triangles and I was interested in the random sequence of shapes. Mannion (1988) studied a much more interesting and difficult problem. His sequence of triangles was obtained by successively picking 3 points, iid uniformly at random in the parent triangle to get a new triangle. The labeling of the vertices is irrelevant for Mannion as it was in my problem and in the next paragraph. As one can guess, his triangles tend to collinear triangles.

Yet another way to get random triangles is to imagine that the vertices represent the three solutions of a cubic equation with random coefficients, $P_3(Z) = a_0 + a_1z + a_2z^2 + a_3z^3 = 0$. Kac (1943) solved the problem of the distribution of the real roots of $P_{n-1}(z) = 0$ when the a_j's are iid $N(0, 1)$. This doesn't seem to be a very practical problem but it is intriguing and one wonders about the distribution of all the roots or the complex roots. It seems very hard to get nontrivial analytic answers. The number of roots inside any closed curve S in the complex plane is given by

$$\frac{1}{2\pi i}\int_s \frac{P_n'(z)}{P_n(z)}\,dz.$$

Hence one needs to find the expectation of $P_n'(z)/P_n(z)$ for various assumptions about the joint distribution of the a_j's, and then to integrate the result. As far as I can see no one has used this approach. Bharucha-Reid and Sambandham (1986) give many references and show simulations with $n = 30$.

Certainly one wants to start with the cubic. Appendix A of Watson (1986) relates the solution of cubics to the shape of the root triangle. With Javier Cabrera, of Rutgers University, I have been studying this case with National Science Foundation support (DMS-84-21301). We have many pictures of both the roots and the shapes of the triangles formed by these roots. Of course, when the coefficients are real, one must get isosceles or collinear triangles. The pictures, especially those for shape, are much more interesting when the coefficients are complex. But the trick is—how to explain the patterns we see!

The analytic behavior of the shape density $m(x, y)$ described in § 4 reminds me of ancient work (Watson, 1956) and references therein) on the joint distribution of the ratios

$$r = \frac{\mathbf{x}'\mathbf{A}\mathbf{x}}{\mathbf{x}'\mathbf{x}},\ s = \frac{\mathbf{x}'\mathbf{B}\mathbf{x}}{\mathbf{x}'\mathbf{x}},$$

where $\mathbf{A}$ and $\mathbf{B}$ commute and $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I})$. In canonical form they are

$$r = \frac{\sum \lambda_i \omega_i}{\sum \omega_i},\quad s = \frac{\sum \mu_i \omega_i}{\sum \omega_i},$$

where the ω_i are independent gamma random variables. Thus (r, s) falls randomly in the convex hull of the points (λ_i, μ_i). Further their joint density changes its analytic form as (r, s) crosses joins of the points (λ_i, μ_i) and has alternative representations. I wonder if there are any connections?

Leaving triangles and returning to the original motivation of Kendall (whether there are too many collinear, or nearly so, points in a picture) we see this as but the first of a series of problems. Earth and planetary scientists often claim to see "lineaments" (linear and circular segments). Sometimes at the points of intersection there is oil or gold, *etc.*. The statistician is at a loss with this sort of data. The biostatistician faces similar problems, e.g., is a cluster of cancer deaths in a neighborhood indicative of a local problem? This public health issue and geology have recently come together in the radon problem (gas comes out from the interior of the earth through faults). The question (could some observed geometrical oddity be due to chance) will raise puzzles forever, I suspect.

In conclusion I would like to urge others in the United States to take an interest in stochastic geometry (at the moment it seems to be solely of interest to Europeans) and to congratulate David Kendall for his immense contributions to the whole field.

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Comment

Dietrich Stoyan

Professor Kendall's paper is an excellent survey on a very important topic and describes many deep and complicated results obtained by himself and his colleagues. It is a pleasure to congratulate him on this success and to wish him further progress. The publication in this journal will help to inform many statisticians of these ideas and methods and so lead to further interesting applications. Because my own work has had until now only weak connections to Professor Kendall's theory of shape (with the nice exception of being a coauthor of a book that contains a chapter on shape theory written by W. S. Kendall), I can give marginal comments only; I take the opportunity to ask some questions.

In my opinion, in some cases the original problem of finding collinearities in point patterns can be solved by means of methods of point process statistics. If the point pattern under study can be interpreted as a sample of a stationary point process, then the orientation analysis of Ohser and Stoyan (1981) can be used to detect orientations and collinearities; see also Stoyan, Kendall and Mecke (1987). More interesting is the case of motion-invariant point processes with "inner orientations"; a nice example is the pattern of self-intersection points of a motion-invariant planar line process. Hanisch and Stoyan (1984) suggested statistical characteristics that are based on third-order moment measures or two-point Palm distributions. An example is the mean number of points in a rhombus with vertices at the members of a "typical" point pair of the point process with distance r (see Figure 1). If the corresponding mean, for which an unbiased estimator was given, is clearly greater than "intensity × area of rhombus" for interesting values of r, then some form of collinearity in the point pattern is detected.

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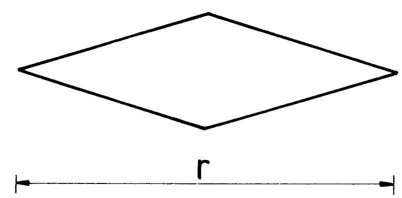

FIG. 1. *A rhombus with vertices of distance r. If the vertices are points of a point process and in the rhombus there are "many" other points of the point process, then this shows some inner orientation.*

Many statisticians and physicists, geographers (see the booklet by Boots, 1987) and others are very much interested in Dirichlet tessellations and the closely related Delaunay tessellations. Therefore the results on the Delaunay tessellation are of great value, both theoretically and practically. In particular, I like the elegant way of simulating "lone" Poisson Delaunay cells.

I think that a promising method for a "shape analysis" of tessellations could be based on the angles at vertices, if all vertices are Y-shaped, with three emanating edges. (This situation very often appears in practical problems, as physicists and materials scientists say.) Then each vertex corresponds to a triangle, which is similar to the Delaunay triangle if the tessellation under study is a Dirichlet tessellation with respect to a point pattern. Most empirical tessellations are not Dirichlet tessellations or, if their generating points are not given, the natural starting points for the shape analysis are the three angles. Therefore it would be helpful to transform shape theory results for triangles into angular coordinates, where, for example, a triangle is described by its maximal and minimal angles.

Perhaps it is of interest to mention a further (additionally to Professor Kendall's findings for PDLY tiles) interesting property of the Dirichlet tessellation, which in future may be better elucidated by the new simulation methods. Together with Dr. H. Hermann,