phenomena can substantially change our way of thinking about time series and systems in general, and the authors of these two papers are to be congratulated for their clear exposition of these issues.

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I am grateful to Steve Ellner and Doug Nychka for recent conversations on some of the topics discussed herein.

Comment: Simplicity and Nonlinearity

Ruey S. Tsay

Chaos is indeed a fascinating subject. It certainly will have some important impact on statistics both in theory and in application. Further, statisticians and probabilists can definitely make significant contributions in chaos. Therefore, I congratulate Professors Chatterjee, Yilmaz and Berliner on their nice and lucid introductions of chaos to the general statistical audience.

I agree with Professor Berliner that chaos is not distinct from mainstream statistics, especially regarding to stochastic processes and time series analysis. The argument between "deterministic" and "stochastic" is misleading. It results from our propensity to dichotomize events surrounding us. From a dynamical system point of view, a "stochastic system" is merely a "deterministic one" with infinite dimension. The difference, if any, is our inability to understand the complexity of a nonlinear system and our preference, justifiably so, to use simple linear models.

Furthermore, there is a close theoretical relation between the stability of a deterministic system and the ergodicity of a stochastic system. For instance, consider the simple deterministic system,

$$y_t = \begin{cases} a y_{t-1}, & \text{if } y_{t-d} \le 0, \\ b y_{t-1}, & \text{if } y_{t-d} > 0, \end{cases} \tag{1}$$

and the stochastic model,

$$x_t = \begin{cases} a x_{t-1} + e_{1,t}, & \text{if } x_{t-d} \le 0, \\ b x_{t-1} + e_{2,t}, & \text{if } x_{t-d} > 0, \end{cases} \tag{2}$$

where a and b are real numbers, d is a positive integer, $\{e_{1,t}\}$ and $\{e_{2,t}\}$ are independent sequences of independently and identically distributed random variates satisfying $E|e_{i,t}| < \infty$. Chen and Tsay (1991) show that the necessary and sufficient condition of geometrical ergodicity of x_t in (2) is

$$\begin{gathered} a < 1, \quad b < 1, \quad ab < 1, \\ a^{r(d)} b^{s(d)} < 1, \quad a^{s(d)} b^{r(d)} < 1, \end{gathered} \tag{3}$$

where $r(d)$ and $s(d)$ are nonnegative integers, depending on d such that $r(d)$ and $s(d)$ are odd and even numbers, respectively. It was shown in Lim (1992) that the condition in (3) is also the stability condition of y_t in (1).

Turning to the impact of chaos on statistics, I believe that the impact is far beyond those discussed by the authors. For example, chaos is an "eye-opener" for statisticians and probabilists. It points out loudly and clearly the need to explore nonlinearity and to develop statistical methods and tools that can adequately analyze nonlinear models. The linear world is very limited. That a "tent-map" can generate a realization with autocorrelations the same as those of a particular first-order autoregressive time series illustrates this point clearly. Linear models will undoubtedly continue to play an important role in statistical analysis, but the time has come for statisticians to see the nonlinear planet.

It is natural to ask the question, can we observe attractors in practice, as raised by Professors Chatterjee and Yilmaz and by many people in studying chaos. However, this is a simple-minded question. It falls again into the dichotomous world I alluded to before. Moreover, that no one has yet observed an attractor does not prove the nonexistence of attractors in practice. The important question is that, given a finite realization, possibly noisy, and some specific objectives of analysis, can we determine the most "appropriate model," within a reasonable class, for the data? This is a pressing problem in chaos. More importantly, it is a typical problem in statistics, and the statistician's job is to provide sound methods and proper tools for answering such a question. Here, I like to emphasize the objectives of the intended analysis, which were not emphasized in the two papers, and the

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fact that only finite observations are available. The class of models I have in mind includes various mixtures of deterministic and stochastic models. An example of such mixtures is the one mentioned by Berliner and studied in Chatterjee and Yilmaz (1991) in which a deterministic system serves as an input to a second system. I also like to mention that, similar to other statistical problems, there are cases in which statistics alone cannot determine the most appropriate model. In these cases, subject matters of the problem under study, such as the implications of a model, should play a more important role in the analysis. All of the above discussions are familiar to statisticians and show that important problems in chaos are no different from those in statistics. The only difference between chaos and statistics is that traditional statistics begins with linearity, whereas chaos is necessarily nonlinear.

In sum, chaos is fascinating because of its mathematical simplicity. It is important, especially to statisticians, because of its nonlinear nature. Theory of chaos and analysis of chaotic data are parts of statistical theory and modeling. Statisticians should be interested in chaos and can make significant contributions in chaos because it is statistics, although not in the traditional and linear sense.

Finally, I like to list some areas in chaos that statisticians and probabilists are well equipped to make significant contributions:

1. Ergodicity conditions of nonlinear dynamical systems, deterministic as well as stochastic.
2. The invariant density of a given dynamical system.
3. Methods (statistical and graphical) for uncovering lower dimensional systems based on noisy data.
4. Nonparametric statistical methods for dynamical system analysis, both for prediction and for structure recovery.
5. Complexity measures of a nonlinear dynamical system based on finite and noisy realizations.

Rejoinder (part 1)

Sangit Chatterjee and Mustafa R. Yilmaz

We are indebted to all six researchers for their stimulating and thoughtful comments on the two surveys. They make it abundantly clear that the theory of nonlinear deterministic chaos is still in its formative stage, and its relationship with statistics is just beginning to be explored by statisticians. We are especially pleased that each comment provides a somewhat different perspective concerning the emerging theory. Collectively, these comments help clarify and sharpen the important issues that will keep researchers busy for a long time to come.

The main motivation for our survey was our belief that the theory of nonlinear deterministic systems provides a different and potentially useful perspective from which statisticians can look at complex processes. We are pleased to observe that this opinion is shared by all but one of the commentators. The basic reason for the recent explosion of interest in this perspective is valid if not yet real: it may enable us to understand and explain the sources of randomness in some processes. No statistician can be indifferent to the exploration of this possibility, no matter how far from reality it might seem presently.

Within the confines of a rejoinder, it is neither possible nor appropriate for us to respond to all comments. With a conscious effort to avoid repetition, we shall briefly touch on some of the issues and points raised, especially in those cases where there is an apparent conflict in viewpoints, a contribution to be recognized or an error to be corrected. For this purpose, we have divided our comments into three sections. First, we briefly respond to each author in alphabetical order, next we provide a brief update of the literature and then conclude with some final thoughts and comments.

DISCUSSION ON COMMENTS

Professor Cutler provides expert discussions of singular and absolutely continuous probability distributions on attractors and their implications for dimension estimation [also see Hunt and Miller (1990) in this context]. Her discussion goes far beyond our review, but contrary to her statement, we do briefly mention lacunarity and nonuniformity in Section 1.2. Professor Cutler also discusses, as does Professor Smith, two basic ways noise enter a dynamic system. First, the errors can be