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ON THE CONVERGENCE RATES OF EMPIRICAL BAYES RULES FOR TWO-ACTION PROBLEMS: DISCRETE CASE[1]

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The purpose of this paper is to investigate the convergence rates of a sequence of empirical Bayes decision rules for the two-action decision problems where the distributions of the observations belong to a discrete exponential family. It is found that the sequence of the empirical Bayes decision rules under study is asymptotically optimal, and the order of associated convergence rates is $O(\exp(-cn))$, for some positive constant c, where n is the number of accumulated past experience (observations) at hand. Two examples are provided to illustrate the performance of the proposed empirical Bayes decision rules. A comparison is also made between the proposed empirical Bayes rules and some earlier existing empirical Bayes rules.

1. Introduction. The empirical Bayes approach in statistical decision theory is appropriate when one is confronted repeatedly and independently with the same decision problem. In such instances, it is reasonable to formulate the component problem in the sequence as a Bayes decision problem with respect to an unknown prior distribution on the parameter space and then use the accumulated observations to improve the decision rule at each stage. This approach is due to Robbins (1956, 1964, 1983). Many such empirical Bayes rules have been shown to be asymptotically optimal in the sense that the risk for the nth decision problem converges to the optimal Bayes risk which would have been obtained if the prior distribution was fully known and the Bayes rule with respect to this prior distribution was used.

The usefulness of empirical Bayes rules in practical applications clearly depends on the convergence rates with which the risks for the successive decision problems approach the optimal Bayes risk. The purpose of this paper is to investigate the convergence rates of a sequence of empirical Bayes rules for two-action decision problems when the distributions of the observations belong to a discrete exponential family.

Let X be a random observation with probability function of the form

$$f(x|\theta) = h(x)\theta^x\beta(\theta), \qquad x = 0, 1, 2, \ldots; 0 < \theta < Q, \tag{1.1}$$

where $h(x) > 0$ for all $x = 0, 1, 2, \ldots,$ and where Q may be finite or infinite. The observation X may be thought of as the value of a sufficient statistic based on several iid observations. Consider the following testing: H_0: $\theta \geq \theta_0$ against H_1: $\theta < \theta_0$, where θ_0 is a known positive constant. For each $i = 0, 1$, let i denote the

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action deciding in favor of H_i. For the parameter θ and action i, the loss function is defined as

$$L(\theta, i) = (1 - i)(\theta_0 - \theta)I_{(0,\theta_0)}(\theta) + i(\theta - \theta_0)I_{[\theta_0, Q)}(\theta), \tag{1.2}$$

where $I_A(\cdot)$ denotes the indicator function of the set A. In (1.2) the first item is the loss due to taking action 0 when $\theta < \theta_0$, and the second item is the loss of taking action 1 when $\theta \geq \theta_0$. It is assumed that θ is the value of a random variable Θ having an unknown prior distribution $G(\theta)$.

For a decision rule d, let $d(x) = P\{\text{accepting } H_0 | X = x\}$. That is, $d(x)$ is the probability of taking action 0 given $X = x$. Let D be the class of all decision rules. For each decision rule d, let $r(G, d)$ denote the associated Bayes risk. Then $r(G) = \inf_{d \in D} r(G, d)$ is the minimum Bayes risk among the class D.

Based on the preceding statistical model, the Bayes risk associated with the decision rule d is

$$r(G, d) = \sum_{x=0}^{\infty} [\theta_0 - \varphi(x)] d(x) f(x) + C, \tag{1.3}$$

where

$$\varphi(x) = \frac{h(x) f(x+1)}{h(x+1) f(x)}, \tag{1.4}$$

$$f(x) = \int_0^Q f(x|\theta)\, dG(\theta), \tag{1.5}$$

$$C = \sum_{x=0}^{\infty} \int_{\theta_0}^{Q} (\theta - \theta_0) f(x|\theta)\, dG(\theta). \tag{1.6}$$

We consider only priors G such that $\int_0^Q \theta\, dG(\theta) < \infty$ to insure that the risk is always finite.

Note that C is a constant which is independent of the decision rule d. Thus, from (1.3), a Bayes decision rule, say d_G, is clearly given by

$$d_G(x) = \begin{cases} 1 & \text{if } \varphi(x) \geq \theta_0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.7}$$

Definition 1.1. A decision rule d is said to be monotone if for $x, y \geq 0$ with $x \leq y$, $d(x) \leq d(y)$.

Since the class $\{f(x|\theta) | 0 < \theta < Q\}$ has monotone likelihood ratio in x, straightforward computation leads to that $\varphi(x)$ is increasing in x. Therefore, by (1.7), the Bayes decision rule $d_G(x)$ is a monotone decision rule.

Since the prior distribution G is unknown, it is not possible to apply the Bayes rule for the decision problem at hand. In this situation, we use the empirical Bayes approach. We note that Johns and Van Ryzin (1971) have studied the preceding decision problem via empirical Bayes approach. In this paper, a sequence of empirical Bayes decision rules $\{d_n^*\}$ is proposed for the decision problem described previously. The associated asymptotic optimality property is investigated. It is found that the order of the rate of convergence of $\{d_n^*\}$ is $O(\exp(-cn))$ for some positive constant c, where n is the number of accumulated past experience (observations) at hand. Two examples are given to illustrate the performance of the proposed empirical Bayes decision rules. A comparison is also made between the proposed empirical Bayes rules and some earlier existing empirical Bayes rules.

2. The proposed empirical Bayes rules and its asymptotic optimality. For each $j = 1, \dots,$ let (X_j, Θ_j) be a pair of random variables, where X_j is observable but Θ_j is not observable. Conditional on $\Theta_j = \theta$, X_j has probability function $f(x|\theta)$. It is assumed that Θ_j, $j = 1, \dots,$ are independently distributed with common unknown prior distribution G. Therefore, (X_j, Θ_j), $j = 1, 2, \dots,$ are iid. Let $\mathbf{X}_n = (X_1, \dots, X_n)$ denote the n past observations and let $X_{n+1} \equiv X$ denote the current random observation.

For each $x = 0, 1, 2, \dots,$ let

$$f_n(x) = \frac{1}{n} \sum_{j=1}^{n} I_{\{x\}}(X_j) + \delta_n, \tag{2.1}$$

where δ_n is a positive value such that $\delta_n = o(1)$. The estimator $f_n(x)$ is analogous to the usual empirical frequency estimator of $f(x)$ with some modification which guarantees that $f_n(x)$ is always positive. Let

$$\varphi_n(x) = \frac{h(x) f_n(x+1)}{h(x+1) f_n(x)}. \tag{2.2}$$

Analogous to the Bayes rule $d_G(x)$ of (1.7), one may obtain an intuitive empirical Bayes rule which decides to take action 0(1) whenever $\varphi_n(x) \geq (<) \theta_0$. See Johns and Van Ryzin (1971), though the estimator $f_n(x)$ given in (2.1) is different from theirs. However, the estimator $\varphi_n(x)$ does not possess the increasing property and therefore the corresponding decision rule is not monotone.

Recall that the class $\{f(x|\theta) | 0 < \theta < Q\}$ has monotone likelihood ratio in x. Under the loss function $L(\theta, i)$ of (1.2), the class of monotone decision rules is essentially complete; see Berger (1985). Thus, it is natural to desire that the proposed empirical Bayes decision rule be monotone. In the following, we propose a monotone empirical Bayes decision rule, say d_n^*, which is obtained on basis of a smoothed version of $\varphi_n(x)$.

Let

$$\varphi_n^*(x) = \left[\max_{0 \leq y \leq x} \varphi_n(y)\right] \wedge Q, \tag{2.3}$$

where $a \wedge b = \min\{a, b\}$. Then the empirical Bayes decision rule d_n^* is defined as

$$d_n^*(x) = \begin{cases} 1 & \text{if } \varphi_n^*(x) \geq \theta_0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Note that the past data $\mathbf{X}_n$ is implicitly contained in the subscript n. From (2.3), $\varphi_n^*(x)$ is nondecreasing in x. Then, by (2.4), we see that $d_n^*(x)$ is a monotone decision rule.

In the following, the asymptotic optimality of the sequence of the proposed empirical Bayes decision rules $\{d_n^*\}$ will be investigated. The monotonicity of the decision rules $\{d_n^*\}$ will be used to obtain the related asymptotic optimality.

Consider an empirical Bayes decision rule $d_n(x)$. Let $r(G, d_n)$ be the Bayes risk associated with the rule d_n. Then

$$r(G, d_n) = \sum_{x=0}^{\infty} [\theta_0 - \varphi(x)] E[d_n(x)] f(x) + C, \tag{2.5}$$

where the expectation E is taken with respect to $\mathbf{X}_n$. Since $r(G)$ is the minimum Bayes risk, $r(G, d_n) - r(G) \geq 0$ for all n. Thus, the nonnegative difference $r(G, d_n) - r(G)$ is used as a measure of the optimality of the empirical Bayes decision rule d_n.

DEFINITION 2.1. A sequence of empirical Bayes decision rules $\{d_n\}_{n=1}^{\infty}$ is said to be asymptotically optimal at least of order α_n relative to the (unknown) prior distribution G if $r(G, d_n) - r(G) \leq O(\alpha_n)$ as $n \to \infty$, where $\{\alpha_n\}$ is a sequence of positive numbers such that $\lim_{n \to \infty} \alpha_n = 0$.

Let $A(\theta_0) = \{x | \varphi(x) > \theta_0\}$ and $B(\theta_0) = \{x | \varphi(x) < \theta_0\}$. Define

$$M = \begin{cases} \min A(\theta_0) & \text{if } A(\theta_0) \neq \varnothing, \\ \infty & \text{if } A(\theta_0) = \varnothing, \end{cases} \tag{2.6}$$

$$m = \begin{cases} \max B(\theta_0) & \text{if } B(\theta_0) \neq \varnothing, \\ -1 & \text{if } B(\theta_0) = \varnothing, \end{cases} \tag{2.7}$$

where $\varnothing$ denotes the empty set.

By the increasing property of $\varphi(x)$ with respect to the variable x, $m \leq M$; also, $m < M$ if $A(\theta_0) \neq \varnothing$. Furthermore,

$$x \leq m \text{ iff } \varphi(x) < \theta_0 \quad \text{and} \quad y \geq M \text{ iff } \varphi(y) > \theta_0. \tag{2.8}$$

The following theorem is our main result.

THEOREM 2.1. *Let $\{d_n^*\}$ be the sequence of empirical Bayes decision rules defined previously. Suppose that $\theta_0 < Q$. Also, assume that*

(a) $\int_0^Q \theta \, dG(\theta) < \infty$ *and*
(b) $m < \infty$.

Then $r(G, d_n^*) - r(G) \leq O(\exp(-cn))$ for some positive constant c.

PROOF. Under assumption (b) and by (2.8), direct computation leads to

$$\begin{aligned} r(G, d_n^*) - r(G) = &\sum_{x=0}^{m} [\theta_0 - \varphi(x)] P\{\varphi_n^*(x) \ge \theta_0\} f(x) \\ &+ \sum_{x=M}^{\infty} [\varphi(x) - \theta_0] P\{\varphi_n^*(x) < \theta_0\} f(x), \end{aligned} \tag{2.9}$$

where $\Sigma_{x=0}^{m} \equiv 0$ if $m = -1$.

The nondecreasing property of $\varphi_n^*(x)$ implies

$$\begin{cases} P\{\varphi_n^*(x) \ge \theta_0\} \le P\{\varphi_n^*(m) \ge \theta_0\} & \text{for all } x \le m, \\ P\{\varphi_n^*(x) < \theta_0\} \le P\{\varphi_n^*(M) < \theta_0\} & \text{for all } x \ge M. \end{cases} \tag{2.10}$$

Combining (2.9) and (2.10), we have

$$r(G, d_n^*) - r(G) \le b_1 P\{\varphi_n^*(m) \ge \theta_0\} + b_2 P\{\varphi_n^*(M) < \theta_0\}, \tag{2.11}$$

where

$$0 \le b_1 = \sum_{x=0}^{m} [\theta_0 - \theta(x)] f(x) < \infty, \qquad 0 \le b_2 = \sum_{x=M}^{\infty} [\varphi(x) - \theta_0] f(x) < \infty$$

and the finiteness of both b_1 and b_2 is guaranteed since $\int \theta \, dG(\theta) < \infty$ by assumption (a).

Therefore, it suffices to consider the asymptotic behavior of both $P\{\varphi_n^*(m) \ge \theta_0\}$ and $P\{\varphi_n^*(M) < \theta_0\}$.

By the definition of $\varphi_n^*(x)$, when $\varphi_n^*(M) < Q$, then $\varphi_n^*(M) \ge \varphi_n(M)$, where $\varphi_n(\cdot)$ is the function defined in (2.2). In view of this fact and by (2.1) and (2.2),

$$\begin{aligned} P\{\varphi_n^*(M) < \theta_0\} &\le P\{\varphi_n(M) < \varphi_0\} \\ &= P\left\{\frac{1}{n} \sum_{j=1}^{n} A_j(M) < -t(M, \theta_0) + \Delta(M, \theta_0, n)\right\}, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} A_j(x) = &h(x)\left[I_{\{x+1\}}(X_j) - f(x+1)\right] \\ &-\theta_0 h(x+1)\left[I_{\{x\}}(X_j) - f(x)\right], \end{aligned} \tag{2.13}$$

$$t(x, \theta_0) = h(x) f(x+1) - \theta_0 h(x+1) f(x), \tag{2.14}$$

$$\Delta(x, \theta_0, n) = \delta_n\left[h(x+1)\theta_0 - h(x)\right]. \tag{2.15}$$

Also, by the definition of $\varphi_n^*(x)$ and (2.1) and (2.2) again,

$$\begin{aligned} &P\{\varphi_n^*(m) \ge \theta_0\} \\ &\quad = P\{\varphi_n(y) \ge \theta_0 \text{ for some } y = 0, 1, \dots, m\} \\ &\quad \le \sum_{y=0}^{m} P\{\varphi_n(y) \ge \theta_0\} \\ &\quad = \sum_{y=0}^{m} P\left\{\frac{1}{n} \sum_{j=1}^{n} A_j(y) \ge -t(y, \theta_0) + \Delta(y, \theta_0, n)\right\}. \end{aligned} \tag{2.16}$$

Note that $A_j(x)$, $j = 1, \dots, n$, are iid; $E[A_j(x)] = 0$ and $a_1(x, \theta_0) \le A_j(x) \le a_2(x, \theta_0)$, where $a_1(x, \theta_0) = -h(x)f(x+1) - h(x+1)\theta_0 + h(x+1)\theta_0 f(x)$ and $a_2(x, \theta_0) = h(x) - h(x)f(x+1) + h(x+1)\theta_0 f(x)$. Also, since $\delta_n = o(1)$ and $m < \infty$, there exists some positive integer n_0 such that for all $n \ge n_0$, $|\Delta(y, \theta_0, n)| \le \frac{1}{2}|t(y, \theta_0)|$ hold for all $0 \le y \le m$ and for $y = M$. Hence, for n sufficiently large, $-t(M, \theta_0) + \Delta(M, \theta_0, n) < 0$ since $t(M, \theta_0) > 0$, and $-t(y, \theta_0) + \Delta(y, \theta_0, n) > 0$ for $0 \le y \le m$ since $t(y, \theta_0) < 0$ for $0 \le y \le m$. In view of the preceding facts and by Theorem 2 of Hoeffding (1963),

$$\begin{aligned} &P\left\{\frac{1}{n}\sum_{j=1}^{n} A_j(M) < -t(M, \theta_0) + \Delta(M, \theta_0, n)\right\} \\ &\quad \le \exp\left\{-2n\left[-t(M, \theta_0) + \Delta(M, \theta_0, n)\right]^2 a_3^{-1}(M, \theta)\right\} \\ &\quad \le \exp\left\{-\frac{n}{2}\left[-t(M, \theta_0)\right]^2 a_3^{-1}(M, \theta_0)\right\} \end{aligned} \tag{2.17}$$

and for $0 \le y \le m$,

$$\begin{aligned} &P\left\{\frac{1}{n}\sum_{j=1}^{n} A_j(y) \ge -t(y, \theta_0) + \Delta(y, \theta_0, n)\right\} \\ &\quad \le \exp\left\{-2n\left[-t(y, \theta_0) + \Delta(y, \theta_0, n)\right]^2 a_3^{-1}(y, \theta_0)\right\} \\ &\quad \le \exp\left\{-\frac{n}{2}\left[-t(y, \theta_0)\right]^2 a_3^{-1}(y, \theta_0)\right\}, \end{aligned} \tag{2.18}$$

where $a_3(x, \theta_0) = a_2(x, \theta_0) - a_1(x, \theta_0) = h(x) + h(x+1)\theta_0$.

Let

$$c = \tfrac{1}{2}\min\left\{t^2(y, \theta_0)a_3^{-1}(y, \theta_0) \mid 0 \le y \le m \text{ or } y = M\right\}. \tag{2.19}$$

It is clear that $c > 0$ since $m < \infty$ from assumption (b) and

$$t^2(y, \theta_0)a_3^{-1}(y, \theta_0) > 0$$

for all $0 \le y \le m$ and for $y = M$. Then from (2.11), (2.12) and (2.16)–(2.19), we have

$$\begin{aligned} r(G, d_n^*) - r(G) &\le b_1 \sum_{y=0}^{m} \exp(-cn) + b_2 \exp(-cn) \\ &= O(\exp(-cn)). \end{aligned} \tag{2.20}$$

Hence, the proof of this theorem is complete. □

3. Examples and remarks. The following two examples have been considered by Johns and Van Ryzin (1971) and used to illustrate the performance of their proposed empirical Bayes decision rules for the two-action problem. We cite them and use the same to illustrate the performance of the proposed empirical Bayes decision rules $\{d_n^*\}$.

EXAMPLE 1 (The geometric distribution). Suppose that

$$f(x|\theta) = \theta^x(1-\theta), \qquad x = 0,1,2,\dots; 0 < \theta < 1,$$

and that the prior distribution has the probability density function $g(\theta)$, where

$$g(\theta) = (\alpha+1)(1-\theta)^{\alpha}, \qquad 0 < \theta < 1, \alpha > -1.$$

Then

$$h(x) \equiv 1 \quad \text{and} \quad f(x) = \frac{(\alpha+1)\Gamma(x+1)\Gamma(\alpha+2)}{\Gamma(x+\alpha+3)}.$$

Thus,

$$\varphi(x) = \frac{h(x)f(x+1)}{h(x+1)f(x)} = \frac{x+1}{x+\alpha+3},$$

which tends to 1 as $x \to \infty$. Taking $0 < \theta_0 < 1$, then $A(\theta_0) = \{x|\varphi(x) \geq \theta_0\} \neq \varnothing$. Therefore, $m < M \equiv \min A(\theta_0) < \infty$. Hence, by Theorem 2.1,

$$r(G, d_n^*) - r(G) \leq O(\exp(-cn))$$

for some positive constant c.

EXAMPLE 2 (The Poisson distribution). Let

$$f(x|\theta) = e^{-\theta}\theta^x/\Gamma(x+1), \qquad x = 0,1,2,\dots; \theta > 0.$$

Letting the prior density function be $g(\theta) = e^{-\theta}$, $\theta > 0$, we then have

$$f(x) = \frac{1}{\Gamma(x+1)}\int_0^{\infty} \theta^x e^{-2\theta}\, d\theta = \left(\frac{1}{2}\right)^{x+1} \quad \text{and} \quad h(x) = \frac{1}{\Gamma(x+1)}.$$

Thus,

$$\varphi(x) = \frac{h(x)f(x+1)}{h(x+1)f(x)} = \frac{x+1}{2},$$

which tends to ∞ as $x \to \infty$. Therefore, for any finite $\theta_0 > 0$, $m < \infty$. Then, by Theorem 2.1, $r(G, d_n^*) - r(G) \leq O(\exp(-cn))$ for some positive constant c.

Johns and Van Ryzin (1971) considered several situations about the behavior of the tail probability of the prior probability density function, under which their proposed empirical Bayes decision rules may achieve the best possible convergence rate $\alpha_n = n^{-1}$. We also apply those conditions to the sequence of the empirical Bayes decision rules $\{d_n^*\}$. We state the result as a corollary without citing the statement of those conditions. The reader is referred to Johns and Van Ryzin (1971) for details.

COROLLARY 3.1. *Let $\{d_n^*\}$ be the sequence of the empirical Bayes decision rules defined in Section 2. Suppose that $\int_0^Q \theta\, dG(\theta) < \infty$. Then, either under the assumptions in Theorem 3 or under the assumptions in Theorem 4 of Johns and Van Ryzin* (1971), *we have $r(G, d_n^*) - r(G) \leq O(\exp(-cn))$ for some positive constant c.*

PROOF. We need only to verify that $A(\theta_0) \neq \varnothing$ under each assumption. This can be done directly by noting Lemmas 4, 5 and 6 of Johns and Van Ryzin (1971). □

REMARKS.

1. One can see that the assumptions given in Theorem 2.1 are simpler and more natural than that of Johns and Van Ryzin (1971), since the prior distribution function $G(\cdot)$ is unknown and therefore, it is hard to verify the behavior of the tail probability of the prior density function. Further, Johns and Van Ryzin's results cannot be applied to the case where the prior distribution function $G(\cdot)$ is not continuous. However, the result of Theorem 2.1 still holds even in this situation.
2. The empirical Bayes tests of Johns and Van Ryzin (1971) have the same pointwise, exponential convergence rate, but poor overall convergence rate. This disappointing fact is due to the behavior of their empirical Bayes tests for the large values of x. To overcome this difficulty, we use the smoothed, monotone estimator $\varphi_n^*(x)$, which simplifies the problem into two points case, see (2.9)–(2.11).

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