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ADDENDUM TO: "LOCAL ASYMPTOTICS FOR LINEAR RANK STATISTICS WITH ESTIMATED SCORE FUNCTIONS"

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In a parametric counterpart of the nonparametric considerations of Neuhaus, a class of quadratic goodness of fit statistics is developed generalizing the usual Cramér–von Mises statistics.

Here a parametric counterpart of the nonparametric considerations of Neuhaus [N] (1987) is presented. The notation of [N] will be kept up.

Let $X_1, \dots, X_n$ be i.i.d. real random variables (rv's) each having a continuous df F and let us start with the simple hypothesis H_0: $F = F_0$ for some fixed continuous df F_0 versus the omnibus alternative. H_0 holds true iff the rv's $U_i = F_0(X_i)$, $1 \le i \le n$, are uniformly distributed on $(0, 1)$, i.e., H_0: $Q = \lambda$ with $Q = \mathscr{L}(F_0(X_i))$. As alternatives H_1 we choose all probability measures $Q \ne \lambda$ on $(0, 1)$ being dominated by λ. Defining $g_n \in L_2^0(0, 1)$ by the equality $dQ/d\lambda = 1 + g_n/\sqrt{n}$, the testing problem rewrites as H_0: $g_n = 0$ versus H_1: $g_n \ne 0$. It is a well known fact from asymptotic testing theory, that for $g_n \to g$ in $L_2(0, 1)$ (always $n \to \infty$) the upper test based on the *linear statistic with score* g_n, i.e.,

$$n^{-1/2} \sum_{i=1}^{n} g_n(U_i), \tag{1}$$

is asymptotically optimal for testing H_0 versus the single sequence of alternatives $\mathscr{L}(U_1, \dots, U_n) = Q^n$ represented by g_n, $n \ge 1$. At this point one is in the same situation as described in [N]: There is an asymptotically optimal test statistic with unknown optimal score function g_n. Following the procedure of [N] we estimate g_n by some estimator $\hat{g}_n$ based on the kernel method and use $n^{-1/2}\sum_{i=1}^{n}\hat{g}_n(U_i)$ as a new test statistic which may be expected to be sensitive over broader ranges of alternatives than the linear statistic (1).

The preceding development immediately extends to composite hypotheses

$$H_0\colon F \in \{F(\cdot, \vartheta)\colon \vartheta \in \Theta\} =: \mathscr{F},$$

where $F(\cdot, \vartheta)$ are continuous df's and $\Theta \subset \mathscr{R}^r$, $r \ge 1$, is some open parameter set: If $\hat{\vartheta}_n = \hat{\vartheta}_n(X_1, \dots, X_n)$ is some estimator of ϑ, we redefine $U_i = F(X_i, \hat{\vartheta}_n)$ and use the test statistic (1) with the new U_i's.

Using the kernel density estimator $(1/n)\sum_{i=1}^{n} K_a(\cdot, U_i)$ of $dQ/d\lambda$ (cf. [N], (2.7)], we get

$$\hat{g}_n = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} K_a(\cdot, U_i) - 1\right)$$

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as an approximation of g_n. Replacing g_n by $\hat{g}_n$ in (1) yields the quadratic test statistic

$$T_n = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(K_a(U_i, U_j) - 1 \right). \tag{2}$$

If $W_n(t) = n^{1/2}(F_n(t) - t)$, $0 \le t \le 1$, is the empirical process of the U_i's, T_n may be rewritten as $T_n = \sum_{\kappa=1}^{\infty} \lambda_\kappa(a) \langle \psi_\kappa', W_n \rangle^2$. Since W_n, $n \ge 1$, converges in distribution under alternatives $g_n \to g$ [see Neuhaus (1976)], the asymptotic distribution of T_n may be derived under suitable regularity conditions as in [N], Theorem 2.5, with the same shape of the limiting distribution.

As already discussed in [N] the choice of (bell-shaped) kernels K is of much less importance than the choice of the bandwidth a. The "Cramér–von Mises kernel" $K(x) = \{2/3 - (|x|/2)(1 - |x|/2)\} 1(|x| \le 1)$ with the very large bandwidth $a = 1$ leads to $\lambda_\kappa(1) = (\pi\kappa)^{-2}$ entailing

$$T_n = \|W_n\|^2 = \int \left(F_n^e(x) - F(x, \hat{\vartheta}_n) \right)^2 dF(x, \hat{\vartheta}_n),$$

which is just the usual Cramér–von Mises statistic; here F_n^e is the empirical df of $X_1, \dots, X_n$. For practical applications in [N] the Parzen-2 kernel is recommended since for all bandwidths $a \in (0, 1]$ the $\lambda_\kappa(a)$'s are nonnegative,

$$\lambda_\kappa(a) = \left(\sin(\pi\kappa a/4)\right)^4 (\pi\kappa a/4)^{-4}, \qquad \kappa \ge 1,$$

rendering the corresponding test asymptotically unbiased, and in the case of simple hypothesis even asymptotically admissible; see [N]. Hall (1985) introduced for the simple hypothesis case (on the circle) a U-statistic which in our setting on the line would correspond to T_n from (2) with triangular kernel

$$K(x) = (1 - |x|) 1(|x| \le 1).$$

Therefore, Hall's statistic may be conceived as a linear statistic with estimated score function in our sense. The "tailoring" in the title of Hall's paper refers to the possiblity of choosing the bandwidth adequately.

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