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COMPARISONS OF OPTIMAL STOPPING VALUES AND PROPHET INEQUALITIES FOR NEGATIVELY DEPENDENT RANDOM VARIABLES

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Let $\mathbf{Y} = (Y_1, \dots, Y_n)$ be random variables satisfying the weak negative dependence condition: $P(Y_i < a_i | Y_1 < a_1, \dots, Y_{i-1} < a_{i-1}) \le P(Y_i < a_i)$ for $i = 2, \dots, n$ and all constants $a_1, \dots, a_n$. Let $\mathbf{X} = (X_1, \dots, X_n)$ have independent components, where X_i and Y_i have the same marginal distribution, $i = 1, \dots, n$. It is shown that $V(\mathbf{X}) \le V(\mathbf{Y})$, where $V(\mathbf{Y}) = \sup\{EY_t: t$ is a stopping rule for $Y_1, \dots, Y_n\}$. Also, the classical inequality which for nonnegative variables compares the expected return of a prophet $E\{Y_1 \vee \cdots \vee Y_n\}$ with that of the statistician $V(\mathbf{Y})$, i.e., $E\{Y_1 \vee \cdots \vee Y_n\} < 2V(\mathbf{Y})$, holds for nonnegative $\mathbf{Y}$ satisfying the negative dependence condition. Moreover, this inequality can be obtained by an explicitly described threshold rule $t(b)$, i.e., $E\{Y_1 \vee \cdots \vee Y_n\} < 2EY_{t(b)}$. Generalizations of this prophet inequality are given. Extensions of the results to infinite sequences are obtained.

1. Introduction. We consider stopping rules for negatively dependent random variables satisfying the following condition:

CONDITION $(*)$. The random variables $Y_1, Y_2, \dots$ are said to be *negatively lower orthant dependent in sequence* [in short: $(*)$] if

$$P(Y_i < a_i | Y_1 < a_1, \dots, Y_{i-1} < a_{i-1}) \le P(Y_i < a_i) \tag{1.1}$$

for $i = 2, 3, \dots$ and all constants $a_1, a_2, \dots$ (whenever the conditional probability is defined). Any finite sequence $Y_1, \dots, Y_n$, satisfying (1.1) for $i = 2, \dots, n$ is also said to satisfy $(*)$.

It is readily seen (see Section 2 for definitions) that negatively associated or negatively dependent in sequence variables [Joag-Dev and Proschan (1983)] satisfy Condition $(*)$. Examples of variables satisfying Condition $(*)$ can, therefore, be obtained from these variables; they include the multinomial, Dirichlet, multivariate hypergeometric distributions, the multivariate normal distribution with negative correlations, permutation distributions including random sampling without replacement and clearly independent random variables.

We consider both finite and possibly infinite stopping rules. For completeness we define the notation needed here.

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DEFINITION. Let $Z_1, \dots, Z_n$ be a sequence of random variables and let F_i be the σ-field generated by $Z_1, \dots, Z_i$, $i = 1, 2, \dots$. The extended positive integer-valued random variable t is an *extended stopping rule* for $\mathbf{Z} = (Z_1, \dots, Z_n)$ if $\{t = i\} \in F_i$ for all $i = 1, \dots, n$ and it is a *stopping rule* for $\mathbf{Z}$ if also $P(t \le n) = 1$. Similarly, if the sequence $Z_1, Z_2, \dots$ is infinite, t is a *stopping rule* if $P(t < \infty) = 1$.

We shall denote the set of stopping rules for a sequence by $T_{\mathbf{Z}}$ and the class of extended stopping rules by $T_{\mathbf{Z}}^*$, where we suppress the dependence on the sequence if no confusion is likely. For any extended stopping rule t, Z_t is defined as in Krengel and Sucheston (1978) by

$$(1.2) \qquad Z_t = \sum_{i=1}^{n} Z_i I(t = i) \quad \text{and} \quad Z_t = \sum_{i=1}^{\infty} Z_i I(t = i)$$

for the finite and infinite sequences, respectively. *The existence of a finite first absolute moment,* i.e., $E|Z_i| < \infty$, *will be assumed for all variables throughout.* The optimal values are defined as

$$(1.3) \qquad V(\mathbf{Z}) = \sup\{EZ_t \colon t \in T\} \quad \text{and} \quad V^*(\mathbf{Z}) = \sup\{EZ_t \colon t \in T^*\}.$$

The values $V(Z_1, Z_2, \dots)$ and $V^*(Z_1, Z_2, \dots)$ are defined correspondingly.

Consider $Y_1, \dots, Y_n$ satisfying Condition $(*)$ and let $X_1, \dots, X_n$ be independent random variables, such that $X_i \sim Y_i$, i.e., X_i and Y_i have the same marginal distribution, $i = 1, \dots, n$. In Section 2 we show that $V(\mathbf{X}) \le V(\mathbf{Y})$, generalizing results of O'Brien (1983) concerning sample with and without replacement from a finite population. These results extend to V^* and to infinite sequences. Section 3 is devoted to "prophet inequalities." A prophet has full foresight of the whole sequence and, hence, his expected return is $E\{\max_{i=1,\dots,n} Z_i\} = E\{Z_1 \vee \cdots \vee Z_n\}$. Krengel and Sucheston (1978) show that for any sequence $X_1, \dots, X_n$ of nonnegative *independent* random variables,

$$(1.4) \qquad E\{X_1 \vee \cdots \vee X_n\} \le 2V(\mathbf{X})$$

and that 2 is the smallest constant for which (1.4) holds for *all* sequences of independent nonnegative random variables. Hill and Kertz (1981) show that the inequality in (1.4) is strict (in all nontrivial cases). We extend the preceding result to nonnegative random variables satisfying Condition $(*)$. Actually, we show a slightly stronger result, namely that there exists a "threshold rule" $t(b)$ such that for any (nontrivial) nonnegative $Y_1, \dots, Y_n$ satisfying Condition $(*)$,

$$(1.5) \qquad E\{Y_1 \vee \cdots \vee Y_n\} < 2EY_{t(b)}.$$

In particular, under Condition $(*)$, we obtain the prophet inequality

$$(1.6) \qquad E\{Y_1 \vee \cdots \vee Y_n\} < 2V(\mathbf{Y}).$$

A result of Kennedy (1985) is also generalized. Similar results hold for infinite sequences. Most of our results are obtained by simple arguments, in fact often simpler than the proofs of the previously known results.

2. Comparison of optimal values. For two random vectors, $\mathbf{X} = (X_1, \dots, X_n)$ and $\mathbf{Y} = (Y_1, \dots, Y_n)$, the notation $X_i \sim Y_i$ indicates that X_i and Y_i have the same marginal distribution.

THEOREM 2.1. *If $Y_1, \dots, Y_n$ satisfy Condition $(*)$ of negative dependence, $X_1, \dots, X_n$ are independent and $X_i \sim Y_i$, $i = 1, \dots, n$, then $V(\mathbf{X}) \le V(\mathbf{Y})$ and $V^*(\mathbf{X}) \le V^*(\mathbf{Y})$.*

REMARK. Throughout this article we write increasing for nondecreasing and decreasing for nonincreasing. For functions of several variables, increasing means nondecreasing in each variable.

LEMMA 2.2. *Let $h(y)$, $y \in R$, be an increasing nonnegative function and $\mathbf{Y}$ satisfy Condition $(*)$. Then assuming the expectations to follow exist,*

$$\begin{aligned} E\{h(Y_i)I(Y_1 < a_1, \dots, Y_{i-1} < a_{i-1})\} \\ \ge E\{h(Y_i)\}E\{I(Y_1 < a_1, \dots, Y_{i-1} < a_{i-1})\}. \end{aligned} \tag{2.1}$$

PROOF. First note that Condition $(*)$ can also be written as

$$P\left(\bigcap_{j=1}^{i-1} \{Y_j < a_j\}, Y_i \ge a_i\right) \ge P(Y_i \ge a_i)P\left(\bigcap_{j=1}^{i-1} \{Y_j < a_j\}\right) \tag{2.2}$$

for $i = 2, \dots, n$. Hence, the lemma is true when h is an indicator function. Now using the standard approximation of h by $\varepsilon\sum_{k=0}^{\infty} I\{v\colon h(v) \ge k\varepsilon\}$, then letting $\varepsilon \to 0$ and noting that $I\{v\colon h(v) \ge k\varepsilon\}$ equals $I\{[b_k, \infty)\}$ or $I\{(b_k, \infty)\}$ for some b_k, yields the result. □

REMARK. Adding a constant to h and standard approximations imply that Lemma 2.2 is true also when h is not necessarily nonnegative.

For any sequence of random variables $Z_1, \dots, Z_n$ and any vector of constants $\mathbf{c} = (c_1, \dots, c_n)$, where possibly $c_i = -\infty$, define the extended stopping rule $t(\mathbf{Z}, \mathbf{c})$ by

$$t(\mathbf{Z}, \mathbf{c}) = \begin{cases} \min\{i \le n : Z_i \ge c_i\}, & \text{if such a value exists,} \\ \infty, & \text{otherwise.} \end{cases} \tag{2.3}$$

We shall write $Z_{t(\mathbf{c})}$ for $Z_{t(\mathbf{Z}, \mathbf{c})}$.

LEMMA 2.3. *Let $Z_1, \dots, Z_n$ be any random variables and define $t(\mathbf{c}) = t(\mathbf{Z}, \mathbf{c})$ by (2.3). If $c_i > -\infty$ for all $1 \le i \le n-1$ and $c_n = 0$, then*

$$Z_{t(\mathbf{c})} = c_1 + [Z_1 - c_1]^+ + \sum_{i=2}^{n} \{c_i - c_{i-1} + [Z_i - c_i]^+\} I(t(\mathbf{c}) > i - 1). \tag{2.4}$$

If $c_i > -\infty$ for $i < k$ and $c_k = -\infty$, where $2 \le k \le n$, the summation in (2.4) should be taken up to $i = k$ only, and the interpretation of the last summand is $(Z_k - c_{k-1})I(t(\mathbf{c}) > k - 1)$. If $c_1 = -\infty$, then $Z_{t(\mathbf{c})} = Z_1$.

PROOF. We consider the case $c_i > -\infty$, $i = 1, \dots, n$ only, since the modifications needed if $c_k = -\infty$ are straightforward:

$$\begin{aligned} Z_{t(\mathbf{c})} &= \sum_{i=1}^{n} c_i I(t(\mathbf{c}) = i) + \sum_{i=1}^{n} [Z_i - c_i] I(t(\mathbf{c}) = i) \\ &= \sum_{i=1}^{n} c_i [I(t(\mathbf{c}) > i-1) - I(t(\mathbf{c}) > i)] + \sum_{i=1}^{n} [Z_i - c_i]^+ I(t(\mathbf{c}) > i-1) \\ &= c_1 + \sum_{i=2}^{n} (c_i - c_{i-1}) I(t(\mathbf{c}) > i-1) + \sum_{i=1}^{n} [Z_i - c_i]^+ I(t(\mathbf{c}) > i-1) \end{aligned}$$

and (2.4) follows. □

PROOF OF THEOREM 2.1. For the sequence $X_1, \dots, X_n$ of independent variables, it is well-known [see, e.g., Chow, Robbins and Siegmund (1971), Theorem 3.2] that the optimal stopping rule has the structure $t(\mathbf{X}, \mathbf{c}^*)$, with $c_n^* = 0$ for extended rules and $c_n^* = -\infty$ if $P(t \le n) = 1$ and

$$c_{i-1}^* = E(X_i \vee c_i^*) = c_i^* + E[X_i - c_i^*]^+ \quad \text{for } i = 2, \dots, n, \tag{2.5}$$

where for $i = n$ and $c_n^* = -\infty$ the interpretation of the right-hand side is EX_n.

Lemma 2.3 implies that for the optimal rule $t(\mathbf{X}, \mathbf{c}^*)$, $V(\mathbf{X}) = EX_{t(\mathbf{c}^*)} = c_1^* + E[X_1 - c_1^*]^+$. For the sequence $Y_1, \dots, Y_n$, the rule $t(\mathbf{Y}, \mathbf{c}^*)$ with $\mathbf{c}^*$ defined previously will not be optimal in general; hence, $V(\mathbf{Y}) \ge EY_{t(\mathbf{c}^*)}$.

Note that $I(t(\mathbf{Y}, \mathbf{c}^*) > i-1) = I(Y_1 < c_1^*, \dots, Y_{i-1} < c_{i-1}^*)$. Therefore, applying (2.4) and then (2.1) we obtain

$$\begin{aligned} EY_{t(\mathbf{c}^*)} &= c_1^* + E[Y_1 - c_1^*]^+ \\ &\quad + \sum_{i=2}^{n} E\{(c_i^* - c_{i-1}^* + [Y_i - c_i^*]^+) I(t(\mathbf{Y}, \mathbf{c}^*) > i-1)\} \\ &\le c_1^* + E[Y_1 - c_1^*]^+ \\ &\quad + \sum_{i=2}^{n} (c_i^* - c_{i-1}^* + E[Y_i - c_i^*]^+) E\{I(t(\mathbf{Y}, \mathbf{c}^*) > i-1)\}. \end{aligned} \tag{2.6}$$

Since $X_i \sim Y_i$, for $i = 1, \dots, n$, (2.5) implies that the latter expression reduces to $c_1^* + E[X_1 - c_1^*]^+$ and we conclude that

$$V(\mathbf{Y}) \ge EY_{t(\mathbf{c}^*)} \ge c_1^* + E[X_1 - c_1^*]^+ = V(\mathbf{X})$$

and similarly for V^*. □

THEOREM 2.1′. *If $Y_1, Y_2, \dots$ satisfy Condition $(*)$ and $X_1, X_2, \dots$ are independent, $X_i \sim Y_i$, $i = 1, 2, \dots$, then*

$$V(X_1, X_2, \dots) \le V(Y_1, Y_2, \dots) \quad \textit{and} \quad V^*(X_1, X_2, \dots) \le V^*(Y_1, Y_2, \dots).$$

PROOF. For any sequence of random variables $Z_1, Z_2, \ldots$ with $E|Z_i| < \infty$, define

$$Z_i(a, b) = \begin{cases} b, & \text{if } Z_i > b, \\ Z_i, & \text{if } a \le Z_i \le b, \\ a, & \text{if } Z_i < a. \end{cases}$$

Set $(Z_1(a, b), \ldots, Z_n(a, b)) = \mathbf{Z}(a, b)$ and consider now the values

$$V(\mathbf{Z}(a, b)) = \sup\{EZ_t(a, b): t \in T_{\mathbf{Z}(a, b)}\},$$
$$V'(\mathbf{Z}(a, b)) = \sup\{EZ_t(a, b): t \in T_{\mathbf{Z}}\}.$$

Theorem 4.8 of Chow, Robbins and Siegmund (1971) states that

$$V(Z_1, Z_2, \ldots) = \lim_{b \to \infty} \lim_{a \to -\infty} \lim_{n \to \infty} V'(Z_1(a, b), \ldots, Z_n(a, b)). \tag{2.7}$$

Note that for each n, $Y_1(a, b), \ldots, Y_n(a, b)$ satisfy Condition $(*)$ and, hence, by Theorem 2.1,

$$V(X_1(a, b), \ldots, X_n(a, b)) \le V(Y_1(a, b), \ldots, Y_n(a, b)).$$

The inequality also holds for V' since $V'(\mathbf{Y}(a, b)) \ge V(\mathbf{Y}(a, b))$ and by independence $V'(\mathbf{X}(a, b)) = V(\mathbf{X}(a, b))$. Taking limits as in (2.7), we now obtain $V(X_1, X_2, \ldots) \le V(Y_1, Y_2, \ldots)$. For V^* the result follows since

$$V^*(X_1, X_2, \ldots) = V^*(X_1^+, X_2^+, \ldots) = V(X_1^+, X_2^+, \ldots)$$
$$\le V(Y_1^+, Y_2^+, \ldots) \le V^*(Y_1, Y_2, \ldots). \qquad \square$$

DEFINITION. The random variables $Y_1, \ldots, Y_n$ are said to be *negatively lower orthant dependent* (NLOD) if

$$P(Y_1 < a_1, \ldots, Y_n < a_n) \le \prod_{i=1}^{n} P(Y_i < a_i) \tag{2.8}$$

for all $a_1, \ldots, a_n$.

REMARK. Note that if $(Y_1, \ldots, Y_n)$ satisfy Condition $(*)$, they are NLOD.

PROPOSITION 2.4. *Let $Y_1, \ldots, Y_n$ satisfy* (2.8) *and let $X_1, \ldots, X_n$ be independent with $X_i \sim Y_i$, $i = 1, \ldots, n$. Then for any $\mathbf{c}$, $t(\mathbf{Y}, \mathbf{c})$ is stochastically smaller than $t(\mathbf{X}, \mathbf{c})$, that is, $P(t(\mathbf{Y}, \mathbf{c}) \ge i) \le P(t(\mathbf{X}, \mathbf{c}) \ge i)$.*

PROOF.

$$P(t(\mathbf{Y}, \mathbf{c}) \ge i) = P(Y_j < c_j: j < i) \le \prod_{j=1}^{i-1} P(Y_j < c_j) = P(t(\mathbf{X}, \mathbf{c}) \ge i). \qquad \square$$

PROPOSITION 2.5. *Let $Y_1, \ldots, Y_n$ satisfy Condition $(*)$ and let $X_1, \ldots, X_n$ be independent with $X_i \sim Y_i$, $i = 1, \ldots, n$. A sufficient condition for $EY_{t(\mathbf{c})} > EX_{t(\mathbf{c})}$*

is that $c_n = 0$ *or* $c_n = -\infty$ *and*

$$c_{i-1} \geq c_i + E[X_i - c_i]^+, \qquad i = 2, \dots, n. \tag{2.9}$$

PROOF. In (2.6), replace $\mathbf{c}^*$ by $\mathbf{c}$ and $E\{I(t(\mathbf{Y}, \mathbf{c}) > i-1)\}$ by $E\{I(t(\mathbf{X}, \mathbf{c}) > i-1)\}$, using Proposition 2.4 and (2.9). By (2.4), the relation (2.6) will then read $EY_{t(\mathbf{c})} \geq EX_{t(\mathbf{c})}$. □

We now briefly discuss the relation of Condition (∗) and other concepts of negative dependence.

DEFINITION [Joag-Dev and Proschan (1983)]. Random variables $Y_1, \dots, Y_n$ are said to be

(a) *negatively associated* (NA) if for every pair of disjoint subsets A_1, A_2 of $\{1, \dots, n\}$,

$$\operatorname{Cov}\{f_1(Y_i, i \in A_1), f_2(Y_j, j \in A_2)\} \leq 0,$$

whenever f_1 and f_2 are increasing;

(b) *negatively dependent in sequence* (NDS) if for $i = 2, \dots, n$, $Y_1, \dots, Y_{i-1} | Y_i = y_i$ is decreasing stochastically in y_i, that is, for any increasing function f $E[f(Y_1, \dots, Y_{i-1}) | Y_i = y_i]$ is decreasing in y_i.

The proof of the following proposition is standard.

PROPOSITION 2.6. *If* $Y_1, \dots, Y_n$ *are either NA or NDS then Condition* (∗) *holds.*

REMARK. The variables (Y_1, Y_2, Y_3) taking the values $(0,0,1)$, $(0,1,0)$, $(1,0,0)$ and $(1,1,1)$, each with probability $\frac{1}{4}$, satisfy Condition (∗); they are neither NA nor NDS.

The next result, due to O'Brien (1983), follows as a special case from our results.

COROLLARY 2.7. *Let* $(I_1, \dots, I_n)$ *and* $(J_1, \dots, J_n)$ *denote random sampling with and without replacement, respectively, from* $\{1, \dots, N\}$, $n \leq N$. *Let* $X_k = r_k(I_k)$ *and* $Y_k = r_k(J_k)$, *where* $r_k(i) \leq r_k(j)$ *if* $i < j$, $k = 1, \dots, n$. *Then* $V(\mathbf{X}) \leq V(\mathbf{Y})$.

PROOF. Joag-Dev and Proschan [(1983), 3.2(a)] show that $J_1, \dots, J_n$ are NA and that increasing functions defined on disjoint subsets of a set of NA random variables are NA. Hence, $Y_1, \dots, Y_n$ are NA. Applying Proposition 2.6 and Theorem 2.1 the result follows. □

For further discussion and examples of NA, NDS and NLOD variables, see Joag-Dev and Proschan (1983) and Block, Savits and Shaked (1985) and references therein.

REMARK. Proposition 2.4 does not imply that if $\mathbf{Y}$ satisfies $(*)$, the *optimal* rule for $\mathbf{Y}$ is always stochastically smaller than that for the corresponding independent $X_1, \dots, X_n$ with $X_i \sim Y_i$ for $i = 1, \dots, n$. This is seen to be false by the following example: Using the notation of Corollary 2.7, with $N = 3$ and $n = 2$, let $r_1(1) = 4$, $r_1(2) = 5$, $r_1(3) = 6$ and $r_2(1) = 1$, $r_2(2) = 3$ and $r_2(3) = 6$. Easy inspection yields that when sampling with replacement the optimal rule always stops with the first observation, while when sampling without replacement the optimal rule stops with $t = 2$ if $J_1 = 1$.

3. Prophet inequalities. For any constant c and $\mathbf{c} = (c, \dots, c)$, write $t(c)$ instead of $t(\mathbf{Z}, \mathbf{c})$. The extended stopping rules $t(c)$ are called *threshold rules* and are simpler to handle in applications than most other rules.

In the present section we shall, for simplicity, consider only nonnegative random variables Y_i, and we also exclude the trivial random variables $Y_i \equiv 0$. All results can be generalized in an obvious way, replacing V by V^* and $Y_1 \vee \cdots \vee Y_n$ by $Y_1^+ \vee \cdots \vee Y_n^+$.

Theorem 3.1, which uses a method developed in Samuel-Cahn (1984), shows that for random variables satisfying Condition $(*)$, the prophet inequality can be obtained by a threshold rule. [In Samuel-Cahn (1984), (3.1) (following) was proved only for independent random variables with weak inequality.]

THEOREM 3.1. *Let $Y_1, \dots, Y_n$ be positive random variables satisfying Condition $(*)$ of negative dependence. Let b be the unique constant satisfying $b = \sum_{i=1}^n E[Y_i - b]^+$. Then*

$$E\{Y_1 \vee \cdots \vee Y_n\} < 2EY_{t(b)} \leq 2V(\mathbf{Y}). \tag{3.1}$$

PROOF. Since $Y_1 \vee \cdots \vee Y_n \leq b + \sum_{i=1}^n [Y_i - b]^+$, it follows that

$$E\{Y_1 \vee \cdots \vee Y_n\} \leq 2b. \tag{3.2}$$

Thus it suffices to show that $EY_{t(b)} > b$. Now

$$\begin{aligned} EY_{t(b)} &= bP(Y_1 \vee \cdots \vee Y_n \geq b) + \sum_{i=1}^n E\{[Y_i - b]I(t(b) = i)\} \\ &= bP(Y_1 \vee \cdots \vee Y_n \geq b) + \sum_{i=1}^n E\{[Y_i - b]^+ I(t(b) > i - 1)\} \\ &\geq bP(Y_1 \vee \cdots \vee Y_n \geq b) + \sum_{i=1}^n E[Y_i - b]^+ P(t(b) > i - 1) \\ &> bP(Y_1 \vee \cdots \vee Y_n \geq b) + P(t(b) = \infty) \sum_{i=1}^n E[Y_i - b]^+ \\ &= bP(Y_1 \vee \cdots \vee Y_n \geq b) + P(Y_1 \vee \cdots \vee Y_n < b)b = b. \end{aligned} \tag{3.3}$$

The first inequality in (3.3) follows from (2.1) and the strict inequality follows, since by assumption there exists an i such that $P(Y_i > b) > 0$ and, for such i, $E[Y_i - b]^+ P(t(b) > i - 1) > E[Y_i - b]^+ P(t(b) = \infty)$.

Note that $t(b)$ is not necessarily a finite stopping rule. We can replace $t(b)$ by $t'(b) \equiv t(\mathbf{Y},(b,\dots,b,0))$ and $EY_{t(b)} \le EY_{t'(b)}$. For $\mathbf{Y}$ nonnegative, $t'(b)$ stops with probability 1. □

The next theorem states that for *any* sequence $\mathbf{Y}$ of nonnegative random variables, the expectation of its maximum is bounded by twice the V value of independent random variables X_i with $X_i \sim Y_i$.

Theorem 3.2. *Let $\mathbf{Y}$ be any positive vector and let X_i be independent, $X_i \sim Y_i$, $i = 1,\dots,n$. Then*

$$E\{Y_1 \vee \cdots \vee Y_n\} < 2V(\mathbf{X}) \tag{3.4}$$

and 2 is the smallest constant for which (3.4) holds for all such $\mathbf{Y}$.

Proof. By Theorem 3.1 applied to $\mathbf{X}$ and (3.2) we have

$$V(\mathbf{X}) \ge EX_{t(b)} > b \ge E\{Y_1 \vee \cdots \vee Y_n\}/2,$$

since the b determined for $\mathbf{X}$ is the same as that for $\mathbf{Y}$. □

Immediate conclusions from Theorems 3.2 and 2.1 are:

Corollary 3.3. *Let $\mathbf{Y}$ be any positive random vector. Let X_i be independent, $X_i \sim Y_i$, $i = 1,\dots,n$, and assume $V(\mathbf{X}) \le V(\mathbf{Y})$. Then $E\{Y_1 \vee \cdots \vee Y_n\} < 2V(\mathbf{Y})$.*

Corollary 3.4. *Let $Y_1, Y_2,\dots$ be any sequence of positive random variables. Let X_i be independent, $X_i \sim Y_i$, $i = 1,2,\dots$, and suppose $V(X_1, X_2,\dots) \le V(Y_1, Y_2,\dots)$. Then $E\{\vee_{i=1}^{\infty} Y_i\} \le 2V(Y_1, Y_2,\dots)$. In particular the conclusion holds for $Y_1, Y_2,\dots$ satisfying Condition (*).*

Proof. This follows immediately since

$$E\{\vee_{i=1}^{\infty} Y_i\} = \lim_{n\to\infty} E\{Y_1 \vee \cdots \vee Y_n\},$$

whereas any random variables $Z_1, Z_2,\dots$ satisfy $V(Z_1, Z_2,\dots) \ge \lim_{n\to\infty} V(Z_1,\dots,Z_n)$. Clearly, in all the preceding results 2 is the best possible constant. □

For $Z_1,\dots,Z_n$, denote the n order statistics by $Z^n_{(1)} \ge Z^n_{(2)} \ge \cdots \ge Z^n_{(n)}$. Kennedy (1985) proved the following generalization of the usual prophet inequality: For any $1 \le k < n$ and any nonnegative independent $X_1,\dots,X_n$,

$$E\left\{\sum_{i=1}^{k} X^n_{(i)}\right\} \le (k+1)V(\mathbf{X}). \tag{3.5}$$

Clearly the usual prophet inequality is obtained by setting $k = 1$. Kennedy also shows that the constant $(k+1)$ in (3.5) is the best possible, by taking $Y_1 = \cdots = Y_{n-1} \equiv \mu$, $Y_n = 1$ or 0 with probability μ and $(1-\mu)$, respectively

$(0 < \mu < 1)$. Then

$$\lim_{\mu \to 0} \left(E\left\{ \sum_{i=1}^{k} Y_{(i)}^{n} \right\} \Big/ V(\mathbf{Y}) \right) = \lim_{\mu \to 0} \left(\mu(k + 1 - \mu)/\mu \right) = k + 1.$$

We strengthen Kennedy's result as follows.

THEOREM 3.5. *Consider positive* $Y_1, Y_2, \dots$ *and let* X_i *be independent,* $X_i \sim Y_i$, $i = 1, 2, \dots$.

(i) *If* $V(X_1, \dots, X_n) \le V(Y_1, \dots, Y_n)$, *then for all* $1 \le k < n$,

$$E\left\{ \sum_{i=1}^{k} Y_{(i)}^{n} \right\} < (k+1) V(Y_1, \dots, Y_n). \tag{3.6}$$

(ii) *Let* $Y_{(i)} = \lim_{n \to \infty} Y_{(i)}^{n}$. *If* $V(X_1, X_2, \dots) \le V(Y_1, Y_2, \dots)$. *Then for* $k = 1, 2, \dots$,

$$E\left\{ \sum_{i=1}^{k} Y_{(i)} \right\} \le (k+1) V(Y_1, Y_2, \dots). \tag{3.7}$$

In particular (3.6) *and* (3.7) *hold when* $Y_1, Y_2, \dots$ *satisfies Condition* $(*)$.

PROOF. Define the constant b as in Theorem 3.1. Clearly $\sum_{i=1}^{k} Y_{(i)}^{n} \le kb + \sum_{i=1}^{n} [Y_i - b]^{+}$ and, hence, $E\{\sum_{i=1}^{k} Y_{(i)}^{n}\} \le (k+1)b$. But by Theorem 3.1, $V(\mathbf{Y}) \ge V(\mathbf{X}) \ge EX_{t(b)} > b$, hence, (3.6). (3.7) is obtained in an obvious way by taking limits. □

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