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CORRECTION

PROPER ACTION IN STEPS, WITH APPLICATION TO DENSITY RATIOS OF MAXIMAL INVARIANTS

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Professor Søren Tolver Jensen (University of Copenhagen) has kindly pointed out two errors in my paper. The first is my statement on page 396 that it is not known whether the notion of proper action is stronger than the Cartan property (under the assumption that the group and space are locally compact). Professor Jensen points out that Palais (1961) has an example in Section 1.1.4 (page 298) of a Cartan G-space X in which X/G is not Hausdorff, and therefore G does not act properly on X. The latter conclusion uses Palais's Theorem 1.2.9(2), together with the equivalence of the definitions "G acts properly on X" and "X is a proper G-space" as seen by Palais'ss Theorem 1.2.9(1)(3) and Proposition 7, subsection 3.4.4 of Bourbaki (1966). Thus, in general proper action is stronger than the Cartan property.

The second error occurs in my proof of proper action in the example of subsection 3.4 since it is not true in general that proper action of a group on two spaces implies proper action on their union (this is demonstrated again by Palais' example quoted above). An elegant proof of the properness of the action $X \to CX$ (X: $p \times n$ of rank p) was given by Professor Jensen by making use of Proposition 5, subsection 3.4.2, of Bourbaki (1966). In that proposition take $G = G'$, $\varphi =$ identity function, $\psi(X) = XX'$, and make use of the known fact that the action of $G = GL(p)$ on the space of $p \times p$ positive definite matrices S, given by $S \to CSC'$, $C \in G$, is proper. Then it follows from part(ii) of that proposition that the action $X \to CX$ is proper. [Parenthetically, Professor Jensen also points out that since ψ is surjective and proper, and since for $n = p$ the properness of the action $X \to CX$ is obvious, the conclusion of part (i) of Bourbaki's Proposition 5 furnishes another proof of the properness of the action $S \to CSC'$.]

A more direct proof of the properness of the action $X \to CX$ that does not use the abovementioned Proposition 5 of Bourbaki, nor the properness of $S \to CSC'$, can be given as follows. As before let $\mathscr{X}$ be the space of all $p \times n$ matrices of rank p and $G = GL(p)$ all nonsingular $p \times p$ matrices. Imbed G in $H =$ all $p \times p$ matrices so that H is identified with Euclidean p^2-space, and let $\| \ \|$ be the Euclidean norm on H. Similarly, let $\| \ \|$ be the Euclidean norm on $\mathscr{X}$ as a subspace of Euclidean pn-space. In order to show the action of G on $\mathscr{X}$ proper it suffices, by Bourbaki (1966), 3.4.5, Theorem 1(c), to show that for any compact subsets A and B of $\mathscr{X}$ the set $K = ((A, B))$ is compact,

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where $((A, B))$ denotes the set of all $g \in G$ for which there exists $x \in A$ and $y \in B$ with $gx = y$. Since K has the relative topology of the Euclidean space H, it is sufficient to show that (a) K is bounded, and (b) K is closed in H. Proof of (a): A simple computation shows $\|gx\|^2 \geq \lambda_{\min}(x)\|g\|^2$ if $g \in H$, $x \in \mathscr{X}$, where $\lambda_{\min}(x)$ is the smallest characteristic root of the positive definite matrix xx'. Let $m > 0$ be the minimum of the positive and continuous function $\lambda_{\min}(\cdot)$ on A and M the maximum of the continuous function $\|\cdot\|^2$ on B. Then $\|g\|^2 > M/m$ implies $gx \notin B$ so that $g \notin K$. Proof of (b): Let $g_n \in K$, $g_n \to g \in H$ as $n \to \infty$. By definition of K there exist sequences $x_n \in A$, $y_n \in B$ such that $g_n x_n = y_n$. By compactness of A, B we may assume $x_n \to x \in A$, $y_n \to y \in B$. By continuity of matrix multiplication, $g_n x_n \to gx$. On the other hand, $g_n x_n = y_n \to y$. By uniqueness of limits, $gx = y$, hence $g \in K$ (note that $y \in \mathscr{X}$ implies $g \in G$).

The same two corrections have to be made in Wijsman [(1990), subsection 13.4, page 209 and subsection 13.4.7, page 210].

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