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THE MAXIMUM LIKELIHOOD METHOD FOR TESTING CHANGES IN THE PARAMETERS OF NORMAL OBSERVATIONS

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We compute the asymptotic distribution of the maximum likelihood ratio test when we want to check whether the parameters of normal observations have changed at an unknown point. The proof is based on the limit distribution of the largest deviation between a d-dimensional Ornstein–Uhlenbeck process and the origin.

1. Introduction. Let $X_1, X_2, \dots, X_n$ be independent normal random variables with parameters $(\mu_1, \sigma_1^2), (\mu_2, \sigma_2^2), \dots, (\mu_n, \sigma_n^2)$, where, as usual,

$$\mu_i = EX_i \quad \text{and} \quad \sigma_i^2 = \operatorname{var} X_i, \qquad 1 \le i \le n. \tag{1.1}$$

We want to test

$$H_0\colon \mu_1 = \mu_2 = \cdots = \mu_n \quad \text{and} \quad \sigma_1 = \sigma_2 = \cdots = \sigma_n \tag{1.2}$$

against the alternative

$$\begin{aligned} H_A\colon\ & \mu_1 = \mu_2 = \cdots = \mu_{[n\tau]} \ne \mu_{[n\tau]+1} = \cdots = \mu_n \\ & \text{or} \quad \sigma_1 = \sigma_2 = \cdots = \sigma_{[n\tau]} \ne \sigma_{[n\tau]+1} = \cdots = \sigma_n \\ & \text{for some } \tau \in (0,1). \end{aligned} \tag{1.3}$$

Under the alternative hypothesis we can have change in the mean or in the variance or in both.

Testing (1.1) against the stronger alternative

$$\begin{aligned} H_A^*\colon\ & \sigma_1 = \sigma_2 = \cdots = \sigma_n \quad \text{and} \quad \mu_1 = \mu_2 = \cdots = \mu_{[n\tau]} \ne \\ & \mu_{[n\tau]+1} = \cdots = \mu_n \text{ for some } \tau \in (0,1) \end{aligned} \tag{1.4}$$

has received considerable attention in the literature. Under H_A^* we can have change only in the mean; the variances remain the same. Page (1954, 1955) considered a very simple procedure for testing H_0 against H_A^*. Sen and Srivastava (1975a, b), Srivastava and Worsley (1986), Yao and Davis (1986), James, James and Siegmund (1987) and Worsley (1986) derived tests for H_0 against H_A^* using the likelihood ratio. Yao and Davis (1986) obtained the asymptotic distribution of the likelihood ratio. For further results the asymptotic distributions of these and related statistics we refer to Csörgő and Horváth (1988) and Gombay and Horváth (1990).

The maximum likelihood method can be used when we want to test H_0 against the more general alternative H_A. It is easy to show that the likelihood

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ratio is

$$\Lambda_n = \max_{1<k<n-1} \frac{\hat{\sigma}_n^n}{\hat{\sigma}_k^k \breve{\sigma}_{n-k}^{n-k}},$$

where

$$\hat{\sigma}_k^2 = \frac{1}{k} \sum_{1\le i\le k} \left(X_i - \hat{X}_k\right)^2, \qquad \breve{\sigma}_{n-k}^2 = \frac{1}{n-k} \sum_{k<i\le n} \left(X_i - \breve{X}_{n-k}\right)^2,$$

$$\hat{X}_k = \frac{1}{k} \sum_{1\le i\le k} X_i, \qquad \breve{X}_{n-k} = \frac{1}{n-k} \sum_{k<i\le n} X_i.$$

The main aim of our paper is the computation of the asymptotic distribution of

$$(1.5) \qquad \lambda_n = \left(\max_{1<k<n-1} \left(n \log \hat{\sigma}_n^2 - k \log \hat{\sigma}_k^2 - (n-k) \log \breve{\sigma}_{n-k}^2\right)\right)^{1/2}.$$

In order to state our main result we must introduce the following functions:

$$(1.6) \qquad a(x) = (2 \log x)^{1/2},$$

$$(1.7) \qquad b(x) = 2 \log x + \log\log x.$$

Let Y be a r.v. satisfying

$$(1.8) \qquad P\{Y \le x\} = \exp(-2e^{-x}) \quad \text{for all } x.$$

THEOREM 1.1. *If H_0 holds, then as $n \to \infty$,*

$$a(\log n)\lambda_n - b(\log n) \to_{\mathscr{D}} Y.$$

The assumption that the observations are normal can be dropped. For example, the conclusion of Theorem 1.1 remains true, if in addition to (1.1) it is also assumed that $E(X_i - \mu)^3 = 0$ and $E\{(X_i - \mu)/\sigma\}^4 = 3$, where μ and σ^2 are the common mean and variances under H_0. This means that the first four moments of $(X_i - \mu)/\sigma$ must be the same as of the standard normal distribution.

The proof of Theorem 1.1 is based on a generalization of the limit theorem in Darling and Erdős (1956) for standardized random vectors. The following section contains the Darling–Erdős–type result and its proof. Both Lemmas 2.1 and 2.2 may have further applications in change-point analysis. Theorem 1.1 is proven in Section 3.

We performed Monte Carlo simulations to demonstrate that the asymptotic distribution in Theorem 1.1 can be used in case of moderate sample sizes. Let $F_n(t) = P\{a(\log n)\lambda_n - b(\log n) \le t\}$ and $F(t) = \exp(-2e^{-t})$. The simulations of $F_{20}(t)$, $F_{50}(t)$ and $F_{100}(t)$ are based on 2000 repetitions in each case, and the standard error is less than 0.001 for all t. The fit is good on the upper tail. The rate of convergence of $F_n(t)$ to $F(t)$ is very slow, but the interval on the upper tail when we have good approximation is increasing when the sample size is increasing. It turns out that the largest deviation between $F_n(t)$ and $F(t)$ occurs about the median. Let $z(\alpha, n)$ and $z(\alpha)$ denote the $100\alpha\%$ quantiles of F_n or F; that is, $F(z(\alpha, n)) = \alpha$ and $F(z(\alpha)) = \alpha$. The Monte Carlo simulation gave the following numerical values: $z(0.9, 20) = 3.12$, $z(0.95, 20) = 3.59$, $z(0.99, 20) = 4.57$; $z(0.9, 50) = 3.14$, $z(0.95, 50) = 3.60$, $z(0.99, 50) = 4.54$; and $z(0.9, 100) = 3.05$, $z(0.95, 100) = 3.60$, $z(0.99, 100) = 4.61$. Using the asymptotic distribution we get that $z(0.9) = 2.94$, $z(0.95) = 3.66$ and $z(0.99) = 5.29$.

2. Darling–Erdős–type results for vectors. Let

$$\{V_1(t), 0 \le t < \infty\}, \dots, \{V_d(t), 0 \le t < \infty\}$$

be independent identically distributed Ornstein–Uhlenbeck processes with $EV_i(t) = 0$ and $EV_i(t)V_i(s) = \exp(-|t-s|/2)$, $1 \le i \le d$. Next we define

$$N(t) = \Big(\sum_{1 \le i \le d} V_i^2(t)\Big)^{1/2}, \tag{2.1}$$

which is just the distance between $(V_1(t), \dots, V_d(t))$ and $\mathbf{0}$. Let Y^* be a random variable with distribution function $\exp(-e^{-x})$ and define

$$b_d(t) = 2\log t + \frac{d}{2}\log\log t - \log\Gamma\Big(\frac{d}{2}\Big), \tag{2.2}$$

where $\Gamma(t)$ is the Gamma function. Since $\Gamma(1/2) = \pi^{1/2}$, the following result reduces to Theorem 2 of Darling and Erdős (1956) when $d = 1$.

LEMMA 2.1. *As* $T \to \infty$,

$$a(T) \sup_{0 \le t \le T} N(t) - b_d(T) \to_{\mathscr{D}} Y^*. \tag{2.3}$$

PROOF. The process $\{N(t), 0 \le t < \infty\}$ is a diffusion process and its initial distribution is the square root of the χ^2-distribution with d degrees of freedom. The backward equation associated with $N(t)$ is

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\Big(\frac{d-1}{x} - x\Big)\frac{\partial u}{\partial x}$$

with boundary condition

$$\lim_{x \downarrow 0} x^{d-1}\frac{\partial u}{\partial x} = 0$$

[cf. Itô and McKean (1965), pages 162–163]. Using Theorem 1 of Mandl [(1968), page 102] we get [cf. also Ronzhin (1985)].

$$\lim_{M \to \infty} P\Big\{\sup_{0 \le t \le u\gamma(M)} N(t) < M\Big\} = e^{-u} \tag{2.4}$$

for all $u \geq 0$, where

$$\gamma(M) = 2^{d/2}\Gamma(d/2)M^{-d}e^{M^2/2}. \tag{2.5}$$

For each $x \in (-\infty, \infty)$ and $T > 0$ there is a unique M such that $\gamma(M) = Te^x$, and therefore by (2.4) we have

$$\lim_{T\to\infty} P\Big\{\sup_{0\le t\le T} N(t) \le \gamma^{-1}(Te^x)\Big\} = \exp(-e^{-x}). \tag{2.6}$$

It is easy to verify that

$$\begin{aligned}\gamma\bigg(\frac{x}{(2\log T)^{1/2}} + (2\log T)^{1/2} + \frac{(d/2)\log\log T}{(2\log T)^{1/2}} - \frac{\log\Gamma(d/2)}{(2\log T)^{1/2}}\bigg)\\ = Te^x\bigg\{1 + O\bigg(\frac{(\log\log T)^2}{\log T}\bigg)\bigg\}\end{aligned} \tag{2.7}$$

as $T \to \infty$, and thus (2.6) yields Lemma 2.1. □

Let $\{(Z_i^{(1)}, \dots, Z_i^{(d)}),\ 1 \le i < \infty\}$ be independent identically distributed random vectors, and define

$$S^{(i)}(k) = \sum_{1\le j\le k} Z_j^{(i)}, \qquad 1 \le i \le d.$$

Lemma 2.2. *Assume that* $EZ_1^{(1)} = EZ_1^{(2)} = \cdots = EZ_1^{(d)} = 0$, *the covariance matrix of* $(Z_1^{(1)}, \dots, Z_1^{(d)})$ *is the identity matrix and*

$$\max_{1\le i\le d} E\big|Z_1^{(i)}\big|^r < \infty \quad \textit{for some } r > 2. \tag{2.8}$$

Then, as $n \to \infty$,

$$a(\log n)\max_{1\le k\le n}\Big(\sum_{1\le i\le d}\big(k^{-1/2}S^{(i)}(k)\big)^2\Big)^{1/2} - b_d(\log n) \to_{\mathscr{D}} Y^*. \tag{2.9}$$

Proof. By Einmahl (1989) we can define d independent Wiener processes $W_1(t), \dots, W_d(t)$ such that

$$\max_{1\le i\le d}\sup_{1\le t\le T}\big|S^{(i)}(t) - W_i(t)\big| =_{\text{a.s.}} o(T^{1/r}). \tag{2.10}$$

Next we observe that

$$\Big\{\Big(\sum_{1\le i\le d}\big(t^{-1/2}W_i(t)\big)^2\Big)^{1/2},\ 1 \le t < \infty\Big\} =_{\mathscr{D}} \{N(\log t),\ 1 \le t < \infty\}, \tag{2.11}$$

where $N(t)$ is defined by (2.1). Let $\nu = 2r/(r-2)$. Now (2.10) and the law of iterated logarithm imply

$$a(\log n) \sup_{(\log n)^\nu \le t \le n} \left| \left(\sum_{1 \le i \le d} \left(t^{-1/2} S^{(i)}(t)\right)^2 \right)^{1/2} - \left(\sum_{1 \le i \le d} \left(t^{-1/2} W_i(t)\right)^2 \right)^{1/2} \right| =_{\text{a.s.}} o(1). \tag{2.12}$$

Using again the law of iterated logarithm it can be established that

$$a(\log n) \sup_{1 \le t \le (\log n)^\nu} \left(\sum_{1 \le i \le d} \left(t^{-1/2} S^{(i)}(t)\right)^2 \right)^{1/2} - \left(b_d(\log n) + x\right) \to_P -\infty \tag{2.13}$$

and

$$a(\log n) \sup_{1 \le t \le (\log n)^\nu} \left(\sum_{1 \le i \le d} \left(t^{-1/2} W_i(t)\right)^2 \right)^{1/2} - \left(b_d(\log n) + x\right) \to_P -\infty \tag{2.14}$$

for all x. Hence Lemma 2.2 follows from Lemma 2.1 and (2.11)–(2.14). □

3. Proof of Theorem 1.1. It is easy to see that linear transformation of the data does not change the values of λ_n and therefore it can be assumed that $\mu = 0$ and $\sigma^2 = 1$. The proof is based on the observation that the asymptotic distribution of λ_n is determined by

$$\max\left\{ \max_{\log n \le k \le n/\log n} \left(n \log \hat{\sigma}_n^2 - k \log \hat{\sigma}_k^2 - (n-k) \log \breve{\sigma}_{n-k}^2\right)^{1/2}, \max_{n - n/\log n \le k \le n - \log n} \left(n \log \hat{\sigma}_n^2 - k \log \hat{\sigma}_k^2 - (n-k) \log \breve{\sigma}_{n-k}^2\right)^{1/2} \right\}.$$

A three-term Taylor expansion gives

$$\begin{aligned} \eta_k &= n \log \hat{\sigma}_n^2 - k \log \hat{\sigma}_k^2 - (n-k) \log \breve{\sigma}_{n-k}^2 \\ &= n\left(\hat{\sigma}_n^2 - 1\right) - k\left(\hat{\sigma}_k^2 - 1\right) - (n-k)\left(\breve{\sigma}_{n-k}^2 - 1\right) \\ &\quad - \tfrac{1}{2} n\left(\hat{\sigma}_n^2 - 1\right)^2 + \tfrac{1}{2} k\left(\hat{\sigma}_k^2 - 1\right)^2 + \tfrac{1}{2}(n-k)\left(\breve{\sigma}_{n-k}^2 - 1\right)^2 \\ &\quad + \tfrac{1}{3}\left(\xi_n^{(1)}\right)^{-3} n\left(\hat{\sigma}_n^2 - 1\right)^3 \\ &\quad - \tfrac{1}{3}\left(\xi_k^{(2)}\right)^{-3} k\left(\hat{\sigma}_k^2 - 1\right)^3 - \tfrac{1}{3}\left(\xi_{n-k}^{(3)}\right)^{-3} (n-k)\left(\breve{\sigma}_{n-k}^2 - 1\right)^3, \end{aligned} \tag{3.1}$$

where $|\xi_n^{(1)} - 1| \le |\hat{\sigma}_n^2 - 1|$, $|\xi_k^{(2)} - 1| \le |\hat{\sigma}_k^2 - 1|$ and $|\xi_{n-k}^{(3)} - 1| \le |\breve{\sigma}_{n-k}^2 - 1|$. Simple algebra shows that

$$\begin{aligned} &n\left(\hat{\sigma}_n^2 - 1\right) - k\left(\hat{\sigma}_k^2 - 1\right) - (n-k)\left(\breve{\sigma}_{n-k}^2 - 1\right) \\ &\quad = k\hat{X}_k^2 + (n-k)\breve{X}_{n-k}^2 - n\hat{X}_n^2 \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&n(\hat{\sigma}_n^2 - 1)^2 - k(\hat{\sigma}_k^2 - 1)^2 - (n-k)(\check{\sigma}_{n-k}^2 - 1)^2 \\
&\quad = \frac{1}{n}\Big(\sum_{1\le i\le n}(X_i^2 - 1)\Big)^2 - \frac{1}{k}\Big(\sum_{1\le i\le k}(X_i^2 - 1)\Big)^2 \\
&\qquad - \frac{1}{n-k}\Big(\sum_{k<i\le n}(X_i^2 - 1)\Big)^2 \qquad (3.3)\\
&\qquad - 2\hat{X}_n^2\sum_{1\le i\le n}(X_i^2 - 1) + n\hat{X}_n^4 + 2\hat{X}_k^2\sum_{1\le i\le k}(X_i^2 - 1) - k\hat{X}_k^4 \\
&\qquad + 2\check{X}_{n-k}^2\sum_{k<i\le n}(X_i^2 - 1) - (n-k)\check{X}_{n-k}^4.
\end{aligned}$$

The law of iterated logarithm yields

$$\max_{1\le k\le n} k^{1/2}(\log\log k)^{-3/2}\left|\xi_k^{(2)}k(\hat{\sigma}_k^2 - 1)^3\right| = O_P(1), \tag{3.4}$$

$$\max_{1\le k<n}(n-k)^{1/2}(\log\log(n-k))^{-3/2}\left|\xi_{n-k}^{(3)}(n-k)(\check{\sigma}_{n-k}^2 - 1)^3\right| = O_P(1), \tag{3.5}$$

$$\max_{1\le k\le n} k^{1/2}(\log\log k)^{-3/2}\hat{X}_k^2\left|\sum_{1\le i\le k}(X_i^2 - 1)\right| = O_P(1), \tag{3.6}$$

$$\max_{1\le k\le n}\left(\frac{k}{\log\log k}\right)^2\hat{X}_k^4 = O_P(1), \tag{3.7}$$

$$\max_{1\le k<n}(n-k)^{1/2}(\log\log(n-k))^{-3/2}\check{X}_{n-k}^2\left|\sum_{k<i\le n}(X_i^2 - 1)\right| = O_P(1) \tag{3.8}$$

and

$$\max_{1\le k<n}\left(\frac{n-k}{\log\log(n-k)}\right)^2\check{X}_{n-k}^4 = O_P(1). \tag{3.9}$$

Thus we can write

$$\eta_k = A_k^{(1)} + A_k^{(2)} + R_k^{(1)} + R_k^{(2)}, \tag{3.10}$$

where

$$\begin{aligned}
A_k^{(1)} &= k\hat{X}_k^2 + (n-k)\check{X}_{n-k}^2 - n\hat{X}_n^2, \\
A_k^{(2)} &= \frac{1}{2k}\Big(\sum_{1\le i\le k}(X_i^2 - 1)\Big)^2 + \frac{1}{2(n-k)}\Big(\sum_{k<i\le n}(X_i^2 - 1)\Big)^2 \\
&\quad - \frac{1}{2n}\Big(\sum_{1\le i\le n}(X_i^2 - 1)\Big)^2, \\
R_k^{(1)} &= \frac{1}{3}\big(\xi_n^{(1)}\big)^{-3} n(\hat{\sigma}_n^2 - 1)^3 - \frac{1}{3}\big(\xi_k^{(2)}\big)^{-3} k(\hat{\sigma}_k^2 - 1)^3 \\
&\quad + \hat{X}_n^2\sum_{1\le i\le n}(X_i^2 - 1) - \frac{n}{2}\hat{X}_n^4 - \hat{X}_k^2\sum_{1\le i\le k}(X_i^2 - 1) + \frac{k}{2}\hat{X}_k^4
\end{aligned}$$

and

$$R_k^{(2)} = \frac{1}{3}\left(\xi_{n-k}^{(3)}\right)^{-3}(n-k)\left(\check{\sigma}_{n-k}^2 - 1\right)^3 + \frac{n-k}{2}\check{X}_{n-k}^4 - \check{X}_{n-k}^2 \sum_{k<i\le n}\left(X_i^2 - 1\right).$$

By (3.4)–(3.9), $R_k^{(1)}$ and $R_k^{(2)}$ satisfy

$$\max_{1\le k<n} k^{1/2}(\log\log k)^{-3/2}\left|R_k^{(1)}\right| = O_P(1), \tag{3.11}$$

$$\max_{1\le k<n} (n-k)^{1/2}(\log\log(n-k))^{-3/2}\left|R_k^{(2)}\right| = O_P(1). \tag{3.12}$$

Combining the law of iterated logarithm with (3.10)–(3.12) we get

$$a^2(\log n)\max_{1\le k\le \log n}\eta_k - (x+b(\log n))^2 \to_P -\infty, \tag{3.13}$$

$$a^2(\log n)\max_{1\le k\le \log n}\left(A_k^{(1)} + A_k^{(2)}\right) - (x+b(\log n))^2 \to_P -\infty, \tag{3.14}$$

$$a^2(\log n)\max_{n-\log n\le k<n}\eta_k - (x+b(\log n))^2 \to_P -\infty, \tag{3.15}$$

$$a^2(\log n)\max_{n-\log n\le k<n}\left(A_k^{(1)} + A_k^{(2)}\right) - (x+b(\log n))^2 \to_P -\infty \tag{3.16}$$

for all x and

$$a^2(\log n)\max_{\log n\le k\le n-\log n}\left|\eta_k - \left(A_k^{(1)} + A_k^{(2)}\right)\right| = o_P(1). \tag{3.17}$$

Thus Theorem 1.1 is proven if

$$\lim_{n\to\infty} P\left\{a^2(\log n)\max_{1\le k<n}\left(A_k^{(1)} + A_k^{(2)}\right) \le (x+b(\log n))^2\right\} = \exp(-2e^{-x}) \tag{3.18}$$

is established.

Observing that

$$(n-k)\check{X}_{n-k}^2 - n\hat{X}_n^2 = \frac{k}{n(n-k)}\left(\sum_{1\le i\le n} X_i\right)^2 - \frac{2}{n-k}\sum_{1\le i\le n} X_i \sum_{1\le j\le k} X_j + \frac{1}{n-k}\left(\sum_{1\le i\le k} X_i\right)^2, \tag{3.19}$$

we obtain

$$a^2(\log n)\max_{1\le k\le n/\log n}\left|(n-k)\check{X}_{n-k}^2 - n\hat{X}_n^2\right| = o_P(1). \tag{3.20}$$

Similar arguments give

$$a^2(\log n) \max_{1\le k\le n/\log n}\left|\frac{1}{n-k}\left(\sum_{k<i\le n}(X_i^2-1)\right)^2 - \frac{1}{n}\left(\sum_{1\le i\le n}(X_i^2-1)\right)^2\right| = o_P(1). \tag{3.21}$$

Using Theorem 2 of Darling and Erdős (1956) we get

$$\max_{n/\log n\le k\le n-n/\log n}\left|A_k^{(1)}+A_k^{(2)}\right| = O_P(\log\log\log n), \tag{3.22}$$

and therefore for all x,

$$a^2(\log n)\max_{n/\log n\le k\le n-n/\log n}\left(A_k^{(1)}+A_k^{(2)}\right) - (x+b(\log n))^2 \to_P -\infty. \tag{3.23}$$

Similarly to (3.20) and (3.21) we obtain

$$a^2(\log n)\max_{n-n/\log n\le k<n}\left|k\hat{X}_k^2 - n\hat{X}_n^2\right| = o_P(1) \tag{3.24}$$

and

$$a^2(\log n)\max_{n-n/\log n\le k<n}\left|\frac{1}{k}\left(\sum_{1\le i\le k}(X_i^2-1)\right)^2 - \frac{1}{n}\left(\sum_{1\le i\le n}(X_i^2-1)\right)^2\right| = o_P(1). \tag{3.25}$$

Now (3.20)–(3.25) yield

$$\begin{aligned}
&\lim_{n\to\infty} P\left\{a^2(\log n)\max_{1\le k<n}\left(A_k^{(1)}+A_k^{(2)}\right)\le (x+b(\log n))^2\right\}\\
&= \lim_{n\to\infty} P\Bigg\{a^2(\log n)\max\Bigg(\max_{1\le k\le n/\log n}\Bigg(\Bigg(k^{-1/2}\sum_{1\le i\le k}X_i\Bigg)^2 + \Bigg(k^{-1/2}\sum_{1\le i\le k}\frac{X_i^2-1}{2^{1/2}}\Bigg)\Bigg)^2,\\
&\qquad \max_{n-n/\log n\le k<n}\Bigg(\Bigg((n-k)^{-1/2}\sum_{k<i\le n}X_i\Bigg)^2 + \Bigg((n-k)^{-1/2}\sum_{k<i\le n}\frac{X_i^2-1}{2^{1/2}}\Bigg)^2\Bigg)\Bigg)\le (x+b(\log n))^2\Bigg\}.
\end{aligned} \tag{3.26}$$

Since $\{X_i,\ 1\le i\le n/\log n\}$ and $\{X_i,\ n-n/\log n\le i\le n\}$ are independent and $\operatorname{var} X_i = \operatorname{var}((X_i^2-1)/2^{1/2}) = 1$, $EX_i(X_i^2-1)=0$, Lemma 2.2 implies that

$$P\left\{a(\log n)\max_{1\le k\le n/\log n}\left(\left(k^{-1/2}\sum_{1\le i\le k}X_i\right)^2+\left(k^{-1/2}\sum_{1\le i\le k}\frac{X_i^2-1}{2^{1/2}}\right)^2\right)^{1/2}-b(\log n),\right.$$

$$(3.27)\quad \left. a(\log n)\max_{n-n/\log n\le k<n}\left(\left((n-k)^{-1/2}\sum_{k<i\le n}X_i\right)^2+\left((n-k)^{-1/2}\sum_{k<i\le n}\frac{X_i^2-1}{2^{1/2}}\right)^2\right)^{1/2}-b(\log n)\right\}\to_{\mathscr{D}}\{Y_1^*,Y_2^*\},$$

where Y_1^* and Y_2^* are independent, identically distributed random variables with distribution function $\exp(-e^{-x})$. This also completes the proof of Theorem 1.1. □

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