The Annals of Statistics
1993, Vol. 21, No. 1, 146–156

ALMOST SURE REPRESENTATIONS OF THE PRODUCT-LIMIT ESTIMATOR FOR TRUNCATED DATA

BY WINFRIED STUTE

Universität Giessen

In the left-truncation model, one observes data (X_i, Y_i) only when $Y_i \le X_i$. Let F denote the marginal d.f. of X_i, the variable of interest. The nonparametric MLE $\hat{F}_n$ of F aims at reconstructing F from truncated data. In this paper an almost sure representation of $\hat{F}_n$ is derived with improved error bounds on the one hand and under weaker distributional assumptions on the other hand.

0. Introduction and main results. Let (X_i, Y_i), $1 \le i \le N$, be a sequence of independent identically distributed random vectors in the plane such that X_i is independent of Y_i. In the left-truncation model, (X_i, Y_i) is observed only when $Y_i \le X_i$. Woodroofe (1985) reviews examples from astronomy and economy where such data may occur. As a consequence of truncation, n, the size of the actually observed sample, is random, with $n \le N$ and N unknown. From the SLLN, as $N \to \infty$,

$$\frac{n}{N} \to \alpha := \mathbb{P}(Y \le X) \qquad \mathbb{P}\text{-a.s.}$$

Now, conditionally on the value of n, the observed data (X_i, Y_i) are still i.i.d., but their joint distribution has changed to become

$$H^*(x, y) = \mathbb{P}(X \le x, Y \le y | Y \le X) = \alpha^{-1} \int_{-\infty}^{x} G(y \wedge z) F(dz),$$

with F and G denoting the d.f. of X and Y, respectively. Write

$$F^*(x) = H^*(x, \infty) = \alpha^{-1} \int_{-\infty}^{x} G(z) F(dz)$$

and

$$G^*(y) = H^*(\infty, y) = \alpha^{-1} \int_{-\infty}^{\infty} G(y \wedge z) F(dz)$$

for the marginals of H^*. The problem now is one of reconstructing F and G from a data set (X_i, Y_i), $1 \le i \le n$, with d.f. H^*. By convention, X is the variable of interest. The nonparametric MLE of F has been derived by Lynden-Bell (1971). For ease of presentation we shall assume throughout that F and G are continuous. Also, we shall assume that both X and Y are nonnegative, though this in no way limits the method. This is only because

Received July 1991; revised April 1992.
AMS 1991 *subject classifications*. Primary 62G05; secondary 62E20, 62G30
Key words and phrases. Product-limit estimator, truncated data, almost sure representation.

typically X and Y are measurements of failure time, length and so on. As in the random censorship model the cumulative hazard function

$$\Lambda(x) = \int_0^x \frac{F(dz)}{1 - F(z)}, \qquad 0 \le x < \infty,$$

serves as a fundamental tool for reconstructing F. Following Woodroofe (1985), put

$$a_F = \inf\{x: F(x) > 0\}, \qquad b_F = \sup\{x: F(x) < 1\}.$$

Similarly, this is true for a_G and b_G. Woodroofe (1985) showed that F can be reconstructed only when $a_G \le a_F$. In this case,

$$\Lambda(x) = \int_{a_F}^x \frac{F^*(dz)}{C(z)}, \qquad 0 \le x < \infty,$$

with

$$\begin{aligned} C(z) &= G^*(z) - F^*(z) \\ &= \alpha^{-1} G(z)[1 - F(z)], \qquad a_F \le z < \infty. \end{aligned}$$

C is consistently estimated by

$$C_n(z) = n^{-1} \sum_{i=1}^n 1_{\{Y_i \le z \le X_i\}},$$

while the above representation of Λ in terms of F^* and C suggests estimation of Λ by

$$\Lambda_n(x) := \int_{a_F}^x \frac{F_n^*(dz)}{C_n(z)} = n^{-1} \sum_{i:\, X_i \le x} C_n^{-1}(X_i).$$

Here and in the following F_n^* and G_n^* will denote the empiricals of $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$, respectively. The MLE of F equals

$$1 - \hat{F}_n(x) = \prod_{i:\, X_i \le x} \left[\frac{nC_n(X_i) - 1}{nC_n(X_i)} \right].$$

REMARK. The following facts about C_n will be used without further mentioning:

(i) $C_n(X_i) \ge 1/n$ for $1 \le i \le n$.
(ii) $C_n(z) = G_n^*(z) - F_n^*(z-)$.

By (ii), the LIL for empirical d.f.'s provides the familiar $(\ln \ln n/n)^{1/2}$ bound for $\|C_n - C\|_\infty$. Note, on the other hand, that C_n is not monotone. Though C is strictly positive on $a_G < z < b_F$, C_n may vanish for some $z \in (a_G, b_F)$. In particular, C_n vanishes for all z less than the smallest Y order statistic. Since

F^* may have positive mass there, we typically get, for example,

$$\int_{a_F}^{b} C_n^{-1}(z) F^*(dz) = \infty, \qquad b > a_F.$$

Similarly, this is true if the lower bound a_F is replaced by some sequence $(a_n)_n$ tending to a_F too fast. We shall come back to that point later.

Now, Woodroofe (1985) in his Theorem 5, proved the weak convergence of

$$W_n(x) = n^{1/2}[\Lambda_n(x) - \Lambda(x)]$$

and

$$Z_n(x) = n^{1/2}\left[\hat{F}_n(x) - F(x)\right]$$

to certain Gaussian processes W and Z (in the space $D[a_F, b]$), provided that $a_G \le a_F \le b < b_F$ and

$$\int_{a_F}^{\infty} 1/G\, dF < \infty. \tag{0.1}$$

Trivially, the last condition is satisfied if $a_G < a_F$. Chao and Lo (1988) obtained an almost sure representation of Λ_n and $\hat{F}_n$ in terms of a sum of i.i.d. processes with remainder $o(n^{-1/2})$. Such results are useful, for example, for a representation of the quantile function of $\hat{F}_n$ or for estimating the density, respectively, hazard function of F. It turns out, however, that bounding the error term by $o(n^{-1/2})$ is insufficient for that purpose. It is the goal of this paper to derive an i.i.d. representation of Λ_n and $\hat{F}_n$ with error term $O(n^{-1})$ (up to a logarithmic factor). Instead of (0.1), we require a little bit more than (0.1), namely (apart from $a_G \le a_F$),

$$\int_{a_F}^{\infty} \frac{F(dz)}{G^2(z)} < \infty. \tag{0.2}$$

Of course, (0.2) is satisfied when $a_G < a_F$. But proofs simplify a lot in this case. The interesting situation is $a_F = a_G \equiv a$, that is, $F(a) = G(a) = 0$ but $F, G > 0$ on (a, ∞). b will always denote a constant less than b_F. Theorems 1 and 2 yield the i.i.d. representations of Λ_n and $\hat{F}_n$. Needless to say, from these functional CLT's and LIL's are easily available.

THEOREM 1. *Assume $a_G \le a_F$ and (0.2). Then uniformly in $a_F \le x \le b < b_F$,*

$$\Lambda_n(x) - \Lambda(x) = \int_{a_F}^{x} C^{-1}(z)[F_n^*(dz) - F^*(dz)] - \int_{a_F}^{x} \frac{C_n(z) - C(z)}{C^2(z)} F^*(dz)$$
$$+ R_n(x) = L_n(x) + R_n(x),$$

where for each $\delta > 3/2$

$$\sup_{a_F \le x \le b} |R_n(x)| = o\left(n^{-1}(\ln n)^{\delta}\right) \quad \textit{with probability } 1.$$

It is easy to check that

$$R_n(x) = \int_{a_F}^{x} \frac{C - C_n}{C^2}(z)(F_n^* - F^*)(dz) + \int_{a_F}^{x} \frac{[C - C_n]^2}{C_n C^2}(z) F_n^*(dz)$$

$$\equiv R_{n1}(x) + R_{n2}(x).$$

THEOREM 2. *Under the assumptions of Theorem* 1, *uniformly in* $a_F \le x \le b < b_F$,

$$\hat{F}_n(x) - F(x) = (1 - F(x)) L_n(x) + R_n^0(x),$$

with

$$\sup_{a_F \le x \le b} |R_n^0(x)| = O\left(\frac{\ln^3 n}{n}\right) \qquad \mathbb{P}\text{-}a.s.$$

Needless to say, L_n *is a sum of i.i.d. (centered) processes in the Skorokhod space* $D[a_F, b]$.

Compared with Chao and Lo (1988) our approach utilizes three new technical tools. First, R_{n1} is dealt with by employing some new results on U-statistic processes as obtained by Stute (1993). Second, as for R_{n2}, C/C_n is bounded on $\{X_i: X_i \le b\}$ by relating it to the sup of a properly weighted bivariate empirical d.f. on the unit square (Lemmas 1.1 and 1.2). Finally, a result of Csáki (1975) on the asymptotic behavior of a properly weighted univariate empirical process is used.

1. Proofs. Our first goal will be to bound the process $(C/C_n)(z)$ for $z = X_i$ and $X_i \le b$. As mentioned in the remark, $C(z) > 0$ on $(a_F, b]$, but $C_n(z)$ may be zero outside the X's. For example, it will not be allowed to consider integrals of the form

$$\int_{a_n}^{x} C_n^{-1}(z) F^*(dz),$$

when a_n is less than the smallest Y order statistic. From the asymptotic theory of order statistics it is therefore possible to create situations for which the last integral on page 666 of Chao and Lo (1988) is infinite with probability 1, as $n \to \infty$. In other words, I have some doubts whether the arguments for Lemmas A.3 and A.4 there are readily adaptable from the censorship model.

Now, in order to incorporate the independence of X_i and Y_i, it is first necessary to consider the (possibly unobservable) full sample (X_i, Y_i), $1 \le i \le N$. It will be seen, that C/C_n on $\{X_i: X_i \le b\}$ is strongly related to the properly weighted bivariate empirical d.f. H_N of the full sample. In such a situation, the independence of X_i and Y_i (in the full sample) is then useful for applying some maximal bounds for two-parameter (strong) martingales, since it guarantees the crucial "conditional independence" property (F4) of, for example, Cairoli and Walsh (1975).

The following lemma presents the two-dimensional analogue of inequality (5) of Shorack and Wellner [(1986), page 415]. For this, let (U_i, V_i), $1 \le i \le N$, be an i.i.d. sample from the uniform distribution on the unit square. Denote with

$$\overline{H}_N(s,t) = \frac{1}{N}\sum_{i=1}^{N} 1_{\{U_i \le s,\, V_i \le t\}}, \qquad 0 \le s,t \le 1,$$

their empirical d.f.

LEMMA 1.1. *For any* $0 < a, b \le 1$ *and* $\lambda \ge 1$

$$\mathbb{P}\Bigl(\sup_{\substack{a \le s \le 1\\ b \le t \le 1}} \overline{H}_N(s,t)/st \ge \lambda\Bigr) \le \exp[-Nab\,h(\lambda) + 1],$$

where

$$h(\lambda) = \lambda(\ln\lambda - 1) + 1 \quad \textit{for } \lambda > 0.$$

PROOF. Set, for $0 \le t_0 \le 1$,

$$\mathscr{F}_{t_0} = \sigma\bigl(1_{\{V_i \le t\}}, 1 \le i \le N, t_0 \le t \le 1, U_1, \dots, U_N\bigr).$$

Then

$$\bigl(\overline{H}_N(s,t)/st, \mathscr{F}_t\bigr)_{b \le t \le 1}$$

is a reverse martingale for each $a \le s \le 1$. Hence for each $r > 0$

$$\sup_{a \le s \le 1} \exp\bigl[r\overline{H}_N(s,t)/st\bigr]$$

is a reverse submartingale. From Doob's maximal inequality

$$\mathbb{P}\Bigl(\sup_{\substack{a \le s \le 1\\ b \le t \le 1}} \overline{H}_N(s,t)/st \ge \lambda\Bigr) \le \exp(-r\lambda)\mathbb{E}\Bigl[\sup_{a \le s \le 1} \exp\bigl[r\overline{H}_N(s,b)/sb\bigr]\Bigr].$$

Another application of Doob's maximal inequality for pth moments yields, since

$$\exp\bigl[r\overline{H}_N(s,b)/sb\bigr], \qquad a \le s \le 1,$$

is a reverse submartingale w.r.t.,

$$\mathscr{G}_{s_0} = \sigma\bigl(1_{\{U_i \le s\}}, 1 \le i \le N, s_0 \le s \le 1, V_1, \dots, V_N\bigr),$$

$$\mathbb{E}\Biggl[\sup_{a \le s \le 1} \exp\Biggl[\frac{r\overline{H}_N(s,b)}{sb}\Biggr]\Biggr] \le \Bigl(\frac{p}{p-1}\Bigr)^p \mathbb{E}\Biggl[\exp\Biggl[\frac{r\overline{H}_N(a,b)}{ab}\Biggr]\Biggr].$$

Letting p tend to infinity yields

$$\mathbb{P}\left(\sup_{\substack{a\le s\le 1\\ b\le t\le 1}} \overline{H}_N(s,t)/st \ge \lambda\right) \le \exp(-r\lambda + 1)\mathbb{E}\left[\exp\left[r\overline{H}_N(a,b)/ab\right]\right].$$

The last expectation equals

$$(1 - ab + ab\exp[r/Nab])^n \le \exp[-Nab(1 - \exp(r/Nab))].$$

Take $r = Nab \ln \lambda$ to get the result. □

We are now in the position to study the ratio between C and C_n.

In the following we shall assume w.l.o.g. that each Y_i has the quantile representation

$$Y_i = G^{-1}(V_i), \qquad 1 \le i \le N,$$

where $V_1, \ldots, V_N$ are i.i.d. from a uniform distribution on the unit interval. As to the X's, we have

$$\{t \le X_i\} = \{-X_i \le -t\}.$$

$-X_i$ has d.f.

$$\mathbb{P}(-X_i \le x) = 1 - F(-x - 0) \equiv L(x).$$

In terms of L^{-1}, we may write

$$-X_i = L^{-1}(U_i), \qquad 1 \le i \le N,$$

the U's again being uniformly distributed and independent of the V's. Hence

$$\{t \le X_i\} = \{L^{-1}(U_i) \le -t\} = \{U_i \le L(-t)\} = \{U_i \le 1 - F(t - 0)\}$$

and therefore

$$\{Y_i \le t \le X_i\} = \{U_i \le 1 - F(t-0), V_i \le G(t)\}.$$

As a consequence,

$$C_n(t) = N\overline{H}_N(1 - F(t-0), G(t))/n. \tag{1.1}$$

Note also that $C_n(X_i) \ge 1/n > 0$. We shall now be concerned with a bound for

$$\mathbb{P}\left(\sup_{i:\, X_i \le b} C(X_i)/C_n(X_i) \ge \lambda\right).$$

First, $C(X_i)/C_n(X_i) \ge \lambda$ implies $C(X_i) \ge \lambda/n$. So

$$\left\{\sup_{i:\, X_i\le b} C(X_i)/C_n(X_i) \ge \lambda\right\} \subset \left\{\sup_{\substack{t:\, C(t)\ge\lambda/n\\ t\le b}} - C_n(t)/C(t) \ge -1/\lambda\right\}.$$

But $C(t) \ge \lambda/n$ implies $G(t) \ge \alpha\lambda/n$, so that in view of (1.1) and $n \le N$ the last event is contained in

$$\left\{ \sup_{\substack{1-F(b)\le s\le 1\\ \alpha\lambda/N\le t\le 1}} -\overline{H}_N(s,t)/st \ge -1/\alpha\lambda \right\}.$$

The same reasoning as in the last proof, but now applied to the reverse martingale $-\overline{H}_N$ rather than $\overline{H}_N$, yields the following lemma.

LEMMA 1.2. *For $b < b_F$ and $\alpha\lambda \ge 1$ one has*

$$\mathbb{P}\Big(\sup_{i:\, X_i \le b} C(X_i)/C_n(X_i) \ge \lambda \Big) \le \exp[-(1-F(b))\alpha\lambda h(1/\alpha\lambda) + 1]. \tag{1.2}$$

Note that

$$\alpha\lambda h(1/\alpha\lambda) = \alpha\lambda - \ln \alpha\lambda - 1,$$

so that the right-hand side of (1.2) is less than or equal to

$$\lambda e^2 \exp[-(1-F(b))\alpha\lambda].$$

From Borel–Cantelli we therefore get the following corollary.

COROLLARY 1.3. *For $b < b_F$, as $N \to \infty$, with probability* 1,

$$\sup_{i:\, X_i \le b} C(X_i)/C_n(X_i) = O(\ln N). \tag{1.3}$$

NOTE. As mentioned before,

$$\frac{n}{N} \to \alpha \quad \text{with probability } 1.$$

Since the theorems have been formulated in terms of n, the size of the actually observed data set, we prefer to reformulate (1.3) so as to become

$$\sup_{i:\, X_i \le b} C(X_i)/C_n(X_i) = O(\ln n) \qquad \mathbb{P}\text{-}a.s. \tag{1.4}$$

Now we come to the representation of Λ_n. Assume $a_F = 0$ w.l.o.g. throughout. Recall that conditionally on the value of n, the observed data form an i.i.d. sequence from H^*. We first deal with the remainder term R_{n1}. Note that

$$\int_0^x \frac{C_n(z)}{C^2(z)} F_n^*(dz) = n^{-2} \sum_{1\le i,\, j\le n} 1_{\{X_i \le x\}} 1_{\{Y_j \le X_i \le X_j\}} C^{-2}(X_i).$$

Split the last sum into its diagonal and off-diagonal part. Write, for $i \ne j$,

$$\begin{aligned} &1_{\{X_i \le x\}} 1_{\{Y_j \le X_i \le X_j\}} C^{-2}(X_i) \\ &\qquad = 1_{\{X_i \le x\}} 1_{\{X_i \le X_j\}} C^{-2}(X_i) - 1_{\{X_i \le x\}} 1_{\{X_i < Y_j\}} C^{-2}(X_i). \end{aligned}$$

Each of these summands contributes to a U-statistic process as studied in

Stute (1993). Theorem 5 there yields a representation in terms of the pertaining Hájek projection and a remainder. In particular, this approach leads to

$$
\begin{aligned}
\int_0^x \frac{C_n(z)}{C^2(z)} F_n^*(dz) &= n^{-2} \sum_{i=1}^{n} 1_{\{X_i \le x\}} C^{-2}(X_i) + \frac{n-1}{n} \int_0^x C^{-1}(z) F_n^*(dz) \\
&\quad + \frac{n-1}{n} \int_0^x \frac{C_n(z)}{C^2(z)} F^*(dz) \qquad (1.5) \\
&\quad - \frac{n-1}{n} \int_0^x C^{-1}(z) F^*(dz) + R_n^0(x),
\end{aligned}
$$

where for each $\delta > 3/2$

$$\sup_{0 \le x \le b} |R_n^0(x)| = o\big(n^{-1}(\ln n)^{\delta}\big) \quad \text{with probability } 1,$$

provided the variable

$$1_{\{Y_2 \le X_1 \le X_2\}} C^{-2}(X_1)$$

has a finite second moment. This, however, follows from

$$\int_0^b C^{-3}(z) F^*(dz) = \alpha^2 \int_0^b \frac{F(dz)}{G^2(z)(1 - F(z))^3} < \infty$$

by (0.2) and $b < b_F$. Application of the SLLN to each of the remaining processes on the right-hand side of (1.5) allows one to replace $(n-1)/n$ by 1, so that the following result is immediate.

LEMMA 1.4. *Under* (0.2), *for each* $\delta > 3/2$,

$$\sup_{0 \le x \le b} |R_{n1}(x)| = o\big(n^{-1}(\ln n)^{\delta}\big) \quad \textit{with probability } 1.$$

LEMMA 1.5. *Under* (0.2),

$$\sup_{0 \le x \le b} |R_{n2}(x)| = O\big(n^{-1}(\ln n)^{3/2}\big) \quad \textit{with probability } 1.$$

PROOF. Immediate consequence of the D-K-W bound for empirical measures, (1.4) and the SLLN applied to

$$\int_0^b C^{-3}(z) F_n^*(dz). \qquad \square$$

Lemmas 1.4 and 1.5 together yield the assertion of Theorem 1. As for Theorem 2, we need a slight modification of $\hat{F}_n$. This is only to safeguard against $\ln 0$ when taking logarithms of $1 - \hat{F}_n(x)$. Without further mentioning, (0.2) will be assumed throughout.

Define $\bar{F}_n$ by

$$1 - \bar{F}_n(x) = \prod_{i:\, X_i \le x} \frac{nC_n(X_i)}{nC_n(X_i) + 1}.$$

LEMMA 1.6. *Uniformly in* $0 \le x \le b < b_F$ *one has*

$$\bar{F}_n(x) - \hat{F}_n(x) = O\left(\frac{\ln^2 n}{n}\right) \quad \textit{with probability } 1.$$

PROOF. For each $0 \le x \le b$, the above difference equals

$$\prod_{i:\, X_i \le x} \frac{nC_n(X_i) - 1}{nC_n(X_i)} - \prod_{i:\, X_i \le x} \frac{nC_n(X_i)}{nC_n(X_i) + 1}. \tag{1.6}$$

In view of

$$\left| \prod_{j=1}^n a_j - \prod_{j=1}^n b_j \right| \le \sum_{j=1}^n |a_j - b_j|, \qquad |a_j|, |b_j| \le 1,$$

(1.6) in absolute values is less than or equal to

$$\sum_{i:\, X_i \le x} n^{-2} C_n^{-2}(X_i) \le n^{-1} \int_0^b C_n^{-2}(z) F_n^*(dz).$$

The assertion follows from (1.4) and the SLLN giving

$$\int_0^b C^{-2}(z) F_n^*(dz) \to \int_0^b C^{-2}(z) F^*(dz). \qquad \square$$

LEMMA 1.7. *Uniformly in* $0 \le x \le b < b_F$ *one has*

$$\ln\left(1 - \bar{F}_n(x)\right) + \Lambda_n(x) = O\left(\frac{\ln^2 n}{n}\right) \quad \textit{with probability } 1.$$

PROOF. The expression on the left-hand side equals

$$\sum_{i:\, X_i \le x} \frac{1}{nC_n(X_i)(nC_n(X_i) + 1)} - \sum_{i:\, X_i \le x} \sum_{m=2}^{\infty} \frac{1}{m[nC_n(X_i) + 1]^m},$$

which in absolute values is less than or equal to

$$2n^{-1} \int_0^b C_n^{-2}(z) F_n^*(dz).$$

The assertion now follows as in the proof of the last lemma. $\square$

LEMMA 1.8. *For* $0 \le x \le b$,

$$F(x) - \bar{F}_n(x) = -[\Lambda_n(x) - \Lambda(x)](1 - F(x)) + \tfrac{1}{2}[\Lambda_n(x) - \Lambda(x)]^2 \exp[-\Lambda_n^*(x)] + \left[\Lambda_n(x) + \ln\left(1 - \bar{F}_n(x)\right)\right]\exp[-\Lambda_n^{**}(x)],$$

with

$$\min(\Lambda_n(x), \Lambda(x)) \le \Lambda_n^*(x) \le \max(\Lambda_n(x), \Lambda(x))$$

and

$$\min\left(\Lambda_n(x), \ln\left(1 - \bar{F}_n(x)\right)\right) \le \Lambda_n^{**}(x) \le \max\left(\Lambda_n(x), \ln\left(1 - \bar{F}_n(x)\right)\right).$$

PROOF. Apply Taylor's expansion. □

LEMMA 1.9. *For* $0 \le x \le b$,

$$\Lambda_n(x) - \Lambda(x) = \int_0^x \frac{F_n^*(dz)}{C_n(z)} - \int_0^x \frac{F^*(dz)}{C(z)}$$

$$(1.7) \qquad = \frac{F_n^*(x) - F^*(x)}{C(x)} + \int_0^x \frac{C(z) - C_n(z)}{C_n(z)C(z)} F_n^*(dz) + \int_0^x \frac{F_n^*(z) - F^*(z)}{C^2(z)} C(dz).$$

PROOF. Use integration by parts. □

LEMMA 1.10. *Under* (0.2), *with probability* 1,

$$\sup_{0 \le x \le b} |\Lambda_n(x) - \Lambda(x)| = O\left(\frac{\ln^{3/2} n}{\sqrt{n}}\right).$$

PROOF. We will utilize Lemma 1.9. From Csáki (1975), with probability 1,

$$(1.8) \qquad \sup_{0 < x} \frac{F_n^*(x) - F^*(x)}{\sqrt{F^*(x)}} = o\left(\sqrt{\frac{\ln^{1+\varepsilon} n}{n}}\right)$$

for each $\varepsilon > 0$. Furthermore, by (0.2), on $0 < x \le b$,

$$\infty > \int_0^b \frac{1}{G^2(z)} F(dz) \ge \frac{\alpha F^*(x)}{G^3(x)},$$

that is,

$$(1.9) \qquad G^3(x) \ge cF^*(x), \quad \text{some } c > 0.$$

This, together with (1.8), yields the desired bound for the first summand in (1.7); take $\varepsilon = 1$. The bound for the second is available from the LIL for empirical measures, (1.4) and the SLLN applied to

$$\int_0^b C^{-2}(z) F_n^*(dz).$$

Finally, the third integral is dealt with by using (1.8), (1.9) and the finiteness of

$$\int_0^b C^{-1/2}(z) C(dz). \qquad \square$$

The proof of Theorem 2 is a simple consequence of Lemmas 1.6–1.8, Lemma 1.10 and Theorem 1.

REFERENCES

CAIROLI, R. and WALSH, J. B. (1975). Stochastic integration in the plane. *Acta Math.* **134** 111–183.

CHAO, M.-T. and LO, S.-H. (1988). Some representations of the nonparametric maximum likelihood estimators with truncated data. *Ann. Statist.* **16** 661–668.

CSÁKI, E. (1975). Some notes on the law of the iterated logarithm for empirical distribution function. In *Colloq. Math. Soc. János Bolyai* **11**. *Limit Theorems of Probability Theory* (P. Révész, ed.) 47–57. North-Holland, Amsterdam.

LYNDEN-BELL, D. (1971). A method of allowing for known observational selection in small samples applied to 3CR quasars. *Monthly Notices Roy. Astronom. Soc.* **155** 95–118.

SHORACK, G. and WELLNER, J. (1986). *Empirical Processes with Applications to Statistics*. Wiley, New York.

STUTE, W. (1993). Integrated U-statistic processes: A martingale approach. Preprint. Univ. Giessen.

WOODROOFE, M. (1985). Estimating a distribution function with truncated data. *Ann. Statist.* **13** 163–177.

MATHEMATISCHES INSTITUT
UNIVERSITÄT GIESSEN
ARNDTSTRASSE 2
D-6300 GIESSEN
GERMANY