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A SEQUENTIAL PROCEDURE WITH ASYMPTOTICALLY NEGATIVE REGRET FOR ESTIMATING A NORMAL MEAN

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A sequential procedure for estimating the mean of a normal distribution is proposed. The procedure is shown to have a negative regret at the origin and the same regret as the usual procedure at the other point asymptotically.

1. Introduction. Let $X_1, X_2, \dots$ be independent and identically distributed normal random variables with unknown mean μ and unknown variance $\sigma^2 > 0$. Given a sample of size n, estimate μ by the sample mean $\overline{X}_n$. Suppose that if one stops with n observations, then the loss incurred is

$$L_n = A\left(\overline{X}_n - \mu\right)^2 + n, \qquad A > 0.$$

If the sample size n is fixed beforehand, then

$$R_n = E(L_n) = A\sigma^2/n + n$$

is minimized by taking $n = n^* = (A\sigma^2)^{1/2}$, and the corresponding minimum fixed sample size risk is

$$R_{n^*} = 2n^*.$$

Since σ^2 is unknown, Robbins (1959) proposed the following type of sequential procedure: Let

$$T = \inf\left\{n \geq m;\, n \geq \left(AV_n/n\right)^{1/2}\right\}, \tag{1.1}$$

where m is the initial sample size and $V_n = \sum_{i=1}^n (X_i - \overline{X}_n)^2$, and estimate μ by $\overline{X}_T$. Woodroofe (1977) showed that if $m \geq 4$, the regret of the procedure

$$R_T - R_{n^*} = 1/2 + o(1)$$

as $A \to \infty$.

The purpose of this article is to show that there does exist a sequential procedure for which the regret is negative at $\mu = 0$ and $1/2$ at $\mu \neq 0$ asymptotically.

Instead of V_n/n as an estimator of σ^2, consider an estimator ϕ_n of the form

$$\phi_n = n^{-1} \min\left\{V_n, \sum_{i=1}^n X_i^2/c_n\right\},$$

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where $c_n \geq 1$ is some constant [cf. Stein (1964)]. We propose the following sequential procedure: Let

$$N = \inf\{n \geq m; n \geq (A\phi_n)^{1/2}\} \tag{1.2}$$

and estimate μ by $\overline{X}_N$. In the sequel we suppose that c_n is nonincreasing and

$$c_n = 1 + c/n + o(1/n)$$

as $n \to \infty$, where c is some nonnegative constant. Then it is shown that the procedure N has the above-stated property. The intuitive explanation of the phenomenon is given in the last section.

For other distributions with negative regret, see Martinsek (1983, 1988).

2. Main result. In this section, an asymptotic expansion is obtained for the regret of the procedure N. In the sequel, without loss of generality, take $\sigma = 1$.

Writing the stopping time N given by (1.2) in the form

$$N = \inf\{n \geq m; Z_n \geq A^{1/2}\}$$

with $Z_n = n\phi_n^{-1/2}$, one may show that Z_n is of the form

$$Z_n = W_n + \xi_n,$$

where

$$W_n = n - 2^{-1}(q_n - n)$$

with $q_n = \sum_{i=1}^n (X_i - \mu)^2$ and

$$\xi_n = 2^{-1}\gamma_n + (3/8)\lambda_n^{-5/2} n(\phi_n - 1)^2$$

with $|\lambda_n - 1| \leq |\phi_n - 1|$ and

$$\gamma_n = \max\left\{n\left(\overline{X}_n - \mu\right)^2, q_n - \frac{1}{c_n}\sum_{i=1}^n X_i^2\right\}.$$

Since W_n, $n \geq 1$, is a random walk, the nonlinear renewal theory can be used to obtain the asymptotic expansion of $E(N)$. It is easy to check the conditions of Theorem 4.5 of Woodroofe (1982) except his condition (4.16), which follows from the next lemma.

LEMMA 1. *For* $0 < \varepsilon < 1$,

$$P(N \leq \varepsilon A^{1/2}) = O(A^{-(m-1)/2}), \quad \textit{as } A \to \infty.$$

PROOF. Let

$$\tilde{N} = \inf\left\{n \geq m; n \geq \left(A\sum_{i=1}^n X_i^2/(nc_n)\right)^{1/2}\right\}.$$

Then it follows from (1.1) and (1.2) that

$$P(N \leq \varepsilon A^{1/2}) \leq P(T \leq \varepsilon A^{1/2}) + P(\tilde{N} \leq \varepsilon A^{1/2}).$$

Note that $P(\tilde{N} \le \varepsilon A^{1/2}) \le P_0(\tilde{N} \le \varepsilon A^{1/2})$, where P_0 denotes the probability under $\mu = 0$. Then the lemma follows from Lemma 2.3 of Woodroofe (1977). □

Theorem 4.5 of Woodroofe (1982) yields that if $m \ge 3$,

$$(2.1)\quad E(N) = \begin{cases} A^{1/2} + \rho - \dfrac{1}{2}\displaystyle\int \max(z^2, c)\, d\Phi(z) - \dfrac{3}{4} + o(1), & \mu = 0, \\ A^{1/2} + \rho - \dfrac{5}{4} + o(1), & \mu \ne 0, \end{cases}$$

as $A \to \infty$, where ρ is the asymptotic mean of $U_A = N\phi_N^{-1/2} - A^{1/2}$ and Φ denotes the standard normal distribution function [see also Theorem A of Aras (1988)].

The main result is given next.

THEOREM. *If* $m > 13$,

$$R_N - R_{n^*} = \begin{cases} \displaystyle\int (z^2 - 1)\max(z^2, c)\, d\Phi(z) - 3/2 + o(1), & \mu = 0, \\ 1/2 + o(1), & \mu \ne 0, \end{cases}$$

as $A \to \infty$.

PROOF. By Wald's lemma,

$$\begin{aligned} R_N - R_{n^*} &= AE\left[(\bar{X}_N - \mu)^2\right] + E(N) - 2A^{1/2} \\ &= E\left[S_N^2\left\{(A^{1/2}/N)^2 - 1\right\}\right] + E(S_N^2) + E(N) - 2A^{1/2} \\ (2.2)\qquad &= E\left[S_N^2\left\{(A^{1/2}/N)^2 - 1\right\}\right] + 2\{E(N) - A^{1/2}\} \\ &= E\left[S_N^2\left\{(A^{1/2}/N)^2 - \phi_N^{-1}\right\}\right] + E\left[S_N^2(\phi_N^{-1} - 1)\right] \\ &\quad + 2\{E(N) - A^{1/2}\}, \end{aligned}$$

where $S_N = \sum_{i=1}^N (X_i - \mu)$. In the next section, it is proved that

$$(2.3)\qquad E\left[S_N^2\left\{(A^{1/2}/N)^2 - \phi_N^{-1}\right\}\right] = -2\rho + o(1)$$

and

$$(2.4)\quad E\left[S_N^2(\phi_N^{-1} - 1)\right] = \begin{cases} \displaystyle\int z^2 \max(z^2, c)\, d\Phi(z) + o(1), & \mu = 0, \\ 3 + o(1), & \mu \ne 0, \end{cases}$$

as $A \to \infty$. Then substituting (2.1), (2.3) and (2.4) into (2.2), the theorem follows. □

Let

$$f(c) = \int (z^2 - 1)\max(z^2, c)\, d\Phi(z).$$

Then it is easy to show that $f(c) \hookrightarrow 0$ as $c \to \infty$. Hence from the theorem the stopping rule N with sufficiently large c has asymptotically a negative regret at $\mu = 0$ and the same regret as T at $\mu \neq 0$.

REMARK. Asymptotically negative regret for any fixed value μ_0 of the mean, not necessarily 0, can be achieved by simply replacing the sum of squares in the stopping rule with the sum of squares about μ_0.

3. Proof of (2.3) and (2.4). In order to prove (2.3) and (2.4), several lemmas are necessary. Using Lemma 1 and the fact that $N \leq T$, the following two lemmas easily follow from Lemmas 1 and 2 of Chow and Yu (1981).

LEMMA 2. *If* $m > t + 1$, $t > 0$, $\{|A^{1/2}/N|^t;\ A \geq 1\}$ *is uniformly integrable.*

LEMMA 3. *For* $t > 0$, $\{|N/A^{1/2}|^t;\ A \geq 1\}$ *is uniformly integrable.*

LEMMA 4. *If* $m > t + 1$, $t > 1$, $\{|\gamma_N|^t;\ A \geq 1\}$ *is uniformly integrable.*

PROOF. Let $A_n = \{V_n < \sum_{i=1}^n X_i^2/c_n\}$. Then

$$\gamma_N = N\big(\overline{X}_N - \mu\big)^2 I_{A_N} + \left(q_N - \sum_{i=1}^N X_i^2/c_N\right) I_{\overline{A}_N}, \tag{3.1}$$

where I_A denotes the indicator function of the set A. Since

$$N\big(\overline{X}_N - \mu\big)^2 = \big(A^{1/2}/N\big)\big(A^{-1/4}S_N\big)^2,$$

by Hölder's inequality, Lemma 5 of Chow and Yu (1981) and Lemmas 2 and 3, $\{|N(\overline{X}_N - \mu)^2|^t;\ A \geq 1\}$ is uniformly integrable if $m > t + 1$. Hence from (3.1) it is enough to prove that $\{|q_N - \sum_{i=1}^N X_i^2/c_N|^t I_{\overline{A}_N};\ A \geq 1\}$ is uniformly integrable.

When $\mu = 0$,

$$\left|q_N - \sum_{i=1}^N X_i^2/c_N\right|^t = \left|\{N(c_N - 1)/c_N\}q_N/N\right|^t \leq B \sup_{n \geq m} |q_n/n|^t$$

for some constant $B > 0$. Hence the uniform integrability follows.

When $\mu \neq 0$,

$$\begin{aligned} q_N - \sum_{i=1}^N X_i^2/c_N = {} & \{N(c_N - 1)/c_N\}q_N/N \\ & - 2\mu N\big(\overline{X}_N - \mu\big)/c_N + N\mu^2/c_N. \end{aligned} \tag{3.2}$$

Since $\bar{A}_n = \{(c_n - 1)V_n - n\bar{X}_n^2 \geq 0\}$,

$$P(\bar{A}_n) \leq E\{\exp(h(c_n - 1)V_n)\}E\{\exp(-hn\bar{X}_n^2)\}$$

for $h > 0$, from which it can be shown that $P(\bar{A}_n) = O(\rho^n)$ as $n \to \infty$ for some constant $0 < \rho < 1$. Hence

$$E\{N^{2t}I_{\bar{A}_N}\} \leq \sum_{n=m}^{\infty} n^{2t}P(\bar{A}_n) < \infty,$$

so that the uniform integrability follows from (3.2). □

LEMMA 5. *If $m > t + 1$, $t > 1$, then $\{U_A^t;\ A \geq 1\}$ is uniformly integrable.*

PROOF. Note that on $\{N > m\}$,

$$\phi_N \geq (N-1)N^{-1}\phi_{N-1} > 2^{-1}\phi_{N-1} > 2^{-1}(N-1)^2/A, \tag{3.3}$$

from which it follows that

$$\begin{aligned} U_A &< N\phi_N^{-1/2} - (N-1)\phi_{N-1}^{-1/2} \\ &\leq (N\phi_N)^{-1/2}\{N^{3/2} - (N-1)^{3/2}\} \\ &< (N\phi_N)^{-1/2}\{(3/2)(N-1)^{1/2} + 3/8\} \\ &\leq ((N-1)\phi_{N-1})^{-1/2}\{(3/2)(N-1)^{1/2} + 3/8\} \\ &< A^{1/2}(N-1)^{-3/2}\{(3/2)(N-1)^{1/2} + 3/8\} \\ &= (3/2)A^{1/2}/(N-1) + (3/8)A^{1/2}/(N-1)^{3/2} \end{aligned}$$

[see Martinsek (1983), page 830]. Hence the lemma follows from Lemma 2. □

LEMMA 6. *If $m > 2$,*

$$E(N) - E(N\phi_N) = \begin{cases} \int \max(z^2, c)\, d\Phi(z) + o(1), & \mu = 0, \\ 1 + o(1), & \mu \neq 0, \end{cases}$$

as $A \to \infty$.

PROOF. Since $N\phi_N = q_N - \gamma_N$, by Wald's lemma

$$E(N) - E(N\phi_N) = E(\gamma_N). \tag{3.4}$$

It is easy to see that γ_n converges in distribution to $\max(Z^2, c)$ for $\mu = 0$ and to Z^2 for $\mu \neq 0$ as $n \to \infty$, where Z denotes the standard normal random variable and that $\{\gamma_n;\ n \geq m\}$ is uniformly continuous in probability. Since $N/A^{1/2} \to 1$ a.s. as $A \to \infty$, by Anscombe's theorem γ_N converges in distribution to $\max(Z^2, c)$ for $\mu = 0$ and to Z^2 for $\mu \neq 0$ as $A \to \infty$. Hence the lemma follows from (3.4) and Lemma 4. □

LEMMA 7. *If* $m > 3$,

$$A^{-1/2}E\left[\left(S_N^2 - N\phi_N\right)\left(N - N\phi_N\right)\right] = \begin{cases} \int (z^2 - 1)\max(z^2, c)\, d\Phi(z) + o(1), & \mu = 0, \\ 2 + o(1), & \mu \neq 0, \end{cases}$$

as $A \to \infty$.

PROOF. Write

$$\begin{aligned} E\left[\left(S_N^2 - N\phi_N\right)\left(N - N\phi_N\right)\right] &= E\left[\left(S_N^2 - q_N\right)\left(q_N - N\phi_N\right)\right] + E\left[\left(S_N^2 - q_N\right)\left(N - q_N\right)\right] \\ &\quad + E\left[\left(q_N - N\phi_N\right)\left(N - q_N\right)\right] + E\left[\left(q_N - N\phi_N\right)^2\right]. \end{aligned} \tag{3.5}$$

It follows from (2.21) of Martinsek (1983) that

$$E\left[\left(S_N^2 - q_N\right)\left(N - q_N\right)\right] = o\left(A^{1/2}\right) \tag{3.6}$$

as $A \to \infty$. By Wald's lemma,

$$\begin{aligned} \left|E\left[\left(q_N - N\phi_N\right)\left(N - q_N\right)\right]\right| &= \left|E\left[\gamma_N\left(N - q_N\right)\right]\right| \\ &\leq E^{1/2}\left(\gamma_N^2\right)E^{1/2}\left[\left(N - q_N\right)^2\right] \\ &= E^{1/2}\left(\gamma_N^2\right)E^{1/2}\left[\left(Z^2 - 1\right)^2\right]E^{1/2}(N) \end{aligned}$$

and

$$E\left[\left(q_N - N\phi_N\right)^2\right] = E\left(\gamma_N^2\right).$$

Then from Lemmas 3 and 4,

$$E\left[\left(q_N - N\phi_N\right)\left(N - q_N\right)\right] = o\left(A^{1/2}\right) \tag{3.7}$$

and

$$E\left[\left(q_N - N\phi_N\right)^2\right] = o\left(A^{1/2}\right) \tag{3.8}$$

as $A \to \infty$. Substituting (3.6), (3.7) and (3.8) into (3.5),

$$\begin{aligned} A^{-1/2}E\left[\left(S_N^2 - N\phi_N\right)\left(N - N\phi_N\right)\right] &= A^{-1/2}E\left[\left(S_N^2 - q_N\right)\left(q_N - N\phi_N\right)\right] + o(1) \\ &= A^{-1/2}E\left[\left(S_N^2 - q_N\right)\gamma_N\right] + o(1) \end{aligned} \tag{3.9}$$

as $A \to \infty$. It is easy to see that $n^{-1}(S_n^2 - q_n)\gamma_n$ converges in distribution to $(Z^2 - 1)\max(Z^2, c)$ for $\mu = 0$ and to $(Z^2 - 1)Z^2$ for $\mu \neq 0$ as $n \to \infty$, and that $\{n^{-1}(S_n^2 - q_n)\gamma_n;\ n \geq m\}$ is uniformly continuous in probability. Then Anscombe's theorem implies that $A^{-1/2}(S_N^2 - q_N)\gamma_N$ converges in distribution to $(Z^2 - 1)\max(Z^2, c)$ for $\mu = 0$ and to $(Z^2 - 1)Z^2$ for $\mu \neq 0$ as $A \to \infty$. Hence from (3.9) to prove the lemma, it is enough to show that $\{A^{-1/2}(S_N^2 -$

$q_N)\gamma_N;\ A \geq 1\}$ is uniformly integrable, which follows from Lemma 5 of Chow and Yu (1981), Lemmas 2, 3 and 4 and Hölder's inequality. □

The proof of the following lemma is similar to that of (2.26) of Martinsek (1983) and is therefore omitted.

LEMMA 8. *If $m > 13$,*

$$A^{-1/2}E\big[(S_N^2 - N\phi_N)(N - N\phi_N)(A^{1/2} - N\phi_N)(N\phi_N)^{-1}\big] = o(1)$$

as $A \to \infty$.

PROOF OF (2.3). Since

$$\begin{aligned}(A^{1/2}/N)^2 - \phi_N^{-1} &= \big[(A^{1/2}/N) - \phi_N^{-1/2}\big]\big[(A^{1/2}/N) + \phi_N^{1/2}\big] \\ &= -N^{-1}U_A\big[(A^{1/2}/N) + \phi_N^{-1/2}\big],\end{aligned}$$

it follows that

$$E\big[S_N^2\{(A^{1/2}/N)^2 - \phi_N^{-1}\}\big] = -E\big[N^{-1}S_N^2U_A\{(A^{1/2}/N) + \phi_N^{-1/2}\}\big]. \tag{3.10}$$

Note that $A^{-1/2}(N - A^{1/2}) \to 0$ a.s. as $A \to \infty$. Then by the same argument as the lemma of Martinsek (1983), it turns out that $N^{-1}S_N^2$ and U_A are asymptotically independent as $A \to \infty$. Using (3.3), it follows from Lemma 5 of Chow and Yu (1981), Lemmas 2, 3 and 5 and Hölder's inequality that $\{N^{-1}S_N^2U_A[(A^{1/2}/N) + \phi_N^{-1/2}];\ A \geq 1\}$ is uniformly integrable. Hence the proof follows from (3.10). □

PROOF OF (2.4). Write

$$\begin{aligned}&E\big[S_N^2(\phi_N^{-1} - 1)\big] \\ &\quad = E\big[(S_N^2 - N\phi_N)(N\phi_N)^{-1}(N - N\phi_N)\big] + E(N) - E(N\phi_N) \\ &\quad = A^{-1/2}E\big[(S_N^2 - N\phi_N)(N - N\phi_N)(A^{1/2} - N\phi_N)(N\phi_N)^{-1}\big] \\ &\qquad + A^{-1/2}E\big[(S_N^2 - N\phi_N)(N - N\phi_N)\big] + E(N) - E(N\phi_N).\end{aligned}$$

Then the proof follows from Lemmas 6, 7 and 8. □

4. Intuitive explanation. The stopping rule N consists of stopping when either (a) the usual rule T would stop or (b) the sum of squares of the observations is sufficiently small. The latter criterion makes sense if the mean is in fact 0, since then it provides an alternative (and better) estimate of the variance.

Note that when the mean is nonzero and A is large, N will be essentially the same as T, because criterion (b) above will rarely be fulfilled before criterion (a) is. That is why the asymptotic answer is the same as for the usual T. If the mean is 0, however, with nonnegligible probability criterion (b) will happen first. Criterion (b) means that the Euclidean distance between the observations and the origin is relatively small. Hence the sample mean is relatively close to 0 (the true mean). In effect, the stopping rule N introduces some "automatic shrinkage" when the mean is 0. The estimator is still the sample mean, but it has (at least with certain probability) been made close to 0 by the action of the stopping rule.

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