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SOME PROPERTIES OF THE KAPLAN–MEIER ESTIMATOR FOR INDEPENDENT NONIDENTICALLY DISTRIBUTED RANDOM VARIABLES

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In this paper we study the Kaplan–Meier estimator in the case where survival and censoring times are not all i.i.d. We prove several results which are analogous to those shown by van Zuijlen in the complete data case.

1. Introduction. The in probability linear bound of the ratio of an empirical distribution function to the true in the i.i.d. random variables case is called Daniels' theorem [Daniels (1945) and Robbins (1954)]. A lot of research effort has been devoted to this type of theorem since then [see, e.g., Wellner (1978)]. Similar inequalities are much harder to prove for the case of independent but nonidentically distributed random variables [see van Zuijlen (1976, 1978, 1982) and Marcus and Zinn (1984)].

In the situation where the observations are censored, the Kaplan–Meier estimator takes over the role of an empirical distribution function. Gill (1980) showed that the analog of Daniels' theorem is also valid for the Kaplan–Meier estimator. He also conjectured that a similar lower bound might also be true. This lower bound was later independently proved by Zhou [(1986), Lemma 2.6] and Shorack and Wellner [(1986), Inequality 7.6.3]. These same inequalities (plus an inequality for the cumulative hazard function) in the situation of independent but nonidentically distributed case are the main results of this paper, although we do not obtain the best bound. Not even van Zuijlen or Marcus and Zinn get the best constant they had hoped for.

The need for this kind of inequality arises, for instance, in the analysis of censored data regression estimators of Koul, Susarla and Van Ryzin (1981), and of Leurgans (1987), where the data are non-i.i.d. and the Kaplan–Meier estimator often appears in the denominator of an expression; see Zhou (1988, 1992) for details.

Recently in analyzing the censored regression estimator of Buckley and James (1979), Lai and Ying (1992) investigated the *limiting behavior* of a *modified* Kaplan–Meier estimator and obtained almost sure rate of various related processes. While their results are certainly important, their modification, among other things, stops the Kaplan–Meier estimator when the risk set is smaller than $Cn^{1-\lambda}$ with $0 < \lambda < 1/16$ and $C > 0$. On the other hand, we

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allow the risk set to go down all the way to one and the bound is good for any n, but our result is in probability, weaker than almost sure. It is well known that almost sure linear bounds do not exist on the interval where the risk set is allowed to be one. The practical implementation of Lai and Ying's modification needs further guidelines on the choice of λ and C. Yang (1990) also studied the bounds of the ratio of a Kaplan–Meier estimator to the true. He obtained mean square as well as probability bounds on the interval from zero to the $(n-k)$th order statistic with k fixed.

Suppose we have two sets of nonnegative random variables: $y_1, y_2, \dots, y_n$ which are i.i.d. with continuous distribution function $G(t)$ and $x_1, x_2, \dots, x_n$ which are independent but nonidentically distributed with continuous distribution functions $F_1(t), F_2(t), \dots, F_n(t)$. We also assume the y_i's are independent of the x_i's.

The data that we observe are

$$z_i = \min(y_i, x_i), \qquad \delta_i = I_{[z_i = y_i]}. \tag{1.1}$$

Estimation of the distribution $G(t)$ based on (1.1) is furnished by the Kaplan–Meier (1958) estimator (and other estimators)

$$1 - \hat{G}_K(t) = \prod_{s \le t} \left(1 - \frac{\Delta N(s)}{R^+(s)}\right) \quad \text{for } t \le T^n, \tag{1.2}$$

here

$$R^+(t) = \Sigma I_{[z_i \ge t]}; \qquad N(t) = \Sigma I_{[z_i \le t,\, \delta_i = 1]};$$

$$\Delta N(s) = N(s) - N(s-) \quad \text{and} \quad T^n = \max_{i \le n} \{z_i\}.$$

Estimation of the cumulative hazard function of the y_i's, $-\log(1 - G(t)) = \Lambda(t)$, is furnished by the Nelson–Aalen estimator

$$\hat{\Lambda}(t) = \sum_{s \le t} \frac{\Delta N(s)}{R^+(s)} \quad \text{for } t \le T^n. \tag{1.3}$$

Altshuler's estimator of $1 - G(t)$ is related to the Nelson–Aalen estimator and is given by

$$1 - \hat{G}_A(t) = e^{-\hat{\Lambda}(t)}. \tag{1.4}$$

We further denote $1 - H_i(t) = P(z_i \ge t) = [1 - G(t)][1 - F_i(t)]$.

REMARK 1.1. If both y_i's and x_i's are nonidentically distributed with arbitrary distributions, then the question of estimation based on observations (1.1) is easily seen to be nonidentifiable. Therefore we always suppose one of the sets of random variables are i.i.d.

It is well known that $\hat{\Lambda}(t) - \Lambda(t)$ is a local martingale on $[0, T^n]$ with respect to the filtration $\mathscr{F}_s$ defined below [see Gill (1980) or Aalen (1978)]:

$$\mathscr{F}_s = \sigma\left\{z_k I_{[z_k \le s]}; \delta_k I_{[z_k \le s]}; k = 1, 2, \dots, n\right\}$$

with predictable variation process

$$\langle \hat{\Lambda} - \Lambda \rangle(t) = \int_0^t \frac{d\Lambda(s)}{R^+(s)}. \tag{1.5}$$

Section 2 contains two basic inequalities with the proofs. Section 3 contains some (more or less) easy corollaries of the theorems in Section 2.

2. Main theorems. In this section, we first look at the in-probability bound of $\hat{\Lambda}(t) - \Lambda(t)$. Theorem 2.1 states that even if $\Lambda(t)$ is unbounded as $t \to \infty$, $\sup_{t < T^n} |\hat{\Lambda}(t) - \Lambda(t)|$ remains bounded in probability for any n. We begin with a lemma.

LEMMA 2.1. *If $t_n(\varepsilon)$ is a real number defined as*

$$\int_0^{t_n(\varepsilon)} \frac{d\Lambda(s)}{\sum_{i=1}^n [1 - H_i(s)]} = \left(\frac{1}{\varepsilon}\right)^{2/3}, \qquad \varepsilon > 0, \tag{2.1}$$

then we have

$$\sum_{i=1}^n [1 - H_i(t_n(\varepsilon))] \le \varepsilon^{2/3}. \tag{2.2}$$

PROOF. Notice that, since $1 - H_i = (1 - F_i)(1 - G)$ and $d\Lambda(s) = dG/1 - G$, we have, by integration by parts,

$$\begin{aligned}
\int_0^{t_n(\varepsilon)} \frac{d\Lambda(s)}{\Sigma[1 - H_i(s)]} &= \int_0^{t_n(\varepsilon)} \frac{1}{\Sigma[1 - F_i]} d\left(\frac{1}{1 - G}\right) \\
&= \frac{1}{\Sigma[1 - F_i]} \frac{1}{1 - G}\bigg|_0^{t_n(\varepsilon)} - \int_0^{t_n(\varepsilon)} \frac{1}{1 - G} d\left(\frac{1}{\Sigma[1 - F_i]}\right) \\
&= \frac{1}{\Sigma[1 - H_i(t_n(\varepsilon))]} - \frac{1}{\Sigma[1 - H_i(0)]} \\
&\quad - \int_0^{t_n(\varepsilon)} \frac{1}{1 - G} d\left(\frac{1}{\Sigma[1 - F_i]}\right) \\
&= \frac{1}{\Sigma[1 - H_i(t_n(\varepsilon))]} - \frac{1}{n} - \int_0^{t_n(\varepsilon)} \frac{1}{1 - G} d\left(\frac{1}{\Sigma[1 - F_i]}\right).
\end{aligned}$$

Plugging this into equation (2.1), we get

$$\frac{1}{\Sigma[1 - H_i(t_n(\varepsilon))]} = \frac{1}{n} + \int_0^{t_n(\varepsilon)} \frac{1}{1 - G} d\left(\frac{1}{\Sigma[1 - F_i]}\right) + \left(\frac{1}{\varepsilon}\right)^{2/3} \ge \left(\frac{1}{\varepsilon}\right)^{2/3}.$$

□

THEOREM 2.1. *If* $\Lambda(t)$ *and* $\hat{\Lambda}(t)$ *are, respectively, the cumulative hazard function of* $G(t)$ *and its Nelson–Aalen estimator defined in* (1.3), *we have, for* $0 < \varepsilon < 1/2$,

$$P\left(\sup_{t<T^n} |\Lambda(t) - \hat{\Lambda}(t)| > \frac{1}{\varepsilon}\right) < C\varepsilon^{2/3}, \tag{2.3}$$

where the constant C can be taken to be 358.

PROOF. Since $\hat{\Lambda}(t \wedge T^n) - \Lambda(t \wedge T^n)$ is an $\mathscr{F}_t$-martingale with predictable variation process

$$\langle \hat{\Lambda} - \Lambda \rangle (t \wedge T^n) = \int_0^{t\wedge T^n} \frac{d\Lambda(s)}{R^+(s)},$$

we have, by using Lenglart's (1977) inequality,

$$P\left(\sup_{t<T^n} |\Lambda(t) - \hat{\Lambda}(t)| > \frac{1}{\varepsilon}\right) \le \eta\varepsilon^2 + P\left(\int_0^{T^n} \frac{d\Lambda(s)}{R^+(s)} > \eta\right), \quad \eta > 0. \tag{2.4}$$

Taking $\eta = (1/\varepsilon)^{4/3}$ gives

$$\le \varepsilon^{2/3} + P\left(\int_0^{T^n} \frac{d\Lambda(s)}{R^+(s)} > \left(\frac{1}{\varepsilon}\right)^{4/3}\right).$$

The process $(1/n)R^+(s)$ is an empirical process of n independent but non-identically distributed random variables. The inequality of van Zuijlen (1978) gives

$$P\left(\sup_{t<T^n} \frac{\Sigma[1 - H_i(t)]}{R^+(t)} > \lambda\right) < \frac{2\pi^2}{3\lambda(1 - (1/\lambda))^4}, \quad \lambda > 1. \tag{2.5}$$

Thus, we have, for $\lambda > 1$,

$$(2.4) \le \varepsilon^{2/3} + \frac{2\pi^2}{3\lambda(1 - (1/\lambda))^4} + P\left(\int_0^{T^n} \frac{\lambda\, d\Lambda(s)}{\Sigma[1 - H_i(s)]} > \left(\frac{1}{\varepsilon}\right)^{4/3}\right).$$

Taking $\lambda = (1/\varepsilon)^{2/3}$ gives

$$= \varepsilon^{2/3} + \varepsilon^{2/3}\frac{2}{3}\frac{\pi^2}{(1 - \varepsilon^{2/3})^4} + P\left(\int_0^{T^n} \frac{d\Lambda(s)}{\Sigma[1 - H_i(s)]} > \left(\frac{1}{\varepsilon}\right)^{2/3}\right). \tag{2.6}$$

Now let us focus on the probability term of the above bound. It is easy to see that the mean of the random variable inside the probability is infinite even when $n = 1$, so we have to work on its probability distribution function.

It is easy to see that

$$P\left(\int_0^{T^n} \frac{d\Lambda(s)}{\Sigma[1 - H_i(s)]} > \left(\frac{1}{\varepsilon}\right)^{2/3}\right) = P(T^n > t_n(\varepsilon)), \tag{2.7}$$

where $t_n(\varepsilon)$ is so defined that

$$\int_0^{t_n(\varepsilon)} \frac{d\Lambda(s)}{\Sigma[1 - H_i(s)]} = \left(\frac{1}{\varepsilon}\right)^{2/3}.$$

Since $T^n = \max\{z_i\}$, we have

$$(2.8)\qquad \begin{aligned}(2.7) &= P(\max\{z_i\} > t_n(\varepsilon)) \le \sum_i P(z_i > t_n(\varepsilon)) \\ &= \sum_i [1 - H_i(t_n(\varepsilon))] \le \varepsilon^{2/3},\end{aligned}$$

where the last inequality follows from Lemma 2.1. Combining this bound of the probability term with (2.6), we have

$$(2.6) \le \varepsilon^{2/3} + \varepsilon^{2/3} \frac{2\pi^2}{3(1 - \varepsilon^{2/3})^4} + \varepsilon^{2/3} = \varepsilon^{2/3}\left(2 + \frac{2\pi^2}{3(1 - \varepsilon^{2/3})^4}\right).$$

For the range $0 \le \varepsilon \le 1/2$, it is again bounded by

$$\varepsilon^{2/3}\left(2 + \frac{2\pi^2}{3\left(1 - \left(\frac{1}{2}\right)^{2/3}\right)^4}\right) \le \varepsilon^{2/3}(358). \qquad \square$$

REMARK 2.1. A better constant can be obtained if we restrict ε to a smaller interval and use van Zuijlen (1982). It can be reduced to a little larger than 3.

Theorem 2.1 deals with the cumulative hazard function $\Lambda(t)$ or $-\log(1 - G)$. In order to apply it to the Kaplan–Meier estimator $\hat{G}_K(t)$, we need the following lemma.

LEMMA 2.2. *Let $\hat{G}_K(t)$, $\hat{G}_A(t)$ be the Kaplan–Meier and Altshuler's estimator of the distribution G based on censored data as in* (1.2) *and* (1.4). *We have, $\forall\ t$, if $1 - \hat{G}_K(t) > 0$, then*

$$\left|\hat{G}_K(t) - \hat{G}_A(t)\right| < \left[1 - \hat{G}_K(t)\right] \frac{4}{R^+(t)}.$$

PROOF. See Cuzick (1985) for a similar inequality and proof. This exact inequality was first stated without proof in Gu [(1987), Lemma 2.2]. $\square$

Using Lemma 2.2, we can get some bounds of $1 - \hat{G}_K$ from Theorem 2.1.

THEOREM 2.2. *For $\delta > e^2$,*

$$P\left(\sup_{t<T^n} \frac{1}{5}\left|\frac{1 - G(t)}{1 - \hat{G}_K(t)}\right| > \delta\right) < C\left(\frac{1}{\log \delta}\right)^{2/3}.$$

For $\lambda > 4$,

$$P\left(\sup_{t<T^n}\left|\log\frac{1-G(t)}{1-\hat{G}_K(t)}\right| > \lambda\right) < C\left(\frac{1}{\lambda-\log 5}\right)^{2/3} < C\left(\frac{1}{\lambda-2}\right)^{2/3},$$

where C is the same constant as in Theorem 2.1.

PROOF. For $t < T^n$, it is easy to see that, by Lemma 2.2 above,

$$\frac{1-G}{1-\hat{G}_K} = \frac{1-G}{1-\hat{G}_A}\frac{1-\hat{G}_A}{1-\hat{G}_K} = \frac{1-G}{1-\hat{G}_A}\left\{\frac{1-\hat{G}_K+\hat{G}_K-\hat{G}_A}{1-\hat{G}_K}\right\}$$

$$(2.9) \qquad = \frac{1-G}{1-\hat{G}_A}\left\{1+\frac{\hat{G}_K-\hat{G}_A}{1-\hat{G}_K}\right\} \le \frac{1-G}{1-\hat{G}_A}\left\{1+\frac{4}{R^+(t)}\right\}$$

$$\le 5\frac{1-G}{1-\hat{G}_A}.$$

On the other hand, we notice that

$$\sup_t\left|\frac{1-G}{1-\hat{G}_A}\right| > \delta \Leftrightarrow \sup_t \log\frac{1-G}{1-\hat{G}_A} > \log\delta$$

and

$$\log\frac{1-G}{1-\hat{G}_A} = \hat{\Lambda}(t) - \lambda(t) = -\log(1-\hat{G}_A) - (-\log 1 - G).$$

Therefore,

$$P\left(\sup_{t<T^n}\frac{1-G}{1-\hat{G}_A} > \delta\right) = P\left(\sup_{t<T^n}\log\frac{1-G}{1-\hat{G}_A} > \log\delta\right)$$

$$= P\left(\sup_{t<T^n}\hat{\Lambda}(t) - \Lambda(t) > \log\delta\right)$$

$$\le P\left(\sup_{t<T^n}|\hat{\Lambda}(t) - \Lambda(t)| > \log\delta\right) \le C\left(\frac{1}{\log\delta}\right)^{2/3},$$

by Theorem 2.1. Thus

$$P\left(\sup_{t<T^n}\frac{1-G(t)}{1-\hat{G}_K(t)} > 5\delta\right) < C\left(\frac{1}{\log\delta}\right)^{2/3}.$$

To prove the second inequality, we first notice that $1-\hat{G}_K(t) \le 1-\hat{G}_A(t)$ [Cuzick (1985)] which, together with (2.9), implies

$$\left|\log\frac{1-G}{1-\hat{G}_K}\right| \le \left|\log\frac{1-G}{1-\hat{G}_A}\right| + \log 5.$$

Now the second inequality can be proved by using the above inequality and Theorem 2.1. □

3. Applications. Although the x_i's in (1.1) are not identically distributed, we can still form its Kaplan–Meier estimator $\hat{F}_K(t)$ by

$$1 - \hat{F}_K(t) = \prod_{s \le t \wedge T^n} \left(1 - \frac{\Delta N^c(s)}{R^+(s)}\right), \tag{3.1}$$

where $N^c(s) = \sum I_{[z_i \le s,\, \delta_i = 0]}$.

This estimator is strongly consistent in the sense that $\lim_n |\hat{F}_K(t) - \overline{F}_n(t)| \to 0$ a.s. where $\overline{F}_n = (1/n)\sum F_i$. However, the ratio of $1 - \hat{F}_K(t)$ to $1 - \overline{F}_n(t)$ is no longer a martingale. So we cannot use the martingale approach.

THEOREM 3.1. *For $\lambda > e^2$, we have, $\forall\, \varepsilon > 0$,*

$$P\left(\sup_{t<T^n} \frac{1 - \hat{F}_K(t)}{1 - \overline{F}_n(t)} > 5\lambda^{1+\varepsilon}\right) \le \frac{(1/\lambda^\varepsilon)\exp(1 - (1/\lambda^\varepsilon))}{1 - (1/\lambda^\varepsilon)\exp(1 - (1/\lambda^\varepsilon))} + C\left(\frac{1}{\log \lambda}\right)^{2/3}, \tag{3.2}$$

where C is the same constant as in Theorem 2.1.

PROOF. One useful fact here is

$$\left[1 - \hat{F}_K(t)\right]\left[1 - \hat{G}_K(t)\right] = \frac{1}{n}\sum I_{[z_i > t]} \triangleq 1 - \hat{H}(t) = \frac{R^+(t+)}{n}.$$

Thus, it is easy to see from $(1 - \overline{F})(1 - G) = 1 - \overline{H} \triangleq (1/n)\sum[1 - H_i(t)]$ that

$$\frac{1 - \hat{F}_K(t)}{1 - \overline{F}(t)} = \frac{1 - \hat{H}}{1 - \overline{H}}\,\frac{1 - G(t)}{1 - \hat{G}_K} \quad \text{for } t < T^n. \tag{3.3}$$

Thus,

$$P\left(\sup_{t<T^n} \left|\frac{1 - \hat{F}_K(t)}{1 - \overline{F}_n(t)}\right| > 5\lambda^{1+\varepsilon}\right) \le P\left(\sup_{t<T^n} \left|\frac{1 - \hat{H}}{1 - \overline{H}}\right| > \lambda^\varepsilon\right) + P\left(\sup_{t<T^n} \left|\frac{1 - G}{1 - \hat{G}_K}\right| > 5\lambda\right).$$

By Theorem 2.2 and van Zuijlen's inequality, we can bound the above by

$$\frac{(1/\lambda^\varepsilon)\exp(1 - (1/\lambda^\varepsilon))}{1 - (1/\lambda^\varepsilon)\exp(1 - (1/\lambda^\varepsilon))} + C\left(\frac{1}{\log \lambda}\right)^{2/3}.$$ □

REMARK 3.1. The slow rate $(1/\log \lambda)^{2/3}$ is presumably caused by the log transformation we used. It pops up exponentially fast near zero.

Gill (1980) showed that $[1 - \hat{G}_K(t)]/[1 - G(t)]$ is a martingale. Using this fact and Doob's inequality, it is not hard to show that

$$P\left(\sup_{t \le T^n} \frac{1 - \hat{G}_K(t)}{1 - G(t)} > \lambda \right) \le \frac{1}{\lambda}, \qquad \lambda > 1. \tag{3.4}$$

COROLLARY 3.1. *For* $\lambda > 1$,

$$\begin{aligned} P\left(\sup_{t<T^n} \frac{1 - \overline{F}_n(t)}{\hat{F}_K(t)} > \lambda^2 \right) &\le \frac{1}{\lambda} + P\left(\sup_{t<T^n} \frac{1 - \overline{H}}{1 - \hat{H}} > \lambda \right) \\ &\le \frac{1}{\lambda} + \frac{2\pi^2}{3\lambda(1 - (1/\lambda))^4}. \end{aligned} \tag{3.5}$$

PROOF. The proof follows easily from (3.3), (3.4), van Zuijlen (1978) and Bonferroni's inequality. □

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