The Annals of Statistics
1991, Vol. 19, No. 4, 2160–2182

TWO-DIMENSIONAL DESIGN FOR CORRELATED ERRORS

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Optimal and highly efficient two-dimensional designs are constructed for correlated errors on the torus and in the plane. The technique uses the method of differences to produce series of connectable planar squares. Efficiency calculations for planner versions of the torus designs show that the torus approximation is very satisfactory.

1. Introduction. A number of recent papers have addressed the problems of analysis and optimality of two-dimensional layouts with correlated errors; see Kiefer and Wynn (1981), Martin (1982, 1986), Gill and Shukla (1985) and Kunert (1987). To this point, exact optimality results are few and have been obtained only for certain correlation structures in conjunction with simplifying assumptions, but recommendations have been made concerning design properties that will give reasonable efficiency and balance across a range of error processes [Martin (1986)]. These are, in fact, in accord with the exact properties Martin (1982) has enumerated for design on the torus, and it is the torus approach that will be considered first.

Let y_{ijk} be the observation in row i and column j of the $m_1 \times m_2$ torus lattice k $(= 1, 2, \dots, s)$ with arbitrary initial cell $(1, 1, k)$ and let $\tau_{[ijk]}$ be the effect of the treatment in cell (i, j, k). Concern will center chiefly on the model

$$y_{ijk} = \mu_k + \tau_{[ijk]} + \varepsilon_{ijk},$$

where the errors follow the completely symmetric second order autonormal process:

$$\sigma^2 \operatorname{var}^{-1}(\varepsilon) = I_s \otimes \left[I_{m_1 m_2} - \alpha\left(I_{m_1} \otimes C_{m_2} + C_{m_1} \otimes I_{m_2}\right) - \gamma\left(C_{m_1} \otimes C_{m_2}\right)\right].$$

Here C_m is the $m \times m$ matrix with

$$(C_m)_{ij} = \begin{cases} 1, & \text{if } i - j \equiv \pm 1 \pmod{m}, \\ 0, & \text{otherwise,} \end{cases}$$

α, γ and σ^2 are constants and $\otimes$ is the Kronecker product. Then it can be shown that if $\alpha, \gamma > 0$, a design is universally optimum for estimation of treatment contrasts if (i) every treatment has each other treatment as first neighbor equally often in rows and columns combined, (ii) every treatment has each other treatment as first neighbor equally often in diagonals, (iii) no

Received September 1989; revised October 1990.
[1]Research supported by the Old Dominion University Research Foundation.
AMS 1980 *subject classifications*. 62K05, 62K99.
Key words and phrases. Correlated errors, efficiency, generalized least squares, neighbor design, optimal design, second order autonormal process, torus design.

treatment immediately neighbors itself in rows, columns, or diagonals and (iv) the set of s toruses is equireplicate, with each pair of distinct treatments appearing together in an equal number of toruses and the treatments being replicated within each torus as nearly equally as possible.

Note that (iv) says that the toruses form a balanced block design as defined by Kiefer (1975). If interest focuses on the treatment means $\mu + \tau_i$ in the simpler model $\mu_k = \mu$ for all k, (iv) is relaxed to equireplication in the s toruses combined. In general, equireplication in each of the toruses is sufficient for (iv) to hold. In the terminology of Martin (1982), (i) and (ii) say the design has second order neighbor balance.

Torus designs are of practical statistical interest mainly in their adaptability to planar applications. In constructing optimal torus designs, it is expected that their planar versions will be near optimal for correlated errors in the plane, though the extent to which this is true depends in part on how the torus error process is adapted to the plane. Martin (1982) discusses two methods. One is to use the equivalent infinite planar process, introducing edge effects that generally make it difficult to attain the complete symmetry of var($\hat{\tau}$) that underlies universal optimality arguments. This is the approach taken in Section 5, where the behavior of planar versions of some of the smaller constructed torus designs (edge effects exert less influence as the design size grows) will be evaluated numerically and seen to be very good. Alternatively, the definition of $\mathrm{var}^{-1}(\varepsilon)$ can be modified so that $\mathrm{var}^{-1}(\varepsilon)_{ij} = 0$ for plots i and j that are neighbors of order 1 or 2 (for the second order process) on the torus but not in the plane, producing a nonstationary planar process, edge sites having variances different from those interior. Gill and Shukla (1985) take this approach, by which it is possible to obtain universally optimum designs for the simpler model mentioned above in reasonably small sets of planar rectangles. The planar components that will be combined in constructing torus designs satisfy their optimality conditions for this case and hence are of some interest in themselves. Further results along this line are given by Uddin and Morgan (1991). Previous results on planar behavior of torus designs may be found in Martin [(1986), pages 272–273].

Regardless of the above considerations, construction of highly efficient and well balanced planar designs is of no importance if the underlying planar correlation model [see (9) of Section 5] cannot adequately describe a variety of real-life phenomena. To some extent the possibilities here are limited since the behavior of optimality criteria can be quite sensitive to the rate at which correlations decay, a point examined at some length by Martin (1986). The second order autonormal model is for long-range correlations and within this class is fairly flexible owing to the two parameter formulation. It includes the process (c_1) and can approximate to second order a variety of other processes such as (c_2) (again see Section 5) of Martin (1986), which are appealing models for field trials. Relevant here too is Martin's [(1986), page 274] conclusion that the reasonable approach for field trials is generalized least squares with correlation specified by a small number of parameters, for which $\hat{\tau}$ will be reasonably robust to the exact form of process chosen.

An $m_1 \times m_2$ torus lattice becomes an $m_1 \times m_2$ planar lattice by cutting the lattice between any two rows and any two columns. If the torus was neighbor balanced of order 2, then the fully bordered version of the planar design is also: Border the first and last rows (columns) by the last and first rows (columns), respectively. Bordered neighbor designs have applications in situations other than that considered here, for instance, in polycross experimentation [see Freeman (1979)], and thus give a separate justification for the study of torus designs. Constructions for fully bordered neighbor designs have been previously given by Freeman (1979), Afsarinejad and Seeger (1988) and Morgan (1990); there seem to be no results for the construction of these designs that, like those given here, balance neighbors to second order and have no like neighbors to second order, which are the desired properties for nondirectional polycrosses. Some unbordered designs which approximate these properties may be found in Freeman (1969).

In summary, infinite families of optimal torus designs are constructed, then demonstrated numerically to have excellent efficiency and balance properties in planar applications. At no point is it proven that planar versions of optimal torus designs are themselves optimal (indeed this is an open and challenging problem), though the calculations indicate that in many cases any further efficiency gain over the designs presented here will of necessity be quite small. Thus the current work provides a variety of two-dimensional plans for experimentation in the presence of long-term correlations that are statistically satisfactory, being to the authors' knowledge the first design work of this type. Section 2 gives the basic construction results and concentrates on optimum designs for prime power numbers of treatments of the form $4t + 1$. In Section 3, two generalizations of the Section 2 results are given, and in Section 4 conditions are relaxed to obtain deigns for nonprime powers. Numerical comparisons for planar versions of the torus designs are made in Section 5. All of the constructions employ the method of differences.

2. Designs with second order neighbor balance. The first result shows how differences can be used to produce a set of $p \times p$ squares with the desired neighbor properties. To set the situation, let G be an abelian group of order v with identity element 0, where v is the number of treatments. For $a = (a_1, a_2, \dots, a_p)$ any p-vector of elements of G, let $a^* = (a_1 - a_2, a_2 - a_3, \dots, a_{p-1} - a_p)$ be the vector of forward neighbor differences. For two such vectors a and b, define $R(a, b)$ as $p \times p$ array with i, j entry $a_i + b_j$.

THEOREM 1. *Suppose there exist p-vectors a and b on an abelian group G such that*

(i) $\pm a^* \cap \pm b^* = \varnothing$,
(ii) $\pm a^* \cup \pm b^*$ *is each nonzero group element* $4(p-1)/(v-1)$ *times*,
(iii) $R(\pm a^*, \pm b^*)$ *is each nonzero group element* $4(p-1)^2/(v-1)$ *times*.

Then $R(a, b)$ *is a balanced neighbor difference array, so that the* v *arrays* $R(a, b) + g$, $g \in G$, *are together balanced for combined first row and column* *neighbors and for first diagonal neighbors. Furthermore, no treatment neighbors itself.*

PROOF. When using the union or intersection notation with vectors of differences, they are treated as lists or sets. So the combined row and column neighbor differences are p copies of $\pm a^* \cup \pm b^*$, which by (ii) are balanced. The diagonal neighbor differences are $R(\pm a^*, \pm b^*)$, which by (iii) are balanced. By (i)–(iii), none of the differences are zero. □

Note that the Theorem 1 designs have nondirectional neighbor balance, and that balance is for rows and columns combined and for diagonal directions combined. Hence they would be appropriate for roughly square plots in which the error process is assumed [in the terminology of Martin (1982)] completely symmetric. Neighbors in each of rows, columns and the two diagonal directions can be balanced by using another v arrays given by a 90° rotation of the v arrays of the theorem. Rotating each of the resulting $2v$ arrays 180° gives a directional design, balanced for neighbors in each of the eight directions. So with additional replication, symmetric and reflection symmetric processes can also be handled, but will not be further pursued here. Directionality is also an issue in the polycross application [Freeman (1979)].

The initial problem here is in finding the vectors a^* and b^*, from which a and b can be reconstructed (though if one wished to consider models with blocking factors, consideration of neighbor differences alone would not be sufficient). It is clear from (ii) of Theorem 1 that $4|(v-1)$. When discussing a^*, b^*, there is no loss of generality in taking $p = (v+3)/4$; larger p will give multiples of these two lists. [In fact, a^* and b^* need not even be of the same size for the torus constructions to follow: each must just be an integer multiple of $(v-1)/4$. Considerations of design size will usually make this of little interest, but compare Example 2b below.] Hence the problem becomes: partition the nonzero elements of G into equal-sized subsets S_1, S_2 satisfying

$$g \in S_i \Rightarrow -g \in S_i, \tag{1}$$

$$\cup S_i = G - 0, \tag{2}$$

(3) $R(S_1, S_2)$ contains each nonzero group element $(v-1)/4$ times.

Perhaps surprising is that these conditions are related to those needed for generation of a balanced incomplete block design, from which a simple solution follows.

THEOREM 2. *Let S_1 and S_2 be sets satisfying* (1) *and* (2). *They satisfy* (3) *if and only if they are initial blocks for a BIBD with $2v$ blocks of size $(v-1)/2$.*

PROOF. The group table for G, after deletion of the zero row and column, can be broken into four subtables: $R(S_1, S_1)$, $R(S_1, S_2)$, $R(S_2, S_1)$ and $R(S_2, S_2)$. Then (3) is satisfied iff $R(S_1, S_1)$ and $R(S_2, S_2)$ together have each nonzero group element with equal frequency. By (1), $S_i = S_{i1} \cup S_{i2}$, where $S_{i2} = -S_{i1}$, so that $R(S_i, S_i)$ can be broken into the four subtables $R(-S_{i1}, S_{i2})$, $R(S_{i1}, -S_{i2})$, $R(S_{i1}, -S_{i1})$ and $R(S_{i2}, -S_{i2})$. Deleting the $(v-1)/4$ zeros in each of the last two tables leaves all the symmetric differences for the set S_i. Hence S_1 and S_2 satisfy (3) if and only if all the symmetric differences within S_1 and S_2 are together each nonzero group element $(v-3)/2$ times and the theorem is proved. □

Corollary 1. *Let $v = 4t+1$ be a prime power. Then sets S_1 and S_2 satisfying* (1)–(3) *exist on the Galois field GF_v of order v.*

Proof. Let S_1 be the set of quadratic residues on GF_v and S_2 the quadratic nonresidues. It is well known that S_1 and S_2 generate the required BIBD [e.g., Raghavarao, (1971), page 84]. The result follows since -1 is quadratic. □

Now a and b of minimal size can be constructed by

$$(4) \qquad a = (1, x^2, x^4, \dots, x^{(v-1)/2}) \quad \text{and} \quad b = (x, x^3, x^5, \dots, x^{(v+1)/2}),$$

where x is any primitive element of GF_v. Hence:

Corollary 2. *Let $v = 4t+1$ be a prime power. Then there exist v $(v+3)/4 \times (v+3)/4$ squares which are balanced for first row and column neighbors combined, are balanced for first diagonal neighbors and have no like first neighbors.*

Since none of the designs in this paper have like first neighbors, mention of this fact will be omitted in succeeding results.

Example 1. The Corollary 2 design for $v = 9$. The nonzero field elements may be written as $x = (1,0)$, $x^2 = (2,1)$, $x^3 = (2,2)$, $x^4 = (0,2)$, $x^5 = (2,0)$, $x^6 = (1,2)$, $x^7 = (1,1)$ and $x^8 = (0,1)$. Adding mod(3, 3), then writing i for x^i, gives these nine squares:

$$R = R(a,b) = \begin{matrix} 7 & 5 & 2 \\ 8 & 1 & 7 \\ 6 & 2 & 3 \end{matrix} \qquad R + (1,0) = \begin{matrix} 2 & 0 & 8 \\ 7 & 5 & 2 \\ 3 & 8 & 4 \end{matrix} \qquad R + (2,0) = \begin{matrix} 8 & 1 & 7 \\ 2 & 0 & 8 \\ 4 & 7 & 6 \end{matrix}$$

$$R + (0,1) = \begin{matrix} 6 & 2 & 3 \\ 4 & 7 & 6 \\ 1 & 3 & 5 \end{matrix} \qquad R + (1,1) = \begin{matrix} 3 & 8 & 4 \\ 6 & 2 & 3 \\ 5 & 4 & 0 \end{matrix} \qquad R + (2,1) = \begin{matrix} 4 & 7 & 6 \\ 3 & 8 & 4 \\ 0 & 6 & 1 \end{matrix}$$

$$R + (0,2) = \begin{matrix} 1 & 3 & 5 \\ 0 & 6 & 1 \\ 7 & 5 & 2 \end{matrix} \qquad R + (1,2) = \begin{matrix} 5 & 4 & 0 \\ 1 & 3 & 5 \\ 2 & 0 & 8 \end{matrix} \qquad R + (2,2) = \begin{matrix} 0 & 6 & 1 \\ 5 & 4 & 0 \\ 8 & 1 & 7 \end{matrix}.$$

Note that one need not use a and b as given in (4). They could be changed, for instance, to impart conventional block properties (not required by our model) to the $R(a,b)$'s. We simply remark that there is considerable scope here for imposing further conditions on a and b depending on the situation at hand.

Gill and Shukla (1985) consider the problem of optimal design in the presence of second order autonormal errors in the plane (their Model 2, which is the nonstationary planar version of the process considered in Section 1), but find no designs. Corollary 2 designs satisfy their conditions, derived for the simpler model $E(y_{ijk}) = \tau_{[ijk]}$. For the same error process and $E(y_{ijk}) = \mu_k + \tau_{[ijk]}$, Uddin and Morgan (1991) are able to choose a and b of Theorem 1 with $p = v$ and satisfying further conditions to generate efficient sets of Latin squares. The only other result of this nature of which we are aware is due to Ipinyomi and Freeman (1988). Their Lemma 3, for odd prime v but easily extended to odd prime powers, gives sets of $v(v-1)/2$ rectangles balanced for neighbors of all orders.

Theorem 1 is the underlying result for all the constructions in this section. We do not know if its conditions can be met for nonprime powers; enumeration shows that it is not possible for $v = 21$ or 33. In Section 4, relaxed versions of Theorem 1 will be used with the merging techniques explained next to obtain designs for other numbers of treatments.

In order to construct designs in single arrays, methods of adjoining the v arrays of a Theorem 1 design are now considered, leading to torus designs and what will be called pseudotorus designs. Suppose R_i and R_j are two $p \times p$ components generated from $R(a, b)$ such that the last column of R_i is the first column of R_j. If a new $p \times (2p-1)$ array is found by merging R_i and R_j at this common end column, one copy of this repeated set of column neighbors is lost, while the diagonal and row neighbors are unaffected. The technique, which is to adjoin all v developed $R(a, b)$'s in this fashion so that a balanced set of neighbors is lost, is first illustrated by Example 2.

EXAMPLE 2a. On $Z_5 \cong GF_5$, take $a = (0, 1, 2, 1, 0)$ and $b = (0, 2, 4, 1, 3)$, where in accordance with Corollary 1, $S_1 = \{1, 4\}$ and $S_2 = \{2, 3\}$. Developing $R(a, b)$ (mod 5) gives

$$\begin{array}{ccc} R_1 & R_2 & R_3 \\ \begin{array}{ccccc} 0&2&4&1&3\\ 1&3&0&2&4\\ 2&4&1&3&0\\ 1&3&0&2&4\\ 0&2&4&1&3 \end{array} & \begin{array}{ccccc} 3&0&2&4&1\\ 4&1&3&0&2\\ 0&2&4&1&3\\ 4&1&3&0&2\\ 3&0&2&4&1 \end{array} & \begin{array}{ccccc} 1&3&0&2&4\\ 2&4&1&3&0\\ 3&0&2&4&1\\ 2&4&1&3&0\\ 1&3&0&2&4 \end{array} \end{array}$$

$$\begin{array}{cc} R_4 & R_5 \\ \begin{array}{ccccc} 4&1&3&0&2\\ 0&2&4&1&3\\ 1&3&0&2&4\\ 0&2&4&1&3\\ 4&1&3&0&2 \end{array} & \begin{array}{ccccc} 2&4&1&3&0\\ 3&0&2&4&1\\ 4&1&3&0&2\\ 3&0&2&4&1\\ 2&4&1&3&0 \end{array} \end{array}.$$

Note two important facts about this set of neighbor balanced squares: (i) they are ordered so that the last column of one square is identical to the first column of the next and (ii) the first and last rows coincide. Using (i), merge the five squares at their common end columns as explained above, giving a cylindrical 5×20 array (R_5 is merged with R_1). This upsets the neighbor balance by deleting one copy of the set of column neighbors generated from a. To correct this, delete one copy of the set of row neighbors generated from b by using (ii): merge the common first and last rows. A planar representation of the resulting 4×20 neighbor balanced torus design, with placement of the R_i's indicated, is

		R_1								R_3								R_5	
0	2	4	1	3	0	2	4	1	3	0	2	4	1	3	0	2	4	1	3
1	3	0	2	4	1	3	0	2	4	1	3	0	2	4	1	3	0	2	4
2	4	1	3	0	2	4	1	3	0	2	4	1	3	0	2	4	1	3	0
1	3	0	2	4	1	3	0	2	4	1	3	0	2	4	1	3	0	2	4
R_5							R_2								R_4				.

The R_i's of Example 2a could be placed in a *mergeable order* because $b_p - b_1 = 3$ generates Z_5 (see Corollary 3 below). If $a_p - a_1 \neq 0$, the first and last rows of the intermediate cylinder design will not coincide, but an obvious consequence of the technique is that the last row will always be a cyclic shift of the first. So by a suitable twisting of the cylinder, the first and last rows can be merged, yielding a *pseudotorus* design.

EXAMPLE 2b. Still on Z_5, take $a = (0,1,2,3,4)$ and $b = (0,3)$. The developed $R(a, b)$'s are

$$
\begin{array}{cc} 0 & 3 \\ 1 & 4 \\ 2 & 0 \\ 3 & 1 \\ 4 & 2 \end{array}
\quad
\begin{array}{cc} 3 & 1 \\ 4 & 2 \\ 0 & 3 \\ 1 & 4 \\ 2 & 0 \end{array}
\quad
\begin{array}{cc} 1 & 4 \\ 2 & 0 \\ 3 & 1 \\ 4 & 2 \\ 0 & 3 \end{array}
\quad
\begin{array}{cc} 4 & 2 \\ 0 & 3 \\ 1 & 4 \\ 2 & 0 \\ 3 & 1 \end{array}
\quad
\begin{array}{cc} 2 & 0 \\ 3 & 1 \\ 4 & 2 \\ 0 & 3 \\ 1 & 4 \end{array}
$$

which merge to give the 4×5 pseudotorus Youden design

$$
\begin{array}{c|ccccc|c}
1 & 4 & 2 & 0 & 3 & 1 & 4 \\
\hline
2 & 0 & 3 & 1 & 4 & 2 & 0 \\
3 & 1 & 4 & 2 & 0 & 3 & 1 \\
4 & 2 & 0 & 3 & 1 & 4 & 2 \\
0 & 3 & 1 & 4 & 2 & 0 & 3 \\
\hline
1 & 4 & 2 & 0 & 3 & 1 & 4
\end{array}
$$

where, while the north neighbor of the (4, 5) element 0 is 4, its south neighbor is 1 (not 2). Borders have been included to clarify the neighbor relationships.

Example 2b also illustrates that a and b need not have the same number of elements to construct a neighbor balanced (pseudo) torus design. Applying the merging and twisting techniques demonstrated by Example 2 to the results of the previous corollaries gives:

COROLLARY 3. *Let $v = 4t + 1$ be a prime. Then there is a $(v - 1)/4 \times v(v - 1)/4$ torus or pseudotorus design balanced for first row and column neighbors combined and balanced for first diagonal neighbors.*

PROOF. Take b as in (4), and a as any $(v+3)/4$ vector with the same set of neighbor differences as a of (4). Since $w_2 = b_{(v+3)/4} - b_1 = x^{(v+1)/2} - x = -2x \neq 0$, it generates $\mathscr{G}_v$, so a mergeable ordering of the developed $R(a, b)$'s is $R_1, R_2, \dots, R_v$, where $R_i = R(a, b) + (i-1)w_2$. The merged components yield a torus or pseudotorus design as $w_1 = a_{(v+3)/4} - a_1$ does or does not equal 0. □

The torus designs of Corollary 3 satisfy conditions (i)–(iv) of Section 1, and so are universally optimum for the second order completely symmetric torus lattice process with $\alpha > 0$ and $\gamma > 0$. If the designs are to be used in the plane, the same neighbors are lost whether a torus or a pseudotorus design is used and the distinction is unimportant. If it is desired to preserve neighbor balance in the plane by bordering, this too can be done for either case. Corollary 3 and all succeeding torus and pseudotorus results could be equivalently stated in terms of fully bordered planar designs; the torus approach is taken for optimality arguments and for the cohesion and simplicity afforded the constructions.

As with the planar components, the torus neighbor balance is such that if two treatments appear as neighbors in columns (rows), they never do in rows (columns). Should the error process lack the assumed row-column symmetry, that is, should it be simply reflection symmetric rather than the assumed completely symmetric, then universal optimality is lost. By replicating the proposed designs with row and column neighbors reversed (multiply the design by x) this can be accommodated and optimality regained. For the designs as given, the efficiency calculations of Appendix 1 indicate that mild departures from row-column symmetry are well tolerated. In particular it is shown there that in the reflection symmetric setting, the proposed designs are two-class partially balanced with A-efficiency no less than $(v-1)/v$.

When $v = q^n$ for prime q and $n > 1$, GF_v does not have an additive generator. In this case the $R(a, b)$'s, upon proper ordering, can be merged in two directions. For instance, the nine components of Example 1 have been set out in a 3×3 array so that the end rows as well as end columns match. Merging these common ends gives the Example 3 design.

EXAMPLE 3. A 6×6 knight's move torus design with second order neighbor balance. Rows and columns together are a PBIB(2) of Latin square type.

$$\begin{array}{cccccc} 7 & 5 & 2 & 0 & 8 & 1 \\ 8 & 1 & 7 & 5 & 2 & 0 \\ 6 & 2 & 3 & 8 & 4 & 7 \\ 4 & 7 & 6 & 2 & 3 & 8 \\ 1 & 3 & 5 & 4 & 0 & 6 \\ 0 & 6 & 1 & 3 & 5 & 4 \end{array}$$

Corollary 4 gives one of the possible series of this type. In terms of differences, the key requirement is that w_1 and w_2, the differences in the end elements of a and b, together generate the additive group.

COROLLARY 4. *Let $v = 4t + 1$ be a squared prime. Then there exists a $v^{1/2}(v - 1)/4 \times v^{1/2}(v - 1)/4$ torus design balanced for first row and column neighbors combined and balanced for first diagonal neighbors.*

PROOF. Take a and b as given in (4) and calculate $w_1 = a_{(v+3)/4} - a_1 = -2$ and $w_2 = b_{(v+3)/4} - b_1 = -2x$. A mergeable $\sqrt{v} \times \sqrt{v}$ arrangement of the v $R(a,b)$'s is given by $R_{ij} = R(a,b) + (i-1)w_1 + (j-1)w_2$. □

The final result of this section is for higher powers of primes. In fact, Corollary 4 is just a special case of this result.

COROLLARY 5. *Let $v = 4t + 1 = q^n$ be a power of the odd prime q, where $n \geq 2$. Then there exist q^{n-2} $q(v-1)/4 \times q(v-1)/4$ toruses together balanced for first row and column neighbors combined and for first diagonal neighbors.*

PROOF. Let a and b be as in (4) for the additive group G of GF_v and form the $q(v-1)/4 \times q(v-1)/4$ torus T be adjoining q^2 of the v Theorem 1 components as in the proof of Corollary 4. Let $A = \{c_1 + c_2 x\colon c_1, c_2 \in Z_q\}$ be the additive subgroup generated by w_1 and w_2. The q^{n-2} toruses are $T + y$, where y takes on one value in each of the q^{n-2} cosets of G/A. □

The individual toruses of Corollary 5 are not generally equireplicate when $n > 2$.

3. Two generalizations. Designs for v not necessarily of the form $4t + 1$ can be obtained by partitioning the nonzero elements of G into more than two subsets. Let $v = 2tm + 1$ and suppose there exist subsets $S_1, S_2, \dots, S_m$ of G, $|S_i| = 2t$, satisfying (1), (2) and

(5) $R(S_i, S_{i+1})$ for $i = 1, 2, \dots, m$ are each nonzero elements of G exactly $2t$ times,

where S_{m+1} is written for S_1. Then construct m $(t+1)$ vectors $a_1, a_2, \dots, a_m$ such that $\pm a_i^* = S_i$. The previous results can then be generalized by applying the techniques of Theorem 1 and Corollaries 2–5, successively taking a, b as each pair a_i, a_{i+1} ($a_{m+1} = a_1$).

THEOREM 3. *Let $v = 2tm + 1$ be a prime power. Sets $S_1, S_2, \dots, S_m$ satisfying* (1), (2) *and* (5) *exist on $G = GF_v$.*

PROOF. Let x be a primitive element of G and

$$S_1 = \{1, x^m, x^{2m}, \dots, x^{(2t-1)m}\} \quad \text{and} \quad S_i = x^{i-1} S_1.$$

S_1 is closed under multiplication and $x^{tm} = -1$ is in S_1, so $g \in S_i \Rightarrow -g \in S_i$. By inspection of $R(S_1, S_2)$, one can see that its entries are those of $R_1 = (x^0, x^m, x^{2m}, \dots, x^{(2t-1)m}) \otimes (1 + x, 1 + x^{m+1}, 1 + x^{2m+1}, \dots, 1 + x^{(2t-1)m+1})$.

Defining $R_i = x^{i-1}R_1$, it follows easily that R_i contains the entries of $R(S_i, S_{i+1})$ and $R_1, R_2, \ldots, R_m$ collectively contain the elements of $(x^0, x^1, x^2, \ldots, x^{2tm-1}) \otimes (1 + x, 1 + x^{m+1},\ 1 + x^{2m+1}, \ldots, 1 + x^{(2t-1)m+1})$, that is, each nonzero element of G $2t$ times. □

Let h be given by $(1 - x^m)^{-1} = x^h$. Then with $a_1 = (x^h, x^{h+m}, \ldots, x^{h+tm})$, $a_i = x^{(i-1)}a_1$ satisfies $\pm a_i^* = S_i$ of Theorem 3. Combining these results yields the following corollaries.

COROLLARY 6. *Let $v = 2tm + 1$ be a prime power, $m > 1$. There exist vm $(t + 1) \times (t + 1)$ squares that are together balanced for combined first row and column neighbors, and for first diagonal neighbors.*

PROOF. Develop the m arrays $R(a_i, a_{i+1})$, $i = 1, 2, \ldots, m$. □

COROLLARY 7. *Let $v = 2tm + 1$, $m > 1$ be prime. Then there exist m $t \times vt$ toruses that are together balanced for combined first row and column neighbors and for first diagonal neighbors.*

PROOF. For each i, connect the v arrays developed from $R(a_i, a_{i+1})$ by the method of Corollary 3. □

COROLLARY 8. *Let $v = 2tm + 1 = q^n$ be a power of the odd prime q, where $m, n > 1$. Then there exist mq^{n-2} $qt \times qt$ toruses that are together balanced for combined first row and column neighbors and for first diagonal neighbors.*

PROOF. q^{n-2} $qt \times qt$ arrays will arise from each $R(a_i, a_{i+1})$ by applying the technique in the proof of Corollary 5. □

The number of separate arrays in Corollaries 6–8 can be halved when $m = 2$ since only one of the two pairs S_1, S_2 and S_2, S_1 need be used, so that Corollaries 2–5 have been generalized. More generally, the number of arrays can be halved if the number of times each S_i is used can be reduced from 2 to 1. This requires that the number of S_i's be even, so write $v = 2tm' + 1 = 4tm + 1$, where $m' = 2m$ is even. The general problem is to find subsets $S_1, S_2, \ldots, S_{2m}$ of G, $|S_i| = 2t$, satisfying (1), (2) and

$$R(S_{2i-1}, S_{2i}) \text{ for } i = 1, 2, \ldots, m \text{ are each nonzero elements of } G \text{ exactly } t \text{ times.} \tag{6}$$

Then apply the Section 2 methods to m pairs a_i, b_i such that $\pm a_i^* = S_{2i-1}$ and $\pm b_i^* = S_{2i}$, $i = 1, 2, \ldots, m$. Under certain conditions the sets of Theorem 3 can be so partitioned.

TABLE 1
r values satisfying (7)

v	t	m	r	Primitive root or polynomial
17	2	2	nonexist.	3
25	2	3	1, 2, 3	$x^2 + x + 2$
	3	2	nonexist.	
37	3	3	2	2
41	2	5	3	11
	5	2	nonexist.	
49	6	2	1, 2	$x^2 + 2x + 5$
	4	3	2	
	3	4	1, 2, 3, 4	
	2	6	2, 5	

THEOREM 4. *For $v = 4tm + 1 = 2tm' + 1$ a prime power, the sets $S_1, S_2, \dots, S_{2m}$ of Theorem 3, in some order, satisfy* (6) *if for some integer r,*

$$(7)\qquad \{1 \pm x^{2r-1}, 1 \pm x^{2m+2r-1}, 1 \pm x^{4m+2r-1}, \dots, 1 \pm x^{2(t-1)m+2r-1}\}$$

is composed of t quadratic residues and t nonresidues.

PROOF. Given an integer r satisfying (7), reorder the sets of Theorem 3 so that $S_1 = \{1, x^{2m}, x^{4m}, \dots, x^{2(2t-1)m}\}$, $S_2 = x^{2r-1}S_1$ and $S_{2i-1} = x^{2(i-1)}S_1$, $S_{2i} = x^{2(i-1)}S_2$ for $i = 1, 2, \dots, m$. Using the relation $x^{2tm} = -1$, the entries of $R(S_1, S_2)$ can be written as $(x^0, x^{2m}, x^{4m}, \dots, x^{2(2t-1)m}) \otimes (1 \pm x^{2r-1}, 1 \pm x^{2m+2r-1}, \dots, 1 \pm x^{2(t-1)m+2r-1})$. $R(S_{2i-1}, S_{2i}) = x^{2(i-1)}R(S_1, S_2)$ implies that the m tables together contain $(x^0, x^2, x^4, \dots, x^{4tm-2}) \otimes (1 \pm x^{2r-1}, 1 \pm x^{2m+2r-1}, 1 \pm x^{4m+2r-1}, \dots, 1 \pm x^{2(t-1)m+2r-1})$. The left-hand vector containing all the quadratic residues gives the result. □

Now write $x^h = (1 - x^{2m})^{-1}$. Then for $a_1 = (x^h, x^{h+2m}, \dots, x^{h+2tm})$ and $b_1 = x^{2r-1}a_1$, $a_i = x^{2(i-1)}a_1$ and $b_i = x^{2(i-1)}b_1$ satisfy $\pm a_i^* = S_{2i-1}$ and $\pm b_i^* = S_{2i}$ in the proof of Theorem 4. Hence the number of arrays in Corollaries 6–8 can be halved whenever (7) holds, which is not always the case (see Table 1). The following are immediate.

COROLLARY 9. *Let $v = 4tm + 1$ be a prime power and suppose* (7) *holds. Then there exist vm $(t + 1) \times (t + 1)$ squares that are together balanced for combined first row and column neighbors and for first diagonal neighbors.*

COROLLARY 10. *Let $v = 4tm + 1$ be prime and suppose* (7) *holds. Then there exist m $t \times vt$ toruses that are together balanced for combined first row and column neighbors and for first diagonal neighbors.*

A small complication arises when connecting the developed $R(a_i, b_i)$ in two directions. The result for even powers of primes is:

COROLLARY 11. *Let $v = 4tm + 1 = q^n$ be a power of the odd prime q, where $n > 1$ is even, and suppose* (7) *holds. Then there exists mq^{n-2} $qt \times qt$ toruses that are together balanced for combined first row and column neighbors and for first diagonal neighbors.*

PROOF. It is sufficient to show that the method of Corollary 5 can be used to form q^{n-2} $qt \times qt$ toruses from the developed $R(a_1, b_1)$'s. Hence it must be shown that $w_1 = x^{h+2tm} - x^h = -2x^h$ and $w_2 = x^{2r-1}w_1$ generate distinct additive subgroups. Now -2 generates the subfield GF_q, which for even n is composed only of quadratic residues and 0, so h and $h + 2r - 1$ having different parities establishes the result. □

The elements of GF_q in GF_{q^n} are powers of x^u, where $u = (q^n - 1)/(q - 1)$. So the proof of Corollary 11 holds for odd $n > 1$ if and only if $2r - 1$ is not an odd multiple of u. Of course the a_i's and b_i's given in the proof are only one of many possibilities and it appears that the corollary will hold for all odd $n > 1$ as well.

Again, it follows from Section 2 that (7) always holds when $m = 1$. Next it will be shown that (7) holds when $t = 1$, that is, that $1 \pm x^{2r-1}$ is one quadratic residue and one nonresidue for some integer r. Multiplying by $y = x^{1-2r}$ this becomes, equivalently, $y \pm 1$ is one quadratic residue and one quadratic nonresidue for some quadratic nonresidue y. The following lemma is proven in Appendix 2.

LEMMA 1. *Let $v = 4m + 1 = q^n$ be a power of the odd prime q. There exists at least one quadratic nonresidue $y \in GF_v$ such that one of $\{y - 1, y + 1\}$ is a quadratic residue and the other a quadratic nonresidue. If $n > 1$, such a y may be chosen so that it is not in the subfield GF_q.*

COROLLARY 12. *Let $v = 4m + 1 = q^n$, where q is an odd prime and $n > 1$. Then there exists $q^{n-2}(v - 1)/4$ $q \times q$ toruses that are together balanced for combined first row and column neighbors and for first diagonal neighbors.*

PROOF. This is just Corollary 11 with $t = 1$ and $x^{2r-1} = y^{-1}$, where y is given by Lemma 1. That y, and hence y^{-1}, is not in the subfield GF_q, removes the Corollary 11 restriction that n be even. □

Note that these designs have only $(v - 1)/4$ replicates.

Proper choice of y for these designs can often give balance of neighbors at higher orders. This property is noted in Example 4, but not explored further here, as higher order torus neighbors are more radically affected by a planar conversion and cannot be preserved by a simple bordering. It may also be shown that the Corollary 12 designs are balanced lattice squares when $n = 2$ and balanced incomplete block designs with nested rows and columns [Singh and Dey (1979)] when n is odd; the similar result for even $n > 2$ requires further restrictions on y.

EXAMPLE 4. A balanced lattice square for 25 treatments that is also balanced for torus neighbors of all orders. This is the Corollary 12 design with $y^{-1} = x^{2r-1} = x$, using the primitive polynomial $x^2 + x + 2$. Again i is written for x^i.

11	16	15	18	2	13	18	17	20	4	15	20	19	22	6
4	23	14	6	3	6	1	16	8	5	8	3	18	10	7
8	21	17	24	22	10	23	19	2	24	12	1	21	4	2
19	7	1	0	13	21	9	3	0	15	23	11	5	0	17
9	20	10	12	5	11	22	12	14	7	13	24	14	16	9

17	22	21	24	8	19	24	23	2	10	21	2	1	4	12
10	5	20	12	9	12	7	22	14	11	14	9	24	16	13
14	3	23	6	4	16	5	1	8	6	18	7	3	10	8
1	13	7	0	19	3	15	9	0	21	5	17	11	0	23
15	2	16	18	11	17	4	18	20	13	19	6	20	22	15

In closing this section it should be pointed out that the approach of (6) is indeed distinct from that of (5) and not simply a technique for halving those designs. To see this, put $v = 25$, $t = 3$ and $m = 2$, for which (7) is not satisfied (Table 1). The following four sets satisfy (1), (2) and (6): $S_1 = \{1, x^4, x^8, x^{12}, x^{16}, x^{20}\}$, $S_2 = x^2 S_1$, $S_3 = xS_1$ and $S_4 = x^3 S_1$. But it can be checked that $R(S_2, S_3)$ and $R(S_1, S_4)$ do not combine to satisfy (5). With these sets a design in two 15×15 toruses can be obtained.

4. Designs for nonprime power v. In this section designs will be constructed using cyclic groups for small v. In most cases the perfect balance of Section 2 will not be attained: the approach here is to keep the range in neighbor counts small, still allowing no like neighbors in rows, columns or diagonals [a general prescription for high efficiency for long-range correlations; see Martin (1986)]. The methods used are those arising from (1)–(3), but without demanding that S_1 and S_2 are equal sized subsets and relaxing (3). The effect of the former will depend on the method of merging the component arrays; the latter relaxes the demand of exact diagonal neighbor balance.

Consider first the cast of $v = 4t + 3$. Let S_1 and S_2 be a partition of $Z_v - 0$, $|S_1| = 2t$, $|S_2| = 2t + 2$, satisfying (1), (2) and

$$(8)\qquad \begin{array}{l} R(S_1, S_2) \text{ contains the nonzero elements of } Z_v \\ \text{with frequencies } f_1 < f_2 < \cdots < f_s. \end{array}$$

Let a, b be such that $\pm a^* = S_1$ and $\pm b^* = S_2$ and write $w_1 = a_{t+1} - a_1$ and $w_2 = b_{t+2} - b_1$. Then in the v arrays $\{R(a, b) + g \colon g \in Z_v\}$ of order $(t + 1) \times (t + 2)$ each pair of distinct treatments occurs as first neighbors $t + 1$ or $t + 2$

times in rows and columns combined and $f_1, f_2, \dots,$ or f_s times in diagonals. If b is chosen such that the greatest common divisor of w_2 and v is $(w_2, v) = 1$, then the $R(a, b)$'s can be merged via common end columns into a $(t + 1) \times (4t + 3)(t + 1)$ cylinder design with first neighbors balanced for rows and columns combined, the merging having deleted the excess column neighbors. Alternatively, as a torus or pseudotorus design, the dimensions are $t \times (4t + 3)(t + 1)$, with combined first row/column neighbor counts of t and $t + 1$. In either case the diagonal first neighbors are the same as in the $R(a, b)$'s. If $(w_1, v) = 1$, a $t(4t + 3) \times (t + 1)$ pseudotorus design with the same counts can be obtained; of the three, the cylinder design is to be preferred in planar applications. For merging in two directions analogous to Corollary 4, if w_1 generates a subgroup G_1 of order v_1 and w_2 is such that $G_1 + iw_2$ for $i = 1, 2, \dots, v_2$ $(v = v_1 v_2)$ are the cosets of G_1 in Z_v, then a $v_1 t \times v_2(t + 1)$ torus or pseudotorus design results again with the same neighbor counts. Of course the size of a (b) can be increased so that $\pm a^*$ $(\pm b^*)$ is multiple copies of S_1 (S_2), multiplying the number of rows (columns) of the design; this may be useful for some of the small treatment numbers (see Table 2), but can further spread the row/column neighbor counts depending on the method of adjoining the $R(a, b)$'s.

Having set the conditions on S_1 and S_2, the problem is to choose a partition that minimizes the dispersion in the f_i's of (8), which will be discussed after the other cases for v are covered.

For $v = 4t + 1$ the procedures are the same except that $|S_1| = |S_2| = 2t$. Hence a pseudotorus design will be balanced for combined first row and column neighbors, while the cylinder row/column neighbor counts will be t and $t + 1$.

When v is even the order 2 element requires that a larger a or b be used if the row/column neighbor counts are to be kept reasonably balanced. For $v = 4t$, partition $Z_v - 0$ as S_1, S_2, $|S_1| = 2t - 1$, $|S_2| = 2t$ satisfying (1), (2) and (8). Now find a and b such that $\pm a^* =$ two copies of S_1 and $\pm b^* = S_2$. Then in the $2t \times (t + 1)$ array $R(a, b)$ (i) the symmetric row differences are $2t$ copies of S_2, (ii) the symmetric column differences are $2(t + 1)$ copies of S_1, (iii) the symmetric diagonal differences are $R(\pm a^*, \pm b^*) =$ two copies of $R(S_1, S_2)$. Since each column of $R(a, b)$ gives two copies of S_1, the $2t \times 4t^2$ cylinder designs will be balanced for combined row and column neighbors. Any torus or pseudotorus design will have row/column neighbor counts of $2t - 1$ and $2t$.

For $v = 4t + 2$ sets S_1, S_2 satisfying (1), (2) and (8) with $|S_1| = 2t + 1$, $|S_2| = 2t$ are required. Then with a, b such that $\pm a^* =$ two copies of S_1 and $\pm b^* = S_2$, in the $2(t + 1) \times (t + 1)$ array $R(a, b)$, (iv) the symmetric row differences are $2(t + 1)$ copies of S_2, (v) the symmetric column differences are $2(t + 1)$ copies of S_1, (vi) the symmetric diagonal differences are $R(\pm a^*, \pm b^*) =$ two copies of $R(S_1, S_2)$. So torus or pseudotorus designs will have combined row and column neighbor counts of $2t$ and $2t + 1$; for cylinder designs the counts are $2t + 1$ and $2(t + 1)$, or $2t$ and $2(t + 1)$, as the arrays are adjoined via rows or columns, respectively. Interesting here is that the row

TABLE 2
Partitions of $Z_v - 0$ and possible design sizes

v	S_1	$f_s - f_1$	Single array minimum design sizes*
7	1, 6	1	$2 \times 14\zeta$ 2×7
8	1, 4, 7	1	$4 \times 16\zeta$ 3×16 6×8 12×4 24×2
10	1, 2, 5, 8, 9 or 1, 4, 5, 6, 9	3	5×20 10×10 25×4 50×2
11	1, 3, 8, 10	2	$3 \times 33\zeta$ 2×33 22×3
12	3, 4, 6, 8, 9†	1	$6 \times 36\zeta$ 5×36 10×18 15×12
	3, 5, 6, 7, 9‡	2	20×9 30×6 60×3
14	1, 2, 5, 7, 9, 12, 13	2	7×42 14×21 49×6 98×3
15	1, 2, 5, 10, 13, 14	2	$4 \times 60\zeta$ 3×60 9×20
	1, 3, 4, 11, 12, 14	2	15×12 45×4
16	1, 2, 4, 8, 12, 14, 15	2	$8 \times 64\zeta$ 7×64 14×32
			28×16 56×8 112×4
18	1, 2, 4, 5, 9, 13, 14, 16, 17†	2	9×72 18×36 27×24
	1, 2, 4, 8, 9, 10, 14, 16, 17	2	54×12 81×8 162×4
	1, 2, 6, 7, 9, 11, 12, 16, 17†	2	
19	1, 2, 6, 8, 11, 13, 17, 18	2	$5 \times 95\zeta$ 4×95 76×5
20	1, 2, 3, 7, 10, 13, 17, 18, 19†	2	$10 \times 100\zeta$ 9×100 18×50 36×25
	1, 2, 4, 9, 10, 11, 16, 18, 19‡	3	45×20 90×10 180×5
21	1, 2, 3, 5, 10, 11, 16, 18, 19, 20	2	5×105 15×35
22	1, 2, 3, 5, 10, 11, 12, 17, 19, 20, 21†	2	11×110 22×55
	1, 3, 4, 5, 8, 11, 14, 17, 18, 19, 21†	2	121×10 242×5
	1, 2, 4, 6, 7, 11, 15, 16, 18, 20, 21‡	3	
23	1, 2, 3, 7, 9, 14, 16, 20, 21, 22	3	$6 \times 138\zeta$ 5×138 115×6
24	1, 2, 4, 5, 10, 12, 14, 19, 20, 22, 23†	2	$12 \times 144\zeta$ 11×144 22×72
	1, 2, 3, 7, 10, 12, 14, 17, 21, 22, 23‡	3	33×48 44×36 66×24
			88×18 132×12 264×6

*ζ = cylinder design; all others are toruses.
†w_1 and w_2 must both be even.
‡Does not minimize $\Sigma n_i f_i^2$.

and column neighbor counts will be balanced if $\pm b^*$ is two copies of S_2 and the $R(a, b)$'s are adjoined by rows.

All sets of S_1, S_2 on Z_v giving the smallest value of $\Sigma n_i f_i^2$, where n_i is the number of elements of Z_v occurring with frequency f_i in $R(S_1, S_2)$, are given in Table 2 for $v \le 24$, along with possible design sizes. On the torus for the second order autonormal process this method gives the MS-optimal design within this class. In all of the cases here at least one of the partitions also minimizes $f_s - f_1$, the value of which is listed for each partition.

As an example of the above techniques, take $v = 8$. Then $a = (0, 1, 5, 4)$ satisfies $\pm a^*$ = two copies of S_1 of Table 2 and $b = (0, 3, 1)$ has $\pm b^* = (2, 6, 3, 5) = S_2$. For these two vectors, the developed $R(a, b)$'s joined in two directions give the 6×8 design displayed in Section 5.

Only two sets S_1, S_2 have been chosen here so as to obtain designs in single arrays. It is not, however, always the case that for given S_1 and S_2, a and b can be found so that w_1 and w_2 are of appropriate orders to generate a single array design. When v is even, if the number of odd elements in S_2 is a multiple of 4, then for the constructions given above, both w_1 and w_2 must be even and thus together generate a subgroup of order no greater than $v/2$. In general if the subgroup G_3 generated by w_1 and w_2 is of order v_3, then the initial torus found by merging the $R(a, b)$'s according to w_1 and w_2 may be developed into v/v_3 toruses by addition of elements of distinct cosets of G_3 (this is analogous to Corollary 5). Alternatively a different partition (with larger $\Sigma n_i f_i^2$) could be used; these are also listed (where necessary) in Table 2.

Designs with size marked ζ in Table 2 are cylinder designs; such designs are given only when they result in exact row/column neighbor balance. Two comments concerning variants on the design sizes are worthy of mention. First, any torus design or cylinder design can be divided into sections with all neighbor counts preserved by bordering. For instance, a 4×16 cylinder design for eight treatments could be layed out as two 4×8 side-bordered arrays. Secondly, as mentioned above, larger designs can be obtained by choosing a and/or b such that their differences give replicates of the required sets, though care must be given to the row and column counts if this is done. Hence, for instance, a 7×16 torus design for eight treatments is a possibility.

Given the sets S_1 and S_2, construction depends only on appropriate choice of a and b. For the second order autonormal process on the torus, this amounts to obtaining w_1 and w_2 of the desired orders. Practically speaking for planar applications, one can be guided by the general recommendations of Martin (1986) for long-term correlation structures: Having already insured no like first and second order neighbors, efficient designs should have as few like third order neighbors as possible, and to keep $\operatorname{var}(\hat{\tau}_i - \hat{\tau}_j)$ as constant as possible balance neighbors to as high an order as possible. The designs in this paper will not be efficient for short-term correlations, which require a large number of like diagonal neighbors.

5. Efficiency calculations. In this section numerical comparisons are used to investigate the behavior in the plane of some of the constructed designs. The model is as given in Section 1, but with planar correlations

$$\operatorname{cov}(\varepsilon_{ij}, \varepsilon_{i'j'}) = \frac{1}{4\pi^2} \times \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{\cos(g\theta_1)\cos(h\theta_2)}{1 - 2\alpha\cos(\theta_1) - 2\alpha\cos(\theta_2) - 4\gamma\cos(\theta_1)\cos(\theta_2)}\, d\theta_1\, d\theta_2, \tag{9}$$

where $|i - i'| = g$, $|j - j'| = h$, in this section only $s = 1$ array is considered and $|\alpha| + |\gamma| < \frac{1}{4}$ [see Moran (1973)]. This is the stationary second order autonormal planar process discussed in Section 1, which is different than the nonstationary planar process considered by Gill and Shukla (1985) and Uddin and Morgan (1991).

Let $\rho_{gh} = \rho_{hg}$ be the correlation for plots separated by g rows and h columns. As mentioned in Section 1, manipulation of α and γ affords considerable flexibility in the ρ_{gh}. In the calculations below, $\rho_{10} = 0.1, 0.2, \dots, 0.5$ and $\rho_{11} \cong \rho_{10}^{\sqrt{2}}$. This diagonal correlation ρ_{11} is the same as that of Martin's

(1986) process c_2, for which $\rho_{gh} = \rho_{10}^{\sqrt{g^2+h^2}}$, a reasonable model for field trials with roughly square plots. The values of α and γ actually used, along with the first few correlations, are

α	γ	ρ_{10}	ρ_{11}	ρ_{20}	ρ_{12}
0.0881	0.0192	0.100	0.039	0.012	0.007
0.1485	0.0298	0.200	0.103	0.048	0.034
0.1890	0.0284	0.300	0.181	0.109	0.084
0.21635	0.02084	0.400	0.274	0.194	0.160
0.23422	0.011822	0.500	0.379	0.300	0.263.

The C matrix for estimation of treatment contrasts is

$$C = X'\left(R^{-1} - \frac{1}{1'R^{-1}1}R^{-1}JR^{-1}\right)X,$$

where X is the plot/treatment incidence matrix, R the correlation matrix for e, and J and 1 are a matrix and vector of ones, respectively. Then

$$\operatorname{tr}(C) \le \operatorname{tr}(R^{-1}) - \frac{1'R^{-1}1}{v} + \sum_{u \ne u'}\sum r^{uu'}I(r^{uu'} > 0) = \frac{v-1}{\lambda^*},$$

say $(R^{-1} = (r^{uu'}))$. Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_{v-1}$ be the nonzero eigenvalues of C^-. A universally optimum design arrived at by the method of Kiefer (1975) would have $\lambda_1 = \lambda_2 = \cdots = \lambda_{v-1} = \lambda^*$, providing a standard against which to evaluate the proposed designs; commonly used are the A $(\sum \lambda_i)$, E (λ_{v-1}) and D $(\prod \lambda_i)$ criteria. In addition, $S = \lambda_{v-1}/\lambda_1$ provides a simple measure of the dispersion in the design. This is just the ratio of the largest and smallest variances of estimated treatment contrasts, which is 1 for the hypothetical universally optimum design.

Consider first the design of Example 2b for five varieties. This design may be extended row-wise by successively adding 1 (mod 5) to the last row; the 5×5 thus obtained is design $D5.185$ of Martin (1986) with rows and columns interchanged and is a Latin square. Values of A, E, D and S for the 4×5 and 5×5 each appear in Table 3. Note in particular that all of the A-

TABLE 3
*Two designs for $v = 5$ relative to universal optimality**

ρ_{10}	A	E	D	S
0.1	(0.9998, 0.99998)	(0.990, 0.994)	(0.9995, 0.9999)	(0.971, 0.991)
0.2	(0.999, 0.9999)	(0.984, 0.988)	(0.998, 0.9998)	(0.951, 0.983)
0.3	(0.999, 0.9998)	(0.979, 0.985)	(0.997, 0.999)	(0.936, 0.978)
0.4	(0.998, 0.9995)	(0.974, 0.983)	(0.995, 0.998)	(0.924, 0.977)
0.5	(0.997, 0.999)	(0.968, 0.983)	(0.991, 0.996)	(0.914, 0.978)

*Within the parentheses, the first entry is for 4×5; the second for 5×5.

TABLE 4
6×8 *for* $v = 8$ *relative to universal optimality*

ρ_{10}	*A*	*E*	*D*	*S*
0.1	0.999	0.952	0.996	0.914
0.2	0.997	0.917	0.988	0.857
0.3	0.995	0.891	0.979	0.818
0.4	0.993	0.873	0.970	0.791
0.5	0.990	0.859	0.958	0.772

efficiencies are greater than 99% and that the loss is small even in terms of S. Similar values are obtained for the 6×5, 7×5 and so on.

Regarding the 5×5 square relative to his process (c_2), which is close to the process considered here, Martin (1986) remarks that it "... has almost perfect second order balance. This was the optimal design found, and it seems unlikely that any better design exists." It is seen that the reason for the near second order balance is that this degree of balance is achieved on the torus and that this design is a member of a family of torus designs for five treatments that exhibit high efficiency and balance.

Examined next is a 6×8 design for eight varieties based on the Section 4 approach (Table 4). The design, constructed using $a = (0, 1, 5, 4)$ and $b = (0, 3, 1)$, is

$$\begin{array}{cccccccc} 0 & 3 & 1 & 4 & 2 & 5 & 3 & 6 \\ 1 & 4 & 2 & 5 & 3 & 6 & 4 & 7 \\ 5 & 0 & 6 & 1 & 7 & 2 & 0 & 3 \\ 4 & 7 & 5 & 0 & 6 & 1 & 7 & 2 \\ 5 & 0 & 6 & 1 & 7 & 2 & 0 & 3 \\ 1 & 4 & 2 & 5 & 3 & 6 & 4 & 7. \end{array}$$

As compared to the designs for five varieties, the behavior here is less satisfactory, reflecting the poorer approximation to neighbor balance. The A-efficiencies still exceed 99% however.

To better see the effect of controlling neighbors, consider this design due to Preece (1976):

$$\begin{array}{cccccc} 4 & 8 & 2 & 1 & 6 & 9 \\ 3 & 7 & 9 & 5 & 2 & 4 \\ 6 & 5 & 1 & 8 & 3 & 7 \\ 2 & 4 & 3 & 7 & 8 & 6 \\ 1 & 9 & 8 & 4 & 5 & 3 \\ 5 & 6 & 7 & 2 & 9 & 1. \end{array}$$

This is a generalized Youden design with the additional property that each 3×3 corner is a complete replicate, so that like varieties are very well separated. There is, however, a single like diagonal neighbor pair. The design of Example 3 is compared to this design (first value) and to the hypothetical universally optimum design (second value) in Table 5. Gains in A-efficiency for

TABLE 5
Comparison of 6×6 *designs for* $v = 9$

ρ_{10}	*A*	*E*	*D*	*S*
0.1	(1.004, 0.999)	(1.095, 0.963)	(1.021, 0.996)	(1.144, 0.931)
0.2	(1.010, 0.997)	(1.166, 0.933)	(1.047, 0.988)	(1.251, 0.880)
0.3	(1.016, 0.995)	(1.212, 0.908)	(1.068, 0.979)	(1.321, 0.838)
0.4	(1.020, 0.993)	(1.236, 0.888)	(1.081, 0.970)	(1.363, 0.805)
0.5	(1.022, 0.991)	(1.248, 0.872)	(1.090. 0.961)	(1.385, 0.782)

the (torus) neighbor-balanced design are small, but are more substantial with respect to the other criteria.

The planar designs so far examined have been obtained by separating torus designs between two rows and two columns and it should be noted that where this is done can affect the planar behavior. It has also been discussed how the balanced neighbor counts of the torus can be maintained in the plane by bordering; in some cases it may be desirable to increase the size of an unbordered planar design by addition of one of these potential borders as an actual row or column of the design. This is especially relevant to Corollary 3 designs, which suffer the greatest departure from neighbor balance in their planar versions because of the repeated set of neighbors lost by the separation of two rows. Adding, say, the row that would serve as the north border of a $(v-1)/4 \times v(v-1)/4$ unbordered design, gives a $(v+3)/4 \times v(v-1)/4$ unbordered design with the property that each pair of neighbors occurs $(v+3)/4$ or $(v-1)/4$ times in rows and columns combined and $(v-1)/4$ or $(v-5)/4$ times in diagonals. This is the closest approximation to exact neighbor balance achievable in an unbordered planar design of this size.

To illustrate this, Table 6 compares 3×39 and 4×39 designs for $v = 13$. The designs are constructed using $a = (0, 1, 4, 8)$ and $b = (0, 6, 11, 9)$ and the column separation is between the first two columns of $R_1 = R(a, b)$. Both designs perform well, the 4×39 design holding a slight advantage in A-efficiency and, as expected, a somewhat stronger edge in the E and S criteria.

TABLE 6
Two designs for $v = 13$ *relative to hypothetical optimum**

ρ_{10}	*A*	*E*	*D*	*S*
0.1	(0.998, 0.9998)	(0.939, 0.981)	(0.991, 0.999)	(0.894, 0.963)
0.2	(0.995, 0.999)	(0.895, 0.967)	(0.973, 0.996)	(0.823, 0.935)
0.3	(0.993, 0.999)	(0.866, 0.956)	(0.956, 0.993)	(0.776, 0.916)
0.4	(0.990, 0.998)	(0.847, 0.948)	(0.941, 0.988)	(0.745, 0.903)
0.5	(0.988, 0.998)	(0.836, 0.943)	(0.928, 0.983)	(0.726, 0.894)

*Within the parentheses, the first entry is for 3×39; the second for 4×49.

In conclusion, the calculations given here relative to an unattainable bound indicate that optimum torus designs can be excellent planar designs.

APPENDIX 1

Efficiency when the correlations lack row-column symmetry. Here the proposed designs are examined for the torus model $E(y_{ijk}) = \mu_k + \tau_{[ijk]}$ but with errors following the reflection symmetric version of the second order autonormal process, for which the row and column correlations are not equal:

$$\sigma^2 \operatorname{var}^{-1}(\varepsilon) = I_s \otimes \left[I_{m_1 m_2} - \alpha_1 I_{m_1} \otimes C_{m_2} - \alpha_2 C_{m_1} \otimes I_{m_2} - \gamma C_{m_1} \otimes C_{m_2} \right]$$

with $\alpha_1 > 0$, $\alpha_2 > 0$ and $\gamma > 0$, and for positive definiteness $1 - 2\alpha_1 - 2\alpha_2 - 4\gamma > 0$. The conditions for universal optimality are the same as for the completely symmetric model except that neighbor counts must be balanced in each of rows and columns rather than just in rows and columns combined. Let $\lambda_0 = \sum_k r_{k,i}^2$ and $\lambda_1 = \sum_k r_{k,i} r_{k,j}$, where $r_{k,i}$ is the replication count for treatment i in torus k. For s $m_1 \times m_2$ toruses satisfying (i)–(iv) of Section 1 but having disjoint sets of row and column neighbors, the C-matrix of the reduced normal equations has diagonal entries

$$c_{ii} = \frac{s m_1 m_2}{v} - \frac{\lambda_0}{m_1 m_2}(1 - 2\alpha_1 - 2\alpha_2 - 4\gamma)$$

and off-diagonal entries

$$c_{ij} = -\frac{4 s m_1 m_2}{v(v-1)}(\alpha_1 + \gamma) - \frac{\lambda_1}{m_1 m_2}(1 - 2\alpha_1 - 2\alpha_2 - 4\gamma)$$

or

$$c_{ij} = -\frac{4 s m_1 m_2}{v(v-1)}(\alpha_2 + \gamma) - \frac{\lambda_1}{m_1 m_2}(1 - 2\alpha_1 - 2\alpha_2 - 4\gamma).$$

Restricting now to Section 2 designs for prime powers, the two values of c_{ij} occur as the difference in the field elements corresponding to treatments i and j is or is not a quadratic residue. The pattern of this C-matrix is just that of a two-class partially balanced incomplete block design of pseudocyclic type [see Raghavarao (1971)], from which the nonzero eigenvalues are easily derived as $(v-1)/2$ copies of each of $e_1 = e + d$ and $e_2 = e - d$, where

$$d = \frac{2(\alpha_1 - \alpha_2) s m_1 m_2}{\sqrt{v}\,(v-1)}$$

and

$$e = \frac{s m_1 m_2 [v - \delta(1 - 2\alpha_1 - 2\alpha_2 - 4\gamma)]}{v(v-1)}$$

with $\delta = 1 + (v^2\theta(1-\theta))/(m_1 m_2)^2$ and $\theta =$ fractional part of $(m_1 m_2)/v$. The common nonzero eigenvalue of a universally optimum design of the same parameters is just e, so that the A-efficiency of the proposed design is

$$\left(\frac{2}{e}\right)\Big/\left[\frac{1}{e+d}+\frac{1}{e-d}\right]=1-\frac{d^2}{e^2}=1-\frac{4v(\alpha_1-\alpha_2)^2}{\left[v-\delta(1-2\alpha_1-2\alpha_2-4\gamma)\right]^2}.$$

As expected, this expression is minimized over the allowable region when all of the correlation is concentrated in one of rows and columns to the exclusion of the other, at $(\alpha_1, \alpha_2, \gamma) = (0, 0.5, 0)$ or $(0.5, 0, 0)$. Thus the A-efficiency is at least $(v - 1)/v$.

The dispersion in the C-matrix has a stronger effect on the balance properties. The S criterion, given by e_1/e_2 or e_2/e_1 as $\alpha_1 < \alpha_2$ or $\alpha_1 > \alpha_2$, is also minimized at $(\alpha_1, \alpha_2, \gamma) = (0, 0.5, 0)$ or $(0.5, 0, 0)$, the lower bound being $(\sqrt{v} - 1)/(\sqrt{v} + 1)$. Likewise the E-efficiency $= 1 - (|d|/e)$ has lower bound $(\sqrt{v} - 1)/\sqrt{v}$ at these points.

The bounds above are approached as the process becomes one-dimensional or nearly so. Though it is not expected that the designs under consideration would be used in such extreme cases, the calculations do indicate that as differences in row and column correlations grow large, designs balanced for neighbors in each of rows and columns will become superior. In so far as such asymmetry is accompanied by small γ, diagonal neighbors become of less concern, so that in the plane designs such as quasicomplete Latin squares and their analogues for m_1 and/or m_2 not a multiple of v [definitions, constructions and references are given by Afsarinejad and Seeger (1988)], when they exist, should be good alternatives [compare model 1 of Gill and Shukla (1985)]. The expressions derived above also indicate that the proposed class of designs is reasonably robust for small departures from row-column symmetry of the process.

APPENDIX 2

Proof of Lemma 1. Write $v = q^n = 4m + 1$, q is prime. Partition GF_v into the q^{n-1} disjoint ordered cycles of length q given by the cosets of $c_o = (0, 1, 2, \dots, q - 1)$, where each cycle $(y_1, y_2, \dots, y_q)$ is ordered so that $y_j - y_{j-1} = 1$. Then replace each y by $\chi(y)$, where

$$\chi(y)=\begin{cases}1, & \text{if } y \text{ is a quadratic residue,}\\ -1, & \text{if } y \text{ is a quadratic nonresidue,}\\ 0, & \text{if } y=0.\end{cases}$$

Now, all q^{n-1} cycles taken together have $4m + 1$ ordered pairs of adjacent elements. Of these, m are $(-1, 1)$, m are $(1, -1)$, m are $(-1, -1)$ and $m - 1$ are $(1, 1)$ [Storer (1967), page 30].

It will first be shown that there is at least one ordered triple $(-1, -1, 1)$ or $(1, -1, -1)$. Suppose this is not so. Then if a cycle contains two consecutive -1's it must contain only -1's and each such cycle will contain q pairs

$(-1,-1)$. Hence the number of $(-1,-1)$ pairs is a multiple of $q \Rightarrow q|m$, that is, $q|(q^n-1)/4$, which is impossible.

Now let $n > 1$. It will be shown that there is a triple $(1,-1,-1)$ or $(-1,-1,1)$ not arising from c_o. The nonzero elements of c_o are in some order x^{iu} for $i = 1, 2, \dots, q-1$, where $u = (q^n-1)/(q-1)$. If n is even, these are all quadratic residues and the result is established. If n is odd, these have $(q-1)/2$ quadratic nonresidues, so that the number of $(-1,-1)$ pairs in $\chi(c_o)$ is j, say, and the number of such pairs in the other $q^{n-1}-1$ cycles is $m-j$, where $0 \le j \le (q-3)/2$. Suppose none of the desired triple occurs outside of $\chi(c_o)$. Then arguing as above, if two consecutive -1's occur in one of these cycles, the cycle contains only -1's. Hence $q|(m-j)$, that is, $4q|(q^n-4j-1) \Rightarrow j = (q-1)/4 \Rightarrow$ there are $(q^{n-1}-1)/4$ cycles composed solely of -1's and $3(q^{n-1}-1)/4$ cycles containing no $(-1,-1)$ pairs.

The proof is completed by counting ordered triples among the $q^{n-1}-1$ cycles excluding $\chi(c_o)$. There are $(q^n-q)/4$ triples $(-1,-1,-1)$, q from each of the cycles composed only of -1's. There are $(q^n-q)/4$ triples $(1,-1,1)$, one for each occurrence of -1 not in $\chi(c_o)$ or the cycles of -1's. To count the number of $(1,1,-1)$ triples, note that since $(1,-1,-1)$ does not occur in these cycles, this is equal to the number of $(1, —, -1)$ triples where the middle element is arbitrary, that is, this is the number of pairs $(w, w+2)$, $w \notin GF_q$, such that w is a quadratic residue and $w+2$ is not. Multiplying by 2^{-1}, this is the number of $(1,-1)$ pairs if 2 is quadratic, or the number of $(-1,1)$ if 2 is not quadratic, among these cycles. In either case this is just the number of -1's in the $3(q^{n-1}-1)/4$ cycles excluding $\chi(c_o)$ and the cycles of -1's, since for these cycles, every -1 is preceded and succeeded by a 1. This number is then easily counted as $(q^n-q)/4$. A similar argument shows that the number of $(-1,1,1)$ triples is also $(q^n-q)/4$.

With the q triples of $\chi(c_o)$, all triples are now accounted for. In particular, after excluding $\chi(c_o)$, there are no triples of the form $(1,1,1)$, $(-1,1,-1)$, $(-1,-1,1)$ or $(1,-1,-1)$. So the q^{n-1} cycles are $\chi(c_o)$, $(q^{n-1}-1)/4$ cycles of -1's and $3(q^{n-1}-1)/4$ cycles composed of consecutive, disjoint triples $(1,1,-1)$. This implies that $3|q$. Since q is prime, $q = 3$. But with odd n, $3^n \not\equiv 1 \pmod 4$, a contradiction.

Acknowledgments. Our thanks to the Associate Editor and referee for their comments, including remarks resulting in Appendix 1.

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