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ON THE MONOTONICITY OF A CERTAIN EXPECTATION

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Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables and let P_θ be a probability measure under which $(X_1, \dots, X_n)$ have joint pdf's $f_n(X_1, \dots, X_n, \theta) = L_n(\theta)$, $n \geq 1$. Suppose $u_n = u_n(X_1, \dots, X_n)$, $n \geq 1$, are statistics such that $(u_n - c)(L_n(\theta') - L_n(\theta)) \geq 0$, $\forall\ (X_1, \dots, X_n)$, $n \geq 1$, for some constant $c = c(\theta, \theta')$, $\theta \neq \theta'$. For any increasing function ψ and stopping time T, it is shown that $E_\theta \psi(u_T) \leq E_{\theta'}\psi(u_T)$, provided that one of the expectations is finite and $P_\theta(T < \infty) = P_{\theta'}(T < \infty) = 1$. The given result holds for a certain monotone likelihood ratio family and an exponential family in particular. This generalizes a result of Chow and Studden and provides a sequential version of a result of Lehmann.

1. Introduction. Let $Y_1, Y_2, \dots$ be iid random variables having a common exponential density $C(\theta)\exp(xQ(\theta))$ with respect to some σ-finite measure μ on $R = (-\infty, \infty)$, where $Q(\theta)$ is continuous and strictly increasing on an open interval $I \subset R$. Let $S_n = Y_1 + Y_2 + \cdots + Y_n$ and T be a stopping time such that $P_\theta(T < \infty) = 1$, $\forall\ \theta \in I$. Chow and Studden [1] have shown that

$$E_\theta(S_T/T) \leq E_{\theta'}(S_T/T) \quad \text{for } \theta < \theta',\ \theta, \theta' \in I. \tag{1}$$

In this article we generalize this result by proving a more general result given by the theorem below. The generalized result holds for a certain monotone likelihood ratio family and an exponential family in particular. Thus the given generalization extends (1) and provides a sequential version of a result of Lehmann [3] ([4], page 74).

2. The result. The main result of this article is given by the following theorem.

THEOREM. *Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables and P_θ a probability measure under which $(X_1, \dots, X_n)$ have joint pdf's $f_n(X_1, \dots, X_n, \theta) = L_n(\theta)$, $n \geq 1$. Suppose $u_n = u_n(X_1, \dots, X_n)$, $n \geq 1$, are statistics such that*

$$(u_n - c)\big(L_n(\theta') - L_n(\theta)\big) \geq 0, \qquad \forall\,(X_1, \dots, X_n),\ n \geq 1, \tag{2}$$

for some constant $c = c(\theta, \theta')$, $\theta \neq \theta'$. Then for any increasing function ψ and

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stopping time T,

$$E_\theta \psi(u_T) \le E_{\theta'} \psi(u_T), \tag{3}$$

provided that one of the expectations is finite and $P_\theta(T < \infty) = P_{\theta'}(T < \infty) = 1$.

PROOF. Clearly, it is enough to prove (3) when both expectations are finite. Let $P = (P_\theta + P_{\theta'})/2$ be the probability measure under which $(X_1, \dots, X_n)$ have joint pdf

$$L_n = (L_n(\theta) + L_n(\theta'))/2,$$

and let E denote the expectation under P. If ψ is bounded, then by (2) we have

$$\begin{aligned} E_{\theta'}\psi(u_T) - E_\theta \psi(u_T) &= E\psi(u_T)(L_T(\theta') - L_T(\theta))/L_T \\ &= E(\psi(u_T) - \psi(c))(L_T(\theta') - L_T(\theta))/L_T \ge 0. \end{aligned}$$

If ψ is unbounded, let $\psi_{a,b} = \max(a, \min(\psi, b))$, and (3) follows by the dominated convergence theorem by first letting $a \to -\infty$ and then letting $b \to \infty$. □

Some extended versions of (3) remain valid without the condition $P_\theta(T < \infty) = P_{\theta'}(T < \infty) = 1$. The following are two possible extensions without this condition. In what follows we assume the remaining conditions of the theorem and I denotes the usual indicator function.

EXTENSION 1. If $\psi(c)[P_{\theta'}(T < \infty) - P_\theta(T < \infty)] \ge 0$, then

$$E_\theta \psi(u_T) I\{T < \infty\} \le E_{\theta'} \psi(u_T) I\{T < \infty\}.$$

PROOF. This follows from (2) and the identity

$$\begin{aligned} &E_{\theta'}\psi(u_T) I\{T < \infty\} - E_\theta \psi(u_T) I\{T < \infty\} \\ &\quad = E(\psi(u_T) - \psi(c)) I\{T < \infty\}(L_T(\theta') - L_T(\theta))/L_T \\ &\qquad + \psi(c)[P_{\theta'}(T < \infty) - P_\theta(T < \infty)]. \end{aligned}$$ □

EXTENSION 2. Suppose that $u_n \to \mu(\theta)$ in probability under P_θ, and let $u_\infty = \mu(\theta)$. Then (3) holds.

PROOF. It is easy to verify that

$$\begin{aligned} E_{\theta'}\psi(u_T) - E_\theta \psi(u_T) &= E(\psi(u_T) - \psi(c)) I\{T < \infty\}(L_T(\theta') - L_T(\theta))/L_T \\ &\quad + (\psi(\mu(\theta')) - \psi(c)) P_{\theta'}(T = +\infty) \\ &\quad - (\psi(\mu(\theta)) - \psi(c)) P_\theta(T = +\infty). \end{aligned}$$

Hence (3) follows from (2) and the fact that $\mu(\theta') \ge c$ and $\mu(\theta) \le c$. □

The following is a special case of the theorem for a certain monotone likelihood ratio family.

COROLLARY 1. *Let* $X_1, X_2, \ldots$ *be iid random variables having a common density* $f(x, \theta)$ *with respect to a* σ*-finite measure* μ *on* $R = (-\infty, \infty)$, *where* $\theta \in \Omega \subset R$ (Ω *is an open interval*). *Let* $u_n = u_n(X_1, \ldots, X_n)$, *and* $L_n(\theta) = \prod_{i=1}^{n} f(X_i, \theta)$, $n \geq 1$. *Assume that* $L_n(\theta)$ *has monotone likelihood ratio in* u_n, $\forall\ n \geq 1$, *and that the interval* $J = \{u_n \colon \log(L_n(\theta')/L_n(\theta)) < 0\}$ *is independent of* n *where* $\theta < \theta'$. *Let* ψ *be an increasing function and* T *a stopping time such that* $E_\theta \psi(u_T)$ *is finite and* $P_\theta(T < \infty) = 1$, $\forall\ \theta \in \Omega$. *Then*

$$E_\theta \psi(u_T) \leq E_{\theta'} \psi(u_T) \quad \text{for } \theta < \theta', \theta, \theta' \in \Omega.$$

That the preceding is true follows from the fact that $J = (-\infty, c)$ and it implies condition (2) of the theorem. An important special case is that of an exponential family with the following result.

COROLLARY 2. *Let* $X_1, X_2, \ldots$ *be iid random variables having density* $f(x, \theta) = \exp(\theta x - b(\theta))$, $\theta \in \Omega$, *with respect to a* σ*-finite measure* μ *on* R, *where* Ω *is an open interval. Let* ψ *be an increasing function and let* T *be a stopping time such that* $E_\theta \psi(S_T/T)$ *is finite and* $P_\theta(T < \infty) = 1$, $\forall\ \theta \in \Omega$, *where* $S_n = X_1 + \cdots + X_n$. *Then*

$$E_\theta \psi(S_T/T) \leq E_{\theta'} \psi(S_T/T) \quad \text{for } \theta < \theta', \theta, \theta' \in \Omega.$$

REMARK. The condition that $E_\theta \psi(u_T)$ be finite would be satisfied if $E_\theta \sup_{n \geq 1} |\psi(u_n)| < \infty$. For example, if $\psi(x) = x$, and $u_n = S_n/n$, then $E_\theta \sup_{n \geq 1} |S_n/n| < \infty$ if $E_\theta X_1^2 < \infty$ (in fact a weaker condition suffices, cf. [2]).

The following is a nonexponential example.

EXAMPLE. Let $X_1, X_2, \ldots$ be iid random variables with the density $f(x, \theta) = \exp(-(x - \theta)) I\{x \geq \theta\}$, where $\Omega = R = (-\infty, \infty)$. It is easy to see that $L_n(\theta)$ has monotone likelihood ratio in $u_n = \min(X_1, \ldots, X_n)$ and that the interval J in Corollary 1 is independent of n. Clearly, $\theta \leq u_n \leq X_1$ a.s. Hence for an increasing function ψ and stopping time T,

$$E_\theta \psi(u_T) \leq E_{\theta'} \psi(u_T) \quad \text{for } \theta < \theta', \theta, \theta' \in \Omega,$$

provided that $E_\theta \psi(X_1)$ is finite and $P_\theta(T < \infty) = 1$, $\forall\ \theta \in \Omega$. In particular, we have

$$E_\theta \min(X_1, \ldots, X_T) \leq E_{\theta'} \min(X_1, \ldots, X_T) \quad \text{for } \theta < \theta'.$$

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