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MONOTONE GAIN, FIRST-ORDER AUTOCORRELATION AND ZERO-CROSSING RATE[1]

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The effect of a linear filter with monotone gain on the first-order autocorrelation of a weakly stationary time series is discussed. When the gain is monotone increasing, the first-order autocorrelation cannot increase. Otherwise, when the gain is monotone decreasing, the correlation cannot decrease. Further, when the gain is strictly monotone, the first-order autocorrelation is unchanged if and only if the process is a pure sinusoid with probability 1. Under the Gaussian assumption, the zero-crossing rate moves oppositely from the first-order autocorrelation.

1. Introduction. In this paper it is shown that when a weakly stationary process is filtered with a time-invariant linear filter possessing a monotone increasing (decreasing) gain, the first-order autocorrelation cannot increase (decrease). If, in addition, the gain is strictly monotone, the first-order autocorrelation is unchanged if and only if the process is a pure sinusoid with probability 1. When the relationship between the autocorrelation and the zero-crossing rate is known, this can be translated into statements concerning the zero-crossing rate. The Gaussian case is the best example [Kedem (1984)].

Let $\{Z_t;\ t = 0, \pm 1, \pm 2, \dots\}$ be a real-valued zero-mean stationary process with autocorrelation ρ_k and spectral distribution function $F(\omega)$, $0 \le \omega \le \pi$. Let $\mathscr{L}(\cdot)$ denote a time-invariant linear filter with transfer function $H(\omega)$ satisfying the conditions that $H(\omega) = \overline{H(-\omega)}$, and that the squared gain $|H(\omega)|^2$ is integrable with respect to the distribution function F. Let $\rho_k(H)$ be the kth-order autocorrelation of the filtered process $\{\mathscr{L}(Z)_t;\ t = 0, \pm 1, \pm 2 \dots\}$.

When the probability

$$\Pr\{\mathscr{L}(Z)_t \mathscr{L}(Z)_{t-1} \le 0\} \tag{1}$$

is independent of t, as for example is the case under strict stationarity, we refer to it as the zero-crossing rate of the filtered process $\{\mathscr{L}(Z)_t\}$, and denote it by $\gamma(H)$. The zero-crossing rate of the original unfiltered process will be denoted simply by γ.

When $\{Z_t\}$ is Gaussian, so is the filtered process $\{\mathscr{L}(Z)_t\}$, and by the well-known *cosine formula*, $\gamma(H)$ can be expressed in terms of $\rho_1(H)$,

$$\gamma(H) = \frac{1}{\pi} \cos^{-1}(\rho_1(H)). \tag{2}$$

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This is the formula that relates the zero-crossing rate to the first-order autocorrelation for Gaussian processes. Similar formulas, that show explicitly the inverse relationship between γ and ρ_1, also exist for some non-Gaussian processes [Barnett and Kedem (1990)].

A slight generalization can be obtained by replacing $\rho_1(H)$ by $\rho_k(H)$ and $\mathscr{L}(Z)_{t-1}$ by $\mathscr{L}(Z)_{t-k}$. However, this point will not be pursued in this paper.

2. Monotone gain functions. Intuitively, when a high-pass filter is applied to a stationary time series, we expect a higher zero-crossing rate, but lower first-order autocorrelation. Similarly, when the filter is low pass, we expect a lower zero-crossing rate, but higher first-order autocorrelation. This intuition can in fact be vindicated with the help of the following basic result.

THEOREM 1. *Let $|H(\omega)|$ be the gain of a linear filter, and let $\rho_1(H)$ be the first-order autocorrelation of the filtered process. Then we have:*

(a) *If $|H(\omega)|$ is monotone increasing in $[0, \pi]$, then*

$$\rho_1 \geq \rho_1(H).$$

If the gain is monotone decreasing the inequality is reversed.

(b) *Assume that $|H(\omega)|$ is strictly monotone. Then*

$$\rho_1 = \rho_1(H)$$

if and only if $\{Z_t\}$ is a pure sinusoid with probability 1.

PROOF. From the spectral representation of the autocorrelation sequence, we have

$$\rho_1 - \rho_1(H) = \frac{\int_0^\pi \int_0^\pi |H(\omega)|^2 [\cos(\lambda) - \cos(\omega)]\, dF(\omega)\, dF(\lambda)}{\int_0^\pi dF(\omega) \int_0^\pi |H(\omega)|^2\, dF(\omega)}. \tag{3}$$

Let I denote the numerator in the right-hand side of (3). Then I can be decomposed into the sum of two integrals:

$$I = \left\{ \iint_T + \iint_{T'} \right\} |H(\omega)|^2 [\cos(\lambda) - \cos(\omega)]\, dF(\omega)\, dF(\lambda),$$

where $T \equiv \{(\lambda, \omega)\colon\ 0 \leq \lambda < \omega \leq \pi\}$, and $T' \equiv \{(\lambda, \omega)\colon\ 0 \leq \omega < \lambda \leq \pi\}$. By switching λ and ω in the second integral, we obtain

$$I = \iint_T \left[|H(\omega)|^2 - |H(\lambda)|^2 \right] [\cos(\lambda) - \cos(\omega)]\, dF(\omega)\, dF(\lambda). \tag{4}$$

If $|H(\cdot)|$ is monotone increasing, the integrand in (4) is always nonnegative on T, and hence $I \geq 0$. On the other hand, if $|H(\cdot)|$ is monotone decreasing, the integrand is nonpositive on T, and therefore $I \leq 0$. Assertion (a) is thus proved.

To prove (b), we note that if $\{Z_t\}$ is, with probability 1, a sinusoid with some frequency $\omega_0 \in (0, \pi)$, then

$$\rho_1 = \rho_1(H) = \cos(\omega_0).$$

Conversely, suppose that $\rho_1 = \rho_1(H)$, and hence $I = 0$, but that $\{Z_t\}$ is not a pure sinusoid. Then we can find in the support of F a constant $\lambda_0 \in [0, \pi]$ such that *both* $[0, \lambda_0]$ and $(\lambda_0, \pi]$ have positive F measure. However, the set T contains $[0, \lambda_0] \times (\lambda_0, \pi]$. Therefore, T contains at least one point (λ', ω') whose neighborhood has a positive $(F \times F)$ measure. Assume, without loss of generality, that $|H(\cdot)|$ is strictly increasing. Then the integrand of I is strictly positive on T. It follows that $I > 0$. This, however, contradicts the fact that $I = 0$, and (b) is proved. □

Suppose $\{Z_t\}$ is Gaussian. Then by the cosine formula (2), the expected zero-crossing rate can be obtained from the first-order autocorrelation by a strictly decreasing transformation. Therefore, we have the following corollary.

COROLLARY 1. *Suppose that the process $\{Z_t\}$ is Gaussian.*

(a) *If $|H(\omega)|$ is monotone increasing in $[0, \pi]$, then*

$$\gamma \le \gamma(H).$$

The inequality is reversed if $|H(\omega)|$ is monotone decreasing.

(b) *Assume $|H(\omega)|$ is strictly monotone. Then*

$$\gamma = \gamma(H)$$

if and only if $\{Z_t\}$ is a pure sinusoid with probability 1. The frequency of the sinusoid is given by $\pi\gamma$.

Since the differencing operator is a high-pass filter with a strictly increasing gain, we can see, under the Gaussian assumption, that the sinusoidal limit given by Kedem (1984) is only a very special case of part (b) of Corollary 1.

Notice that Theorem 1 has no restrictions on the spectral distribution function F. In particular, F does not need to have a density. However, if F does have a (spectral) density with respect to Lebesgue measure, and if this density is positive almost everywhere on $[0, \pi]$, then formula (4) for I implies that $\rho_1 \ne \rho_1(H)$ whenever the gain is monotonic and is not equal to a constant almost everywhere. This is weaker than strict monotonicity, and covers ideal high-pass and low-pass filters.

THEOREM 2. *Suppose that the process $\{Z_t\}$ has an absolutely continuous spectral distribution function F and that its density $f(\cdot)$ is positive on $[0, \pi]$ almost everywhere with respect to Lebesgue measure. Then $\rho_1 \ne \rho_1(H)$ as long as $|H(\cdot)|$ is monotone on $[0, \pi]$ and does not coincide with a function which is a constant almost everywhere. If, in addition, $\{Z_t\}$ is Gaussian, then $\gamma \ne \gamma(H)$.*

PROOF. Suppose that $|H(\cdot)|$ is increasing. Since $|H(\cdot)|$ is not equal to a constant, except on a set of Lebesgue measure 0, it follows that for any $\lambda_0 \in (0, \pi)$, we can either find a $\lambda_1 \in (\lambda_0, \pi)$ such that $|H(\lambda)| \le |H(\lambda_0)| < |H(\lambda_1)| < |H(\omega)|$, for all $(\lambda, \omega) \in [0, \lambda_0] \times [\lambda_1, \pi] \subset T$, or we can find a $\lambda_2 \in (0, \lambda_0)$ such that $|H(\lambda)| \le |H(\lambda_2)| < |H(\lambda_0)| \le |H(\omega)|$, for all $(\lambda, \omega) \in [0, \lambda_2] \times [\lambda_0, \pi] \subset T$. From this, and the fact that $f(\cdot)$ is positive almost everywhere, it follows that the set on which the integrand in (4) is greater than 0 has a positive Lebesgue measure. Therefore, $I > 0$ and $\rho_1 > \rho_1(H)$. Similarly, we can obtain the reversed strict inequality if $|H(\cdot)|$ is decreasing and not equal to a constant. The result for the zero-crossing rate follows from the cosine formula. □

Based on Theorems 1 and 2, we can also compare the effect of two different filters on a time series. For convenience let $\mathscr{L}_\alpha$ and $\mathscr{L}_\beta$ be two linear time-invariant filters with transfer function $H(\omega; \alpha)$ and $H(\omega; \beta)$. We denote the first-order autocorrelations and zero-crossing rates, respectively, by $\rho_1(\alpha)$, $\rho_1(\beta)$ and $\gamma(\alpha)$, $\gamma(\beta)$. Then we have the following corollary.

COROLLARY 2. *Consider two filters* $\mathscr{L}_\alpha(\cdot)$ *and* $\mathscr{L}_\beta(\cdot)$, *and assume that* $\mathscr{L}_\alpha(\cdot)$ *has a well-defined inverse* $\mathscr{L}_\alpha^{-1}(\cdot)$.

(a) *If the function*

$$G(\omega; \alpha, \beta) \equiv \frac{|H(\omega; \beta)|^2}{|H(\omega; \alpha)|^2}$$

is monotone increasing in $\omega \in [0, \pi]$, *then* $\rho_1(\alpha) \ge \rho_1(\beta)$. *If, in addition,* $\{Z_t\}$ *is Gaussian, then* $\gamma(\alpha) \le \gamma(\beta)$. *The inequalities are reversed when* $G(\omega; \alpha, \beta)$ *is monotone decreasing in* $\omega \in [0, \pi]$.

(b) *If* $G(\omega; \alpha, \beta)$ *is strictly monotone, the inequalities in part* (a) *are strict unless* $\{Z_t\}$ *is a pure sinusoid with probability* 1.

(c) *Suppose that* $\{Z_t\}$ *has an absolutely continuous spectral distribution F and that its density f is positive almost everywhere with respect to Lebesgue measure. If* $G(\omega; \alpha, \beta)$ *is monotone and not equal to a constant almost everywhere, then the inequalities in part* (a) *are strict.*

PROOF. The results follow immediately from Theorems 1 and 2 upon noting the fact that $G(\omega; \alpha, \beta)$ is the squared gain of the filter $\mathscr{L}_\beta \mathscr{L}_\alpha^{-1}(\cdot)$ and

$$\mathscr{L}_\beta \mathscr{L}_\alpha^{-1}(\mathscr{L}_\alpha(Z))_t = \mathscr{L}_\beta(Z)_t. \qquad \square$$

3. Applications.

3.1. *Ideal low-pass filters.* Let us first consider the ideal low-pass filter $\mathscr{L}_\lambda(\cdot)$ which has the gain function $|H(\omega; \lambda)| \equiv 1$ if $|\omega| < \lambda$, and $|H(\omega; \lambda)| \equiv 0$ if $\lambda < |\omega| \le \pi$, where $\lambda \in (0, \pi)$. The first-order correlation of the filtered process is denoted by $\rho_1(\lambda)$, and the corresponding expected zero-crossing rate is denoted by $\gamma(\lambda)$. A slight modification of (4) shows that $\rho_1(\lambda)$ is monotone decreasing in λ. In the Gaussian case $\gamma(\lambda)$, is monotone increasing in λ.

3.2. *Exponential smoothing*. The exponentially weighted moving average (EWMA) filter $\mathscr{L}_\alpha(\cdot)$ is defined by

$$\mathscr{L}_\alpha(Z)_t = Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} + \cdots,$$

where $\alpha \in (-1, 1)$. The squared gain is given by

$$|H(\omega;\alpha)|^2 = \frac{1}{1 - 2\alpha\cos(\omega) + \alpha^2}.$$

Since

$$G(\omega;\alpha,\beta) = \frac{1 - 2\alpha\cos(\omega) + \alpha^2}{1 - 2\beta\cos(\omega) + \beta^2}$$

is strictly decreasing in $\omega \in [0, \pi]$ for any $-1 < \alpha < \beta < 1$, we have the following corollary.

COROLLARY 3. *Let $\rho_1(\alpha)$ and $\gamma(\alpha)$ be the first-order correlation and zero-crossing rate of the process $\mathscr{L}_\alpha(Z)_t$, respectively. Then it follows that $\rho_1(\alpha)$ is strictly increasing and, in the Gaussian case, $\gamma(\alpha)$ is strictly decreasing in $\alpha \in (-1, 1)$ unless $\{Z_t\}$ is a pure sinusoid with probability* 1.

PROOF. See Corollary 2. □

Corollary 3 was applied in a detection problem in Kedem and Li (1989).

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