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LARGE SAMPLE THEORY OF A MODIFIED BUCKLEY–JAMES ESTIMATOR FOR REGRESSION ANALYSIS WITH CENSORED DATA[1]

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Buckley and James proposed an extension of the classical least squares estimator to the censored regression model. It has been found in some empirical and Monte Carlo studies that their approach provides satisfactory results and seems to be superior to other extensions of the least squares estimator in the literature. To develop a complete asymptotic theory for this approach, we introduce herein a slight modification of the Buckley–James estimator to get around the difficulties caused by the instability at the upper tail of the associated Kaplan–Meier estimate of the underlying error distribution and show that the modified Buckley–James estimator is consistent and asymptotically normal under certain regularity conditions. A simple formula for the asymptotic variance of the modified Buckley–James estimator is also derived and is used to study the asymptotic efficiency of the estimator. Extensions of these results to the multiple regression model are also given.

1. Introduction. Consider the linear regression model

$$y_i = \beta x_i + \varepsilon_i, \qquad i = 1, \dots, n, \tag{1.1}$$

where $\varepsilon, \varepsilon_1, \varepsilon_2, \dots$ are i.i.d. random variables with a continuous distribution function F such that $E|\varepsilon| < \infty$ (but $E\varepsilon$ need not be 0) and the x_i are either nonrandom or are independent of $\{\varepsilon_n\}$. Suppose that the responses y_i are not completely observable and that the observations are (z_i, δ_i, x_i), where

$$z_i = \min\{y_i, c_i\}, \qquad \delta_i = I_{\{y_i \le c_i\}} \tag{1.2}$$

and (c_i, x_i) are independent random vectors that are independent of $\{\varepsilon_i\}$. We call c_i censoring variables and (1.1)–(1.2) the censored regression model.

When the y_i are completely observable, the least squares estimate b_n of β is the solution $b = b_n$ of the equation

$$\sum_{i=1}^{n} (x_i - \bar{x}_n)(y_i - bx_i) = 0, \tag{1.3}$$

where $\bar{x}_n = n^{-1}\sum_{i=1}^{n} x_i$. In the case of censored data (z_i, δ_i, x_i) with nonrandom x_i and c_i, Buckley and James (1979) proposed to modify (1.3) as follows.

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Let

$$(1.4)\quad y_i(b) = y_i - bx_i, \qquad c_i(b) = c_i - bx_i, \qquad z_i(b) = \min\{y_i(b), c_i(b)\}.$$

Noting that

$$E(y_i|\delta_i, z_i) = \delta_i z_i + (1-\delta_i)E[\varepsilon - (c_i - \beta x_i) + c_i|\varepsilon > c_i - \beta x_i]$$
$$= z_i + (1-\delta_i)\int_{z_i - \beta x_i}^{\infty}(1 - F(s))\,ds/(1 - F(z_i - \beta x_i)),$$

the Buckley–James method is to replace the $y_i - bx_i$ in (1.3) by the following estimate $y_i^*(b)$ of $E(y_i|\delta_i, z_i) - bx_i$:

$$(1.5)\quad y_i^*(b) = z_i(b) + (1-\delta_i)\int_{z_i(b)}^{\infty}\left(1 - \hat{F}_{n,b}(s)\right)ds/\left\{1 - \hat{F}_{n,b}(z_i(b))\right\},$$

where $\hat{F}_{n,b}$ is the Kaplan–Meier estimate of F based on $\{(z_i(b), \delta_i): i \le n\}$, that is,

$$(1.6)\qquad \hat{F}_{n,b}(u) = 1 - \prod_{i:\, z_i(b) \le u}\{1 - 1/Z_n(b, z_i(b))\}^{\delta_i},$$

$$(1.7)\qquad Z_n(b,t) = \#\{j \le n: z_j(b) \ge t\} = \sum_{j=1}^{n} I_{\{z_j(b) \ge t\}}.$$

Thus, the Buckley–James estimator is defined by the equation

$$(1.8)\qquad \sum_{i=1}^{n}(x_i - \bar{x}_n)y_i^*(b) = 0,$$

or more precisely, is defined as a zero-crossing of the random function $\sum_{i=1}^{n}(x_i - \bar{x}_n)y_i^*(b)$. We say that $\hat{b}$ is a zero-crossing of a function $\psi(b)$ if the right- and left-hand limits $\psi(\hat{b}+)$ and $\psi(\hat{b}-)$ do not have the same sign, that is, if $\psi(\hat{b}+)\psi(\hat{b}-) \le 0$.

Buckley and James (1979) and Miller and Halpern (1982) have reported satisfactory performance of the Buckley–James estimator in some simulation and empirical studies. In particular, Miller and Halpern (1982) compared the performance of the Buckley–James estimator with two other extensions of the least squares method to censored data by Miller (1976) and by Koul, Susarla and Van Ryzin (1981) and with the Cox (1972) regression analysis that assumes a proportional hazards model instead of the linear regression model (1.1). From the results of these different methods applied to the Stanford heart transplant data, Miller and Halpern (1982) concluded that the Cox and the Buckley–James estimators are "the two most reliable regression estimates to use with censored data" and that "the choice between them should depend on the appropriateness of the proportional hazards model or the linear model for the data." Leurgans (1987) recently proposed an improvement of the Koul, Susarla and Van Ryzin approach and found it to be competitive with the Cox and Buckley–James estimators in her analysis of the Stanford heart transplant data.

While there is now a complete asymptotic theory for the Cox regression method in the proportional hazards model, a corresponding theory for the Buckley–James estimator is lacking. As a first step towards such a theory, James and Smith (1984) studied the weak consistency of the estimator. Assuming that $F^{-1}(1) < \infty$, that c_i and x_i are nonrandom with $\sum_{i=1}^n (x_i - \bar{x}_n)^2 \to \infty$ and $\sum_{i=1}^n (1 - F(c_i - \beta x_i))|x_i - \bar{x}_n| = O(\sum_{i=1}^n (x_i - \bar{x}_n)^2)$ as $n \to \infty$ and that $Z_n(\beta, t) \to_P \infty$ as $n \to \infty$ for every $t < F^{-1}(1)$, their main results are that

$$\sum_{i=1}^n (x_i - \bar{x}_n) y_i^*(\beta) \Big/ \sum_{i=1}^n (x_i - \bar{x}_n)^2 \to_P 0 \tag{1.9}$$

and that under considerably more stringent assumptions,

$$P\left\{ \sum_{i=1}^n (x_i - \bar{x}_n) y_i^*(b) \text{ has a zero-crossing in } (\beta - \delta, \beta + \delta) \right\} \to 1 \quad \text{for all } \delta > 0. \tag{1.10}$$

An important step in their proof of (1.10) is to show that under these assumptions, one of which is that $H(u) = \lim_{n\to\infty} n^{-1} \sum_{i=1}^n I_{\{x_i \le u\}}$ exists for every u,

$$\sup_{u < F^{-1}(1)} \left| \hat{F}_{n,b}(u) - \int F(u - (\beta - b)x)\, dH(x) \right| \to_P 0, \tag{1.11}$$

see James and Smith [(1984), page 594]. They obtained (1.11) by regarding $\hat{F}_{n,b}$ as the usual product-limit estimator based on i.i.d. random variables $\varepsilon_i^* = \varepsilon_i - (b - \beta)x_i$ censored by independent random variables $c_i^* = c_i - bx_i$ when the x_i are i.i.d. with a common distribution function H. However, the usual independence condition between $\{\varepsilon_i^*\}$ and the censoring sequence $\{c_i^*\}$ to guarantee consistency of the product-limit estimator is not satisfied unless $b = \beta$. We shall show in Section 3 that (1.11) in general does not hold unless $b = \beta$ and that the limit of $\hat{F}_{n,b}$ is in fact considerably more complicated when $b \neq \beta$. Our derivation of this limit is based on a direct asymptotic analysis of (1.6) using the approximation lemmas developed in Lai and Ying (1988) for stochastic integrals of empirical-type processes.

Making use of the approximation lemmas of Lai and Ying (1988), we also establish in Section 3 the asymptotic linearity of a slightly modified form of the random function $\xi_n^*(b) = \sum_{i=1}^n (x_i - \bar{x}_n) y_i^*(b)$ in some neighborhood of β. The idea behind our modification $\xi_n(b)$, whose precise definition will be given in Section 2, of the Buckley–James statistics $\xi_n^*(b)$ is to get around the difficulties caused by the instability of the Kaplan–Meier estimate $\hat{F}_{n,b}(u)$ at the upper tail where $n^{-1}Z_n(b, u)$ is small. In Section 2, we also develop a stochastic integral representation of $\xi_n(b)$. Making use of this stochastic integral representation and martingale central limit theorems, we establish in Section 4 the asymptotic normality of $\xi_n(\beta)$. Combining the asymptotic normality of $\xi_n(\beta)$ with the asymptotic linearity of $\xi_n(b)$ established in Section 3, we obtain in Section 4 the asymptotic normality of the modified Buckley–James estimator and use this result to study the asymptotic efficiency of the estimator in Section 5, where extensions to multiple regression models are also given.

2. A modification of the Buckley–James estimator and related stochastic integral representations. Throughout the sequel we shall assume knowledge of an upper bound $\rho > |\beta|$ and restrict b to a bounded interval $[-\rho, \rho]$. An earlier modification of the Buckley–James estimator has been given in Lai and Ying (1988) under the assumption that $\sup_n |x_n| < \infty$ and

$$F^{-1}(1) < \infty, \qquad \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} P\{c_i > F^{-1}(1)\} > 0. \tag{2.1}$$

The basic idea is to introduce a smooth weight function of the form

$$p_n(t) = p(n^{\lambda}(t - cn^{-\lambda})), \qquad 0 \le t \le 1, \tag{2.2}$$

with $c > 0$, $0 < \lambda < 1$, and p being a twice-continuously differentiable and nondecreasing function on the real line such that

$$p(u) = 0 \quad \text{for } u \le 0, \qquad p(u) = 1 \quad \text{for } u \ge 1. \tag{2.3}$$

The weight function p_n is used in Lai and Ying (1988) to redefine the Kaplan–Meier estimator $\hat{F}_{n,b}$ in (1.6) as

$$\tilde{F}_{n,b}(u) = 1 - \prod_{i:\, z_i(b) \le u} \{1 - p_n(n^{-1}Z_n(b, z_i(b)))/Z_n(b, z_i(b))\}^{\delta_i}, \tag{2.4}$$

and to replace the $y_i^*(b)$ in (1.8) by

$$y_i'(b) = z_i(b) + (1 - \delta_i)\frac{\int_{z_i(b)}^{\infty}(1 - \tilde{F}_{n,b}(s))p_n(n^{-1}Z_n(b,s))\,ds}{1 - \tilde{F}_{n,b}(z_i(b))}. \tag{2.5}$$

As shown in Lai and Ying (1988), the weight function p_n smoothly removes the instability associated with small value of $n^{-1}Z_n(b, s)$. However, the bias it introduces into $y_i'(b)$ as an estimate of $E(y_i|\delta_i, z_i) - bx_i$ causes additional difficulties without the assumption (2.1). In particular, since we would like our asymptotic theory to be applicable to unbounded ε_i, as in the Gaussian case for which the classical least squares estimator based on completely observable y_i is known to be asymptotically efficient, we do not assume (2.1) herein and therefore have to further modify (2.5) to remove the bias caused by introducing the weight function p_n.

Useful insights into such modification and its analysis are provided by certain stochastic integrals with respect to empirical-type processes and by Lemma 1 for these integrals. With $z_i(b)$ defined by (1.4), define the empirical-type processes

$$
\begin{aligned}
&Z_n(b,t) = \sum_{i=1}^{n} I_{\{z_i(b)\ge t\}}, && Z_n^x(b,t) = \sum_{i=1}^{n} (x_i - \bar{x}_n) I_{\{z_i(b)\ge t\}},\\
(2.6)\quad &J_n(b,t) = \sum_{i=1}^{n} I_{\{z_i(b)\ge t,\,\delta_i=0\}}, && J_n^x(b,t) = \sum_{i=1}^{n} (x_i - \bar{x}_n) I_{\{z_i(b)\ge t,\,\delta_i=0\}},\\
&N_n(b,t) = \sum_{i=1}^{n} I_{\{z_i(b)\le t,\,\delta_i=1\}}, && N_n^x(b,t) = \sum_{i=1}^{n} (x_i - \bar{x}_n) I_{\{z_i(b)\le t,\,\delta_i=1\}}.
\end{aligned}
$$

In view of (1.5) and (2.6), the Buckley–James statistics $\sum_{i=1}^{n}(x_i - \bar{x}_n)y_i^*(b)$ can be written in the form

$$
(2.7)\qquad
\begin{aligned}
&-\int_{t=-\infty}^{\infty} t\,dZ_n^x(b,t)\\
&\quad-\int_{t=-\infty}^{\infty}\left\{\int_t^{\infty}\left[\left(1-\hat{F}_{n,b}(s)\right)/\left(1-\hat{F}_{n,b}(t)\right)\right]ds\right\}dJ_n^x(b,t).
\end{aligned}
$$

For $b=\beta$, replacing Z_n^x and J_n^x by their expected values and $\hat{F}_{n,b}$ by F in (2.7) leads to the integral on the left-hand side of (2.8) below with $l(t)\equiv 1$. The following lemma suggests that if we suitably modify the term $z_i(b)$ in (2.5), then we may be able to remove the bias caused by introducing the weight function $p_n(n^{-1}Z_n(b,s))$ into the last term of (2.5).

LEMMA 1. *Let $l(t)$ be a function of bounded variation on the real line. If $\int_{-\infty}^{\infty}|t|\,dF(t)<\infty$ and $\int_{-\infty}^{\infty}|t||dl(t)|<\infty$, then*

$$
(2.8)\qquad
\begin{aligned}
&-\int_{t=-\infty}^{\infty} t\,d\left[l(t)EZ_n^x(\beta,t)\right]\\
&\quad-\int_{t=-\infty}^{\infty}\left\{\int_t^{\infty}\left[(1-F(s))/(1-F(t))\right]l(s)\,ds\right\}dEJ_n^x(\beta,t)=0.
\end{aligned}
$$

PROOF. Let $Z^i(t)=I_{\{z_i-\beta x_i\ge t\}}$, $J^i(t)=I_{\{z_i-\beta x_i\ge t,\,\delta_i=0\}}$. Let $\mathbf{x}=(x_1,\dots,x_n)$. It suffices to show that for $i=1,\dots,n$,xp

$$
(2.9)\qquad
\begin{aligned}
&-\int_{-\infty}^{\infty} t\,d\left[l(t)E(Z^i(t)|\mathbf{x})\right]-\int_{-\infty}^{\infty}\frac{\int_t^{\infty}(1-F(s))l(s)\,ds}{1-F(t)}\,dE(J^i(t)|\mathbf{x})\\
&\quad=-\int_{-\infty}^{\infty} t\,d\left[l(t)(1-F(t))\right].
\end{aligned}
$$

Let $G_i(t)=P\{c_i-\beta x_i\ge t|x_i\}$. Since $z_i-\beta x_i=\min\{\varepsilon_i, c_i-\beta x_i\}$,

$$
dE\left(J^i(t)|\mathbf{x}\right)=(1-F(t))\,dG_i(t)\quad\text{and}\quad E\left(Z^i(t)|\mathbf{x}\right)=(1-F(t))G_i(t).
$$

Integration by parts gives

$$\begin{aligned}
-\int_u^\infty t\,d[l(t)(1-F(t))G_i(t)] &- \int_u^\infty\left\{\int_t^\infty (1-F(s))l(s)\,ds\right\}dG_i(t)\\
&= G_i(u)\left\{ul(u)(1-F(u)) + \int_u^\infty (1-F(s))l(s)\,ds\right\}\\
&\quad + \int_u^\infty l(t)(1-F(t))G_i(t)\,dt + \int_u^\infty G_i(t)[-(1-F(t))l(t)]\,dt\\
&= -G_i(u)\int_u^\infty s\,d[(1-F(s))l(s)]
\end{aligned}$$

for every u that is a continuity point of both G_i and l. Letting $u \to -\infty$ gives the desired conclusion (2.9). □

Setting $l(t) = p_n(n^{-1}Z_n(b,t))$, Lemma 1 suggests the following modification of the Buckley–James statistics (2.7):

$$\begin{aligned}
\xi_n(b) &= -\int_{t=-\infty}^{\infty} t\,d\big[p_n(n^{-1}Z_n(b,t))Z_n^x(b,t)\big]\\
&\quad -\int_{t=-\infty}^{\infty} \frac{\int_t^\infty \big(1-\hat F_{n,b}(s)\big)p_n\big(n^{-1}Z_n(b,s)\big)\,ds}{1-\hat F_{n,b}(t)}\,dJ_n^x(b,t) \qquad (2.10)\\
&= \sum_{i=1}^n (x_i-\bar x_n)\hat y_i(b) - \sum_{i=1}^n z_i(b)Z_n^x(b,z_i(b))\Delta p_n\big(n^{-1}Z_n(b,z_i(b))\big),
\end{aligned}$$

where

$$\begin{aligned}
\hat y_i(b) &= z_i(b)p_n\big(n^{-1}Z_n(b,z_i(b))\big)\\
&\quad + (1-\delta_i)\frac{\int_{z_i(b)}^\infty \big(1-\hat F_{n,b}(s)\big)p_n\big(n^{-1}Z_n(b,s)\big)\,ds}{1-\hat F_{n,b}(z_i(b))}, \qquad (2.11)\\
\Delta p_n\big(n^{-1}Z_n(b,t)\big) &= p_n\big(n^{-1}Z_n(b,t+)\big) - p_n\big(n^{-1}Z_n(b,t)\big).
\end{aligned}$$

Throughout the sequel we shall consider $\xi_n(b)$ instead of the previous modification $\sum_{i=1}^n(x_i-\bar x_n)y_i'(b)$ [with $y_i'(b)$ given by (2.5)] in Lai and Ying (1988). By a modified Buckley–James estimator we mean a zero-crossing of $\xi_n(b)$.

It is well known that the classical Kaplan–Meier estimator $\hat F_{n,\beta}(t)$ of $F(t)$ is quite unstable when $n^{-1}\sum_{i=1}^n(1-F(t))G_i(t)$ is near 0. Replacing the unknown β by b in $\hat F_{n,b}$ causes additional difficulties. Because of this instability, the integral $\int_t^\infty[(1-\hat F_{n,b}(s))/(1-\hat F_{n,b}(t))]\,ds$ that appears in the Buckley–James statistic (2.7) is an unreliable estimate of $\int_t^\infty[(1-F(s))/(1-F(t))]\,ds$ when $n^{-1}\sum_{i=1}^n(1-F(t))G_i(t)$ is small, as reflected from the sample data by a small value of $n^{-1}Z_n(b,t)$. Our idea to circumvent this difficulty is to multiply the integrand $(1-\hat{F}_{n,b}(s))/(1-\hat{F}_{n,b}(t))$ by a smooth weight function

$$p_n(n^{-1}Z_n(b,s)) = \begin{cases} 1 & \text{if } Z_n(b,s) \geq (c+1)n^{1-\lambda}, \\ 0 & \text{if } Z_n(b,s) \leq cn^{1-\lambda}. \end{cases} \tag{2.12}$$

Lemma 1 suggests that we can compensate the bias due to introducing this weight function into the second term of (2.7) by also including the same weight function in the first term. This is the heuristic idea behind our modification of the Buckley–James estimator. Moreover, instead of straightforward trimming [i.e., with $I_{\{Z_n(b,t) \geq cn^{1-\lambda}\}}$ in place of $p_n(n^{-1}Z_n(b,t))$], we use here a smooth version analogous to the kernel method in density estimation. Since a key idea of our analysis is to approximate the random function $\xi_n(b)$ by a nonrandom function $\zeta_n(b)$ [which basically replaces the empirical-type processes Z_n, Z_n^x, J_n and J_n^x that appear in $\xi_n(b)$ by their expectations], our use of a smooth trimming function leads to smooth $\zeta_n(b)$ which is essential to the asymptotic linearity result in Theorem 1(ii) of the next section. Making use of asymptotic linearity and other asymptotic properties of $\xi_n(b)$, it will be shown that the modified Buckley–James estimator is asymptotically efficient when F is normal and therefore the estimator is indeed a natural extension of the classical least squares method to censored data.

3. Consistency of the modified Buckley–James estimator and asymptotic linearity of $\xi_n(b)$. The following assumptions will be made in the sequel for the analysis of the modified Buckley–James statistics $\xi_n(b)$:

$$|x_i| \leq B \text{ for all } i \text{ and some nonrandom constant } B, \tag{3.1}$$

F has a twice-continuously differentiable density f such that

$$\int_{-\infty}^{\infty} t^2\,dF(t) < \infty \text{ and } \int_{-\infty}^{\infty} (f'(t)/f(t))^2\,dF(t) < \infty, \tag{3.2}$$

$$\int_{-\infty}^{\infty} \sup_{|h|\leq\delta} \{|f'(t+h)| + |f''(t+h)|\}\,dt < \infty \quad \text{for some } \delta > 0, \tag{3.3}$$

$$E\exp(\theta\varepsilon_1^-) + \sup_n E\exp(\theta c_n^-) < \infty \quad \text{for some } \theta > 0, \tag{3.4}$$

where a^- denotes the negative part of a $\left(\text{i.e., } a^- = |a| I_{\{a\leq 0\}}\right)$,

$$\sup_{\substack{|b|\leq\rho,\\ -\infty<t<\infty}} \sum_{i=1}^{n} P\{t \leq c_i - bx_i \leq t+h\} = O(nh) \quad \text{as } h\to 0 \text{ and } nh\to\infty, \tag{3.5}$$

$$n^{-1}\sum_{i=1}^{n}(x_i-\bar{x}_n)^r P[c_i - \beta x_i \geq s|x_i] \to_P \Gamma_r(s), \tag{3.6}$$

some nonrandom function, for $r = 0, 1, 2$ and for every s with $F(s) < 1$.

The following lemma approximates $\hat{F}_{n,b}(t)$ and $\xi_n(b)$ essentially by replacing Z_n, Z_n^x, J_n, J_n^x in (1.6) and (2.10) by their expectations. Its proof is given in Appendix A and depends on the approximation lemmas for stochastic integrals of empirical-type processes developed in Lai and Ying (1988), where assumptions of the type (3.1), (3.3) and (3.5) are discussed and play a basic role in the development of the stochastic integral approximation lemmas.

LEMMA 2. *Under the assumptions* (3.1)–(3.5), *define*

$$F_n(b,t) = 1 - \exp\left\{-\int_{u \le t} \frac{dE N_n(b,u)}{EZ_n(b,u)}\right\}, \tag{3.7}$$

$$\begin{aligned}\zeta_n(b) = &-\int_{t=-\infty}^{\infty} t\, d\left[p_n(n^{-1}EZ_n(b,t))EZ_n^x(b,t)\right]\\ &-\int_{t=-\infty}^{\infty} \frac{\int_t^\infty (1-F_n(b,s))p_n(n^{-1}EZ_n(b,s))\,ds}{1-F_n(b,t)}\, dE J_n^x(b,t).\end{aligned} \tag{3.8}$$

Then for every $\varepsilon > 0$,

$$\begin{aligned}&\sup\left\{\left|\hat{F}_{n,b}(t) - F_n(b,t)\right| : |b| \le \rho, Z_n(b,t) \ge n^{1-\varepsilon}\right\}\\ &\quad = O(n^{-1/2+4\varepsilon}) \quad a.s.,\end{aligned} \tag{3.9}$$

$$\sup_{|b|\le\rho} |\xi_n(b) - \zeta_n(b)| = O(n^{1/2+4\lambda+\varepsilon}) \quad a.s., \tag{3.10}$$

$$\begin{aligned}&\sup\left\{\left|\frac{1-\hat{F}_{n,b}(s)}{1-\hat{F}_{n,b}(t)} - \frac{1-F_n(b,s)}{1-F_n(b,t)}\right| : t \le s, |b| \le \rho, Z_n(b,s) \ge \frac{cn^{1-\lambda}}{2}\right\}\\ &\quad = \mathrm{O}(\mathrm{n}^{-1/2+3\lambda+\varepsilon}) \quad \text{a.s.},\end{aligned} \tag{3.11}$$

where $0 < \lambda < 1$ *and* $c > 0$ *are given in* (2.2). *Moreover, for every* $0 < \gamma < 1$ *and* $\varepsilon > 0$,

$$\begin{aligned}&\sup_{|b-\beta|\le n^{-\gamma}} |\xi_n(b) - \xi_n(\beta) - \zeta_n(b) + \zeta_n(\beta)|\\ &\quad = O(n^{(1-\gamma)/2+4\lambda+\varepsilon} + n^{1/2-\gamma+6\lambda+\varepsilon} + n^{7\lambda+\varepsilon}) \quad a.s.,\end{aligned} \tag{3.12}$$

$$\begin{aligned}&\sup\left\{\left|\frac{1-\hat{F}_{n,b}(s)}{1-\hat{F}_{n,b}(t)} - \frac{1-\hat{F}_{n,\beta}(s)}{1-\hat{F}_{n,\beta}(t)} - \frac{1-F_n(b,s)}{1-F_n(b,t)} + \frac{1-F(s)}{1-F(t)}\right| :\right.\\ &\qquad \left. |b-\beta| \le n^{-\gamma}, t \le s, \min(Z_n(b,s), Z_n(\beta,s)) \ge \frac{cn^{1-\lambda}}{2}\right\}\\ &\quad = O(n^{-(1+\gamma)/2+3\lambda+\varepsilon} + n^{-1/2-\gamma+5\lambda+\varepsilon} + n^{-1+6\lambda+\varepsilon}) \quad a.s.\end{aligned} \tag{3.13}$$

Furthermore, with probability 1,

$$\xi_n(b) = -\int_{-n^\lambda}^{n^\lambda} t\, d\big[p_n(n^{-1}Z_n(b,t))Z_n^x(b,t)\big]$$

$$(3.14)\qquad -\int_{-n^\lambda}^{n^\lambda}\left\{\int_t^{n^\lambda}\frac{1-\hat{F}_{n,b}(s)}{1-\hat{F}_{n,b}(t)}p_n(n^{-1}Z_n(b,s))\,ds\right\}dJ_n^x(b,t)$$

for all large n,

$$\zeta_n(b) = \varepsilon_n(b) - \int_{-n^\lambda}^{n^\lambda} t\, d\big[p_n(n^{-1}EZ_n(b,t))EZ_n^x(b,t)\big]$$

$$(3.15)\qquad -\int_{-n^\lambda}^{n^\lambda}\left\{\int_t^{n^\lambda}\frac{1-F_n(b,s)}{1-F_n(b,t)}p_n(n^{-1}EZ_n(b,s))\,ds\right\}dEJ_n^x(b,t),$$

with $\sup_{|b|\le\rho}|\varepsilon_n(b)|\to 0.$

REMARKS. (i) Let $G_i(x,s) = P\{c_i - \beta x_i \ge s | x_i = x\}$. Suppose that the x_i are i.i.d. with a common distribution function H. Then in view of (2.6), we can express (3.7) in the form

$$(3.16)\qquad F_n(b,t) = 1-\exp\left\{-\int_{-\infty}^t \frac{\int_{-\infty}^\infty f(u+(b-\beta)x)\times\bar{G}_n(x,u+(b-\beta)x)\,dH(x)}{\int_{-\infty}^\infty[1-F(u+(b-\beta)x)]\times\bar{G}_n(x,u+(b-\beta)x)\,dH(x)}\,du\right\},$$

where $\bar{G}_n = n^{-1}\sum_{i=1}^n G_i$ and $f = F'$. When $b=\beta$, $F_n(b,t) = F(t)$. When $b\ne\beta$, even if $\lim_{n\to\infty}\bar{G}_n = G$ exists, the right-hand side of (3.16) with $\bar{G}_n$ replaced by G cannot be simplified to the expression of James and Smith in (1.11), noting that

$$1-\int_{-\infty}^\infty F(t+(b-\beta)x)\,dH(x) = \exp\left\{-\int_{-\infty}^t\frac{\int_{-\infty}^\infty f(u+(b-\beta)x)\,dH(x)}{\int_{-\infty}^\infty[1-F(u+(b-\beta)x)]\,dH(x)}\,du\right\}.$$

(ii) Since $F(s) = F_n(\beta, s)$, it follows from (3.1), (3.3), (3.5) and (3.7) that $\sup\{|F_n(b,s)-F(s)|\colon |b-\beta|\le n^{-3\delta}, EZ_n(b,s)\ge n^{1-\delta}\} = O(n^{-\delta})$ a.s., for every $0<\delta<1$. Combining this with (3.9) and (3.4) shows that if $\liminf_{n\to\infty} n^{-1}\sum_{i=1}^n P\{c_i-\beta x_i > F^{-1}(1)\} > 0$, then for every $0<\gamma<\frac{3}{10}$,

$$\sup_{\substack{|b-\beta|\le n^{-\gamma}\\ -\infty<t<\infty}}|\hat{F}_{n,b}(t)-F(t)| = O(n^{-\gamma/3})\quad\text{a.s.},$$

$$\sup_{\substack{|b-\beta|\le n^{-\gamma}\\ -\infty<t<\infty}}\left|\left\{\int_0^\infty(1-\hat{F}_{n,b}(t))\,dt - \int_0^\infty(1-F(t))\,dt\right\} - \left\{\int_{-\infty}^0\hat{F}_{n,b}(t)\,dt-\int_{-\infty}^0F(t)\,dt\right\}\right| = O(n^{-\gamma/4})\quad\text{a.s.}$$

Therefore, in this case $\int_{-\infty}^{\infty} t\,d\hat{F}_{n,\hat{\beta}_n}(t)$ provides a strongly consistent estimate of the common mean $\int_{-\infty}^{\infty} t\,dF(t)$ of the ε_i if $\hat{\beta}_n - \beta = o(n^{-\gamma})$ a.s. for some $\gamma > 0$. Theorem 3 in the next section gives such consistency results (with $\gamma = \frac{1}{4}$) for the modified Buckley–James estimator $\hat{\beta}_n$. On the other hand, if $\limsup_{i \to \infty}(c_i - \beta x_i) < F^{-1}(1)$ a.s., then it is not possible to estimate $\int_{-\infty}^{\infty} t\,dF(t)$ consistently from the censored data (1.2) even when β is known, since the sample contains little information about $F(t)$ for $t > \limsup_{i \to \infty}(c_i - \beta x_i)$.

Lemma 2 shows that the modified Buckley–James statistics $\xi_n(b)$ can be approximated by their nonrandom counterparts $\zeta_n(b)$ with two kinds of error bounds for the approximation. The first kind, given by (3.10), implies the uniform strong law (3.17) in Theorem 1(i) below if $\lambda < \frac{1}{8}$. To establish the asymptotic normality of the modified Buckley–James estimator $\hat{\beta}_n$, which is a zero-crossing of $\xi_n(b)$, we need the result (3.12), which implies in the case $\lambda < \frac{1}{16}$ the asymptotic linearity result (3.20) of Theorem 1(ii). It will be shown in Section 4 that $n^{-1/2}\xi_n(\beta)$ has a limiting normal distribution. Combining this with (3.20) gives the asymptotic normality of $\hat{\beta}_n$, as will be discussed in detail in Section 4.

THEOREM 1. *Define* $\xi_n(b)$ *by* (2.10) *and* $\zeta_n(b)$ *by* (3.8).

(i) *Suppose that* $\lambda > 0$ *in the weight function* (2.2) *is so chosen that* $\lambda < \frac{1}{8}$. *Then under the assumptions* (3.1)–(3.5),

(3.17) $$\sup_{|b| \le \rho} n^{-1} |\xi_n(b) - \zeta_n(b)| \to 0 \quad a.s.$$

(ii) *Suppose that* $\lambda > 0$ *in* (2.2) *is so chosen that* $\lambda < \frac{1}{16}$ *and assume* (3.1)–(3.6). *Defining the* $\Gamma_r(s)$ *for* $r = 0, 1, 2$ *as in* (3.6), *let*

$$\tau = \sup\{t\colon (1 - F(t))\Gamma_0(t) > 0\},$$

(3.18)
$$A = \int_{-\infty}^{\tau} \left\{\Gamma_2(t) - \frac{\Gamma_1^2(t)}{\Gamma_0(t)}\right\} \frac{\int_t^{\tau}(1 - F(s))\,ds}{1 - F(t)} \left\{\frac{f'(t)}{f(t)} + \frac{f(t)}{1 - F(t)}\right\} dF(t).$$

Assume furthermore that

(3.19) $$\lim_{n \to \infty} n^{-(1-\lambda)} \sum_{i=1}^{n} P\{c_i - \beta x_i > \tau + \varepsilon\} = 0 \quad \textit{for every } \varepsilon > 0, \textit{if } F(\tau) < 1.$$

Then with probability 1,

(3.20) $$\xi_n(b) = \xi_n(\beta) - An(b - \beta) + o\big(\max\{n^{1/2}, n|b - \beta|\}\big) \quad \textit{uniformly in } |b - \beta| \le n^{-\lambda}.$$

REMARK. Note that A defined by (3.18) is indeed finite. Since Γ_2 and Γ_1^2/Γ_0 are bounded by $4B^2$ and since $\int_{-\infty}^{\infty}\{(f'/f)^2 + f^2/(1 - F)^2\}\,dF < \infty$ by (3.2) and Lemma 2 of Lai and Ying (1989), the finiteness of A follows from the

Schwarz inequality and the finiteness of

$$\lim_{T\uparrow\tau}\int_a^T\left[\int_t^T(1-F(s))\,ds\right]^2(1-F(t))^{-2}\,dF(t)$$

$$=\lim_{T\uparrow\tau}\left\{-\left[\int_a^T(1-F(s))\,ds\right]^2(1-F(a))^{-1}\right.$$

$$\left.+2\int_a^T(1-F(t))^{-1}\left[\int_t^T(1-F(s))\,ds\right](1-F(t))\,dt\right\}$$

$$=2\int_a^\tau(s-a)(1-F(s))\,ds-\left[\int_a^\tau(1-F(s))\,ds\right]^2(1-F(a))^{-1},$$

noting that $\int_{-\infty}^a[\int_t^\infty(1-F(s))\,ds]^2\,dF(t)<\infty$ by (3.2) for every $a\in(-\infty,\tau)$.

The proof of Theorem 1(ii) makes use of (3.12) and the following lemma, whose proof consists of an asymptotic analysis of the nonrandom function $\zeta_n(b)$ and is given in Appendix A.

Lemma 3. *With the same notation and assumptions as in Theorem* 1(ii), *as* $n\to\infty$,

(i) $\sup_{|b|\le\rho,|b'-b|\le n^{-\gamma}}|\zeta_n(b')-\zeta_n(b)|=O(n^{1-\gamma+4\lambda})$ *for any* $1>\gamma>\lambda$.

(ii) $\zeta_n(\beta)=0$ *and* $\zeta_n(b)\sim -An(b-\beta)$ *uniformly in* $|b-\beta|\le n^{-\lambda}$.

(iii) *If furthermore* $F(\tau)<1$, *then* $\zeta_n(b)\sim -An(b-\beta)$ *as* $n\to\infty$ *and* $b\to\beta$.

Proof of Theorem 1(ii). In view of (3.12) and Lemma 3(ii), (3.20) holds in the interval $|b-\beta|\le n^{-1/2+\delta}$ for $\delta>0$ with $4\lambda+\delta<\frac{1}{4}$. To establish (3.20) in the interval $n^{-1/2+\delta}\le|b-\beta|\le n^{-\lambda}$, take any $0<\gamma<\frac{1}{2}$. For $|b-\beta|\ge n^{-\gamma-\delta}$, $n|b-\beta|\ge n^{1-\gamma-\delta}$ with $1-\gamma-\delta>\max\{(1-\gamma)/2+4\lambda,\frac{1}{2}-\gamma+6\lambda,7\lambda\}$, since $4\lambda+\delta<\frac{1}{4}$ and $1-\gamma>\frac{1}{2}$. Hence (3.12) implies that for any $0<\gamma<\frac{1}{2}$,

$$\sup_{n^{-\gamma-\delta}\le|b-\beta|\le n^{-\gamma}}\{|\xi_n(b)-\xi_n(\beta)-\zeta_n(b)+\zeta_n(\beta)|/(n|b-\beta|)\}\to 0\quad\text{a.s.}$$

Combining this with Lemma 3(ii) then shows that (3.20) holds in the interval $n^{-1/2+j\delta}\le|b-\beta|\le\min\{n^{-1/2+(j+1)\delta},n^{-\lambda}\}$ for $j=1,2,\dots$. □

4. Asymptotic normality of $\xi_n(\beta)$ and of the modified Buckley–James estimator. In this section we prove the asymptotic normality of the modified Buckley–James estimator $\hat\beta_n$ by first establishing in Theorem 2 the asymptotic normality of $\xi_n(\beta)$ and then combining this result with Theorem 1(ii) on the asymptotic linearity of $\xi_n(b)-\xi_n(\beta)$.

THEOREM 2. *Suppose that* $\lambda > 0$ *in the weight function* (2.2) *is so chosen that* $\lambda < \frac{1}{3}$. *Define* $\hat{F}_{n,b}$ *by* (1.6), $\xi_n(b)$ *by* (2.10) *and let*

$$\begin{aligned}\xi_{n,1} = -\sum_{i=1}^{n}(x_i - \bar{x}_n)\bigg\{&\int_{-\infty}^{\infty} t\, d\big[p_n(n^{-1}EZ_n(\beta,t))I_{\{z_i - \beta x_i \geq t\}}\big]\\ &+\int_{-\infty}^{\infty}\bigg[\int_t^{\infty}\frac{1-F(s)}{1-F(t)}p_n(n^{-1}EZ_n(\beta,s))\,ds\bigg]dI_{\{c_i-\beta x_i \geq t,\, c_i - \beta x_i < \varepsilon_i\}}\bigg\},\end{aligned}\tag{4.1}$$

$$W_n(t) = \frac{\big(\hat{F}_{n,\beta}(t) - F(t)\big)}{(1 - F(t))},\tag{4.2}$$

$$\begin{aligned}\xi_{n,2} = \int_{-n^{\lambda}}^{n^{\lambda}}&\bigg[E\sum_{i=1}^{n}(x_i - \bar{x}_n)I_{\{c_i - \beta x_i \geq t\}}\bigg]\\ &\times\bigg[\int_t^{n^{\lambda}}(1 - F(s))p_n(n^{-1}EZ_n(\beta,s))\,ds\bigg]dW_n(t).\end{aligned}\tag{4.3}$$

(i) *Under the assumptions* (3.1)–(3.5),

$$n^{-1/2}\{\xi_n(\beta) - (\xi_{n,1} + \xi_{n,2})\} \to_P 0.$$

(ii) *Under the assumptions* (3.1)–(3.6) *and* (3.19), $n^{-1/2}\xi_n(\beta)$ *converges in distribution to a normal random variable with mean* 0 *and variance*

$$v = \int_{-\infty}^{\tau}\bigg\{\Gamma_2(t) - \frac{\Gamma_1^2(t)}{\Gamma_0(t)}\bigg\}\bigg\{\frac{\int_t^{\tau}(1 - F(s))\,ds}{1 - F(t)}\bigg\}^2 dF(t),\tag{4.4}$$

where τ *is defined in* (3.18) *and the* $\Gamma_r(s)$ *are defined in* (3.6).

Note that $v < \infty$ (see Remark following Theorem 1). The definition (4.1) of $\xi_{n,1}$ corresponds to substituting the terms $p_n(n^{-1}Z_n(\beta, u))$ and $\hat{F}_{n,\beta}(u)$ in the definition (2.10) of $\xi_n(\beta)$ by the nonrandom functions $p_n(n^{-1}EZ_n(\beta, u))$ and $F(u)$. As will be shown in the proof of Theorem 2, $n^{-1/2}\xi_{n,1}$ has a limiting normal distribution with mean 0 and variance

$$\tilde{v} = \int_{-\infty}^{\tau}\Gamma_2(t)\bigg\{\frac{\int_t^{\tau}(1 - F(s))\,ds}{1 - F(t)}\bigg\}^2 dF(t),\tag{4.5}$$

see (4.23). Making use of the limit in (1.11) and under more restrictive assumptions, Smith (1988) recently concluded from certain variance computations that $\sum_{i=1}^{n}(x_i - \bar{x}_n)y_i^*(\beta)$ [the left-hand side of the Buckley–James equation (1.8)] is asymptotically equivalent in probability to

$$\sum_{i=1}^{n}(x_i - \bar{x}_n)\bigg\{z_i(\beta) + (1 - \delta_i)\frac{\int_{z_i(\beta)}^{\infty}(1 - F(s))\,ds}{1 - F(z_i(\beta))}\bigg\},\tag{4.6}$$

which is tantamount to replacing $\hat{F}_{n,\beta}(s)$ in (1.5) by $F(s)$ and which by the classical central limit theorem is asymptotically normal $N(0, n\tilde{v})$ under the assumption (3.6) and is therefore asymptotically equivalent to $\xi_{n,1}$. However, Theorem 2 and its proof suggest that, except in the special case $\Gamma_1(t) = 0$ for almost every t (with respect to F), we need to add another term that is asymptotically equivalent to $\xi_{n,2}$ when we replace $\hat{F}_{n,\beta}(s)$ in (1.5) by $F(s)$ to approximate $\sum_{i=1}^n (x_i - \bar{x}_n) y_i^*(\beta)$. Indeed, $\tilde{v}$ is larger than the variance (4.4) of the limiting normal distribution in Theorem 2 unless $\Gamma_1 = 0$ a.e. (F), which is essentially the setting considered by Smith (1988).

Under the additional assumption that (c_i, x_i) are i.i.d., Ritov (1990) recently proved the asymptotic normality of a class of statistics $\Psi_n(\beta)$ that look similar to $\xi_n(\beta)$. Specifically, let ψ be a real-valued function on the real line such that $\int_{-\infty}^{\infty} \psi^2(t)\, dF(t) < \infty$ and such that there exist $K > 0$ and $\delta > 0$ for which (i) $\psi(t) = \psi(\min(t, K))$ for all t, (ii) $P\{\min(y_1, c_1) - bx_1 < K\} < 1$ for all $b \in [\beta - \delta, \beta + \delta]$, (iii) $\lim_{d \to 0} \int \sup\{[\psi(t+u) - \psi(t)]^2 : |u| \le d\}\, dF(t) = 0$. Define

$$\Psi_n(b) = n^{-1/2} \sum_{i=1}^n (x_i - \bar{x}_n) \left\{ \delta_i \psi(z_i(b)) + (1 - \delta_i) \frac{\int_{z_i(b)}^{\infty} \psi(t)\, d\hat{F}_{n,b}(t)}{1 - \hat{F}_{n,b}(z_i(b))} \right\}.$$

By establishing in his Proposition 4.1 that $\Psi_n(\beta)$ can be approximated by certain censored linear rank statistics Γ_n in the sense that $|\Psi_n(\beta) - \Gamma_n| \to_P 0$, he then obtained the limiting normal distribution of $\Psi_n(\beta)$ from the corresponding well known results for Γ_n; see Gill (1980). For the function $\psi(t) = t$, which clearly violates condition (i), $n^{1/2}\Psi_n(b)$ reduces to the Buckley–James statistic $\sum_{i=1}^n (x_i - \bar{x}_n) y_i^*(b)$. It is natural to ask whether the proof of Ritov's (1990) Proposition 4.1 can be modified to make it also work for the function $\psi(t) = t$. We have looked into this possibility and have found the answer to be negative; see Lai and Ying (1991).

We preface the proof of Theorem 2 by the following three lemmas. Making use of the approximation in Theorem 2(i) and Lemma 6, we obtain the asymptotic normality result in Theorem 2(ii). To prove Theorem 2(i), let $p_{n,\beta}(s) = p_n(n^{-1} E Z_n(\beta, s))$ and $\hat{p}_{n,\beta}(s) = p_n(n^{-1} Z_n(\beta, s))$. By (4.2) for $s \ge t$,

$$\frac{1 - \hat{F}_{n,\beta}(s)}{1 - \hat{F}_{n,\beta}(t)} - \frac{1 - F(s)}{1 - F(t)} = - \frac{(W_n(s) - W_n(t))(1 - F(s))}{1 - \hat{F}_{n,\beta}(t)},$$

and therefore

$$\begin{aligned}
&\int_t^{n^\lambda} \frac{1 - \hat{F}_{n,\beta}(s)}{1 - \hat{F}_{n,\beta}(t)} \hat{p}_{n,\beta}(s)\, ds - \int_t^{n^\lambda} \frac{1 - F(s)}{1 - F(t)} \hat{p}_{n,\beta}(s)\, ds \\
&\qquad = -\left(1 - \hat{F}_{n,\beta}(t)\right)^{-1} \int_t^{n^\lambda} (1 - F(s)) \hat{p}_{n,\beta}(s) \int_{t+}^{s} dW_n(u)\, ds \qquad (4.7)\\
&\qquad = -\left(1 - \hat{F}_{n,\beta}(t)\right)^{-1} \int_{t+}^{n^\lambda} \int_u^{n^\lambda} (1 - F(s)) \hat{p}_{n,\beta}(s)\, ds\, dW_n(u).
\end{aligned}$$

For notational simplicity we let $Z_n^x(t)$, $J_n^x(t)$, $N_n(t)$ denote $Z_n^x(\beta, t)$, $J_n^x(\beta, t)$ and $N_n(\beta, t)$. From (3.14) and (4.7), it follows that with probability 1, for all large n,

$$\xi_n(\beta) = \hat{\xi}_{n,1} + \hat{\xi}_{n,2}, \tag{4.8}$$

where

$$\hat{\xi}_{n,1} = -\int_{-n^\lambda}^{n^\lambda} t\, d\big[\hat{p}_{n,\beta}(t) Z_n^x(t)\big] - \int_{-n^\lambda}^{n^\lambda}\int_t^{n^\lambda} \frac{1-F(s)}{1-F(t)} \hat{p}_{n,\beta}(s)\, ds\, dJ_n^x(t),$$

$$\hat{\xi}_{n,2} = \int_{-n^\lambda}^{n^\lambda}\big(1-\hat{F}_{n,\beta}(t)\big)^{-1}\int_{t+}^{n^\lambda}\int_u^{n^\lambda}(1-F(s))\hat{p}_{n,\beta}(s)\, ds\, dW_n(u)\, dJ_n^x(t)$$

$$= \int_{-n^\lambda}^{n^\lambda}\int_{-n^\lambda}^{u-}\big(1-\hat{F}_{n,\beta}(t)\big)^{-1} dJ_n^x(t)\int_u^{n^\lambda}(1-F(s))\hat{p}_{n,\beta}(s)\, ds\, dW_n(u).$$

Lemma 4 shows that we can approximate $\hat{p}_{n,\beta}(t)$ in $\hat{\xi}_{n,1}$ by $p_{n,\beta}(t)$, leading to the term $\xi_{n,1}$ defined by (4.1). Lemma 5 shows that we can approximate $\hat{p}_{n,\beta}(t)$ in $\hat{\xi}_{n,2}$ by $p_{n,\beta}(t)$ and a further approximation of $\int_{-n^\lambda}^{u-}(1-\hat{F}_{n,\beta}(t))^{-1}\, dJ_n^x(t)$ in Lemma 5 leads to the term $\xi_{n,2}$ defined by (4.3). From Lemmas 4 and 5, whose proofs are given in Appendix B, Theorem 2(i) follows.

LEMMA 4. *With the same notation and assumptions as in Theorem* 2(i),

$$n^{-1/2}\big(\hat{\xi}_{n,1} - \xi_{n,1}\big) \to_P 0.$$

LEMMA 5. *With the same notation and assumptions as in Theorem* 2(i), *let* $T_n = \max_{i\le n} z_i(\beta)$ *and let* $\Lambda(t) = -\log(1-F(t))$. *Let* $\mathscr{F}_n(t)$ *be the complete σ-field generated by*

$$x_i,\qquad I_{\{z_i(\beta)\le t\}},\qquad \delta_i I_{\{z_i(\beta)\le t\}},\qquad z_i(\beta) I_{\{z_i(\beta)\le t\}},\qquad i=1,\dots,n.$$

Then $\{W_n(t), \mathscr{F}_n(t), -\infty < t \le T_n\}$ *is a martingale with predictable variation process*

$$\langle W_n\rangle(t) = \int_{-\infty}^t \left[\frac{1-\hat{F}_{n,\beta}(s-)}{1-F(s)}\right]^2 \frac{d\Lambda(s)}{Z_n(s)}. \tag{4.9}$$

In fact, $\{N_n(t) - \int_{-\infty}^t Z_n(s)\, d\Lambda(s), \mathscr{F}_n(t), -\infty < t \le T_n\}$ *is a martingale and*

$$W_n(t) = \int_{-\infty}^t \frac{1-\hat{F}_{n,\beta}(s-)}{(1-F(s))Z_n(s)}\big[dN_n(s) - Z_n(s)\, d\Lambda(s)\big]. \tag{4.10}$$

Moreover, letting $C_n(t) = \sum_{i=1}^n P\{c_i - \beta x_i \ge t\}$, *we have for every* $\delta > 0$,

$$\begin{aligned}&\sup_{t\le T_n}(1-F(t))^{(1+\delta)/2} C_n^{1/2}(t)|W_n(t)|\\ &\qquad + \sup_{t\le T_n}(1-F(t))C_n(t)/Z_n(t) = O_p(1),\end{aligned} \tag{4.11}$$

$$n^{-1/2}\big(\hat{\xi}_{n,2} - \xi_{n,2}\big) \to_P 0. \tag{4.12}$$

LEMMA 6. *With the same notation and assumptions as in Theorem* 2(ii), *let* $C_n^x(t) = E[\sum_{i=1}^n (x_i - \bar{x}_n) I_{\{c_i - \beta x_i \geq t\}}]$, $C_n(t) = \sum_{i=1}^n P\{c_i - \beta x_i \geq t\}$. *For* $u \geq -n^\lambda$, *define*

$$\xi_{n,2}(u) = \int_{-n^\lambda}^u C_n^x(t) \left[\int_t^{n^\lambda} (1 - F(s)) p_{n,\beta}(s)\, ds \right] dW_n(t),$$

$$\tilde{\xi}_{n,2}(u) = \int_{-n^\lambda}^u C_n^x(t) \frac{\int_t^{n^\lambda} (1 - F(s)) p_{n,\beta}(s)\, ds}{(1 - F(t)) C_n(t)}\, dM_n(t),$$

where $M_n(t) = N_n(t) - \int_{-\infty}^t Z_n(s)\, d\Lambda(s)$.

(i) *For every* $t^* < \tau$,

$$\sup_{u \leq t^*} n^{-1/2} |\xi_{n,2}(u) - \tilde{\xi}_{n,2}(u)| \to_P 0.$$

(ii) *For every* $\varepsilon > 0$,

$$\lim_{t^* \uparrow \tau} \limsup_{n \to \infty} P\left\{ \sup_{t^* \leq u \leq n^\lambda} n^{-1/2} |\xi_{n,2}(u) - \xi_{n,2}(t^*)| \geq \varepsilon \right\} = 0.$$

PROOF. Take any $t^* < \tau$. In view of (4.10),

$$\begin{aligned} \xi_{n,2}(u) - \tilde{\xi}_{n,2}(u) = \int_{-n^\lambda}^u C_n^x(t) \left[\int_t^{n^\lambda} (1 - F(s)) p_{n,\beta}(s)\, ds \right] \\ \times \left\{ \frac{1 - \hat{F}_{n,\beta}(t-)}{(1 - F(t)) Z_n(t)} - \frac{1}{EZ_n(t)} \right\} dM_n(t). \end{aligned} \tag{4.13}$$

Since $\{M_n(t), \mathscr{F}_n(t), -\infty < t < \infty\}$ is a martingale with predictable variation process $\langle M_n \rangle(t) = \int_{-\infty}^t Z_n(s)\, d\Lambda(s)$ [cf. Gill (1980)], $\{\xi_{n,2}(u) - \tilde{\xi}_{n,2}(u), \mathscr{F}_n(u), u \leq t^*\}$ is a martingale with

$$\begin{aligned} \langle \xi_{n,2} - \tilde{\xi}_{n,2} \rangle (t^*) = \int_{-n^\lambda}^{t^*} \left(\frac{C_n^x(t)}{n} \right)^2 \left[\int_t^{n^\lambda} (1 - F(s)) p_{n,\beta}(s)\, ds \right]^2 \\ \times \left\{ \frac{1 - \hat{F}_{n,\beta}(t-)}{1 - F(t)} \frac{n}{Z_n(t)} - \frac{n}{EZ_n(t)} \right\}^2 \\ \times Z_n(t) (1 - F(t))^{-1} dF(t) = O_p(1), \end{aligned} \tag{4.14}$$

noting that $\int_{t^*}^\infty (1 - F(s))\, ds + \int_{-\infty}^{t^*} u^2\, dF(u) < \infty$ and that

$$\sup_{t \leq t^*} \left\{ \left| \frac{1 - \hat{F}_{n,\beta}(t-)}{1 - F(t)} - 1 \right| + \left| \frac{n}{Z_n(t)} - \frac{n}{EZ_n(t)} \right| \right\} = O_p(n^{-1/2}). \tag{4.15}$$

From (4.13), (4.14) and Lenglart's inequality [cf. Gill (1980)], (i) follows.

To prove (ii), note that by (3.6) and the dominated convergence theorem,

$$\lim_{n \to \infty} n^{-1} C_n(t) = \Gamma_0(t), \qquad \lim_{n \to \infty} n^{-1} C_n^x(t) = \Gamma_1(t). \tag{4.16}$$

By (4.9) and Lenglart's inequality, for every $\eta > 0$,

$$P\left\{\sup_{t^*<u\le n^\lambda} n^{-1/2}\left|\int_{t^*+}^{u} C_n^x(t)\left[\int_t^{n^\lambda}(1-F(s))p_{n,\beta}(s)\,ds\right]dW_n(t)\right| \ge \varepsilon\right\}$$

$$\le \frac{\eta}{\varepsilon^2} + P\left\{\int_{t^*}^{n^\lambda}\left(\frac{C_n^x(t)}{n}\right)^2\left[\int_t^{n^\lambda}(1-F(s))p_{n,\beta}(s)\,ds\right]^2\right.$$

$$\left.\times\left\{\frac{1-\hat{F}_{n,\beta}(t-)}{1-F(t)}\right\}^2\frac{n}{Z_n(t)}\frac{dF(t)}{1-F(t)} \ge \eta\right\}$$

$$\to \frac{\eta}{\varepsilon^2} + P\left\{\int_{t^*}^{\tau}\frac{\Gamma_1^2(t)}{\Gamma_0(t)}\left[\int_t^{\tau}\frac{1-F(s)}{1-F(t)}\,ds\right]^2 dF(t) \ge \eta\right\} \quad \text{as } n\to\infty,$$

by (4.11), (4.15), (4.16) and the finiteness of $\int_{-\infty}^{\tau}[\int_t^{\tau}(1-F(s))\,ds/(1-F(t))]^2\,dF(t)$ (see Remark following Theorem 1), noting that $\sup_{t\le T_n}\{(1-\hat{F}_{n,\beta}(t-))/(1-F(t))\} = O_p(1)$ by Theorem 3.2.1 of Gill (1980). Letting $t^*\uparrow\tau$ and then $\eta\downarrow 0$ gives the desired conclusion. □

PROOF OF THEOREM 2. (i) follows from Lemmas 4 and 5. To prove (ii), we shall show that for every $u<\tau$ and every a,

$$P\left\{n^{-1/2}\left(\xi_{n,1}+\tilde{\xi}_{n,2}(u)\right)\le a\,\middle|\,x_1,\dots,x_n\right\} \to_P P\left\{N(0,\sigma_u^2)\le a\right\}, \tag{4.17}$$

where

$$\sigma_u^2 = \int_{-\infty}^{\tau}\Gamma_2(t)\left[\int_t^{\tau}\frac{1-F(s)}{1-F(t)}\,ds\right]^2 dF(t)$$

$$-\int_{-\infty}^{u}\frac{\Gamma_1^2(t)}{\Gamma_0(t)}\left[\int_t^{\tau}\frac{1-F(s)}{1-F(t)}\,ds\right]^2 dF(t).$$

From (4.17) and Lemma 6(i), it follows that $\lim_{n\to\infty}P\{n^{-1/2}(\xi_{n,1}+\tilde{\xi}_{n,2}(u))\le a\} = P\{N(0,\sigma_u^2)\le a\}$ for every $u<\tau$ and for every a. Combining this result with Lemma 6(ii) and Theorem 2(i) and noting that $\sigma_u^2\to v$ as $u\uparrow\tau$, we then obtain $N(0,v)$ as the limiting distribution of $n^{-1/2}\xi_n(\beta)$.

To prove (4.17), let $Z^i(t) = I_{\{z_i-\beta x_i\ge t\}}$, $J^i(t) = I_{\{c_i-\beta x_i\ge t,\, c_i-\beta x_i<\varepsilon_i\}}$ and define

$$U_{ni} = \int_{-\infty}^{\infty} t\,d\left[p_{n,\beta}(t)\left\{Z^i(t)-E\left(Z^i(t)\,|\,x_1,\cdots,x_n\right)\right\}\right]$$

$$+\int_{-\infty}^{\infty}\left\{\int_t^{\infty}(1-F(s))p_{n,\beta}(s)\,ds/(1-F(t))\right\}$$

$$\times d\left[J^i(t)-E\left(J^i(t)\,|\,x_1,\cdots,x_n\right)\right].$$

By (4.1) and (2.9), noting that $\Sigma_{i=1}^n(x_i - \bar{x}_n) = 0$,

$$\xi_{n,1} = -\sum_{i=1}^{n} (x_i - \bar{x}_n) U_{ni}. \tag{4.18}$$

Let $G_i(x_i, t) = P\{c_i - \beta x_i \ge t | x_i\}$. By (2.2), (2.3), (3.6) and (3.19),

$$\lim_{n\to\infty} p_{n,\beta}(t) = \begin{cases} 1 & \text{if } t < \tau, \\ 0 & \text{if } t > \tau \text{ and } F(\tau) < 1. \end{cases} \tag{4.19}$$

Conditional on $x_1, \dots, x_n$, the $U_{ni}(i = 1, \dots, n)$ are independent zero-mean random variables with

$$\begin{aligned}
&\operatorname{Var}(U_{ni}|x_1, \dots, x_n) \\
&\quad\to E\left[\left\{\min(z_i - \beta x_i, \tau) + I_{\{c_i - \beta x_i < \min(\varepsilon_i, \tau)\}} \frac{\int_{c_i-\beta x_i}^{\tau}(1 - F(s))\, ds}{1 - F(c_i - \beta x_i)}\right\}^2 \middle| x_i\right] \\
&\qquad - \left\{-\int_{-\infty}^{\tau} t\, d(1 - F(t)) + \tau(1 - F(\tau))\right\}^2 \\
&\hspace{6cm} [\text{as } n \to \infty, \text{ by (4.32) and (2.9)}] \\
&= -\left\{\int_{-\infty}^{\tau} t\, dF(t) + \tau(1 - F(\tau))\right\}^2 + \int_{-\infty}^{\tau} t^2 G_i(x_i, t)\, dF(t) \\
&\qquad + \tau^2 \int_{\tau}^{\infty} G_i(x_i, t)\, dF(t) - \tau^2 \int_{\tau}^{\infty} (1 - F(t))\, dG_i(x_i, t) \\
&\qquad - \int_{-\infty}^{\tau-} \left\{t + \int_t^{\tau} \frac{1 - F(s)}{1 - F(t)}\, ds\right\}^2 (1 - F(t))\, dG_i(x_i, t) \\
&= \int_{-\infty}^{\tau} \left\{\int_t^{\tau} \frac{1 - F(s)}{1 - F(t)}\, ds\right\}^2 G_i(x_i, t)\, dF(t).
\end{aligned} \tag{4.20}$$

The last equality can be shown by applying integration by parts to the last two terms of the preceding expression and noting that G_i is left continuous and that

$$\lim_{t\to-\infty} \left\{t + \int_t^{\tau} (1 - F(s))\, ds/(1 - F(t))\right\} = \tau(1 - F(\tau)) + \int_{-\infty}^{\tau} s\, dF(s).$$

Moreover, the convergence in (4.20) is uniform in $i \le n$ and the last term in (4.20) is bounded by $\int_{-\infty}^{\tau}\{\int_t^{\tau}(1 - F(s))\, ds/(1 - F(t))\}^2\, dF(t) < \infty$ (cf. Remark following Theorem 1).

Let $N^i(t) = I_{\{\varepsilon_i \le t, \varepsilon_i \le c_i - \beta x_i\}}$. Fix $u < \tau$ and define

$$V_{ni}(u) = \int_{-n^{\lambda}}^{u} C_n^x(t) \frac{\int_t^{n^{\lambda}}(1 - F(s)) p_{n,\beta}(s)\, ds}{(1 - F(t)) C_n(t)} [\, dN^i(t) - Z^i(t)\, d\Lambda(t)],$$

where C_n^x and C_n are defined in Lemma 6. Then

$$\tilde{\xi}_{n,2}(u) = \sum_{i=1}^{n} V_{ni}(u). \tag{4.21}$$

Conditional on $x_1, \dots, x_n$, the $V_{ni}(u)$ $(i = 1, \dots, n)$ are independent zero-mean random variables with

$$\begin{aligned}
&\operatorname{Var}(V_{ni}(u)|x_1, \dots, x_n) \\
&\quad = \int_{-n^\lambda}^{u} (C_n^x(t))^2 \left[\frac{\int_t^{n^\lambda}(1 - F(s))p_{n,\beta}(s)\,ds}{(1 - F(t))C_n(t)}\right]^2 E(Z^i(t)|x_i)\,d\Lambda(t) \\
&\quad = \int_{-n^\lambda}^{u} \left[\frac{C_n^x(t)}{C_n(t)}\right]^2 \left[\frac{\int_t^{n^\lambda}(1 - F(s))p_{n,\beta}(s)\,ds}{1 - F(t)}\right]^2 G_i(x_i, t)\,dF(t) \qquad (4.22) \\
&\quad \le 4B^2(1 - F(u))^{-2} \int_{-\infty}^{u} \left[\int_0^\infty (1 - F(s))\,ds + |t|\right]^2 dF(t) < \infty.
\end{aligned}$$

Since $n^{-1}\sum_{i=1}^n (x_i - \bar{x}_n)^2 G_i(x_i, t) \to_P \Gamma_2(t)$, it follows from (4.18) and (4.20) that

$$\begin{aligned}
n^{-1}\operatorname{Var}(\xi_{n,1}|x_1, \dots, x_n) &= n^{-1} \sum_{i=1}^{n} (x_i - \bar{x}_n)^2 \operatorname{Var}(U_{ni}|x_1, \dots, x_n) \qquad (4.23) \\
&\to_P \int_{-\infty}^{\tau} \Gamma_2(t) \left[\int_t^\tau \frac{1 - F(s)}{1 - F(t)}\,ds\right]^2 dF(t).
\end{aligned}$$

Since $n^{-1}\sum_{i=1}^n G_i(x_i, t) \to_P \Gamma_0(t)$ by (3.6), it follows from (4.21), (4.22) and (4.16) that

$$\begin{aligned}
n^{-1}\operatorname{Var}\big(\tilde{\xi}_{n,2}(u)\big|x_1, \dots, x_n\big) &= n^{-1} \sum_{i=1}^{n} \operatorname{Var}(V_{ni}(u)|x_1, \dots, x_n) \qquad (4.24) \\
&\to_P \int_{-\infty}^{u} \left[\int_t^\tau \frac{1 - F(s)}{1 - F(t)}\,ds\right]^2 \frac{\Gamma_1^2(t)}{\Gamma_0(t)}\,dF(t).
\end{aligned}$$

It will be shown later that

$$\begin{aligned}
&2n^{-1}\operatorname{Cov}\big(\xi_{n,1}, \tilde{\xi}_{n,2}(u)\big|x_1, \dots, x_n\big) \\
&\quad = 2n^{-1} \sum_{i=1}^{n} (x_i - \bar{x}_n) E(U_{ni}V_{ni}(u)|x_1, \dots, x_n) \qquad (4.25) \\
&\quad \to_P -2\int_{-\infty}^{u} \left[\int_t^\tau \frac{1 - F(s)}{1 - F(t)}\,ds\right]^2 \frac{\Gamma_1^2(t)}{\Gamma_0(t)}\,dF(t).
\end{aligned}$$

In view of (4.18) and (4.21), we can apply the central limit theorem for sums of (triangular array of) independent random variables to conclude that the conditional distribution of $n^{-1/2}(\xi_{n,1} + \tilde{\xi}_{n,2}(u))$ given $x_1, \dots, x_n$ converges to the normal distribution with mean 0 and variance

$$\int_{-\infty}^{\tau} \Gamma_2(t)\left[\int_t^{\tau} \frac{1-F(s)}{1-F(t)}\,ds\right]^2 dF(t)$$

$$-\int_{-\infty}^{u}\left[\int_t^{\tau} \frac{1-F(s)}{1-F(t)}\,ds\right]^2 \frac{\Gamma_1^2(t)}{\Gamma_0(t)}\,dF(t) = \sigma_u^2,$$

given by (4.23)–(4.25), thus establishing (4.17).

To prove (4.25), we note that similar to (4.20), for any $u < \tau$, as $n \to \infty$,

$$E[U_{ni}V_{ni}(u)|x_1,\dots,x_n]$$

$$\to E\left[\left\{\min(z_i - \beta x_i, \tau) + (1-\delta_i)\frac{\int_{z_i-\beta x_i}^{\tau}(1-F(s))\,ds}{1-F(z_i-\beta x_i)}\right\}\right.$$
$$\left.\times\left\{\int_{-\infty}^{u} \frac{C_n^x(t)\int_t^{\tau}(1-F(s))\,ds}{C_n(t)(1-F(t))}(dN^i(t) - Z^i(t)\,d\Lambda(t))\right\}\middle|x_i\right] \tag{4.26}$$

uniformly in $i \le n$. For $t < \tau$,

$$E\left[N^i(t)\left\{\min(z_i-\beta x_i,\tau) + (1-\delta_i)\frac{\int_{z_i-\beta x_i}^{\tau}(1-F(s))\,ds}{1-F(z_i-\beta x_i)}\right\}\middle|x_i\right]$$
$$= E\left[N^i(t)\varepsilon_i|x_i\right] = \int_{-\infty}^{t} sG_i(x_i,s)\,dF(s), \tag{4.27}$$

$$E\left[Z^i(t)\left\{\min(z_i-\beta x_i,\tau) + (1-\delta_i)\frac{\int_{z_i-\beta x_i}^{\tau}(1-F(s))\,ds}{1-F(z_i-\beta x_i)}\right\}\middle|x_i\right]$$
$$= -\int_{s=t}^{\infty} \min(s,\tau)\,d[(1-F(s))G_i(x_i,s)]$$
$$-\int_{u=t}^{\tau} \frac{\int_u^{\tau}(1-F(s))\,ds}{1-F(u)}(1-F(u))\,dG_i(x_i,u)$$
$$= \tau(1-F(\tau))G_i(x_i,\tau) - \int_{s=t}^{\tau} s\,d[(1-F(s))G_i(x_i,s)]$$
$$-\int_t^{\tau}(1-F(s))\left[\int_{u=t}^{s} dG_i(x_i,u)\right]ds$$
$$= t(1-F(t))G_i(x_i,t) + G_i(x_i,t)\int_t^{\tau}(1-F(s))\,ds. \tag{4.28}$$

The last equality in (4.28) follows by applying integration by parts to the second term of the preceding expression. In view of (4.27) and (4.28), the right-hand side of (4.26) can be expressed as

$$\begin{aligned}&\int_{-\infty}^{u}\frac{C_n^x(t)\int_t^\tau(1-F(s))\,ds}{C_n(t)(1-F(t))}\Bigg\{tG_i(x_i,t)\,dF(t)\\ (4.29)\qquad &\qquad -G_i(x_i,t)\left[t(1-F(t))+\int_t^\tau(1-F(s))\,ds\right]\frac{dF(t)}{1-F(t)}\Bigg\}\\ &=\int_{-\infty}^{u}\frac{C_n^x(t)G_i(x_i,t)}{C_n(t)}\left[\frac{\int_t^\tau(1-F(s))\,ds}{1-F(t)}\right]^2 dF(t).\end{aligned}$$

Since $n^{-1}\sum_{i=1}^n(x_i-\bar{x}_n)G_i(x_i,t)\to_P\Gamma_1(t)$, (4.25) follows from (4.16), (4.26) and (4.29). □

Combining Theorems 1 and 2, we obtain the following theorem on the consistency and asymptotic normality of the modified Buckley–James estimator $\hat{\beta}_n$.

THEOREM 3. *Under the assumptions* (3.1)–(3.6) *and* (3.19), *define* $\xi_n(b)$ *by* (2.10) *and* $\zeta_n(b)$ *by* (3.8) *and define the modified Buckley–James estimator as a zero-crossing of* $\xi_n(b)$ *in the interval* $[-\rho,\rho]$ *with* $\rho>|\beta|$. *Suppose that* $\lambda>0$ *in the weight function* (2.2) *is so chosen that* $\lambda<\frac{1}{16}$. *Assume that*

$$(4.30)\qquad \lim_{n\to\infty} n^{-3/4}\left\{\inf_{\substack{|b|\le\rho,\\ |b-\beta|\ge n^{-\lambda}}}|\zeta_n(b)|\right\}=\infty,$$

and that $A\neq 0$, *where* A *is defined in* (3.18). *Then* $\hat{\beta}_n-\beta=o(n^{-1/4})$ *a.s. and* $n^{1/2}(\hat{\beta}_n-\beta)$ *has a limiting normal distribution with mean* 0 *and variance*

$$(4.31)\qquad A^{-2}\int_{-\infty}^{\tau}\left\{\Gamma_2(t)-\frac{\Gamma_1^2(t)}{\Gamma_0(t)}\right\}\left\{\frac{\int_t^\tau(1-F(s))\,ds}{1-F(t)}\right\}dF(s).$$

PROOF. Putting $\gamma=1-\lambda(>\lambda)$ in Lemma 3(i), we obtain that

$$(4.32)\qquad \sup_{\substack{b,b'\in[-\rho,\rho],\\ |b-b'|\le n^{-1+\lambda}}}|\zeta_n(b)-\zeta_n(b')|=O(n^{5\lambda})=o(n^{3/4}).$$

Since $4\lambda<\frac{1}{4}$, it follows from (3.10) that

$$(4.33)\qquad \sup_{|b|\le\rho}|\xi_n(b)-\zeta_n(b)|=o(n^{3/4})\quad\text{a.s.}$$

In view of (4.30), (4.32) and (4.33),

$$(4.34)\qquad \begin{aligned}&P\{\xi_n(b)\text{ does not have a zero-crossing on }|b-\beta|\ge n^{-\lambda}\\ &\qquad\qquad \text{and }|b|\le\rho\text{ for all large }n\}=1.\end{aligned}$$

From (4.33), (4.34) and Lemma 3(ii), it follows that $\hat{\beta}_n=\beta+o(n^{-1/4})$ a.s.

Hence by Theorem 1(ii), given $0 < \varepsilon < 1$, there exists n_0 such that $P(\Omega_n) \ge 1 - \varepsilon$ for all $n \ge n_0$, where

$$\Omega_n = \left\{ \sup_{|b-\beta| \le n^{-1/4}} |\xi_n(b) - \xi_n(\beta) + An(b-\beta)| \le \varepsilon^2 \max(n^{1/2}, n|b-\beta|) \right.$$

$$\left. \text{and } |\hat{\beta}_n - \beta| \le n^{-1/4} \right\}.$$

On Ω_n, if $|b - \beta| \le \varepsilon^{-1} n^{-1/2}$, then $|\xi_n(b)| \ge \frac{1}{2}|A|\varepsilon^{-1} n^{1/2} - |\xi_n(\beta)|$ for all large n and sufficiently small ε. Since $|\hat{\beta}_n - \beta| \le n^{-1/4}$ on Ω_n and since $\hat{\beta}_n$ is a zero-crossing of $\xi_n(b)$, the desired conclusion then follows from the limiting normal distribution of $n^{-1/2}\xi_n(\beta)$ established in Theorem 2. □

REMARK. Note that condition (4.30) of Theorem 3 is satisfied if for some $\delta > 0$,

$$\liminf_{n \to \infty} n^{-(1-\lambda)} \left\{ \inf_{n^{-\lambda} \le |b-\beta| \le \delta} |\zeta_n(b)| \right\} > 0, \tag{4.35}$$

$$\liminf_{n \to \infty} n^{-1} \left\{ \inf_{|b| \le \rho, |b-\beta| \ge \delta} |\zeta_n(b)| \right\} > 0. \tag{4.36}$$

Suppose that $F(\tau) < 1$. Then by Lemma 3(iii), $\zeta_n(b) \sim -An(b - \beta)$ as $n \to \infty$ and $b \to \beta$. For $A \ne 0$, this implies that (4.35) holds for all sufficiently small $\delta > 0$. When $(x_i, c_i, \varepsilon_i)$ are i.i.d., $n^{-1}\zeta_n(b) \to \zeta(b)$ as $n \to \infty$ under (3.1)–(3.5), the convergence being uniform in $|b| \le \rho$, where

$$\zeta(b) = E\Bigg((x_1 - Ex_1)\Bigg\{E[\min(\varepsilon_1, c_1 - \beta x_1) - (b-\beta)x_1 | x_1]$$

$$- \int_{-\infty}^{\infty} h(t)\, dP[\varepsilon_1 + (\beta - b)x_1 > c_1 - bx_1 \ge t | x_1]\Bigg\}\Bigg),$$

$$h(t) = \int_t^{\infty} \exp\left\{ -\int_t^s \frac{E[f(u + (b-\beta)x_1) P(c_1 \ge u + bx_1 | x_1)]}{E[(1 - F(u + (b-\beta)x_1)) P(c_1 \ge u + bx_1 | x_1)]}\, du \right\} ds.$$

Moreover, $\zeta(b)$ is continuous for $|b| \le \rho$. Hence if $\zeta(b) \ne 0$ for $b \ne \beta$ (with $|b| \le \rho$), then (4.36) holds for all $\delta > 0$.

5. Extension to multiple regression models, asymptotic efficiency of the modified Buckley–James estimator and concluding remarks. Suppose that the β and x_i in (1.1) are replaced by $(p \times 1)$ vectors $\beta = (\beta_1, \dots, \beta_p)^T$ and $\mathbf{x}_i = (x_{i1}, \dots, x_{ip})^T$ and that by $\beta \mathbf{x}_i$ we mean $\beta^T \mathbf{x}_i$, where β^T denotes the transpose of β. Let $\bar{\mathbf{x}}_n = n^{-1}\sum_{i=1}^n \mathbf{x}_i = (\bar{x}_{n1}, \dots, \bar{x}_{np})^T$. As in Section 1, the observations are $(z_i, \delta_i, \mathbf{x}_i^T)$, $i = 1, \dots, n$, where z_i and δ_i are defined in (1.2) and $(c_i, \mathbf{x}_i^T)$ are independent random vectors that are independent of $\{\varepsilon_i\}$. For $\mathbf{b} = (b_1, \dots, b_p)^T$, define $y_i(\mathbf{b})$, $c_i(\mathbf{b})$ and $z_i(\mathbf{b})$ by (1.4) in which $\mathbf{b}\mathbf{x}_i$ refers to $\mathbf{b}^T\mathbf{x}_i$ and define scalars $Z_n(\mathbf{b}, t)$, $J_n(\mathbf{b}, t)$, $N_n(\mathbf{b}, t)$ and vectors $\mathbf{Z}_n^x(\mathbf{b}, t)$, $\mathbf{J}_n^x(\mathbf{b}, t)$, $\mathbf{N}_n^x(\mathbf{b}, t)$ as in (2.6). We also use $|\mathbf{a}|$ to denote $\sqrt{\mathbf{a}^T\mathbf{a}}$ for the vector **a**. Finally define the modified Buckley–James statistics $\zeta_{\mathbf{n}}(\mathbf{b}) = (\xi_{n1}(\mathbf{b}), \dots, \xi_{np}(\mathbf{b}))^T$ by (2.10) and define $\zeta_{\mathbf{n}}(\mathbf{b})$ by (3.8). As before, it will be assumed that an upper bound $\rho > \max_{j \le p}|\beta_j|$ is known so that the b_j can be restricted to $[-\rho, \rho]$. The multivariate version of assumption (3.6) takes the following form:

For $j, k \in \{1, \dots, p\}$ and for $t < F^{-1}(1)$,

$$
\begin{aligned}
&n^{-1} \sum_{i=1}^{n} P\{c_i - \beta^T \mathbf{x}_i \ge t | \mathbf{x}_i\} \to_P \Gamma_0(t),\\
(5.1)\qquad &n^{-1} \sum_{i=1}^{n} (x_{ik} - \bar{x}_{nk}) P\{c_i - \beta^T \mathbf{x}_i \ge t | \mathbf{x}_i\} \to_P \Gamma_k(t),\\
&n^{-1} \sum_{i=1}^{n} (x_{ij} - \bar{x}_{nj})(x_{ik} - \bar{x}_{nk}) P\{c_i - \beta^T \mathbf{x}_i \ge t | \mathbf{x}_i\} \to_P \Gamma_{jk}(t),
\end{aligned}
$$

where $\Gamma_0(t)$, $\Gamma_k(t)$ and $\Gamma_{kj}(t)$ are nonrandom functions. Let $\tau = \sup\{t\colon (1 - F(t))\Gamma_0(t) > 0\}$, as in (3.18).

Following James and Smith (1984), we shall define the multivariate Buckley–James estimator $\hat{\beta}_n$ as a minimizer of $|\xi_{\mathbf{n}}(\mathbf{b})|$. By using the same arguments in their proofs, Theorems 1–3 can be extended to the multiple regression setting. This is the content of:

THEOREM 4. *Assume* (3.1)–(3.5).

(i) *Suppose that* $\lambda > 0$ *in the weight function* (2.2) *is so chosen that* $\lambda < \frac{1}{8}$. *Then* $\sup_{|b_1| \le \rho, \dots, |b_p| \le \rho} n^{-1}|\xi_{\mathbf{n}}(\mathbf{b}) - \zeta_{\mathbf{n}}(\mathbf{b})| \to_P 0$.

(ii) *Suppose that* $0 < \lambda < \frac{1}{16}$ *and that* (5.1) *and* (3.19) *also hold. Then with probability* 1,

$$
(5.2)\qquad \begin{aligned}
\xi_{\mathbf{n}}(\mathbf{b}) = \xi_{\mathbf{n}}(\beta) - \mathbf{A}n(\mathbf{b} - \beta) + o\big(\max\{n^{1/2}, n|\mathbf{b} - \beta|\}\big)&\\
\textit{uniformly in } |\mathbf{b} - \beta| \le n^{-\lambda}&,
\end{aligned}
$$

where $\mathbf{A} = (a_{jk})_{1 \le j, k \le p}$ *is defined by*

$$
(5.3)\qquad \begin{aligned}
a_{jk} = \int_{-\infty}^{\tau} &\left\{\Gamma_{jk}(t) - \frac{\Gamma_j(t)\Gamma_k(t)}{\Gamma_0(t)}\right\} \frac{\int_t^{\tau}(1 - F(s))\, ds}{1 - F(t)}\\
&\times \left\{\frac{f'(t)}{f(t)} + \frac{f(t)}{1 - F(t)}\right\} dF(t).
\end{aligned}
$$

Moreover, $n^{-1/2}\xi_{\mathbf{n}}(\beta)$ *converges in distribution to a normal random vector with mean* 0 *and covariance matrix* $\mathbf{V} = (v_{jk})_{1 \le j, k \le p}$ *defined by*

$$
(5.4)\qquad v_{jk} = \int_{-\infty}^{\tau} \left\{\Gamma_{jk}(t) - \frac{\Gamma_j(t)\Gamma_k(t)}{\Gamma_0(t)}\right\} \left\{\frac{\int_t^{\tau}(1 - F(s))\, ds}{1 - F(t)}\right\}^2 dF(t).
$$

(iii) *Suppose that* $0 < \lambda < \frac{1}{16}$ *and that* (5.1) *and* (3.19) *hold. Assume furthermore that the matrix* $\mathbf{A}$ *defined by* (5.3) *is nonsingular and that* (4.30) *also holds. Then* $\hat{\beta}_n - \beta = o(n^{-1/4})$ *a.s., and* $n^{-1/2}(\hat{\beta}_n - \beta)$ *has a limiting normal distribution with mean* 0 *and covariance matrix* $\mathbf{A}^{-1}\mathbf{V}\mathbf{A}^{-1}$.

We now apply Theorem 4(iii) to study the asymptotic efficiency of the modified Buckley–James estimator $\hat{\beta}_n$. To begin with, consider the univariate case $p = 1$. Application of the Schwarz inequality to the integral in (3.18) defining A gives

$$A^2 \le D \int_{-\infty}^{\tau} \left\{\Gamma_2(t) - \frac{\Gamma_1^2(t)}{\Gamma_0(t)}\right\}\left\{\frac{\int_t^\tau (1 - F(s))\, ds}{1 - F(t)}\right\}^2 dF(t),$$
$$(5.5)\qquad D = \int_{-\infty}^{\tau} \left\{\Gamma_2(t) - \frac{\Gamma_1^2(t)}{\Gamma_0(t)}\right\}\left\{\frac{f'(t)}{f(t)} + \frac{f(t)}{1 - F(t)}\right\}^2 dF(t).$$

Consequently, the variance (4.31) of the limiting normal distribution of $n^{1/2}(\hat{\beta}_n - \beta)$ is bounded below by D^{-1}. The inequality in (5.5) is strict unless there exists $\alpha \ne 0$ for which

$$(5.6)\qquad \frac{\int_t^\tau (1 - F(s))\, ds}{1 - F(t)} = \alpha\left\{\frac{f'(t)}{f(t)} + \frac{f(t)}{1 - F(t)}\right\} \quad \text{a.e. } (F).$$

When $f(t) = (2\pi\sigma^2)^{-1/2}\exp(-\frac{1}{2}(t - \mu)^2/\sigma^2)$, the normal density, and $\tau = \infty$, (5.6) holds with $\alpha = \sigma^2$. Hence in the normal case with $\tau = \infty$, the variance of the limiting normal distribution of $n^{1/2}(\hat{\beta}_n - \beta)$ attains the lower bound D^{-1}.

The lower bound D^{-1} for the asymptotic variance of the modified Buckley–James estimator derived from the Schwarz inequality in (5.5) is in fact an asymptotic lower bound for the variances of the limiting normal distributions of regular estimators [cf. Begun, Hall, Huang and Wellner (1983)], for the semiparametric problem of estimating β in the censored regression model (1.1)–(1.2) when the common distribution of the ε_i and the distribution of the censoring variables c_i are unknown. In the case of i.i.d. (c_i, x_i), the general theory of asymptotic lower bounds in semiparametric estimation developed by Begun, Hall, Huang and Wellner (1983) can be applied to the present problem. More generally, for i.i.d. $(c_i, \mathbf{x}_i^T)$ in the censored multiple regression model with p-dimensional vectors β and $\mathbf{x}_i$, define $\mathbf{D} = (d_{jk})_{1 \le j, k \le p}$ by

$$(5.7)\qquad d_{jk} = \int_{-\infty}^{\tau} \left\{\Gamma_{jk}(t) - \frac{\Gamma_j(t)\Gamma_k(t)}{\Gamma_0(t)}\right\}\left\{\frac{f'(t)}{f(t)} + \frac{f(t)}{1 - F(t)}\right\}^2 dF(t),$$

where Γ_{jk}, Γ_k and Γ_0 are given by (5.1), the results of Begun, Hall, Huang and Wellner (1983) show that the limiting distribution of $n^{1/2}(T_n - \beta)$ for a sequence of regular estimators $\{T_n\}$ is a convolution of $N(0, \mathbf{D}^{-1})$ with some distribution. Extension of this theory to the setting of Theorem 4, in which $(c_i, \mathbf{x}_i^T)$ need not be identically distributed, together with estimation methods that attain the asymptotic $N(0, \mathbf{D}^{-1})$ distribution, will be presented elsewhere. Note that if (5.6) holds for some $\alpha \ne 0$, as in the case of $\tau = \infty$ and normal f, then $\mathbf{D}^{-1} = \mathbf{A}^{-1}\mathbf{V}\mathbf{A}^{-1}$ equals the covariance matrix of the asymptotic distribution of the modified Buckley–James estimator $\hat{\beta}_n$ in Theorem 4(iii).

APPENDIX A

Throughout the sequel we shall let $p_{n,b}(s) = p_n(n^{-1}EZ_n(b, s))$ and $\hat{p}_{n,b}(s) = p_n(n^{-1}Z_n(b, s))$.

Proof of Lemma 2. By Lemma 4 of Lai and Ying (1988), with probability 1,

$$\begin{aligned} &Z_n(b, s) \geq cn^{1-\lambda} \Rightarrow EZ_n(b, s) \geq cn^{1-\lambda}/2, \\ &EZ_n(b, s) \geq cn^{1-\lambda} \Rightarrow Z_n(b, s) \geq cn^{1-\lambda}/2 \end{aligned} \tag{A.1}$$

for all large n. From (3.1), (3.3), (3.5), (3.7) and the mean value theorem, it follows that

$$\begin{aligned} \sup\Bigg\{\left|\frac{1 - F_n(b, s)}{1 - F_n(b, t)} - \frac{1 - F_n(\beta, s)}{1 - F_n(\beta, t)}\right| &: t \leq s,\ |b - \beta| \leq n^{-\gamma}, \\ \min(EZ_n(b, s), EZ_n(\beta, s)) \geq \frac{cn^{1-\lambda}}{2}\Bigg\} &= O(n^{-\gamma+2\lambda}) \quad \text{a.s.} \end{aligned} \tag{A.2}$$

Note that $F_n(\beta, s) = F(s)$. Making use of (A.1), (A.2) and an argument similar to the proof of Theorem 3 of Lai and Ying (1988), we obtain the desired conclusions (3.9), (3.11) and (3.13).

Since $1 - F(n^\lambda) \leq n^{-2\lambda}\int_{n^\lambda}^\infty t^2\, dF(t) = o(n^{-2\lambda})$ by (3.2), we can apply (3.1) to conclude that $\sup_{|b| \leq \rho} n^{-1}Z_n(b, n^\lambda) \leq 1 - F(n^\lambda - B\rho) = o(n^{-2\lambda})$. Therefore by Lemma 4 of Lai and Ying (1988), $P\{\sup_{|b| \leq \rho} Z_n(b, n^\lambda) < cn^{1-\lambda}$ for all large $n\} = 1$. From (3.4), it follows that $\sum_{n=1}^\infty P\{\inf_{|b| \leq \rho, i \leq n} z_i(b) \leq -n^\lambda\} \leq \sum_{n=1}^\infty [n \max_{i \leq n} P\{c_i^- \geq n^\lambda - B\rho\} + nP\{\varepsilon_1^- \geq n^\lambda - B\rho\}] < \infty$. Hence, applying the Borel–Cantelli lemma, we obtain that $P(\Omega_0) = 1$, where

$$\begin{aligned} \Omega_0 = &\left\{\inf_{|b|\leq\rho, i\leq n} z_i(b) > -n^\lambda \text{ for all large } n\right\} \\ &\cap \left\{\sup_{|b|\leq\rho} Z_n(b, n^\lambda) < cn^{1-\lambda} \text{ for all large } n\right\}. \end{aligned} \tag{A.3}$$

On Ω_0, for all large n, $p_{n,b}(t) = 0$ for $t \geq n^\lambda$ by (2.12) and therefore (2.10) reduces to the desired representation (3.14) for $\xi_n(b)$. Similarly, making use of (3.4) and the fact that $n^{-1}EZ_n(b, n^\lambda) = o(n^{-2\lambda})$, it can be shown that (3.8) can be rewritten as (3.15). For $-n^\lambda \leq t \leq n^\lambda$, define

$$\begin{aligned} U_{n,b}(t) &= \int_t^{n^\lambda} \frac{1 - \hat{F}_{n,b}(s)}{1 - \hat{F}_{n,b}(t)}\hat{p}_{n,b}(s)\, ds, \\ u_{n,b}(t) &= \int_t^{n^\lambda} \frac{1 - F_n(b, s)}{1 - F_n(b, t)} p_{n,b}(s)\, ds. \end{aligned} \tag{A.4}$$

It follows from (1.6) [cf. Appendix 4 and Lemma 3.2.1 of Gill (1980)] that

$$\begin{aligned} dU_{n,b}(t) &= -\hat{p}_{n,b}(t)\,dt \\ &\quad + \frac{d\hat{F}_{n,b}(t)}{(1-\hat{F}_{n,b}(t))(1-\hat{F}_{n,b}(t-))}\int_t^{n^\lambda}(1-\hat{F}_{n,b}(s))\hat{p}_{n,b}(s)\,ds \\ &= -\hat{p}_{n,b}(t)\,dt + \frac{dN_n(b,t)}{Z_n(b,t)}\int_t^{n^\lambda}\frac{1-\hat{F}_{n,b}(s)}{1-\hat{F}_{n,b}(t)}\hat{p}_{n,b}(s)\,ds. \end{aligned}$$

Therefore applying integration by parts to both integrals in (3.14) gives for all large n,

$$\begin{aligned} \xi_n(b) &= -n^\lambda \hat{p}_{n,b}(-n^\lambda)Z_n^x(b,-n^\lambda) + U_{n,b}(-n^\lambda)J_n^x(b,-n^\lambda) \\ &\quad + \int_{-n^\lambda}^{n^\lambda}\hat{p}_{n,b}(t)\{Z_n^x(b,t)-J_n^x(b,t)\}\,dt \\ &\quad + \int_{-n^\lambda}^{n^\lambda}\frac{J_n^x(b,t)}{Z_n(b,t)}U_{n,b}(t)\,dN_n(b,t). \end{aligned} \tag{A.5}$$

Similarly, applying integration by parts to (3.15) gives

$$\begin{aligned} \zeta_n(b) &= \varepsilon_n(b) - n^\lambda p_{n,b}(-n^\lambda)EZ_n^x(b,-n^\lambda) + u_{n,b}(-n^\lambda)EJ_n^x(b,-n^\lambda) \\ &\quad + \int_{-n^\lambda}^{n^\lambda}p_{n,b}(t)\{EZ_n^x(b,t)-EJ_n^x(b,t)\}\,dt \\ &\quad + \int_{-n^\lambda}^{n^\lambda}\frac{EJ_n^x(b,t)}{EZ_n(b,t)}u_{n,b}(t)\,dEN_n(b,t). \end{aligned} \tag{A.6}$$

By (4.7) and (2.17) of Lai and Ying (1988), for every $\varepsilon > 0$,

$$\begin{aligned} &\sup_{\substack{|b-\beta|\le n^{-\gamma}\\ -\infty<s<\infty}}\left|\hat{p}_{n,b}(s)-\hat{p}_{n,\beta}(s)-p_{n,b}(s)+p_{n,\beta}(s)\right| \\ &\qquad = O(n^{-(1+\gamma)/2+2\lambda+\varepsilon}) \quad \text{a.s.} \end{aligned} \tag{A.7}$$

Moreover, making use of (2.2) together with (3.1), (3.3), (3.5) and Lemma 4 of Lai and Ying (1988), it can be shown that

$$\begin{aligned} &\sup_{\substack{|b-\beta|\le n^{-\gamma}\\ -\infty<s<\infty}}\left|p_{n,b}(s)-p_{n,\beta}(s)\right| = O(n^{\lambda-\gamma}), \\ &\sup_{\substack{|b|\le\rho\\ -\infty<s<\infty}}\left|\hat{p}_{n,b}(s)-p_{n,b}(s)\right| = O(n^{-1/2+\lambda+\varepsilon}) \quad \text{a.s.} \end{aligned} \tag{A.8}$$

Combining (3.11), (3.13) and (A.2), (A.7), (A.8) via the identity

$$\begin{aligned} a_1b_1 - a_2b_2 - a_3b_3 + a_4b_4 &= (a_1-a_2-a_3+a_4)b_1 + a_3(b_1-b_2-b_3+b_4) \\ &\quad + (a_2-a_4)(b_1-b_2) + (a_3-a_4)(b_2-b_4), \end{aligned}$$

and making use of (2.12) and (A.1), we obtain that

$$\sup_{\substack{|b-\beta|\le n^{-\gamma},\\ |t|\le n^{\lambda}}} \int_t^{n^\lambda} \left| \frac{1-\hat{F}_{n,b}(s)}{1-\hat{F}_{n,b}(t)} \hat{p}_{n,b}(s) - \frac{1-F_n(b,s)}{1-F_n(b,t)} p_{n,b}(s) \right.$$

$$\left. - \frac{1-\hat{F}_{n,\beta}(s)}{1-\hat{F}_{n,\beta}(t)} \hat{p}_{n,\beta}(s) + \frac{1-F_n(\beta,s)}{1-F_n(\beta,t)} p_{n,\beta}(s) \right| ds \tag{A.9}$$

$$= O\left(n^{-(1+\gamma)/2+4\lambda+\varepsilon} + n^{-1/2-\gamma+6\lambda+\varepsilon} + n^{-1+7\lambda+\varepsilon}\right) \quad \text{a.s.},$$

$$\sup_{\substack{|b|\le\rho,\\ |t|\le n^{\lambda}}} \int_t^{n^\lambda} \left| \frac{1-\hat{F}_{n,b}(s)}{1-\hat{F}_{n,b}(t)} \hat{p}_{n,b}(s) - \frac{1-F_n(b,s)}{1-F_n(b,t)} p_{n,b}(s) \right| ds \tag{A.10}$$

$$= O\left(n^{-1/2+4\lambda+\varepsilon}\right) \quad \text{a.s.},$$

$$\sup_{\substack{|b-\beta|\le n^{-\gamma},\\ |t|\le n^{\lambda}}} \int_t^{n^\lambda} \left| \frac{1-F_n(b,s)}{1-F_n(b,t)} p_{n,b}(s) - \frac{1-F_n(\beta,s)}{1-F_n(\beta,t)} p_{n,\beta}(s) \right| ds \tag{A.11}$$

$$= O\left(n^{-\gamma+3\lambda}\right).$$

In view of (2.12), $U_{n,b}(t) = 0$ if $Z_n(b,t) \le cn^{1-\lambda}$ and $u_{n,b}(t) = 0$ if $EZ_n(b,t) \le cn^{1-\lambda}$. Moreover, by (3.1) and (2.6), $J_n^x / Z_n \le 2B$. Hence, by arguments similar to those in the proof of Theorem 3 of Lai and Ying (1988), we obtain from (A.4)–(A.6) and (A.9)–(A.11) the desired conclusions (3.10) and (3.12). □

PROOF OF LEMMA 3(i). The desired conclusion follows from (A.6), (A.8) together with (A.12) and (A.14) below. By (3.1), (3.5) and the boundedness of f implied by (3.3), as $h \to 0$ and $nh \to \infty$,

$$\sup_{|b'-b|+|t'-t|\le h} n^{-1}\{|EZ_n(b,t) - EZ_n(b',t')| + |EZ_n^x(b,t) - EZ_n^x(b',t')|$$

$$+ |EJ_n^x(b,t) - EJ_n^x(b',t')|\} = O(h). \tag{A.12}$$

As shown in the proof of (3.4) of Lai and Ying (1988), there exists $K > 0$ such that

$$\sup_{\substack{|b'-b|\le n^{-\gamma},\\ -\infty<y<\infty}} \left| \int_{-\infty}^{y} h(s)\, d(EN_n(b,s) - EN_n(b',s)) \right| \le K\left(\sup_s |h(s)|\right) n^{1-\gamma} \tag{A.13}$$

for all bounded functions h. Noting that $\gamma > \lambda$ and that $u_{n,b}(t) = 0$ if

$EZ_n(b,t) \le cn^{1-\lambda}$, it can be shown from (A.4) and (A.11)–(A.13) that

$$\begin{aligned}\text{(A.14)}\quad &\sup_{|b'-b|\le n^{-\gamma}}\left|\int_{-n^\lambda}^{n^\lambda}\frac{n^{-1}EJ_n^x(b,t)}{n^{-1}EZ_n(b,t)}u_{n,b}(t)\,dEN_n(b,t)\right.\\ &\qquad\left.-\int_{-n^\lambda}^{n^\lambda}\frac{n^{-1}EJ_n^x(b',t)}{n^{-1}EZ_n(b',t)}u_{n,b'}(t)\,dEN_n(b',t)\right| = O(n^{1-\gamma+4\lambda}). \quad \square\end{aligned}$$

PROOF OF LEMMA 3(ii). Let $t_b = \inf\{t\colon 1 - F(t) = c|b-\beta|/2\}$. Since by (3.2),

$$\begin{aligned}\text{(A.15)}\quad f(x) &= \left|\int_x^\infty f'(t)\,dt\right| \le \left\{\int_x^\infty (f'/f)^2\,dF\right\}^{1/2}(1-F(x))^{1/2}\\ &= o\big((1-F(x))^{1/2}\big)\end{aligned}$$

as $F(x)\uparrow 1$ and since $1 - F(t_b) = c|b-\beta|/2$, we obtain by (3.1) that

$$\begin{aligned}\sup_n n^{-1}EZ_n(b,t_b) &\le \sup_n \frac{1}{n}\sum_{i=1}^n \{1 - F(t_b + (b-\beta)x_i)\}\\ &= \frac{c|b-\beta|}{2} + o(b-\beta)\end{aligned}$$

as $b \to \beta$, and therefore by (2.12),

$$\text{(A.16)}\qquad \sup_{|b-\beta|\le n^{-\lambda},\, u\ge t_b} p_{n,b}(u) = 0 \quad \text{for all large } n.$$

Let $G_{ib}(t) = P\{c_i - bx_i \ge t|x_i\}$. By (2.6), (3.7), (3.8) and (A.16), $\zeta_n(b) = \zeta_{n,1}(b-\beta,b) + \zeta_{n,2}(b-\beta,b)$ for $|b-\beta| \le n^{-\lambda}$ and all large n, where

(A.17)

$$\zeta_{n,1}(a,b) = -\int_{-\infty}^{t_b} t\,d\left[p_{n,b}(t)\sum_{i=1}^n E\{(x_i-\bar{x}_n)G_{ib}(t)[1-F(t+ax_i)]\}\right],$$

$$\begin{aligned}\zeta_{n,2}(a,b) = -\int_{-\infty}^{t_b}&\left(\int_t^{t_b}\exp\left\{-\int_t^s\frac{\sum_{i=1}^n E\{G_{ib}(u)f(u+ax_i)\}\,du}{\sum_{i=1}^n E\{G_{ib}(u)[1-F(u+ax_i)]\}}\right\}\right.\\ &\left.\times p_{n,b}(s)\,ds\right)\sum_{i=1}^n E[(x_i-\bar{x}_n)(1-F(t+ax_i))\,dG_{ib}(t)].\end{aligned}$$

Moreover, the same argument as that used in the proof of Lemma 1 shows that $\zeta_{n,1}(0,b) + \zeta_{n,2}(0,b) = 0$.

Making use of (3.1), (3.3), (3.4), (A.15), (A.17) and Taylor's expansions for f and $1-F$, it can be shown by arguments similar to those given in the

Appendix of Lai and Ying (1989) that as $b \to \beta$,

$$
\begin{aligned}
&\zeta_{n,2}(b-\beta, b) - \zeta_{n,2}(0,b) \\
&\quad = (b-\beta)E\Bigg\{\int_{-\infty}^{t_b}\left(\int_t^{t_b}(1-F(s))p_{n,b}(s)\,ds\right) \\
&\qquad\qquad \times \frac{f(t)}{1-F(t)}\sum_{i=1}^n (x_i-\bar{x}_n)x_i\,dG_{ib}(t) \\
&\quad + \int_{-\infty}^{t_b}\int_t^{t_b}(1-F(s))\left[\int_t^s\left(\frac{f'(u)}{f(u)}+\frac{f(u)}{1-F(u)}\right)\frac{\sum_{i=1}^n Ex_iG_{ib}(u)}{\sum_{i=1}^n EG_{ib}(u)}\frac{dF(u)}{1-F(u)}\right] \\
&\qquad\qquad \times p_{n,b}(s)\,ds\sum_{i=1}^n (x_i-\bar{x}_n)\,dG_{ib}(t)\Bigg\} + o(n(b-\beta)),
\end{aligned}
$$

uniformly in n. As in (A.15), $f(t) = O((F(t))^{1/2})$ as $F(t) \to 0$, and therefore by (3.2), $tf(t) = O(\{t^2F(t)\}^{1/2}) \to 0$ as $t \to -\infty$. Noting that $\sum_{i=1}^n (x_i - \bar{x}_n) = 0$, we can apply integration by parts to the first integral and interchange the order of integration in the second integral ($\int_{t=-\infty}^{t_b}\int_{s=t}^{t_b}\int_{u=t}^{s} = \int_{u=-\infty}^{t_b}\int_{s=u}^{t_b}\int_{t=-\infty}^{u}$) to rewrite the previous expression as

$$
\begin{aligned}
&\zeta_{n,2}(b-\beta,b) - \zeta_{n,2}(0,b) - o(n(b-\beta)) \\
&\quad = (b-\beta)E\sum_{i=1}^n\Bigg\{\int_{-\infty}^{t_b}(x_i-\bar{x}_n)x_iG_{ib}(t)[p_{n,b}(t)-\psi_{n,b}(t)]f(t)\,dt \\
&\qquad\qquad + \int_{-\infty}^{t_b}(x_i-\bar{x}_n)x_iG_{ib}(u)\psi_{n,b}(u) \\
&\qquad\qquad\qquad \times\left[\sum_{i=1}^n Ex_iG_{ib}(u)\Big/\sum_{i=1}^n EG_{ib}(u)\right]dF(u)\Bigg\},
\end{aligned}
\tag{A.18}
$$

where $\psi_{n,b}(t) = \{\int_t^{t_b}(1-F(s))p_{n,b}(s)\,ds\}[f'(t)/f(t) + f(t)/(1-F(t))]/(1-F(t))$. In view of (3.3), integration by parts and (A.17) give that

$$
\begin{aligned}
&\zeta_{n,1}(b-\beta,b) - \zeta_{n,1}(0,b) \\
&\quad = \int_{-\infty}^{t_b} p_{n,b}(t)\sum_{i=1}^n E\{(x_i-\bar{x}_n)G_{ib}(t)[F(t)-F(t+(b-\beta)x_i)]\}\,dt \\
&\quad = -(b-\beta)\int_{-\infty}^{t_b} p_{n,b}(t)\sum_{i=1}^n E\{x_i(x_i-\bar{x}_n)G_{ib}(t)\}f(t)\,dt \\
&\qquad + O\big(n(b-\beta)^2\big).
\end{aligned}
$$

Combining this with (A.18) and the identity $\zeta_{n,1}(0,b) + \zeta_{n,2}(0,b) = 0$ gives

$$\begin{aligned}
&\zeta_{n,1}(b-\beta,b) + \zeta_{n,2}(b-\beta,b) + o(n(b-\beta)) \\
&\quad = -n(b-\beta)\int_{-\infty}^{t_b} \psi_{n,b}(t) E\Bigg\{ n^{-1}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2 G_{ib}(t) \\
&\qquad\qquad + n^{-1}\sum_{i=1}^{n}(x_i - \bar{x}_n) G_{ib}(t)\bar{x}_n \\
&\qquad\qquad - \left[n^{-1}\sum_{i=1}^{n}(x_i - \bar{x}_n) G_{ib}(t) \right] \frac{\sum_{i=1}^{n} E x_i G_{ib}(t)}{\sum_{i=1}^{n} E G_{ib}(t)} \Bigg\}\, dF(t).
\end{aligned} \tag{A.19}$$

By (3.6) and (3.19) together with (3.1) and (3.5), as $n \to \infty$ and $b \to \beta$,

$$p_{n,b}(t) \to 1 \text{ if } t < \tau, \text{ and } \to 0 \text{ if } t > \tau \text{ and } F(\tau) < 1,$$

$$\begin{aligned}
&n^{-1}\sum_{i=1}^{n} G_{ib}(t) \to_P \Gamma_0(t), \\
&n^{-1}\sum_{i=1}^{n} (x_i - \bar{x}_n) G_{ib}(t) \to_P \Gamma_1(t), \\
&n^{-1}\sum_{i=1}^{n} (x_i - \bar{x}_n)^2 G_{ib}(t) \to_P \Gamma_2(t),
\end{aligned} \tag{A.20}$$

$$\begin{aligned}
n^{-1}\sum_{i=1}^{n} E x_i G_{ib}(t) &= n^{-1}\sum_{i=1}^{n} E\{(x_i - \bar{x}_n) G_{ib}(t) + \bar{x}_n G_{ib}(t)\} \\
&= \Gamma_1(t) + E\bar{x}_n \Gamma_0(t) + o(1),
\end{aligned}$$

where the last equality follows from the dominated convergence theorem since $|x_i| \le B$. From (A.19) and (A.20) and the definition of $\psi_{n,b}(t)$, Lemma 3(ii) follows. □

PROOF OF LEMMA 3(iii). Since $F(\tau) < 1$, (3.19) implies that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sup_{|b-\beta| \le \delta,\, u > \tau+\varepsilon} p_{n,b}(u) = 0$ for all large n. Using this in place of (A.16) and replacing t_b by $\tau + \varepsilon$ with $F(\tau + \varepsilon) < 1$ in (A.17)–(A.20) give the desired conclusion. □

APPENDIX B

As in Lemmas 4 and 5, we shall denote $Z_n^x(\beta, t)$ by $Z_n^x(t)$, etc.

PROOF OF LEMMA 4. Applying Lemma 1 with $l(t) = \hat{p}_{n,\beta}(t) - p_{n,\beta}(t)$ and making use of an argument similar to the proof of (3.14) and (3.15) in

Appendix A, it suffices for the proof of Lemma 4 to show that

$$-n^{-1/2}\left\{\int_{-n^{\lambda}}^{n^{\lambda}} t\,d\big[(\hat{p}_{n,\beta}(t) - p_{n,\beta}(t))(Z_n^x(t) - EZ_n^x(t))\big]\right.$$
$$\left.+\int_{-n^{\lambda}}^{n^{\lambda}}\left[\int_t^{n^{\lambda}}\frac{1-F(s)}{1-F(t)}(\hat{p}_{n,\beta}(s) - p_{n,\beta}(s))\,ds\right]d(J_n^x(t) - EJ_n^x(t))\right\} \to_P 0.$$

Integration by parts then reduces the problem to showing that

$$\text{(B.1)}\quad \begin{aligned} n^{-1/2}\Bigg\{&\int_{-n^{\lambda}}^{n^{\lambda}}(\hat{p}_{n,\beta}(t) - p_{p,\beta}(t)) \\ &\times\big[(Z_n^x(t) - EZ_n^x(t)) - (J_n^x(t) - EJ_n^x(t))\big]\,dt \\ &+\int_{-n^{\lambda}}^{n^{\lambda}}\left[\int_t^{n^{\lambda}}(1-F(s))(\hat{p}_{n,\beta}(s) - p_{n,\beta}(s))\,ds\right] \\ &\times(J_n^x(t) - EJ_n^x(t))(1-F(t))^{-2}\,dF(t)\Bigg\} \to_P 0. \end{aligned}$$

From (2.2), (2.3), (2.12) and (A.1), it follows that with probability 1, for all large n,

$$\text{(B.2)}\quad \begin{aligned} &\sup_t|\hat{p}_{n,\beta}(t) - p_{n,\beta}(t)| \\ &\quad= \sup_t|p_n(n^{-1}Z_n(t)) - p_n(n^{-1}EZ_n(t))| \\ &\quad\le n^{\lambda}\Big(\sup_x|p'(x)|\Big) \\ &\qquad\times\sup\{n^{-1}|Z_n(t) - EZ_n(t)|: cn^{1-\lambda}/2 \le EZ_n(t) \le 2(c+1)n^{1-\lambda}\}. \end{aligned}$$

By Lemma 4 of Lai and Ying (1988), for every $\varepsilon > 0$,

$$\text{(B.3)}\quad \begin{aligned} &\sup_{EZ_n(t)\le 2(c+1)n^{1-\lambda}}|Z_n(t) - EZ_n(t)| = O(n^{(1-\lambda)/2+\varepsilon}) \quad \text{a.s.} \\ &\sup_s\{|Z_n^x(t) - EZ_n^x(t)| + |J_n^x(t) - EJ_n^x(t)|\} = O(n^{1/2+\varepsilon}) \quad \text{a.s.} \end{aligned}$$

Since $\int_{1-F(t)\ge cn^{-\lambda}/2}(1 - F(t))^{-2}\,dF(t) = O(n^{\lambda})$ and $\int_{-\infty}^0|t|\,dF(t) + \int_0^{\infty}(1 - F(s))\,ds < \infty$, we obtain the desired conclusion (B.1) from (B.2) and (B.3), noting that $3\lambda/2 < \frac{1}{2}$ and that $1 - F(t) \ge n^{-1}EZ_n(t)$. □

Proof of Lemma 5. For the martingale structure of $W_n(t)$ and the formulas (4.9) and (4.10), see Gill [(1980), pages 26–27 and 34–38]. To prove (4.11), first note that since $Z_n(t) = \sum_{i=1}^n I_{\{\min(\varepsilon_i, c_i-\beta x_i)\ge t\}}$ has expectation $(1 - F(t))C_n(t)$, $\sup_{t\le T_n}\{(1 - F(t))C_n(t)/Z_n(t)\} = O_p(1)$, by Theorem 1.1.1 and Corollary 1.3.1 of van Zuijlen (1977). Moreover, since $(1 - F(t))^{(1+\delta)/2}C_n^{1/2}(t)$ is nonnegative and nonincreasing in t, we obtain by Lemma 2.9 of Gill (1983) that for every $a > 0$,

$$
\begin{aligned}
&P\left\{\sup_{t\le T_n}(1-F(t))^{(1+\delta)/2}C_n^{1/2}(t)|W_n(t)|\ge 2a\right\}\\
&\quad\le P\left\{\sup_{t\le T_n}\left|\int_{-\infty}^{t}(1-F(s))^{(1+\delta)/2}C_n^{1/2}(s)\,dW_n(s)\right|\ge a\right\}\\
&\quad\le a^{-1}+P\left\{\int_{-\infty}^{T_n}\frac{(1-F(s))^{(1+\delta)}C_n(s)}{Z_n(s)}\right.\\
&\qquad\left.\times\left[\frac{1-\hat F_{n,\beta}(s-)}{1-F(s)}\right]^2\frac{dF(s)}{1-F(s)}\ge a\right\}\to 0\quad\text{as } a\to\infty,
\end{aligned}\tag{B.4}
$$

where the last inequality follows from (4.9) and Lenglart's inequality [cf. Gill (1980), page 18].

To prove (4.12), let $l_n(t)=\hat p_{n,\beta}(t)-p_{n,\beta}(t)$ and use integration by parts to obtain from (4.8) that

$$
\begin{aligned}
&\hat\xi_{n,2}-\int_{-n^\lambda}^{n^\lambda}\int_{-n^\lambda}^{u-}\left(1-\hat F_{n,\beta}(t)\right)^{-1}dJ_n^x(t)\int_u^{n^\lambda}(1-F(s))p_{n,\beta}(s)\,ds\,dW_n(u)\\
&\quad=-\int_{-n^\lambda}^{n^\lambda}W_n(u)\left\{\int_u^{n^\lambda}(1-F(s))l_n(s)\,ds\right\}\frac{dJ_n^x(u)}{1-\hat F_{n,\beta}(u-)}\\
&\qquad+\int_{-n^\lambda}^{n^\lambda}\left\{\int_{-n^\lambda}^{u-}\frac{dJ_n^x(t)}{1-\hat F_{n,\beta}(t)}\right\}W_n(u)(1-F(u))l_n(u)\,du.
\end{aligned}\tag{B.5}
$$

Letting $s_n=\sup\{s\colon EZ_n(s)\ge cn^{1-\lambda}/2\}$, it follows from (B.2) that $l_n(s)=0$ for $s>s_n$ and all large n. By (4.11), $\sup_{s\le s_n}|W_n(s)|(1-F(s))^\varepsilon=O_p(n^{-(1-\lambda)/2})$ for every $\varepsilon>0$, and therefore

$$
\sup_{s\le s_n}|W_n(s)|=O_p(n^{-(1-\lambda)/2+\delta})\quad\text{for all }\delta>0,\tag{B.6}
$$

noting that $1-F(s)\ge n^{-1}EZ_n(s)$. In view of (3.4), there exists t^* with $F(t^*)<1$ such that $\inf_{n\ge1}n^{-1}C_n(t^*)>0$. Since $dN_n(s)\le -dZ_n(s)$ and $|dJ_n^x(u)|\le -2B\,dZ_n(u)$,

$$
\begin{aligned}
&\int_{-\infty}^{s_n}\left(1-\hat F_{n,\beta}(u)\right)^{-1}|dJ_n^x(u)|\\
&\quad=O\left(\int_{-\infty}^{s_n}\exp\left(-\int_{-\infty}^{u}\frac{dZ_n(s)/n}{Z_n(s)/n}\right)|dJ_n^x(u)|\right)\\
&\quad=O\left(-n\int_{-\infty}^{s_n}\frac{dZ_n(u)/n}{Z_n(u)/n}\right)=O(n\log n)\quad\text{a.s.,}
\end{aligned}\tag{B.7}
$$

by the definition of s_n and (A.2). Noting that $\sup_{u\le t^*}|W_n(u)|=O_p(n^{-1/2})$ by

(4.11), that $\sup_s |l_n(s)| = o(n^{-1/2+\lambda/2+\varepsilon})$ a.s. for every $\varepsilon > 0$ by (B.2) and (B.3) and that $\int_{t^*}^{\infty}(1 - F(s))\,ds < \infty$, we obtain from (B.5)–(B.7) that

$$\hat{\xi}_{n,2} - \int_{-n^\lambda}^{n^\lambda}\int_{-n^\lambda}^{u-}\left(1 - \hat{F}_{n,\beta}(t)\right)^{-1} dJ_n^x(t) \times \int_u^{n^\lambda}(1 - F(s))p_{n,\beta}(s)\,ds\,dW_n(u) = O_p(n^{3\lambda/2+\varepsilon}) \quad \text{for every } \varepsilon > 0. \tag{B.8}$$

Since $3\lambda < 1$, (4.12) follows from (B.8) and (A.3) [with $P(\Omega_0) = 1$] if it can be shown that

$$n^{-1/2}\int_{-n^\lambda}^{n^\lambda}\left\{\int_{-\infty}^{u-}\left(1 - \hat{F}_{n,\beta}(t)\right)^{-1} dJ_n^x(t) - E\sum_{i=1}^{n}{}^{`}(x_i - \bar{x}_n)I_{\{c_i-\beta x_i \geq u\}}\right\} \times \left\{\int_u^{n^\lambda}(1 - F(s))p_{n,\beta}(s)\,ds\right\} dW_n(u) \to_P 0. \tag{B.9}$$

To prove (B.9), note that since $dE\,J_n^x(t) = (1 - F(t))\,dE[\sum_{i=1}^n(x_i - \bar{x}_n)I_{\{c_i-\beta x_i \geq t\}}]$,

$$\int_{-\infty}^{u-}(1 - F(t))^{-1}\,dE\,J_n^x(t) = E\left[\sum_{i=1}^{n}(x_i - \bar{x}_n)I_{\{c_i-\beta x_i \geq u\}}\right]. \tag{B.10}$$

Since $(1 - \hat{F}_{n,\beta}(t))^{-1} - (1 - F(t))^{-1} = W_n(t)/(1 - \hat{F}_{n,\beta}(t))$, it follows from (B.6), (B.7) and (B.3) that

$$\begin{aligned}
&\sup_{u \leq s_n}\left|\int_{-\infty}^{u-}\left(1 - \hat{F}_{n,\beta}(t)\right)^{-1} dJ_n^x(t) - \int_{-\infty}^{u-}(1 - F(t))^{-1}\,dE\,J_n^x(t)\right| \\
&\quad = \sup_{u \leq s_n}\left|\int_{-\infty}^{u-}\frac{W_n(t)}{1 - \hat{F}_{n,\beta}(t)}\,dJ_n^x(t) + \frac{J_n^x(u-) - EJ_n^x(u-)}{1 - F(u)} - \int_{-\infty}^{u-}\frac{J_n^x(t) - EJ_n^x(t)}{(1 - F(t))^2}\,dF(t)\right| \\
&\quad = O_p(n^{1/2+\lambda+\varepsilon}) \quad \text{for every } \varepsilon > 0.
\end{aligned}$$

Since $p_{n,\beta}(s) = 0$ for $s > s_n$ and $\int_{t^*}^{\infty}(1 - F(s))\,ds + \int_{-\infty}^{t^*}u^2\,dF(u) < \infty$, it follows from (4.11) that

$$n^{-1}\int_{-n^\lambda}^{n^\lambda}\left\{\int_{-\infty}^{u-}\left(1 - \hat{F}_{n,\beta}(t)\right)^{-1} dJ_n^x(t) - \int_{-\infty}^{u-}(1 - F(t))^{-1}\,dE\,J_n^x(t)\right\}^2 \times\left\{\int_u^{n^\lambda}(1 - F(s))p_{n,\beta}(s)\,ds\right\}^2\left\{\frac{1 - \hat{F}_{n,\beta}(u-)}{1 - F(u)}\right\}^2 \times\frac{dF(u)}{(1 - F(u))Z_n(u)} \to_P 0. \tag{B.11}$$

Since $\{W_n(t), \mathscr{F}_n(t), t \leq T_n\}$ is a martingale with predictable variation process (4.9), the desired conclusion (B.9) follows from (B.10), (B.11) and Lenglart's inequality. □

REFERENCES

BEGUN, J. M., HALL, W. J., HUANG, W.-M. and WELLNER, J. A. (1983). Information and asymptotic efficiency in parametric-nonparametric models. *Ann. Statist.* **11** 432–452.

BUCKLEY, J. and JAMES, I. (1979). Linear regression with censored data. *Biometrika* **66** 429–436.

COX, D. R. (1972). Regression models and life-tables (with discussion). *J. Roy. Statist. Soc. Ser. B* **34** 187–202.

GILL, R. D. (1980). *Censoring and Stochastic Integrals*. Math. Centre Tract **124**. Mathematisch Centrum, Amsterdam.

GILL, R. D. (1983). Large sample behavior of the product-limit estimator on the whole line. *Ann. Statist.* **11** 49–58.

JAMES, I. R. and SMITH, P. J. (1984). Consistency results for linear regression with censored data. *Ann. Statist.* **12** 590–600.

KOUL, H., SUSARLA, V. and VAN RYZIN, J. (1981). Regression analysis with randomly right-censored data. *Ann. Statist.* **9** 1276–1288.

LAI, T. L. and YING, Z. (1988). Stochastic integrals of empirical-type processes with applications to censored regression. *J. Multivariate Anal.* **27** 334–358.

LAI, T. L. and YING, Z. (1989). Linear rank statistics in regression analysis with censored or truncated data. Technical Report, Dept. Statistics, Stanford Univ.

LAI, T. L. and YING, Z. (1991). A missing information principle and M-estimators in regression analysis with censored and truncated data. Technical Report, Dept. Statistics, Stanford Univ.

LEURGANS, S. (1987). Linear models, random censoring and synthetic data. *Biometrika* **74** 301–309.

MILLER, R. G. (1976). Least squares regression with censored data. *Biometrika* **63** 449–464.

MILLER, R. G. and HALPERN, J. (1982). Regression with censored data. *Biometrika* **69** 521–531.

RITOV, Y. (1990). Estimation in a linear regression model with censored data. *Ann. Statist.* **18** 303–328.

SMITH, P. J. (1988). Asymptotic properties of linear regression estimators under a fixed censorship model. *Austral. J. Statist.* **30** 52–66.

VAN ZUIJLEN, M. C. A. (1977). *Empirical Distribution and Rank Statistics*. Math. Centre Tract **79**. Mathematisch Centrum, Amsterdam.

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