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EMPIRICAL LIKELIHOOD IS BARTLETT-CORRECTABLE[1]

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It is shown that, in a very general setting, the empirical likelihood method for constructing confidence intervals is Bartlett-correctable. This means that a simple adjustment for the expected value of log-likelihood ratio reduces coverage error to an extremely low $O(n^{-2})$, where n denotes sample size. That fact makes empirical likelihood competitive with methods such as the bootstrap which are not Bartlett-correctable and which usually have coverage error of size n^{-1}. Most importantly, our work demonstrates a strong link between empirical likelihood and parametric likelihood, since the Bartlett correction had previously only been available for parametric likelihood. A general formula is given for the Bartlett correction, valid in a very wide range of problems, including estimation of mean, variance, covariance, correlation, skewness, kurtosis, mean ratio, mean difference, variance ratio, etc. The efficacy of the correction is demonstrated in a simulation study for the case of the mean.

1. Introduction. The method of empirical likelihood was introduced by Owen (1988, 1990) as an alternative to the bootstrap for constructing confidence regions in nonparametric problems. It is in the spirit of Efron (1981), who showed that nonparametric inference can be conducted by applying parametric techniques to suitable families of distributions supported on the data. Empirical likelihood has some conceptual advantages over the bootstrap, most noticeably in two dimensions, where it uses only the data to determine the shape of a confidence region. However, one could be excused for thinking of empirical likelihood as simply one of many bootstrap competitors. In this paper we show that empirical likelihood has a major advantage over the bootstrap and related techniques: It admits a Bartlett correction. Thus, a simple empirical correction for the expected value of the empirical log-likelihood ratio reduces coverage error to an impressively low $O(n^{-2})$, where n is sample size. This is achieved without forcing any constraints of symmetry on the confidence region. Indeed, the Bartlett-corrected region enjoys all the conceptual advantages of its noncorrected counterpart; for a discussion of those advantages see Owen (1988, 1990). Furthermore, *the bootstrap is not Bartlett-correctable*, and so the coverage accuracy of bootstrap methods cannot be enhanced by a simple correction. Usually, the bootstrap can only be

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corrected by resorting to highly computer-intensive methods such as bootstrap iteration.

As we show, Bartlett correction is available very generally for a wide range of parameters, including means, variances, covariances, correlation coefficients, skewness, kurtosis, ratios of means, differences between means, ratios of variances, etc. We demonstrate that the correction is applicable to all parameters which admit the "smooth function model," introduced by Bhattacharya and Ghosh (1978).

Various authors, including Barndorff-Nielsen and Cox (1984), Lawley (1956) and McCullagh [(1987), page 212], have discussed Bartlett corrections for parametric models, but this is the first time they have been considered in purely nonparametric circumstances. Indeed, it is striking that they should be available for empirical likelihood, and the fact they do exist demonstrates an unexpectedly close relationship between parametric and nonparametric likelihood. Thus, there is more to empirical likelihood than first meets the eye.

Nevertheless, it is not possible to obtain our results on Bartlett correction from classical ones for parametric likelihood. Empirical likelihood requires the fitting of $n-1$ parameters to a data set of size n, and this setting is not even countenanced by classical parametric statistical theory!

In Section 2 we briefly review the method of empirical likelihood and describe our main results. Section 3 discusses a simulation study which confirms the efficacy of Bartlett correction, and Section 4 outlines the derivation of the Bartlett correction.

2. Results.

2.1. *Empirical likelihood and the smooth function model.* We begin by reviewing the method of empirical likelihood. Let $X_1, \dots, X_n$ denote a sample from an unknown r-variate distribution F_0 having mean μ_0 and nonsingular covariance matrix Σ_0. Let $\theta = (\theta^1, \dots, \theta^q)^T$ be a q-dimensional parameter $(q \le r)$ that can be expressed as a function of the mean of the underlying distribution, and put $\theta_0 = \theta(\mu_0)$. This is the smooth function model, introduced by Bhattacharya and Ghosh (1978). The empirical likelihood function L for this parameter is defined by considering distributions F_p, $p = (p^1, \dots, p^n)$, supported on the sample, where X_i is assigned mass p^i. For a specified value θ_1 of the parameter of interest, the empirical likelihood $L(\theta_1)$ is defined to be the maximum value of $\prod p^i$ over all such distributions that satisfy $\theta(\Sigma X_i p^i) = \theta_1$. If no distribution F_p satisfying the constraint exists, then by definition $L(\theta_1) = 0$. Since $\prod p^i$ attains its overall maximum when $p^i = n^{-1}$, $i = 1, \dots, n$, it follows that the empirical likelihood function is maximized at $\hat{\theta} = \theta(\overline{X})$, where $\overline{X} = n^{-1}\Sigma X_i$ is the sample mean. The empirical likelihood ratio statistic is

$$W_0 = -2\log\{L(\theta_0)/L(\hat{\theta})\} = -2\log\{n^n L(\theta_0)\}.$$

To appreciate the implications and extreme generality of the smooth function model in nonparametric inference, let us consider the case where θ is the correlation coefficient of a bivariate random variable (X, Y). (Thus, $q = 1$). Now, θ is a function of five means, $E(X)$, $E(Y)$, $E(X^2)$, $E(Y^2)$ and $E(XY)$. Therefore, applying the argument in the previous paragraph requires taking $q = 1$ and $r = 5$ (not $r = 2$). A problem involving a variance ratio requires $q = 1$ and $r = 4$ [since four means are involved, $E(X)$, $E(Y)$, $E(X^2)$ and $E(Y^2)$]. A problem involving the bivariate pair (mean, standard deviation) for a univariate population has $q = 2$ and $r = 2$, and so on. The essential assumption is that θ be a smooth function of the mean μ, having sufficiently many derivatives in a neighborhood of μ_0. Usually the function has, in fact, an infinite number of derivatives; this is the case in all the examples considered previously, and in many other circumstances (ratios of means or variances, skewness, etc.).

2.2. *Coverage accuracy.* It follows from the argument which we shall give in Section 4 that

$$P(W_0 \le z) = P\left(\chi_q^2 \le z\right) + O(n^{-1}),$$

and thus the error in coverage level of confidence regions obtained by using the chi-squared approximation to the distribution of the empirical likelihood ratio statistic is of order $O(n^{-1})$. Empirical likelihood regions share this size of error with most confidence regions constructed by bootstrap methods [Hall (1988)]. However, unlike bootstrap methods, empirical likelihood admits a Bartlett correction. That is,

$$P\left[W_0\{E(nR^TR)/q\}^{-1} \le z\right] = P\left(\chi_q^2 \le z\right) + O(n^{-2}), \tag{2.1}$$

where R is a q-dimensional vector, defined in Section 4, such that $W_0 = nR^TR + O_p(n^{-3/2})$. The error in the coverage level of confidence regions obtained using the Bartlett adjustment is of order $O(n^{-2})$. These expansions follow from the usual assumptions which guarantee the existence of certain Edgeworth expansions, namely the existence of sufficiently many moments of F_0, that F_0 satisfies Cramér's condition and that $\theta(\cdot)$ is smooth.

The ratio $\{E(nR^TR)/q\}^{-1}$ admits a simple expansion,

$$\{E(nR^TR)/q\}^{-1} = 1 - an^{-1} + O(n^{-2}), \tag{2.2}$$

where a is a fixed constant. We shall give a general formula for a in Section 2.4. In practice, one may estimate a by replacing unknown population moments by their empirical counterparts. Replacing a by an estimate $\hat{a}$ and ignoring the $O(n^{-2})$ term in (2.2) does not upset the validity of (2.1):

$$P\{W_0(1 - \hat{a}n^{-1}) \le z\} = P\left(\chi_q^2 \le z\right) + O(n^{-2}).$$

This is the Bartlett correction: a simple, empirical adjustment for the expected value of log-likelihood ratio, reducing coverage error by an order of magnitude.

2.3. *Why does the Bartlett correction work, and why is the bootstrap not Bartlett-correctable?* In general, statistics S which give rise to two-sided confidence intervals have distributions which admit Edgeworth expansions whose terms decrease in powers of n^{-1}. For example, if $S = \{n^{1/2}(\hat{\theta} - \theta_0)/\hat{\sigma}\}^2$, where $n^{-1}\hat{\sigma}^2$ denotes an estimate of the variance of $\hat{\theta}$, then

$$(2.3)\quad P(S \le z) = P(\chi_1^2 \le z) + n^{-1}p(z^{1/2})(2\pi)^{-1/2}e^{-z/2} + O(n^{-2}),$$

where p is an odd polynomial of degree 5. If the terms in p of degrees 3 and 5 vanish identically, so that $p(u) \equiv cu$ for a constant c, then it is clear from (2.3) that a simple adjustment for the expected value of S will remove the term of order n^{-1} from the right-hand side of (2.3):

$$\begin{aligned} P\{S/(1 - n^{-1}c) \le z\} &= P\{-z^{1/2}(1 - \tfrac{1}{2}n^{-1}c) \le N(0,1) \le z^{1/2}(1 - \tfrac{1}{2}n^{-1}c)\} \\ &\quad + n^{-1}cz^{1/2}(2\pi)^{-1/2}e^{-z/2} + O(n^{-2}) \\ &= P(\chi_q^2 \le z) + O(n^{-2}). \end{aligned}$$

It is also clear that if either of the terms of degrees 3 and 5 in p does not vanish, then the term of order n^{-1} in (2.3) cannot be removed by a simple adjustment of this form. That is precisely the reason why Bartlett correction is available for empirical likelihood but not, in general, for the bootstrap: When $S = W_0$, the resulting polynomial p does not contain terms of degrees 3 or 5, but such terms are usually present in other cases. [The reader is referred to Hall (1988), where versions of p are given in the case of general bootstrap problems. Those p's involve terms of degree 3, although not necessarily of degree 5.]

Our proof of the efficacy of Bartlett correction reduces, essentially, to a demonstration that p contains no terms of orders 3 or 5 in the case $S = W_0$. Exactly why this should be the case is not clear to us.

2.4. *Formula for the Bartlett correction.* We first treat the case where $\Sigma_0 = I$, the identity. In the notation of Section 1, put

$$(2.4)\qquad \alpha^{j_1 \cdots j_k} = E(X_1^{j_1} \cdots X_1^{j_k}), \qquad \theta^i_{j_1 \cdots j_k} = \frac{\partial^k \theta^i(\mu)}{\partial \mu^{j_1} \cdots \partial \mu^{j_k}}\bigg|_{\mu = \mu_0},$$

where X_1^j and μ^j denote the jth elements of X_1 and μ, respectively. Define $\Theta = (\theta^i_j)$, a $q \times r$ matrix, and let $Q = (\Theta\Theta^T)^{-1}$, $M = \Theta^T Q \Theta$, $N = \Theta^T Q$,

$$t_1 = \alpha^{jkl}\alpha^{mno}M^{jm}M^{kn}M^{lo}, \qquad t_2 = \alpha^{jkl}\alpha^{mno}M^{jk}M^{lm}M^{no},$$

$$t_3 = \alpha^{jklm}M^{jk}M^{lm}, \qquad t_4 = \alpha^{jkl}N^{ju}\theta^u_{mn}(I - M)^{mk}(I - M)^{nl},$$

$$t_5 = Q^{uv}\theta^u_{jk}\theta^v_{lm}\{(I - M)^{jk}(I - M)^{lm} + 2(I - M)^{jl}(I - M)^{km}\},$$

where repeated subscripts are summed over in the usual summation convention. Then the Bartlett correction a introduced in Section 2.2 is given by

$$(2.5)\qquad a = q^{-1}(\tfrac{5}{3}t_1 - 2t_2 + \tfrac{1}{2}t_3 - t_4 + \tfrac{1}{4}t_5).$$

In the case where $\Sigma_0 \neq I$, apply the preceding definitions to the function ψ defined by

$$\psi(\lambda) = \theta(\Sigma_0^{1/2}\lambda)$$

instead of to $\theta(\lambda)$. Note that the derivative in (2.4) should now be evaluated at $\Sigma_0^{-1/2}\mu_0$, not at μ_0. Of course, $\hat{a}$ is obtained from a on substituting estimates for unknowns.

2.5. *Concluding remarks.* It should be noted that an asymptotic expansion for the expectation of W_0 does not exist. For example, suppose $q = 1$ and the functional of interest is the mean. Then, the likelihood ratio statistic W_0 is infinity when the true mean θ_0 falls outside the range of the data. If F_0 is not degenerate, this happens with positive probability, and so $E(W_0) = \infty$. Therefore, the problem of adjusting W_0 by a factor depending on its mean is a subtle one. Nevertheless, (2.1) is true. The reason is that $n^{-1}W_0$ can be approximated in a distributional sense by $R^T R$ (to order $n^{-5/2}$), and an adjustment determined by the mean of $R^T R$ is effective in improving the accuracy of a chi-squared approximation.

3. The case of the mean. The case $r = q = 1$ is that of the mean, $\theta = \mu = E(X)$. Here, formula (2.5) reduces to $a = \frac{1}{2}\mu_4\mu_2^{-2} - \frac{1}{3}\mu_3^2\mu_2^{-3}$, where $\mu_j = E(X - \mu)^j$ denotes the jth central moment of the population. Our estimate of a, based on a random sample $X_1, \dots, X_n$ drawn from the distribution of X, is $\hat{a} = \frac{1}{2}\hat{\mu}_4\hat{\mu}_2^{-2} - \frac{1}{3}\hat{\mu}_3^2\hat{\mu}_2^{-3}$, where $\hat{\mu}_j = n^{-1}\Sigma(X_i - \bar{X})^j$. To assess the improvement in coverage accuracy which results from these Bartlett corrections, using either a or $\hat{a}$, a modest simulation study was performed. For various distributions, sample sizes and nominal coverage levels, 5000 simulated data sets were generated and two-sided confidence intervals were constructed by using three different methods. The first method is the uncorrected empirical likelihood method which employs a simple chi-squared critical value $c_{1-\alpha}$. The second method utilizes the *theoretical* Bartlett correction $1 - an^{-1}$ and involves replacing the chi-squared critical value by $c_{1-\alpha}/(1 - an^{-1})$. This method, of course, assumes knowledge of the actual population moments. The final method uses the *estimated* Bartlett correction.

Table 1 reports the results based on simulated standard normal data at sample sizes 10 and 20. Table 2 reports the results for chi-squared data with one degree of freedom for sample sizes of 20 and 40. Finally, Table 3 reports the results for data sampled from the t-distribution with five degrees of freedom, at sample sizes 15 and 30. The actual theoretical values of a for these three situations are $\frac{3}{2}$, $\frac{29}{6}$ and $\frac{9}{2}$.

Overall, the results are more than satisfactory. Even for normal data, the uncorrected empirical likelihood intervals have coverage levels significantly different from the nominal level. In fact, the observed coverage of empirical likelihood is always below the nominal level. In all cases, the theoretical and estimated Bartlett corrections substantially improve coverage accuracy. The most difficult situation is the chi-squared distribution. In this case, the

TABLE 1
Normal data

	Nominal level			
	80%	**90%**	**95%**	**99%**
$n = 10$:				
Empirical Likelihood	0.7470	0.8446	0.8964	0.9542
Theoretical Bartlett	0.7796	0.8706	0.9182	0.9650
Estimated Bartlett	0.7938	0.8802	0.9246	0.9696
$n = 20$:				
Empirical Likelihood	0.7834	0.8830	0.9342	0.9804
Theoretical Bartlett	0.8006	0.8962	0.9418	0.9844
Estimated Bartlett	0.8034	0.8980	0.9424	0.9848

TABLE 2
χ_1^2 *data*

	Nominal level			
	80%	**90%**	**95%**	**99%**
$n = 20$:				
Empirical Likelihood	0.7314	0.8288	0.8878	0.9536
Theoretical Bartlett	0.7872	0.8772	0.9262	0.9706
Estimated Bartlett	0.7634	0.8546	0.9034	0.9616
$n = 40$:				
Empirical Likelihood	0.7644	0.8680	0.9236	0.9740
Theoretical Bartlett	0.7910	0.8896	0.9418	0.9800
Estimated Bartlett	0.7804	0.8788	0.9334	0.9774

TABLE 3
t_5 *data*

	Nominal level			
	80%	**90%**	**95%**	**99%**
$n = 15$:				
Empirical Likelihood	0.7516	0.8502	0.9094	0.9692
Theoretical Bartlett	0.8266	0.9106	0.9544	0.9862
Estimated Bartlett	0.7898	0.8884	0.9348	0.9794
$n = 30$:				
Empirical Likelihood	0.7768	0.8784	0.9308	0.9780
Theoretical Bartlett	0.8114	0.9042	0.9496	0.9866
Estimated Bartlett	0.7954	0.8928	0.9422	0.9832

theoretical adjustment performs quite well; however, the estimated adjustment is only a modest improvement over the unadjusted empirical likelihood method. The difficulty in this case arises because the sample estimate of a is quite skewed and biased, with the result being that the estimated correction does not have as big an effect as the theoretical adjustment. The success of the estimated Bartlett correction method is especially surprising in the t-distribution situation as it involves a sample estimate of the fourth moment of the underlying population.

4. Derivation of formula (2.5). We give an outline of the argument, referring the reader to the technical report of DiCiccio, Hall and Romano (1988) for further details. It may be assumed without loss of generality that $\mu_0 = 0$ and $\Sigma_0 = I$. Let $\overline{W}_0$ denote the empirical log-likelihood ratio for the *mean* (not for θ). Then it may be shown that

$$\overline{W}_0 = 2\sum_{i=1}^{n} \log\{1 + t^T(X_i - \mu_0)\},$$

where $t = t(\mu_0)$ is determined by the equation

$$\sum_{i=1}^{n} \{1 + t^T(X_i - \mu_0)\}^{-1}(X_i - \mu_0) = 0.$$

Thence it may be proved that

$$\begin{aligned} n^{-1}\overline{W}_0 = {} & A^jA^j - A^{jk}A^jA^k + \tfrac{2}{3}\alpha^{jkl}A^jA^kA^l + A^{jl}A^{kl}A^jA^k \\ & + \tfrac{2}{3}A^{jkl}A^jA^kA^l - 2\alpha^{jkm}A^{lm}A^jA^kA^l \\ & + \alpha^{jkn}\alpha^{lmn}A^jA^kA^lA^m - \tfrac{1}{2}\alpha^{jklm}A^jA^kA^lA^m + O_p(n^{-5/2}), \end{aligned} \tag{4.1}$$

where $A^{j_1 \cdots j_k} = n^{-1}\sum_i(X_i^{j_1} \cdots X_i^{j_k} - \alpha^{j_1 \cdots j_k})$ and the summation notation convention is implicit in formulae such as (4.1). From this it may be shown, after very extensive algebra, that $n^{-1}W_0 = R^TR + O_p(n^{-5/2})$, where $R = R_1 + R_2 + R_3$ is a q-vector, and for $O = \Theta^TQ^{1/2}$, $P = Q^{1/2}$, $R_1 = O^T\mu_1$,

$$\begin{aligned} R_2^u = {} & -\tfrac{1}{2}O^{ju}M^{kl}A^{jk}A^l + \tfrac{1}{3}\alpha^{jkl}O^{ju}M^{km}M^{ln}A^mA^n \\ & + \tfrac{1}{2}P^{uv}\theta_{jk}^v(I - M)^{jl}(I - M)^{km}A^lA^m, \end{aligned}$$

$$\begin{aligned} R_3^u = {} & -\tfrac{1}{4}O^{ju}N^{kv}\theta_{lm}^v(I - M)^{ln}(I - M)^{mo}A^{jk}A^nA^o + \tfrac{3}{8}O^{ju}M^{lm}M^{kn}A^{jl}A^{km}A^n \\ & + \tfrac{1}{3}O^{ju}\alpha^{jkl}M^{km}N^{lv}\theta_{no}^v(I - M)^{np}(I - M)^{oq}A^mA^pA^q \\ & + \tfrac{1}{3}O^{ju}M^{km}M^{ln}A^{jkl}A^mA^n - \tfrac{5}{12}O^{ju}\alpha^{jkm}M^{mn}M^{ko}M^{lp}A^{ln}A^oA^p \\ & - \tfrac{5}{12}O^{ju}\alpha^{kln}M^{mn}M^{ko}M^{lp}A^{jm}A^oA^p \\ & + \tfrac{4}{9}O^{ju}\alpha^{jkn}\alpha^{lmo}M^{no}M^{kp}M^{lq}M^{mr}A^pA^qA^r \\ & - \tfrac{1}{4}O^{ju}\alpha^{jklm}M^{kn}M^{lo}M^{mp}A^nA^oA^p \\ & - \tfrac{1}{2}P^{uv}\theta_{jk}^vN^{jw}\theta_{mn}^w(I - M)^{kl}(I - M)^{mo}(I - M)^{np}A^lA^oA^p \\ & - P^{uv}\theta_{jk}^v(I - M)^{jl}(I - M)^{km}M^{no}A^lA^{mn}A^o \\ & + P^{uv}\theta_{jk}^v(I - M)^{jl}(I - M)^{km}\alpha^{mno}M^{np}M^{oq}A^lA^pA^q. \end{aligned}$$

From these formulae, and after very lengthy algebra, it may be proved that the sth cumulant of nR^TR [Johnson and Kotz (1970), page 153] is

$$\kappa_s = 2^{s-1}(s-1)!q\{E(nR^TR)/q\}^s + O(n^{-3/2}).$$

Ignoring terms of order $O(n^{-3/2})$, the sth cumulant of $(nR^TR)\cdot\{E(nR^TR)/q\}^{-1}$ is $2^{s-1}(s-1)!q$, which is the sth cumulant of χ_q^2.

By the validity of Edgeworth expansions for $n^{1/2}R$ in this situation and since $W_0 = nR^TR + O_p(n^{-3/2})$, it follows that

$$\begin{aligned} P\Big[W_0\{E(nR^TR)/q\}^{-1} \le z\Big] &= P\Big[(nR^TR)\{E(nR^TR)/q\}^{-1} \le z\Big] \\ &= P\big(\chi_q^2 \le z\big) + O(n^{-3/2}). \end{aligned} \tag{4.2}$$

Moreover, by an argument based on the oddness and evenness of polynomials in Edgeworth expansions that is given, for example, by Barndorff-Nielsen and Hall (1988), the $O(n^{-3/2})$ term in (4.2) is actually $O(n^{-2})$. Therefore,

$$P\Big[W_0\{E(nR^TR)/q\}^{-1} \le z\Big] = P\big(\chi_q^2 \le z\big) + O(n^{-2}).$$

With $t_1, \dots, t_5$ defined as in Section 2.4, and with

$$t_6 = \alpha^{jkl}N^{ju}\theta^u_{mn}(I-M)^{mn}M^{kl},$$

we may prove that

$$\begin{aligned} E(R_1^uR_1^u) &= n^{-1}q, \\ E(R_1^uR_2^u) &= n^{-2}\big(\tfrac{1}{3}t_1 - \tfrac{1}{2}t_3 + \tfrac{1}{2}t_4 + \tfrac{1}{2}q\big) + O(n^{-3}), \\ E(R_1^uR_3^u) &= n^{-2}\big(\tfrac{43}{72}t_1 - \tfrac{73}{72}t_2 + \tfrac{5}{8}t_3 - t_4 + \tfrac{1}{12}t_6 - \tfrac{3}{8}q\big) + O(n^{-3}), \\ E(R_2^uR_2^u) &= n^{-2}\big(-\tfrac{7}{36}t_1 + \tfrac{1}{36}t_2 + \tfrac{1}{4}t_3 + \tfrac{1}{4}t_5 - \tfrac{1}{6}t_6 - \tfrac{1}{4}q\big) + O(n^{-3}). \end{aligned}$$

Hence,

$$\begin{aligned} E(nR^TR) &= nE(R^uR^u) \\ &= n\{E(R_1^uR_1^u) + 2E(R_1^uR_2^u) \\ &\qquad + 2E(R_1^uR_3^u) + E(R_2^uR_2^u)\} + O(n^{-2}) \\ &= q + n^{-1}\big(\tfrac{5}{3}t_1 - 2t_2 + \tfrac{1}{2}t_3 - t_4 + \tfrac{1}{4}t_5\big) + O(n^{-2}), \end{aligned}$$

whence

$$\{E(nR^TR)/q\}^{-1} = 1 - (qn)^{-1}\big(\tfrac{5}{3}t_1 - 2t_2 + \tfrac{1}{2}t_3 - t_4 + \tfrac{1}{4}t_5\big) + O(n^{-2}),$$

as had to be shown.

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