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SECOND ORDER ANALYSIS OF TWO-STAGE RANK TESTS FOR THE ONE-SAMPLE PROBLEM

BY WILLEM ALBERS

University of Twente

In this paper we present a rank analogue to Stein's two-stage procedure. We analyze its behavior to second order using existing asymptotic expansions for fixed sample size rank tests and recent results on combinations of independent rank statistics.

1. Introduction. Consider the problem of testing H_0: $\theta = 0$ based on a sample from a sequence $X_1, X_2, \dots$ of independent identically distributed (iid) random variables (rv's) from a continuous distribution function (df) $F(x - \theta)$, where $F(-x) = 1 - F(x)$ for all x. For the special case $F(x) = \Phi(x/\sigma)$, in which Φ is the standard normal df, Stein proposed a by now well-known two-stage t-test [see, e.g. Lehmann (1986), pages 258–260] which makes it possible for each given alternative to guarantee at least a given power, independent of the unknown scale parameter σ. But for rank tests, which, being distribution-free, already have a level independent of F, it would be even more worthwhile to have a power independent of certain aspects of F, such as the scale parameter. Consequently, the purpose of the present paper is to obtain a rank analogue to Stein's procedure and to study its behavior to second order, using the results on asymptotic expansions for one-sample rank tests from Albers, Bickel and van Zwet (1976) (denoted by ABZ henceforth).

Although ABZ provides an excellent starting point, the derivation of expansions for the two-stage case still poses tedious technical problems. Therefore we shall use a device which simplifies matters considerably. Instead of evaluating the rank statistic for the total sample, we let the two-stage character persist in that we evaluate separate rank statistics for the initial and the second sample. These two statistics are then combined to a final test statistic in an optimal manner.

At first sight this approach might seem to lead to an inferior procedure, but fortunately this is not the case. A similar device was applied successfully by Albers and Akritas (1987) in the context of censored rank tests. Moreover, Albers (1988) studied the effects of splitting to second order and demonstrated that for rank tests the loss incurred can typically be compensated by as little as a single additional observation. In Section 2 we shall collect the results from that paper which are relevant for the present application.

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The desired expansion is obtained in Section 3 along the following lines. A suitable conditioning argument allows application of the results from ABZ to each of the two separate rank statistics. The conditional expansions thus obtained in their turn lead through application of the results from Section 2 to a conditional expansion for the combined statistic. The final result follows by taking expectations. A specific choice, closely related to Stein's procedure, is considered in Section 4. Finally, some examples and the conclusions of a small simulation study are presented. The results indicate that the proposed two-stage rank procedure works quite well. In particular, it performs much better than the simple version based on mere first order approximations.

2. Combined rank tests. Suppose that in testing a certain parameter θ the total sample of size n has been divided into r ($r \geq 2$) subsamples of sizes n_ν, $\nu = 1, \dots, r$, for each of which a suitable statistic T_ν has been evaluated. Then it is of some interest to obtain the best combination T^* of these T_ν and to evaluate the loss with respect to the standard statistic T which would have been used if the total sample were undivided. For one-sample rank statistics we typically have for certain μ and σ^2 that T_ν is $\mathrm{AN}(\theta\mu n_\nu, \sigma^2 n_\nu)$, where "AN" stands for asymptotically normal. As $\sum_{\nu=1}^{r} n_\nu = n$, it immediately follows that $\sum_{\nu=1}^{r} T_\nu$ is $\mathrm{AN}(\theta\mu n, \sigma^2 n)$. But obviously T itself is also $\mathrm{AN}(\theta\mu n, \sigma^2 n)$. Hence $T^* = \sum_{\nu=1}^{r} T_\nu$ and the loss incurred is negligible to first order. Note that this result extends to the case of two-sample rank statistics, provided that the two samples of sizes m_1 and $m_2 = n - m_1$ are divided such that $m_{1\nu}/m_1 = m_{2\nu}/m_2 = n_\nu/n$, $\nu = 1, \dots, r$ holds to first order.

In view of the above it makes sense to compare the performance of T^* and T to second order, which requires asymptotic expansions to $o(N^{-1})$ rather than mere asymptotic normality results. Suppose that the df G of a suitably standardized version $S = (T - \zeta)/\beta$ for certain η and δ satisfies $\sup_x |G(x) - \tilde{G}(x - \eta)| \leq \delta$, where

$$\tilde{G}(x) = \Phi(x) + \phi(x) \sum_{k=0}^{p} b_k H_k(x), \tag{2.1}$$

in which $\phi = \Phi'$, H_k is the Hermite polynomial of degree k and the b_k are coefficients, which for convenience are supposed to satisfy $|b_k| \leq 1$. Moreover, suppose for each T_ν similar results are available, which will be denoted by adding subscripts ν wherever appropriate. Then $T^* = \sum_{\nu=1}^{r} T_\nu$ has a standardized version $S^* = \sum_{\nu=1}^{r} \gamma_\nu S_\nu$, where $\gamma_\nu = \beta_\nu / (\sum_{g=1}^{r} \beta_g^2)^{1/2}$ satisfies $\sum_{\nu=1}^{r} \gamma_\nu^2 = 1$. Now it is not difficult to show [see Albers (1988) for details] that the df G^* of S^* satisfies, for example,

$$\begin{aligned} &\sup_x \left| G^*(x) - \tilde{G}^*\left(x - \sum_{\nu=1}^{r} \gamma_\nu \eta_\nu\right) \right| \\ &\quad = O\left(\left(\sum_{\nu=1}^{r} \delta_\nu \right) \left\{ 1 + \sum_{\nu=1}^{r} \sum_{k=0}^{p} |b_{k\nu}| \right\} + \sum_{\nu=1}^{r} \left(\sum_{k=0}^{p} |b_{k\nu}| \right)^3 \right), \end{aligned} \tag{2.2}$$

where

$$\tilde{G}^*(x) = \Phi(x) + \phi(x)\left[\sum_{k=0}^{p}\sum_{\nu=1}^{r} b_{k\nu}\gamma_\nu^{k+1}H_k(x) \right.$$

$$(2.3)\qquad \left. - \sum_{k=0}^{p}\sum_{l=0}^{p}\sum_{\substack{\nu=1 \\ \nu<g}}^{r}\sum_{g=1}^{r} b_{k\nu}b_{lg}\gamma_\nu^{k+1}\gamma_g^{l+1}H_{k+l+1}(x)\right].$$

Unfortunately, this result does not seem to be very useful, as $\tilde{G}^*(x - \Sigma_{\nu=1}^{r}\gamma_\nu\eta_\nu)$ in (2.3) differs widely from $\tilde{G}(x - \eta)$ in (2.1) and hence in no way suggests G^* and G, and thus T^* and T, to be close. Nor does it help to observe that $b_{k\nu}$ is related to $b_k = b_k(\theta, n)$ through $b_{k\nu} = b_k(\theta, n_\nu)$ and that for η_ν and δ_ν similar results hold. It does help, however, to check for which k (and l) one of the following criteria holds:

$$(2.4)\qquad b_k(\theta, n_\nu) = \gamma_\nu^{(1-k)}b_k(\theta, n),$$

$$(2.5)\qquad b_{k+l+1}(\theta, n_\nu) = -\{1 - \delta(k,l)/2\}b_k(\theta, n_\nu)b_l(\theta, n_\nu),$$

where $\delta(\cdot,\cdot)$ stands for Kronecker's delta. For such k and l, the corresponding parts of $\tilde{G}^*$ and $\tilde{G}$ indeed agree, for example, under (2.4), we obtain that $\Sigma_{\nu=1}^{r}b_k(\theta, n_\nu)\gamma_\nu^{k+1} = b_k(\theta, n)\Sigma_{\nu=1}^{r}\gamma_\nu^2 = b_k(\theta, n)$. It turns out that for the simple case where T is the sample sum, (2.4) or (2.5) holds for all k (and l). Hence $\tilde{G}^* \equiv \tilde{G}$, which should be the case as $T^* \equiv T$ and thus $G^* \equiv G$. An explanation of the at first peculiar form of the two criteria is easily obtained by looking at the expansions of the characteristic functions for this case. See once more Albers (1988) for details.

Summarizing the above, we have shown how to derive from the expansion $\tilde{G}$ in (2.1) for T the expansion $\tilde{G}^*$ in (2.3) for T^* and how to eliminate apparent differences between these two using criteria (2.4) and (2.5). Next we apply these results to one-sample rank statistics. Let $X_1, \dots, X_n$ be iid rv's with continuous df $F(x - \theta)$, such that $F(-x) = 1 - F(x)$ for all x. Denote the order statistics of $|X_1|, \dots, |X_n|$ by $0 < Z_1 < \cdots < Z_n$ and let $V_j = 1$ if the X_i corresponding to Z_j is positive and $V_j = 0$ otherwise. Finally, introduce the exact scores $a_j = a_{jn} = EJ(U_{j:n})$ for a continuous function J on $(0,1)$ and order statistics $U_{1:n} < \cdots < U_{n:n}$ of a sample of size n from the uniform distribution on $(0,1)$. Then the one-sample linear rank statistic for testing $H_0 = \theta = 0$ is given by

$$(2.6)\qquad T = \sum_{j=1}^{n} a_j V_j.$$

An expansion $\tilde{G}(x-\eta)$ to order n^{-1} for the df $G(x)$ of $S = (2T - \Sigma_{j=1}^{n} a_j)/(\Sigma_{j=1}^{n} a_j^2)^{1/2}$, both under H_0 and contiguous alternatives $\theta = O(n^{-1/2})$, is available from Theorem 4.1 of ABZ, which we shall now quote.

First we give the conditions on the df F and the score function J. Let $\mathscr{Q}$ be the class of twice continuously differentiable functions Q on (0, 1) that satisfy

$$\limsup_{t\to 0,1} t(1-t)\left|\frac{Q''(t)}{Q'(t)}\right| < \frac{3}{2}.$$

Let $\mathscr{F}$ be the class of df's on $\mathbb{R}^1$ with positive densities that are symmetric about zero, four times differentiable and such that, for $\psi_i = f^{(i)}/f$, $\Psi_i(t) = \psi_i(F^{-1}((1+t)/2))$, $m_1 = 6$, $m_2 = 3$, $m_3 = \frac{4}{3}$, $m_4 = 1$, we have $\Psi_1 \in \mathscr{Q}$ and

$$\limsup_{y\to 0} \int_{-\infty}^{\infty} |\psi_i(x+y)|^{m_i} f(x)\, dx < \infty, \qquad i = 1,\dots,4.$$

Let $\mathscr{J}$ be the class of nonconstant functions J on (0, 1) that satisfy $J \in \mathscr{Q}$ and $\int_0^1 J^4(t)\, dt < \infty$. Moreover, we define using the convention that integration will be over (0, 1), unless stated otherwise,

$$\eta(n) = -\frac{n^{1/2}\theta \int J\Psi_1}{\left(\int J^2\right)^{1/2}},$$

$$\bar{b}_0 = -\frac{\int J\left(3\Psi_1^3 - 6\Psi_1\Psi_2 + \Psi_3\right)\int J^2}{6\left(\int J\Psi_1\right)^3},$$

$$\bar{b}_1 = \frac{\int J^2\Psi_1^2 - \int\int J(s)\Psi_1'(s)J(t)\Psi_1'(t)(s\wedge t - st)\, ds\, dt}{2\left(\int J\Psi_1\right)^2}, \tag{2.7}$$

$$\bar{b}_2 = \frac{\int J^3\Psi_1}{3\left(\int J^2\right)\left(\int J\Psi_1\right)}, \qquad \bar{b}_3 = \frac{\int J^4}{12\left(\int J^2\right)^2},$$

$$\tilde{b}_0(n) = \frac{2\Sigma_{j=1}^{n} \operatorname{cov}\left(J(U_{j:n}), \Psi_1(U_{j:n})\right)}{\int J\Psi_1} - \frac{\Sigma_{j=1}^{n} \sigma^2\left(J(U_{j:n})\right)}{\int J^2}.$$

Finally, let

$$\tilde{G}(x) = \Phi(x) + \frac{1}{n}\phi(x)\left\{\frac{1}{2}\eta(n)\tilde{b}_0(n) + \sum_{k=0}^{3} \eta(n)^{(3-k)}\bar{b}_k H_k(x)\right\}. \tag{2.8}$$

Then we have:

LEMMA 2.1. *Let $F \in \mathscr{F}$, $J \in \mathscr{J}$ and $0 \le \theta \le Cn^{-1/2}$ for some $C > 0$. Then T from* (2.6) *satisfies*

$$\sup_x \left| P\left(\frac{2T - \sum_{j=1}^n a_j}{\left(\sum_{j=1}^n a_j^2\right)^{1/2}} \le x \right) - \tilde{G}(x - \eta(n)) \right| = o(n^{-1}).$$

PROOF. This result is contained in Theorem 4.1 of ABZ. □

Inspection of (2.7) and (2.8) shows that the terms involving $n^{-1}\eta(n)^{(3-k)}$ all satisfy (2.4) if we replace γ_ν by $\tilde{\gamma}_\nu = (n_\nu/n)^{1/2}$. As it can be verified that $\gamma_\nu - \tilde{\gamma}_\nu = o(n^{-1/2})$, it follows that $\tilde{G}^*$ and $\tilde{G}$ agree to the desired order for all these coefficients. In a similar way one checks that $\sum_{\nu=1}^r \gamma_\nu \eta_\nu = \eta + o(n^{-1})$. Hence the only difference between $\tilde{G}^*(x - \sum_{\nu=1}^r \gamma_\nu \eta_\nu)$ and $\tilde{G}(x - \eta)$ to order n^{-1} is caused by the $\tilde{b}_0(n)$-term. In fact we obtain that

$$(2.9) \quad G^*(x) = G(x) + \phi(x)\frac{\eta(n)}{2n}\left\{ \sum_{\nu=1}^r \tilde{b}_0(n_\nu) - \tilde{b}_0(n) \right\} + o(n^{-1}),$$

[see Theorem 3.2 of Albers (1988)].

To make this result more transparent, we translate it into terms of deficiencies. It follows from (2.9) that the deficiency d_n of the test based on T^* with respect to the one based on T, which simply equals the additional number of observations required by the former test to match the power of the latter based on n observations, is given by $d_n = \sum_{\nu=1}^r \tilde{b}_0(n_\nu) - \tilde{b}_0(n) + o(1)$. If, for example, F is logistic and $J(t) = t$ (i.e., Wilcoxon scores), $d_n \to (r-1)/2$. Hence one additional observation already suffices to pay for two additional subgroups. A simulation study for this case shows close agreement with these theoretical values. If F is normal and $J(t) = \Phi^{-1}((1+t)/2)$ (i.e., normal scores), we have to pay a penalty $\frac{1}{2}(\log\log n + \gamma) + o(1)$ for each additional subgroup, where γ is Euler's constant $\lim_{k\to\infty}(\sum_{i=1}^k i^{-1} - \log k) = 0.577216\ldots$.

For the application in the next section the one-sample case suffices. In passing we note that in Albers (1988) the considerably more complicated two-sample case is covered as well. Here the number of terms is much larger and some of these terms are of order $n^{-1/2}$ rather than n^{-1}, which necessitates the use of (2.5) in addition to (2.4). Nevertheless, the final result is again relatively simple to interpret.

3. The expansion for the two-stage test. In the two-stage situation we have an initial sample of size m and a second sample of size $N - m$, in which $N = N(X_1, \ldots, X_m)$ in general. We shall require that $N = N(Z_{(m)})$, where $Z_{(m)}$ is the vector of absolute order statistics of the first sample. This ensures that the two-stage test remains distribution-free, since the ranks and $Z_{(m)}$ are independent under H_0. As announced, we shall not consider a single rank statistic T of the form (2.6) for the total sample, but instead analyze $T^* =$

$T_1 + T_2$, where T_1 and T_2 are separate rank statistics for the first and second samples, respectively.

The first step toward an expansion for the df of T^* consists of conditioning on $Z_{(m)}$. As T_2 depends on the first sample only through $N = N(Z_{(m)})$, it is immediate that conditionally this statistic is independent of T_1 and moreover has the same distribution as T from (2.6) for sample size $N(z_{(m)}) - m$. Hence Lemma 2.1 directly applies to T_2.

For T_1 the situation is less simple, as it obviously depends on $Z_{(m)}$ in a more complicated way than T_2. Hence, rather than the final result from Lemma 2.1,we shall have to use as our starting point the expansion conditional on $Z_{(m)}$ (Theorem 2.1 of ABZ), from which $\tilde{G}$ is derived. This conditional expansion has to be modified in such a way that the combination results of the previous section become applicable. This is achieved by dividing it into a deterministic part, for which we can simply use $\tilde{G}(x - \eta(m))$, and a stochastic correction part, which keeps track of the dependence on $Z_{(m)}$.

To be more specific, in analogy to (2.6), define a_{1j} and V_{1j} for $j = 1, \dots, m$ and a_{2j} and V_{2j} for $j = 1, \dots, N - m$. Let $P_{1j} = P(V_{1j} = 1|Z_{(m)})$ and $\pi_{1j} = EP_{1j}$. The leading term in the conditional expansion is $\Phi(x - \Sigma_{j=1}^m a_{1j}(2P_{1j} - 1)/(\Sigma_{j=1}^m a_{1j}^2)^{1/2})$, from which the leading term for the deterministic part is obtained by replacing P_{1j} by π_{1j}. The difference is then expanded in powers of

$$(3.1) \qquad U = \frac{\Sigma_{j=1}^m a_{1j}(P_{1j} - \pi_{1j})}{\left(\Sigma_{j=1}^m a_{1j}^2\right)^{1/2}}.$$

Let

$$\tilde{K}(x) = \tilde{G}(x) + \phi(x)\left\{-2U + \left[-2(U^2 - EU^2) + \frac{1}{2}\frac{\Sigma_{j=1}^m a_{1j}^2\left\{(2P_{1j} - 1)^2 - E(2P_{1j} - 1)^2\right\}}{\Sigma_{j=1}^m a_{1j}^2}\right]H_1(x) + \frac{2}{3}\left(\frac{\Sigma_{j=1}^m a_{1j}^3(P_{1j} - \pi_{1j})}{\left(\Sigma_{j=1}^m a_{1j}^2\right)^{3/2}}\right)H_2(x)\right\}.$$

We then have:

LEMMA 3.1. *Under the assumptions of Lemma* 2.1, T_1 *satisfies*

$$\sup_x \left| P\left(\frac{2T_1 - \Sigma_{j=1}^m a_{1j}}{\left(\Sigma_{j=1}^m a_{1j}^2\right)^{1/2}} \le x \middle| Z_{(m)}\right) - \tilde{K}(x - \eta(m))\right| = o(m^{-1}) + O\left(\sum_{j=1}^m |2P_{1j} - 1|^5 + |U|^3\right),$$

except on a set of $z_{(m)}$*-values with probability* $o(m^{-1})$.

PROOF. See Albers (1989), Lemma 2.2. □

It is the term involving U which has to be handled with care. On one hand, it has zero expectation and as such disappears eventually in ABZ. But on the other hand, it is in probability of order $m^{-1/2}$ and hence here it will produce a nonnegligible interaction term with $N = N(Z_{(m)})$.

Conditional on $Z_{(m)} = z_{(m)}$, the situation now is as follows: T_1 and T_2 are independent, Lemma 2.1 applies to T_2 for sample size $N(z_{(m)}) - m$ and Lemma 3.1 applies to T_1. Application of (2.1)–(2.3) and in particular of (2.7)–(2.9), then leads to a conditional expansion for $T^* = T_1 + T_2$. The result is the expansion $\tilde{G}(x - \eta(N))$ from (2.8) for sample size N, with, in the first place, the correction for splitting from (2.9) with $r = 2$, $n_1 = m$ and $n_2 = N - m$, and, in the second place, a stochastic correction part, of which the only term that matters is the one involving

$$\tilde{U} = \frac{\sum_{j=1}^{m} a_{1j}(P_{1j} - \pi_{ij})}{\left(\sum_{j=1}^{m} a_{1j}^2 + \sum_{j=1}^{N-m} a_{2j}^2\right)^{1/2}}. \tag{3.2}$$

To be more precise, we introduce, adapting the convention that summation involving a_{1j} or a_{2j} runs from 1 to m or $N - m$, respectively,

$$\begin{aligned}\tilde{H}(x) = \Phi(x) + \phi(x)\Bigg\{ & \frac{1}{2}N^{-1}\eta(N)\left(\bar{b}_0(m) + \bar{b}_0(N-m)\right) \\ & + N^{-1}\sum_{k=0}^{3}\eta(N)^{(3-k)}\bar{b}_k H_k(x) - 2\tilde{U} \\ & + \frac{\left(-2\{\sum a_{1j}(P_{1j} - \pi_{1j})\}^2 + 2E\{\sum a_{1j}(P_{1j} - \pi_{1j})\}^2 + \frac{1}{2}\sum a_{1j}^2\left\{(2P_{1j} - 1)^2 - E(2P_{1j}^2 - 1)^2\right\}\right)}{\sum a_{1j}^2 + \sum a_{2j}^2} H_1(x) \\ & + \frac{2}{3}\frac{\sum a_{1j}^3(P_{1j} - \pi_{1j})}{\left(\sum a_{1j}^2 + \sum a_{2j}^2\right)^{3/2}} H_2(x)\Bigg\}.\end{aligned} \tag{3.3}$$

Then we obtain:

LEMMA 3.2. *Let $F \in \mathscr{F}$ and $J \in \mathscr{J}$. Suppose that $0 \le \theta \le Cm^{-1/2}$ for some $C > 0$ and N satisfies $P((1 + \varepsilon) \le N/m \le \varepsilon^{-1}) = 1 - o(m^{-1})$ for some*

$\varepsilon > 0$. *Then* $T^* = T_1 + T_2$ *satisfies*

$$\sup_x \left| P\left(\frac{2T^* - \Sigma a_{1j} - \Sigma a_{2j}}{\left(\Sigma a_{1j}^2 + \Sigma a_{2j}^2\right)^{1/2}} \le x \middle| Z_{(m)} \right) - \tilde{H}(x - \eta(N)) \right| \tag{3.4}$$
$$= o(N^{-1}) + O\left(\Sigma |2P_{1j} - 1|^5 + |\tilde{U}|^3\right),$$

except on a set of $z_{(m)}$-*values with probability* $o(m^{-1})$.

PROOF. See Albers (1989), Lemma 2.3. □

Under the hypothesis we have $\theta = 0$, $\eta(N) = 0$, $P_{1j} \equiv \frac{1}{2}$, $\pi_{1j} = \frac{1}{2}$ and $\tilde{U} = 0$ in (3.2). Hence under H_0 the expansion $\tilde{H}(x - \eta(N))$ in (3.3) boils down to $\Phi(x) + \phi(x)N^{-1}\bar{b}_3 H_3(x)$. Hence the test based on $S^* = (2T^* - \Sigma_{j=1}^{m} a_{1j} - \Sigma_{j=1}^{N-m} a_{2j})/(\Sigma_{j=1}^{m} a_{1j}^2 + \Sigma_{j=1}^{N-m} a_{2j}^2)^{1/2}$ has critical value $\xi_\alpha = \tilde{\xi}_\alpha + o(N^{-1})$, where

$$\tilde{\xi}_\alpha = u_\alpha - N^{-1}\bar{b}_3 H_3(u_\alpha), \tag{3.5}$$

with $u_\alpha = \Phi^{-1}(1 - \alpha)$. Not only does this result agree with that for the fixed sample case, but moreover (3.5) also shows that, once T^* has been standardized to S^* through the conditional mean and the conditional standard deviation, the remaining dependence of the critical value of the conditioning is very limited. In fact, replacement of N^{-1} in (3.5) by, for example, $(EN)^{-1}$ will typically result in changes of $o(m^{-1})$. Consequently, the unconditional distribution of S^* can be used in studying the behavior of the exact conditional test to $o(m^{-1})$.

In view of the above, the next step consists of taking the expectation with respect to $Z_{(m)}$ of the conditional expansion for S^*. To ensure that a second order analysis is possible (and moreover, that it makes sense to go beyond first order results) we need a condition like: For certain $\varepsilon > 0$ and $\beta > 1$,

$$P((1 + \varepsilon) \le N/m \le \varepsilon^{-1}) = 1 - o(m^{-1}), \qquad E|N - EN|^{2\beta} = O(m^\beta). \tag{3.6}$$

Let $r \ge (1 + 2\varepsilon)m$ and define $\eta(r) = -r^{1/2}\theta \int J\Psi_1/(\int J^2)^{1/2}$. Moreover, let $\tilde{G}_r$ be $\tilde{G}$ from (2.8) with n and $\eta(n)$ replaced by r and $\eta(r)$, respectively. Finally, define $\bar{U} = (N/r)^{1/2} - 1$, and

$$\bar{H}(x) = \tilde{G}_r(x) + \phi(x)\Bigg\{\frac{1}{2}r^{-1}\eta(r)\left(\tilde{b}_0(m) + \tilde{b}_0([r] - m) - \tilde{b}_0([r])\right)$$
$$-\eta(r)E\bar{U} - \frac{1}{2}\eta(r)^2 E\bar{U}^2 H_1(x) \tag{3.7}$$
$$+ \frac{r^{-1}\eta(r)E\left[\bar{U}\Sigma_{j=1}^{m} a_{1j}\left(\psi_1(Z_{1j}) - E\psi_1(Z_{1j})\right)\right]}{\int J\Psi_1}\Bigg\},$$

where $[r]$ denotes the largest integer less than or equal to r. Then we finally arrive at:

THEOREM 3.1. *Let $F \in \mathscr{F}$ and $J \in \mathscr{J}$. Suppose that $\theta = O(m^{-1/2})$ and N satisfies* (3.6). *If r is chosen such that $r = EN + o(m^{1/2})$, we have*

$$\sup_x |P(S^* \le x) - \overline{H}(x - \eta(r))| = o(m^{-1}). \tag{3.8}$$

PROOF. Here we shall indicate the steps involved. For details, consult Albers (1989). We need to show that the expectation of $\tilde{H}(x - \eta(N))$ in (3.3) produces $\overline{H}(x - \eta(r))$ in (3.7) to the desired order $o(m^{-1})$. Since $\eta(N) = \eta(r)(1 + \overline{U})$, expansion of $E\Phi(x - \eta(N))$ explains the leading term Φ in $\overline{H}$, as well as the terms involving $E\overline{U}$ and $E\overline{U}^2$. Moreover, condition (3.6) ensures that the term with $E|\overline{U}|^3$ is $o(m^{-1})$.

As concerns the second order part, we note in the first place that it is easily verified that replacement of N by r causes negligible differences in those terms which involve $\eta(\cdot)$ or $\tilde{b}_0(\cdot)$. The remaining terms in (3.3) all have a numerator with zero expectation. If they are moreover $O_P(m^{-1})$, their contribution to $\overline{H}(x - \eta(r))$ clearly is $o(m^{-1})$ and thus these terms vanish. The sole exception is $\tilde{U}$ from (3.2), which is of order $m^{-1/2}$ rather than m^{-1} in probability. Replacement of the denominator in (3.2) by $N^{1/2}(\int J^2)^{1/2} = r^{1/2}(1 + \overline{U})(\int J^2)^{1/2}$ causes a difference of $o(m^{-1})$, while expansion of the resulting expression in terms of $\overline{U}$ produces a leading term which indeed has expectation zero, as well as a mixed term involving $\overline{U}$, which corresponds to the last term in (3.7). [Note that $2(P_{1j} - \pi_{1j})$ to first order equals $-\theta(\psi_1(Z_{1j}) - E\psi_1(Z_{1j}))$.]

It remains to show that the set of exceptional $z_{(m)}$-values, which is required to have probability $o(m^{-1})$, causes no trouble. This verification requires some care, but otherwise can be executed completely along the lines of ABZ. □

The interpretation of this result is straightforward: (3.7) begins with the fixed sample size result for n equal [to $o(m^{1/2})$] to EN, followed by a correction term to account for the splitting and three terms involving $\overline{U}$ due to the sample size being random. The first two of these three terms simply result from expanding $E\Phi(x - \eta(N))$. The complicated last one reflects the interaction between the two stages of the procedure.

4. Tests with guaranteed power. Theorem 3.1 in the previous section enables us to select $N = N(Z_{(m)})$ such that the power of the test based on S^* satisfies $\pi^*(\kappa m^{-1/2}) = \pi_1 + o(m^{-1})$ for given κ and π_1. We shall concentrate on the situation corresponding to Stein's procedure, but the general case can be dealt with in exactly the same way. Hence $F \in \{\tilde{F}(\cdot/\sigma),\ \sigma > 0\}$ with $\int_{-\infty}^{\infty} x^2\, d\tilde{F}(x) = 1$. As $\int J\Psi_1 = \int J\tilde{\Psi}_1/\sigma$, the condition $\pi^*(\kappa m^{-1/2}) = \pi_1$ will be met to first order for

$$r = \frac{m(u_\alpha - u_\pi)^2 \sigma^2 \int J^2}{\left(\kappa \int J \tilde{\Psi}_1\right)^2}, \tag{4.1}$$

where $u_\pi = \Phi^{-1}(1 - \pi_1)$.

An initial estimator N_1 is obtained by replacing σ^2 in (4.1) by a suitable estimator $\bar{S}_m^2$. Essentially we use the sample variance, but as N should depend on $X_1, \dots, X_m$ only through $Z_{(m)} = (Z_{11}, \dots, Z_{1m})$, we shall select the modified version $\bar{S}_m^2 = m^{-1}\sum_{j=1}^m Z_{1j}^2 = m^{-1}\sum_{i=1}^m X_i^2$. Now a correction term $\hat{f}_r$ is added to N_1, selected through (3.7) such that it precisely cancels the lower order terms. The final touch then consists of setting, again denoting the largest integer less than or equal to y by $[y]$,

$$N = \max\left(m, \left[N_1 + \hat{f}_r + \tfrac{1}{2}\right]\right). \tag{4.2}$$

Consult Albers (1989) for the actual evaluation of $\hat{f}_r$ and the corresponding proof. For sake of brevity we just present some explicit examples. Write

$$\begin{aligned} N = \max\Bigg(m, \Bigg[c_1 \frac{m(u_\alpha - u_\pi)^2}{\kappa^2}\left(\bar{S}_m^2 - \frac{\kappa^2}{m}\right) \\ \times\left(1 + c_2 \frac{(1 - u_\alpha u_\pi + u_\pi^2)}{m}\right) + c_3\Bigg]\Bigg). \end{aligned} \tag{4.3}$$

Then we achieve $\pi^*(\kappa m^{-1/2}) = \pi_1 + o(m^{-1})$ for the normal case $\tilde{F} = \Phi$, $J(t) = \Phi^{-1}((1 + t)/2)$ if we choose $c_1 = 1$, $c_2 = \frac{1}{2}$ and $c_3 = \frac{1}{2}\log\log m + \frac{1}{2}\log\log([N_1] - m) + \gamma + \frac{1}{2}u_\alpha^2 + 1$. For the logistic case $\tilde{F}(x) = (1 + \exp(\pi 3^{-1/2}x))^{-1}$ and $J(t) = t$, we need $c_1 = (3/\pi)^2$, $c_2 = 4/5$ and $c_3 = (5u_\alpha^2 - 3u_\alpha u_\pi + u_\pi^2 + 2)/10 + 6/\pi^2 + 1$. A small simulation study for the latter example shows a quite satisfactory agreement with the theoretical results. With 10^4 simulations for each case, $0.01 \le \alpha \le 0.05$, $0.5 \le \pi_1 \le 0.9$ and $10 \le m \le 20$ we find that the realized power typically falls short of the prescribed π_1 by approximately 0.01. By contrast, the simple first order solution, for which $c_2 = c_3 = 0$ in (4.3), is rather bad as it leads to power values which are typically about 0.10 too low.

To conclude this section, we briefly compare the normal example above to Stein's procedure. There $N = \max(m, [S_m^2/c] + 1$, where S_m^2 is the sample variance and $c^{-1} = m(t_{m-1,\alpha} - t_{m-1,\pi})^2/\kappa^2$, in which $t_{m-1,\alpha}$ is the upper α-point of the $t(m - 1)$-distribution. After some calculation we obtain for the expected difference between the sample sizes of the rank procedure and Stein's procedure the expression

$$\begin{aligned} \tfrac{1}{2}\log\log m + \tfrac{1}{2}\log\log([N_1] - m) + \gamma + \tfrac{1}{2}u_\alpha^2 + \tfrac{1}{2} \\ -(u_\alpha - u_\pi)^2\sigma^2\left(\tfrac{1}{2}u_\alpha^2 + u_\alpha u_\pi\right)/\kappa^2, \end{aligned}$$

which will typically be quite small.

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FACULTY OF APPLIED MATHEMATICS
UNIVERSITY OF TWENTE
P.O. BOX 217
7500 AE ENSCHEDE
THE NETHERLANDS