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SOME RESULTS ON s^{n-k} FRACTIONAL FACTORIAL DESIGNS WITH MINIMUM ABERRATION OR OPTIMAL MOMENTS

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The minimum aberration criterion is commonly used for selecting good fractional factorial designs. In this paper we obtain minimum aberration 2^{n-k} designs for $k = 3, 4$ and any n. For $k > 4$ analogous results are not available for general n since the resolution criterion is not periodic for general n and $k > 4$. However, it can be shown that for any fixed k, both the resolution criterion and the minimum aberration criterion have a periodicity property in n for s^{n-k} designs with large n. Furthermore, the optimal-moments criterion is periodic for any n and k.

1. Introduction and definitions. Fractional factorial designs are among the most commonly used plans for designed experiments [see, e.g., Box, Hunter and Hunter (1978)]. The many successful applications of this method in the quest of industrial quality and productivity are a recent testimony to its importance. A key question in selecting such designs is to develop a goodness criterion of a design. It had been a standard practice to choose a fractional factorial design with maximum resolution. Since designs with the same *resolution* are not equally good, a more refined criterion called *minimum aberration* was introduced by Fries and Hunter (1980). When the experimenter has little knowledge about the relative sizes of the factorial effects, the minimum aberration criterion selects designs with good overall properties.

With some exceptions [Fries and Hunter (1980); Franklin (1984)], there are few theoretical results on minimum aberration designs in the literature. In Section 3 we obtain minimum aberration 2^{n-k} designs for $k = 3$ and 4 and any n. Formally an s^{n-k} design is a fractional factorial design with n factors each of s levels and s^{n-k} runs. For $k > 4$, analogous results to those for $k = 3$ and 4 are not possible because it is known in the literature on coding theory that the resolution criterion is not periodic in n for 2^{n-k} designs with general n [see Verhoeff (1987)]. However for *large n* and any fixed k, we show in Section 4 that both the resolution criterion and the minimum aberration criterion have a periodicity property in n for s^{n-k} designs (Theorems 1 and 2). For another criterion called optimal-moments [Franklin (1984)], we show in Section 5 that a periodicity property holds for any s^{n-k} designs (Theorem 4). We

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also give a characterization of a minimum-variance s^{n-k} design in Theorem 3. A minimum-variance design is a special case of an optimal-moments design.

In the following we use a simple example to explain and motivate the definitions described before. The general definitions are formally given at the end of the section.

A 2^{n-k} fractional factorial design has n variables each of 2 levels and 2^{n-k} runs. When $k = 0$, it is a full factorial design, in which the 2^n runs consist of all possible level combinations of the n variables. These combinations form a $2^n \times n$ matrix with n independent columns.

For example, when $n = 5$, we have 32 runs and five independent columns denoted by 1, 2, 3, 4, 5. If we define columns 6 and 7 by

$$6 = 123, \qquad 7 = 234, \tag{1}$$

where column 123 is formed by adding columns 1, 2, 3 (mod 2), the resulting design has 7 variables with 32 runs and is called 2^{7-2} fractional factorial design.

Rewrite (1) as $I = 1236 = 2347$, where I is the column of 0's. Another relation among the factors (columns) is

$$I = 1236 \times 2347 = 1467.$$

All together we have

$$I = 1236 = 2347 = 1467,$$

which forms a *defining contrasts subgroup* of the 2^{7-2} design with I being its identity element. The elements 1236, 2347 and 1467 in the subgroup are called *words*. The symbols 1, 2 and so on are called *letters*. The number of letters in a word is called the length of the word or wordlength.

Denote this design by d_1. Let $A_i(d_1)$ be the number of words of length i in the defining contrasts subgroup for d_1 and the vector

$$W(d_1) = (A_1(d_1), A_2(d_1), \dots) = (0, 0, 0, 3, \dots)$$

be its *wordlength pattern*. The ith moment of d_1 is defined to be

$$M_i(d_1) = \sum_{j=1}^{\infty} j^i A_j(d_1).$$

The ith central moment of the design for $i > 1$ is defined to be

$$\mu_i(d_1) = \sum_{j=1}^{\infty} (j - M_1(d_1))^i A_j(d_1).$$

The *resolution* of d_1 is the smallest i with positive $A_i(d_1)$. In this case, d_1 has resolution IV.

Given n and k, a 2^{n-k} fractional factorial design is not uniquely determined by its resolution. Consider two other designs:

$$d_2\colon\quad I = 1236 = 1457 = 234567,$$

$$d_3\colon\quad I = 12346 = 12357 = 4567.$$

Both have resolution IV, but have different wordlength patterns

$$W(d_2) = (0,0,0,2,0,1,0,\dots),$$
$$W(d_3) = (0,0,0,1,2,0,0,\dots),$$

and hence different moments. Since the first unequal components of $W(d_2)$ and $W(d_3)$ (fourth component) have the relation

$$W(d_2)[4] = 2 > W(d_3)[4] = 1,$$

we say that d_3 has *less aberration* than d_2, which in turn has less aberration than d_1. Compare their moments, the first unequal moments are

$$M_2(d_2) = 68 > M_2(d_3) = 66.$$

We say d_3 is better than d_2 in moments when this moment is of even order and d_3 is worse than d_2 when this moment is of odd order. Note that this relation will not be affected when we compare central moments instead of moments.

In general, an s^{n-k} fractional factorial design at s levels, s a prime power, can be defined as follows. A word is written as an n-dimensional vector with components in a finite field of s elements F_s. For two words

$$w_1 = (a_1, a_2, \dots, a_n) \quad \text{and} \quad w_2 = (b_1, b_2, \dots, b_n),$$

with $a_i, b_i \in F_s$, their product is a word defined by

$$w_1 \times w_2 = (a_1 + b_1, a_2 + b_2, \dots, a_n + b_n).$$

An s^{n-k} fractional factorial design is defined by its defining contrasts subgroup, which is a subgroup of words generated under the multiplication $\times$. The word w_1 and all its constant multiples

$$\lambda w_1 = (\lambda a_1, \lambda a_2, \dots, \lambda a_n) \quad \text{for any } \lambda \neq 0,\ \lambda \in F_s$$

are considered to be the same in the subgroup. The length of a word is the number of its nonzero components. The wordlength pattern, resolution and moments are defined in the same way as in two levels.

DEFINITIONS. An s^{n-k} fractional factorial design has maximum resolution, if no other s^{n-k} fractional factorial design has larger resolution.

Let d_1 and d_2 be two s^{n-k} fractional factorial designs and r be the smallest i such that $A_i(d_1) \neq A_i(d_2)$. d_1 has less aberration than d_2 if $A_r(d_1) < A_r(d_2)$. An s^{n-k} fractional factorial design has minimum aberration, if no other s^{n-k} fractional factorial design has less aberration.

Let m be the first i such that $M_i(d_1) \neq M_i(d_2)$. If m is odd and $M_m(d_1) < M_m(d_2)$, d_2 has better moments; if m is even and $M_m(d_1) < M_m(d_2)$, d_1 has better moments. An s^{n-k} fractional factorial design has optimal moments, if no other s^{n-k} fractional factorial design has better moments. An s^{n-k} fractional factorial design is a minimum-variance design if it maximizes the first moment $M_1(d)$ and minimizes the second moment $M_2(d)$.

Without loss of generality, we assume throughout the paper that each of the n letters in an s^{n-k} design must *appear in at least one word* of its defining relations. Indeed, this is equivalent to maximizing the first moment.

The term minimum aberration and its formal definition are due to Fries and Hunter (1980). The optimal-moments criterion can be found in Franklin (1984).

2. A technical lemma. Denote the maximum resolution of an s^{n-k} design by $R_s(n, k)$ and define the m-lag of a vector W to be

$$\mathrm{lag}(W, m) = \big(\underbrace{0, \dots, 0}_{m}, W\big),$$

where W is preceded by m zeros. We have the following lemma.

LEMMA 1. (i) *For any s^{n-k} fractional factorial design D_1 with wordlength pattern W_1, there exists an $s^{(n+(s^k-1)/(s-1))-k}$ design D_2 with wordlength pattern W_2, such that $W_2 = \mathrm{lag}(W_1, s^{k-1})$.*

(ii) $R_s(n + (s^k - 1)/(s - 1), k) \geq R_s(n, k) + s^{k-1}$.

PROOF. Part (ii) is a consequence of (i). The resolution of D_2 is $R_s(n, k) + s^{k-1}$. Hence the maximum resolution $s^{(n+(s^k-1)/(s-1))-k}$ design has a resolution greater than or equal to $R_s(n, k) + s^{k-1}$.

To prove (i), we first consider the special case $s = 2$. A 2^{n-k} design D_1 can be represented by a $(2^k - 1) \times n$ matrix with its rows being the $2^k - 1$ defining relations and its columns being the n letters. This matrix consists of two submatrices A_1 and B_1, where A_1 has k rows corresponding to k independent generators and the $2^k - k - 1$ rows in B_1 are generated from those in A_1. A $2^{(n+2^k-1)-k}$ design D_2 can be constructed by adding the matrix $\begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$ to $\begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$ as follows:

$$\begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix}, \qquad A_2 = [I_k, C_2], \tag{2}$$

where A_2 is $k \times (2^k - 1)$, I_k is the identity matrix of order k and B_2 is $(2^k - k - 1) \times (2^k - 1)$. The $2^k - 1$ columns in A_2 are given by

$$\sum_{i=1}^{k} \lambda_i c_i, \qquad \lambda_i \in F_2', \qquad \text{at least one } \lambda_i = 1,$$

where F_2 is the finite field of two elements and $\{c_i\}$ are the k column vectors in I_k. The rows of B_2 are generated from those of A_2 in exactly the same way as B_1 is generated from A_1. The resulting matrix of order $(2^k - 1) \times (n + 2^k - 1)$ in (2) gives $2^k - 1$ defining relations for $n + 2^k - 1$ letters, that is a $2^{(n+2^k-1)-k}$ design. It is easy to show that each row of A_2 or of B_2 has 2^{k-1} components with 1's and therefore each word in D_2 is 2^{k-1} longer than that of the corresponding word in D_1. This proves (i) for $s = 2$. Extension to general prime power s is straightforward. The finite field F_2 is replaced by F_s, the finite field of s elements, the first nonzero λ_i in $\sum_{i=1}^{k}\lambda_i c_i$ is chosen to be 1, $2^k - 1$ replaced by $(s^k - 1)/(s - 1)$ and the number of nonzero elements in any row of A_2 or B_2 is s^{k-1}. □

According to Lemma 1, we expect to find optimal designs of large size through those of small size. This is true for $k \le 4$ and $s = 2$, which will be studied in the next section.

3. Minimum aberration 2^{n-k} designs for $k = 3$ and $k = 4$. The minimum aberration 2^{n-1} designs are obtained by defining $n = 12 \cdots (n-1)$. For $k = 2$, Robillard (1968) gave the following method for constructing minimum aberration 2^{n-2} designs.

Let $n - 2 = 3m + r$, where $0 \le r < 3$. For $r = 0$, define

$$(n-1) = 12 \cdots (2m), \qquad n = (m+1)(m+2) \cdots (3m);$$

for $r = 1$, define

$$(n-1) = 12 \cdots (2m+1), \qquad n = (m+1)(m+2) \cdots (3m+1);$$

for $r = 2$, define

$$(n-1) = 12 \cdots (2m+1), \qquad n = (m+1)(m+2) \cdots (3m+2).$$

Then it is easy to show that these designs have the maximum resolution

$$R_2(n,2) = \left[\frac{2n}{3}\right]$$

and minimum aberration.

For 2^{n-3} designs, we find minimum aberration designs using the following rule. Let $n = 7m + r$, $0 \le r \le 6$. For $i = 1, \dots, 7$, define

$$B_i = \begin{cases} (im - m + 1)(im - m + 2) \cdots (im)(7m + i), & \text{when } i \le r, \\ (im - m + 1)(im - m + 2) \cdots (im), & \text{otherwise.} \end{cases}$$

These B_i divide the n letters into 7 approximately equal blocks. Let the defining contrasts subgroup be

$$\begin{aligned} I &= B_7B_6B_4B_3 = B_7B_5B_4B_2 = B_6B_5B_4B_1 \\ &= B_6B_5B_3B_2 = B_7B_5B_3B_1 = B_7B_6B_2B_1 = B_4B_3B_2B_1. \end{aligned} \tag{3}$$

We will prove that the previous designs have minimum aberration and therefore maximum resolution. Their resolutions can be summarized by the formula

$$R_2(n,3) = \begin{cases} \left[\dfrac{4n}{7}\right], & \text{when } r \ne 2, \\ \left[\dfrac{4n}{7}\right] - 1, & \text{otherwise.} \end{cases}$$

TABLE 1
Minimum aberration 2^{n-4} designs

n	Defining contrasts
5	$I = 12 = 13 = 14 = 15$
6	$I = 13 = 24 = 125 = 126$
7	$I = 124 = 135 = 236 = 1237$
8	$I = 1235 = 1246 = 1347 = 2348$
9	$I = 12346 = 12357 = 2458 = 3459$
10	$I = 23457 = 23468 = 13569 = 1456t_0$
11	$I = 34568 = 134579 = 12467t_0 = 2567t_1$
12	$I = 145679 = 24568t_0 = 23578t_1 = 13678t_2$
13	$I = 25678t_0 = 135679t_1 = 34689t_2 = 124789t_3$
14	$I = 136789t_1 = 24678t_0t_2 = 14579t_0t_3 = 23589t_0t_4$
15	$I = 124789t_0t_2 = 135789t_1t_3 = 12568t_0t_1t_4 = 13469t_0t_1t_5$
16	$I = 123589t_0t_1t_3 = 24689t_0t_2t_4 = 23679t_1t_2t_5 = 2457t_0t_1t_2t_6$
17	$I = 23469t_0t_1t_2t_4 = 13579t_0t_1t_3t_5 = 3478t_0t_2t_3t_6 = 3568t_1t_2t_3t_7$
18	$I = 3457t_0t_1t_2t_3t_5 = 2468t_0t_1t_2t_4t_6 = 14589t_1t_3t_4t_7 = 4679t_2t_3t_4t_8$
19	$I = 4568t_1t_2t_3t_4t_6 = 3579t_1t_2t_3t_5t_7 = 2569t_0t_2t_4t_5t_8 = 1578t_0t_3t_4t_5t_9$

Note: $t_0, t_1, \dots, t_9$ denote factors $10, 11, \dots, 19$.

To construct 2^{n-4} minimum aberration designs, we use the same idea as in $k = 3$. Let $n = 15m + r$, $0 \le r < 15$. Divide the n letters into 15 approximately equal blocks given by

$$B_i = \begin{cases} (im - m + 1)(im - m + 2) \cdots (im)(15m + i), & \text{for } i \le r, \\ (im - m + 1)(im - m + 2) \cdots (im), & \text{otherwise}. \end{cases}$$

When $r \ne 5$, let

$$\begin{aligned} I &= B_{15}B_{14}B_{12}B_9B_8B_7B_6B_1 = B_{15}B_{13}B_{11}B_9B_8B_7B_5B_2 \\ &= B_{15}B_{14}B_{11}B_{10}B_8B_6B_5B_3 = B_{15}B_{13}B_{12}B_{10}B_7B_6B_5B_4. \end{aligned} \tag{4}$$

When $r = 5$, switch B_{15} and B_5 in the previous defining contrasts subgroup.

For $5 \le n \le 19$, the designs are given in Table 1 and their wordlength patterns are in Table 2. The resolutions can be summarized by

$$R_2(n,4) = \begin{cases} \left[\dfrac{8n}{15}\right], & \text{when } r \ne 2,3,4,6,10, \\ \left[\dfrac{8n}{15}\right] - 1, & \text{otherwise}. \end{cases}$$

By using computer search, Franklin (1984) obtained minimum aberration 2^{n-k} designs for $k = 3$, $7 \le n \le 14$ and $k = 4$, $8 \le n \le 15$. Although he did not show the minimum aberration property for larger n, he did suggest (page 229) that for larger n "the matrix (which he used to define the design) should be repeated as often as necessary". As we show later, this indeed gives the minimum aberration designs with large n from the smaller ones. Tables 1 and

TABLE 2
Wordlength patterns of minimum aberration 2^{n-4} designs

n	**Wordlength pattern**
5	(0, 10, 0, 5, 0, ...)
6	(0, 3, 8, 3, 0, 1, 0, ...)
7	(0, 0, 7, 7, 0, 0, 1, 0, ...)
8	(0, 0, 0, 14, 0, 0, 0, 1, 0, ...)
9	(0, 0, 0, 6, 8, 0, 0, 1, 0, ...)
10	(0, 0, 0, 2, 8, 4, 0, 1, 0, ...)
11	(0, 0, 0, 0, 6, 6, 2, 1, 0, ...)
12	(0, 0, 0, 0, 0, 12, 0, 3, 0, ...)
13	(0, 0, 0, 0, 0, 4, 8, 3, 0, ...)
14	(0, 0, 0, 0, 0, 0, 8, 7, 0, ...)
15	(0, 0, 0, 0, 0, 0, 0, 15, 0, ...)
16	(0, 0, 0, 0, 0, 0, 0, 7, 8, 0, ...)
17	(0, 0, 0, 0, 0, 0, 0, 3, 8, 4, 0, ...)
18	(0, 0, 0, 0, 0, 0, 0, 1, 6, 6, 2, 0, ...)
19	(0, 0, 0, 0, 0, 0, 0, 0, 4, 6, 4, 1, 0, ...)

2 give the defining contrasts and the wordlength patterns of the minimum aberration 2^{n-4} designs for a complete cycle of $n (5 \le n \le 19)$, which have some overlaps with Franklin's tables.

To prove the minimum aberration property, we need the following results [Brownlee, Kelly and Loraine (1948)], for any 2^{n-k} design.

R0 $$\sum A_i = 2^k - 1.$$

R1 $$\sum iA_i = n2^{k-1}.$$

R2 Either all the words have even length or 2^{k-1} words have odd length.

First we consider the case of $k = 3$. For $n = 4$, the design defined by (3) has the wordlength pattern

$$(0, 6, 0, 1, 0, \dots).$$

Since we do not allow any words of length 1, we have

$$\text{R0:} \quad A_2 + A_3 + A_4 = 7,$$

$$\text{R1:} \quad 2A_2 + 3A_3 + 4A_4 = 16,$$

which together with R2 make $A_2 = 6$, $A_4 = 1$. Therefore the proposed design is the unique solution.

For $5 \le n \le 10$, the proofs are similar. See Chen and Wu (1989). For $n \ge 11$, the proofs are essentially the same as those for $4 \le n \le 10$ because, from Lemma 1, they involve the same type of equations. For example, when $n = 4 + 7m$, the equations become

$$\text{R0:} \quad A_{2+4m} + A_{3+4m} + A_{4+4m} + \cdots = 7, \tag{5}$$

$$\text{R1:} \quad 2A_{2+4m} + 3A_{3+4m} + 4A_{4+4m} + \cdots = 16. \tag{6}$$

By subtracting 2 × (5) from (6), we get

$$A_{3+4m} + 2A_{4+4m} + 3A_{5+4m} + \cdots = 2,$$

which forces $A_{5+4m} = A_{6+4m} = \cdots = 0$. Therefore the proof for $n = 4 + 7m$ reduces to that for $n = 4$.

Next we consider the case of $k = 4$. For $n = 5$, according to R0–R2, there are two other patterns with less aberration

$$W = (0, 6, 8, 1, 0, \dots),$$

$$W = (0, 7, 7, 0, 1, \dots).$$

Consider the first one. Let l be a letter shared by a shortest word and the longest word. By deleting all the words that contain l, the wordlength pattern of the resulting $2^{n'-3}$ design ($n' \le 4$) must be (using R0 and R2)

$$W = (0, 3, 4, 0, \dots),$$

which violates R1. The proof of the second one is the same.

For $n = 6$, according to R1 and R2, only two wordlength patterns have less aberration

(a) $$W = (0, 2, 8, 5, 0, \dots),$$

(b) $$W = (0, 3, 7, 4, 1, \dots).$$

In case (a), suppose letter l occurs in a shortest word. Let us delete all the words which contain l. The remaining words define a $2^{n'-3}$ design with $n' \le 5$. According to R1, its first moment is at most $5 \times 4 = 20$. On the other hand, $A_2 \le 1$ and $A_4 \le 5$, implying $A_3 \ge 1$ and from R2, A_3 must be 4. So we have $(A_2, A_4) = (1, 2)$ or $(0, 3)$. Therefore

$$2A_2 + 3A_3 + 4A_4 = 22 \quad \text{or} \quad 24 > 20,$$

violating R1. In case (b), notice that there is a letter which occurs in at least two of the three shortest words. This is because the product of these three words has at most length 5 (note this is also true for $n = 6 + 7m$). Deleting the words containing this letter would lead to the same violation as in the case (a). Thus, we prove that no other design can have less aberration.

For $n = 7$, there are two other patterns satisfying R0–R2 with less aberration:

$$W = (0, 0, 6, 7, 2, 0, \dots),$$

$$W = (0, 0, 7, 6, 1, 1, 0, \dots).$$

To prove that there is no design with either of the wordlength patterns, we assume the contrary. Suppose D_1 is a 2^{7-4} design with $W = (0, 0, 6, 7, 2, 0, \dots)$. The lengths of its generators are not all even. Otherwise, since the product of two even length words has even length, it is impossible to have six words of length 3. By adding a new letter to all odd length generators, we have a 2^{8-4}

design D_2. Every word in the defining contrasts subgroup of D_2 has even length since all its generators have even length. Because the corresponding words in D_2 are longer than the words in D_1, from R0 and R1, the wordlength pattern of D_2 has to be

$$\text{(c)} \qquad W = (0,0,0,13,0,2,0,\dots).$$

Now we prove the impossibility of (c). Assume that the two longest words are w_1 and w_2. There is a letter l in w_1 but not in w_2. By deleting all the words containing l, the remaining words should define a $2^{n'-3}$ design with $n' \le 7$. From R0 and R2, the wordlength pattern of this design must be $(0,0,0,6,0,1,0,\dots)$ with its first moment being 30, which violates R1. Therefore such a design does not exist. Proof of the impossibility of $W = (0,0,7,6,1,1,0,\dots)$ is similar.

For the remaining cases ($8 \le n \le 19$), the minimum aberration property can be proved by using R0–R2 and the methods in (a)–(c); see Chen and Wu (1989). Using the same argument as in $k = 3$ [see (5) and (6)], we can show that the proofs for $n \ge 20$ are the same as those for $5 \le n \le 19$.

4. Periodicity of the resolution and minimum aberration criteria for large n. The results in Section 3 on minimum aberration 2^{n-k} designs for $k = 3$ and 4 hold for *any* n. Such is not true for $k \ge 5$ because the periodicity property of resolution breaks down at $k = 5$ [see Table I, Verhoeff (1987)]. In this section, we will prove a periodicity property for two criteria, resolution and minimum aberration for *large* n and any fixed k.

THEOREM 1. *For any fixed k, there exists a positive integer N_k such that for $n > N_k$,*

$$R_s\left(n + \frac{s^k - 1}{s - 1}, k\right) = R_s(n,k) + s^{k-1}.$$

PROOF. For any fixed n, we consider the sequence $\{R_s(n + m(s^k - 1)/(s-1), k)\}_{m=0}^{\infty}$. The stated result amounts to proving that this sequence is an arithmetic sequence for large m.

Assuming the contrary, there would be infinitely many m, say $\{m_i\}_{i=0}^{\infty}$, such that

$$\text{(7)} \qquad R_s\left(n + \frac{s^k - 1}{s - 1}m_i, k\right) \ne R_s\left(n + \frac{s^k - 1}{s - 1}(m_i - 1), k\right) + s^{k-1}.$$

From Lemma 1, the inequality in (7) must satisfy

$$\text{(8)} \qquad R_s\left(n + \frac{s^k - 1}{s - 1}m_i, k\right) \ge R_s\left(n + \frac{s^k - 1}{s - 1}(m_i - 1), k\right) + s^{k-1} + 1.$$

Applying (8) and Lemma 1 repeatedly, we have

$$
\begin{aligned}
R_s\left(n+\frac{s^k-1}{s-1}m_i, k\right) &\geq R_s\left(n+\frac{s^k-1}{s-1}(m_i-1), k\right)+s^{k-1}+1 \\
&\geq R_s\left(n+\frac{s^k-1}{s-1}m_{i-1}, k\right)+(m_i-m_{i-1})s^{k-1}+1 \qquad (9)\\
&\geq R_s(n,k)+m_i s^{k-1}+i.
\end{aligned}
$$

On the other hand, Plotkin (1960) proved that

$$R_s(n,k) \leq \left[\frac{s^{k-1}(s-1)n}{s^k-1}\right].$$

So

$$R_s\left(n+\frac{s^k-1}{s-1}m_i, k\right) \leq m_i s^{k-1}+\left[\frac{s^{k-1}(s-1)n}{s^k-1}\right]. \qquad (10)$$

From (9) and (10) we have

$$\left[\frac{s^{k-1}(s-1)n}{s^k-1}\right] \geq R_s(n,k)+i,$$

which is impossible as $i \to \infty$. □

A similar result holds for the minimum aberration criterion.

THEOREM 2. *For any fixed k, there exists a positive integer M_k such that for $n \geq M_k$, the minimum aberration property is periodic, that is, if a minimum aberration s^{n-k} design has the wordlength pattern W, then there exists a minimum aberration design $s^{(n+(s^k-1)/(s-1))-k}$ with the wordlength pattern* $\mathrm{lag}(W, s^{k-1})$.

PROOF. Let v_{ni} be the number of words with the shortest length in a minimum aberration $s^{(n+i(s^k-1)/(s-1))-k}$ design. From Lemma 1(i), Theorem 1 and the definition of minimum aberration, v_{ni} has the property:

$$v_{ni} \geq v_{nj} \quad \text{for } i \leq j \text{ and } n \geq N_k,\ N_k \text{ given in Theorem 1.}$$

Therefore there is a positive integer v_1, such that $v_{ni} = v_1$ for sufficiently large i. Similarly, each of the sequences of the number of words with the second shortest wordlength, third shortest wordlength and so on, has a positive integer as a limit. Now if we have only finitely many such sequences, there would be a finite $M_k > N_k$ such that for $n \geq M_k$,

$$W_{si} = \mathrm{lag}(W_{s0}, s^{k-1}i),$$

where W_{si} is the wordlength pattern of an $s^{(n+i(s^k-1)/(s-1))-k}$ minimum aberration design.

To complete the proof, we prove that the wordlengths of a minimum aberration $s^{(n+i(s^k-1)/(s-1))-k}$ design are in a finite range for any i. Note the following fact for s^{n-k} fractional factorial design with minimum aberration,

$$\sum_{i=1}^{n} iA_i = ns^{k-1}.$$

So if the design has resolution R, then the longest possible wordlength

$$L \leq ns^{k-1} - \left(\frac{s^k - 1}{s - 1} - 1\right) R. \tag{11}$$

For a minimum aberration $s^{(n+i(s^k-1)/(s-1))-k}$ design, using Lemma 1(ii), we have the following bound for its shortest length,

$$R_s\left(n + \frac{s^k - 1}{s - 1} i, k\right) \geq R_s(n, k) + s^{k-1} i. \tag{12}$$

From (11) and (12), its longest wordlength is bounded above by

$$\left(n + \frac{s^k - 1}{s - 1} i\right) s^{k-1} - \left(\frac{s^k - 1}{s - 1} - 1\right)\left(R_s(n, k) + s^{k-1} i\right) \leq s^{k-1} i + s^{k-1} n.$$

So the wordlengths of a minimum aberration $s^{(n+i(s^k-1)/(s-1))-k}$ design must be in the range

$$[s^{k-1} i, s^{k-1} i + s^{k-1} n],$$

which has finite length independent of i. This proves the theorem. □

5. Periodicity of the optimal-moments criterion. Unlike the resolution and minimum aberration criteria, the optimal-moments criterion enjoys a periodicity property for *any* s^{n-k} designs. In order to prove this result in Theorem 4, we employ the following representation of an s^{n-k} design.

For simplicity we first consider the case of $s = 2$. As in the proof of Lemma 1, a 2^{n-k} design can be represented by a $(2^k - 1) \times n$ matrix $F = \begin{bmatrix} A_1 \\ B_1 \end{bmatrix}$ [see (2)]. Since the $2^k - 1$ columns of the matrix A_2 in (2) contain every possible nonzero $k \times 1$ vector, each of the n columns of A_1 must be part of A_2. Noting that B_2 in (2) is generated from A_2 in the same way as B_1 is generated from A_1, we see that each of the n columns of F must be part of $G = \begin{bmatrix} A_2 \\ B_2 \end{bmatrix}$. It is easy to show that G is a submatrix of a $2^k \times 2^k$ Hadamard matrix H_{2^k} in which ± 1 are replaced by 0 and 1. Denote the columns of G by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{2^k-1}$. Then each column of the matrix F for a 2^{n-k} design must be equal to one of the $\mathbf{v}_j$. Let f_i be the frequency of these columns taking $\mathbf{v}_i$. The 2^{n-k} design can thus be represented by the frequency vector $\mathbf{f} = (f_1, f_2, \ldots, f_{2^k-1})$.

For example, a 2^{3-2} fractional factorial design with the defining contrasts subgroup

$$I = 123 = 12 = 3$$

can be represented by $\mathbf{f} = (1, 0, 2)$, where

$$F = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad G = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

For general s, we noted in the end of the proof of Lemma 1 that the matrix in (2) has a natural extension. In each column vector of this matrix the first nonzero element is set to be 1. Then each column of the matrix F can be found from the columns of G. The same representation in terms of the frequency vector $\mathbf{f}$ holds for any s^{n-k} design.

With this representation, we find a simple expression for the moments of a 2^{n-k} design $\mathbf{d}$. It is easy to show that for $\mathbf{d}$, the length of its ith word is equal to the ith component of the vector $\sum_{j=1}^{2^k-1} f_j \mathbf{v}_j$, where $(f_1, f_2, \dots f_{2^k-1})$ is the frequency vector representation for $\mathbf{d}$. Therefore the mth moment of a 2^{n-k} design

$$M_m = \Big\| \sum_j f_j \mathbf{v}_j \Big\|_m, \tag{13}$$

where $\|\mathbf{v}\|_m = \sum_i v_i^m$, $\mathbf{v} = (v_1, v_2, \dots, v_{2^k-1})'$. It is easy to see $M_0 = n$, $M_1 = n2^{k-1}$. For an s^{n-k} design, the expression M_m still holds by replacing all the nonzero value in $\mathbf{v}_j$ by 1 for the calculation in (13).

With the expression for M_2, we are ready to give a characterization of any minimum-variance s^{n-k} design with the frequency vector

$$\mathbf{f} = (f_1, f_2, \dots, f_{2^k-1}).$$

THEOREM 3. *For any minimum-variance s^{n-k} design, the frequencies f_i can only take two neighboring values. Suppose $n = q(s^k - 1)/(s - 1) + r$, $0 \le r < (s^k - 1)/(s - 1)$. Then $f_i = q$ or $q + 1$.*

PROOF. First we prove the case of $s = 2$. Recall that the matrix G can be augmented by a column vector of 0's and a row vector of 0's to form a Hadamard matrix H_{2^k}. Therefore

$$\|\mathbf{v}_j\|_2 = 2^{k-1},$$

$$\langle \mathbf{v}_j, \mathbf{v}_k \rangle = 2^{k-2} \quad \text{for } j \ne k,$$

$$\begin{aligned} M_2 &= \Big\langle \sum f_j \mathbf{v}_j, \sum f_j \mathbf{v}_j \Big\rangle \\ &= 2^{k-1} \sum f_j^2 + 2^{k-2} \sum_{i \ne j} f_i f_j \\ &= 2^{k-2} \Big[\sum f_j^2 + \Big(\sum f_j \Big)^2 \Big] \\ &= 2^{k-2} \Big[\sum f_j^2 + n^2 \Big], \end{aligned}$$

which is minimized by taking $f_j = q$ or $q + 1$. For general s, $\|\mathbf{v}_j\|_2 = s^{k-1}$, $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = (s-1)/s\|\mathbf{v}_i\|_2$ and the rest of the proof is the same. □

The minimum-variance designs given in the proof of Theorem 3 are not unique since there are $\binom{2^k - 1}{n}$ ways of choosing the frequencies f_i. A natural question is: When do these designs have optimal moments? Without loss of generality, we assume $q = 0$ and write $M_m = \|\Sigma_j f_j \mathbf{v}_j\|_m$, where $f_j = 0$ or 1 and $\Sigma f_j = n$. The vector $\Sigma_j f_j \mathbf{v}_j$ is the sum of n column vectors of the Hadamard matrix H_{2^k} given at the beginning of this section. If for this H_{2^k}, all its $2^k \times n$ submatrices are isomorphic (i.e., identical after permutations of rows, columns or $\{0, 1\}$ in F_2), then the $\Sigma f_j \mathbf{v}_j$ vector will be identical for any minimum-variance designs. That is, under the stated condition, any of the minimum-variance designs has the same set of moments and therefore has optimal moments.

THEOREM 4. *For any optimal-moments s^{n-k} design d_1, there exists an optimal-moments $s^{(n+(s^k-1)/(s-1))-k}$ design d_2, which has the same central moments as d_1.*

PROOF. For any optimal-moments s^{n-k} design d_1 with the frequency vector $\mathbf{f} = (f_1, f_2, \dots)$, we can construct as in the proof of Lemma 1 an $s^{(n+(s^k-1)/(s-1))-k}$ design with the frequency vector $\mathbf{f} + 1 = (f_1 + 1, f_2 + 1, \dots)$. It is easy to show that d_2 has the same central moments as d_1. If d_2 is not an optimal-moments design, there is a frequency vector $(g_1, g_2, \dots)$ which gives an optimal-moments $s^{(n+(s^k-1)/(s-1))-k}$ design d_3. From Theorem 3, since $\Sigma g_i = n + (s^k - 1)/(s-1)$, all $g_i \geq 1$. So the frequency vector $(g_1 - 1, g_2 - 1, \dots)$ determines an s^{n-k} design called d_4. Note that d_4 has the same central moments as d_3. If d_3 has better moments than d_2, d_4 has better moments than d_1, which contradicts the assumption that d_1 is an optimal-moments design. □

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