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COVERAGE PROBABILITIES OF BOOTSTRAP-CONFIDENCE INTERVALS FOR QUANTILES

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An asymptotic expansion of length 2 is established for the coverage probabilities of confidence intervals for the underlying q-quantile which are derived by bootstrapping the sample q-quantile. The corresponding level error turns out to be of order $O(n^{-1/2})$ which is unexpectedly low.

A confidence interval of even more practical use is derived by using backward critical points. The resulting confidence interval is of the same length as the one derived by ordinary bootstrap but it is distribution free and has higher coverage probability.

Introduction and main results. Let $X_1, \dots, X_n$ be a sample of n independent and identically distributed random variables (r.v.'s) with common distribution function (d.f.) F and denote by F_n the empirical d.f. pertaining to $X_1, \dots, X_n$. The bootstrap method, introduced by Efron (1979), often provides a quite reasonable approach for the problem of determining a confidence interval (c.i.) for an unknown parameter of interest of the underlying d.f., say $T(F)$.

To this end, consider a random sample $X_1^*, \dots, X_n^*$ of size n generated independently according to F_n and denote the resulting empirical d.f. by F_n^*. Under suitable regularity conditions [see, for example, Bickel and Freedman (1981) and Beran (1984a, b)]

$$\sup_{x \in \mathbb{R}} \Big| P_n\Big\{n^{1/2}\Big(T\big(F_n^*\big) - T\big(F_n\big)\Big) \le x\Big\} - P\big\{n^{1/2}\big(T(F_n) - T(F)\big) \le x\big\}\Big| \to_{n\to\infty} 0 \tag{1}$$

with probability 1 or in probability, where $G_n^*(x) := P_n\{n^{1/2}(T(F_n^*) - T(F_n)) \le x\}$ denotes the bootstrap estimate of $G_n(x) := P\{n^{1/2}(T(F_n) - T(F)) \le x\}$, $x \in \mathbb{R}$. We add the index n to the first probability to indicate its dependence on the outcome of $X_1, \dots, X_n$.

Denote by H^{-1} the generalized inverse of a d.f., that is, $H^{-1}(\alpha) := \inf\{t \in \mathbb{R}: H(t) \ge \alpha\}$, $\alpha \in (0,1)$. If G_n converges to a continuous and strictly increasing d.f. G, it follows from (1) that

$$G_n^{*-1}(\alpha) \to_{n\to\infty} G^{-1}(\alpha) \quad \text{in probability, } \alpha \in (0,1).$$

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Consequently, for $\alpha \in (0, 1/2)$,

$$I_n := \left(T(F_n) - G_n^{*-1}(1-\alpha)n^{-1/2}, T(F_n) - G_n^{*-1}(\alpha)n^{-1/2}\right]$$

defines a c.i. of asymptotic level $1 - 2\alpha$ for $T(F)$. Notice that this is an *unconditional* c.i., that is, we have

$$P\{T(F) \in I_n\} \to_{n\to\infty} 1 - 2\alpha, \tag{2}$$

and the question suggests itself at which rate this c.i. attains its level.

Such rates can usually not be derived from results on the speed of convergence of the bootstrap estimate in (1) as the present paper shows. Explicit rates of convergence in (2) were established by Hall (1986) and Singh (1986). While Singh (1986) [see also Liu and Singh (1987)] compares the coverage probabilities of c.i. based on the normal approximation with those derived by the (nonstudentized and studentized) bootstrap in case of the sample mean, Hall (1986) obtains an explicit formula for the first error term in (2) for a class of studentized statistics which are expressible as functions of vector means. In particular, it turns out that this first error term is of order $O(n^{-1})$.

On the other hand, by the poor rate of convergence of the bootstrap quantile estimate itself which is $O_P(n^{-1/4})$ [Singh (1981) and Falk and Reiss (1989)], one might conjecture that c.i.'s for the unknown q-quantile which are based on bootstrapping the sample q-quantile, are only of minor practical importance. However, the following result contradicts this conjecture.

We denote by Φ the standard normal d.f., $\phi = \Phi'$ and $\sigma_q := (q(1-q))^{1/2}$. Put $\langle x \rangle := \min\{m \in \mathbb{Z}: m \geq x\}$ and $[x] := \max\{m \in \mathbb{Z}: m \leq x\}$, $x \in \mathbb{R}$.

THEOREM. *Suppose that F is twice continuously differentiable near $F^{-1}(q)$ and that $f = F'$ is positive. Then we have for $x \in (0,1)$,*

$$\begin{aligned} &P\left\{n^{1/2}\left(F_n^{-1}(q) - F^{-1}(q)\right) < G_n^{*-1}(x)\right\} \\ &\qquad = x + C_n(F, q, x)n^{-1/2} + o(n^{-1/2}), \end{aligned}$$

where

$$\begin{aligned} C_n(F,q,x) = &\frac{\phi(\Phi^{-1}(x))}{(q(1-q))^{1/2}} \\ &\times\left(\frac{\operatorname{sign}(\frac{1}{2} - x)(\Phi^{-1}(x))^2}{2} - g(\Phi^{-1}(x)) + R_n(\Phi^{-1}(x))\right) \end{aligned}$$

with

$$g(u) = \begin{cases} -u^2(1-q)\left(1 - q\dfrac{f'(F^{-1}(q))}{f^2(F^{-1}(q))}\right), & u \leq 0, \\ u^2 q\left(1 + (1-q)\dfrac{f'(F^{-1}(q))}{f^2(F^{-2}(q))}\right), & u > 0, \end{cases}$$

and

$$R_n(u) = \langle \sigma_q n^{1/2} u + (2 - q + (1 - 2q)u^2)/3 + O(n^{-1/2}) \rangle$$
$$- \sigma_q n^{1/2} u - (2 - q + (1 - 2q)u^2)/3 + \delta_n - 1,$$

$$\delta_n = \begin{cases} 1, & nq \in \mathbb{N}, \\ nq - [nq], & nq \notin \mathbb{N}. \end{cases}$$

Consequently, the bootstrap c.i. I_n has unexpectedly low level error $O(n^{-1/2})$. Observe that $R_n(\Phi^{-1}(x))$ oscillates in the interval $[-1, 1]$. Hence, also in this case the bootstrap approach can cope with large sample competitors [see Section 2.6 of the book by Serfling (1980) and the comparison in the sequel].

The proof of our theorem shows that the preceding expansion holds uniformly for those d.f. in the class $\mathscr{F}(\varepsilon, D_1, D_2) := \{F$ d.f.: F^{-1} is three times differentiable in $I(\varepsilon) = (q - \varepsilon, q + \varepsilon)$ with $\sup_{p \in I(\varepsilon)} |(F^{-1})^{(i)}(p)| \le D_1$, $i = 1, 2, 3$, $(F^{-1})'(q) \ge D_2\}$, where D_1, D_2 are given positive constants.

In the following we show that the bootstrap c.i. can be improved by a simple backward procedure. Consider $n^{1/2}(T(F) - T(F_n))$ in place of $n^{1/2}(T(F_n) - T(F))$. If $n^{1/2}(T(F_n) - T(F))$ has a limiting distribution which is symmetric to the origin, then this is the limiting distribution of $n^{1/2}(T(F) - T(F_n))$ as well. The resulting confidence interval for $T(F)$ is then

$$I_{nb} := \left[T(F_n) + G_n^{*-1}(\alpha) n^{-1/2}, T(F_n) + G_n^{*-1}(1 - \alpha) n^{-1/2}\right)$$
$$= \left\{\vartheta : \alpha \le G_n^*\left(n^{1/2}(\vartheta - T(F_n))\right) < 1 - \alpha\right\},$$

that is, the endpoints of I_{nb} are *backward critical points* in the sense of Hall (1988). Note that this c.i. coincides with the one derived by Efron's (1982) percentile method.

The theoretical arguments in the paper by Hall (1988) amount to a strong case against the backward method if $T(F)$ is a smooth functional of the mean of F. In the case of the q-quantile, however, it turns out that the backward method outperforms the ordinary one.

Obviously, I_n and I_{nb} have the same length but I_{nb} is distribution free and has higher coverage probability in case of $T(F) = F^{-1}(q)$ as is shown below.

Since for any d.f. H the r.v. $H^{-1}(U)$ has d.f. H, where U is uniformly on $(0, 1)$ distributed, we have for $T(F) = F^{-1}(q)$ the representation

$$G_n^*\left(n^{1/2}(T(F) - T(F_n))\right) = P_n\left\{F_n^{*-1}(q) \le F^{-1}(q)\right\} = H_n\left(F_n(F^{-1}(q))\right),$$

where $\overline{F}_n$ denotes the empirical d.f. pertaining to a sample of n independent and uniformly on $(0, 1)$ distributed r.v.'s independent of $X_1, \dots, X_n$ and $H_n(x) := P\{\overline{F}_n^{-1}(q) \le x\}$, $x \in [0, 1]$. Hence, $G_n^*(n^{1/2}(F^{-1}(q) - F_n^{-1}(q)))$ equals the d.f. H_n evaluated at the random point $F_n(F^{-1}(q))$.

Now, $F_n(F^{-1}(q))$ equals $\bar{F}_n(F(F^{-1}(q)))$ in distribution and therefore, if we assume that $F(F^{-1}(q)) = q$, we have

$$G_n^*\big(n^{1/2}\big(F^{-1}(q) - F_n^{-1}(q)\big)\big) =_{\mathscr{D}} H_n\big(\bar{F}_n(q)\big),$$

where $=_{\mathscr{D}}$ denotes distributional equality.

Consequently, $G_n^*(n^{1/2}(F^{-1}(q) - F_n^{-1}(q)))$ is actually a pivot and the c.i. I_{nb} is therefore distribution free if only $F(F^{-1}(q)) = q$. In addition, I_{nb} has higher coverage probability as is immediate from the following result and our theorem.

LEMMA. *For $x \in (0,1)$,*

$$\begin{aligned}P\{H_n(\bar{F}_n(q)) < x\} &\\ = x + n^{-1/2}&\frac{\phi(\Phi^{-1}(x))}{(q(1-q))^{1/2}}\\ \times&\left((1-2q)\frac{(\Phi^{-1}(x))^2}{6} + \frac{4-2q}{3} - 2\delta_n + R_n(\Phi^{-1}(x))\right)\\ + o(n^{-1/2}),&\end{aligned}$$

where $R_n(\Phi^{-1}(x))$ coincides with the remainder term in the theorem.

COROLLARY. *Under the assumptions of the theorem,*

$$P\{F^{-1}(q) \in I_n\} = P\{F^{-1}(q) \in I_{nb}\} - 2n^{-1/2}\frac{\phi(\Phi^{-1}(\alpha))(\Phi^{-1}(\alpha))^2}{(q(1-q))^{1/2}} + o(n^{-1/2}).$$

Observe further that

$$\begin{aligned}I_{nb} &= \{\vartheta\colon \alpha \le H_n(F_n(\vartheta)) < 1-\alpha\}\\ &= \{\vartheta\colon F_n^{-1}(H_n^{-1}(\alpha)) \le \vartheta < F_n^{-1}(H_n^{-1}(1-\alpha))\} = [X_{r_1:n}, X_{r_2:n}),\end{aligned}$$

where $r_1 := r_1(n) := \langle nH_n^{-1}(\alpha)\rangle$, $r_2 := r_2(n) := \langle nH_n^{-1}(1-\alpha)\rangle$ and $X_{1:n} \le \cdots \le X_{n:n}$ are the order statistics pertaining to $X_1, \dots, X_n$.

Consequently, I_{nb} turns out to be a c.i. for $F^{-1}(q)$ based on the classical asymptotic (distribution free) approach which uses order statistics [see Section 2.6.3 in the book by Serfling (1980)]; observe that

$$n^{1/2}\left(\frac{r_1}{n} - q\right) \to_{n\to\infty} \Phi^{-1}(\alpha)(q(1-q))^{1/2},$$

$$n^{1/2}\left(\frac{r_2}{n} - q\right) \to_{n\to\infty} \Phi^{-1}(1-\alpha)(q(1-q))^{1/2}.$$

The length of I_{nb} is $X_{r_2:n} - X_{r_1:n}$, which is therefore also the length of I_n derived by ordinary bootstrap. Hence, it follows from Section 2.6.4 in Serfling

(1980) that

$$n^{1/2}\left(X_{r_2:n} - X_{r_1:n}\right) \to_{n\to\infty} 2\Phi^{-1}(1-\alpha)(F^{-1})'(q)(q(1-q))^{1/2}$$

with probability 1.

The computation of H_n^{-1} or r_i, respectively, nevertheless requires numerical methods if n is large and therefore, this look towards I_{nb} as a classical c.i. alternatively to the bootstrap approach demands for its practical calculation the assistance of a computer as well.

Extensive numerical simulations, which have been carried out by Dohmann (1989), support the theoretical advantage of using backward critical points in the case of $T(F) = F^{-1}(q)$. Figures 1 and 2 exemplify the gain of relative performance typically obtained by using the backward method. The vertical

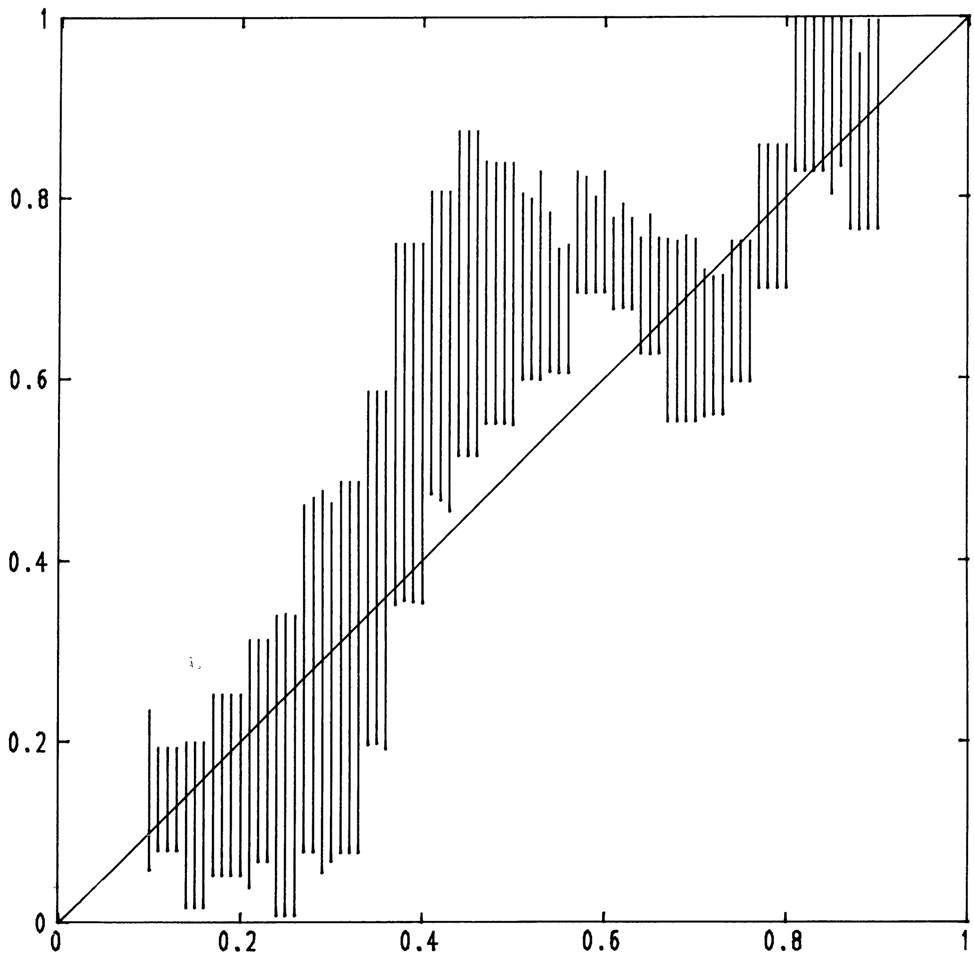

FIG. 1. *Confidence intervals derived by ordinary bootstrap.*

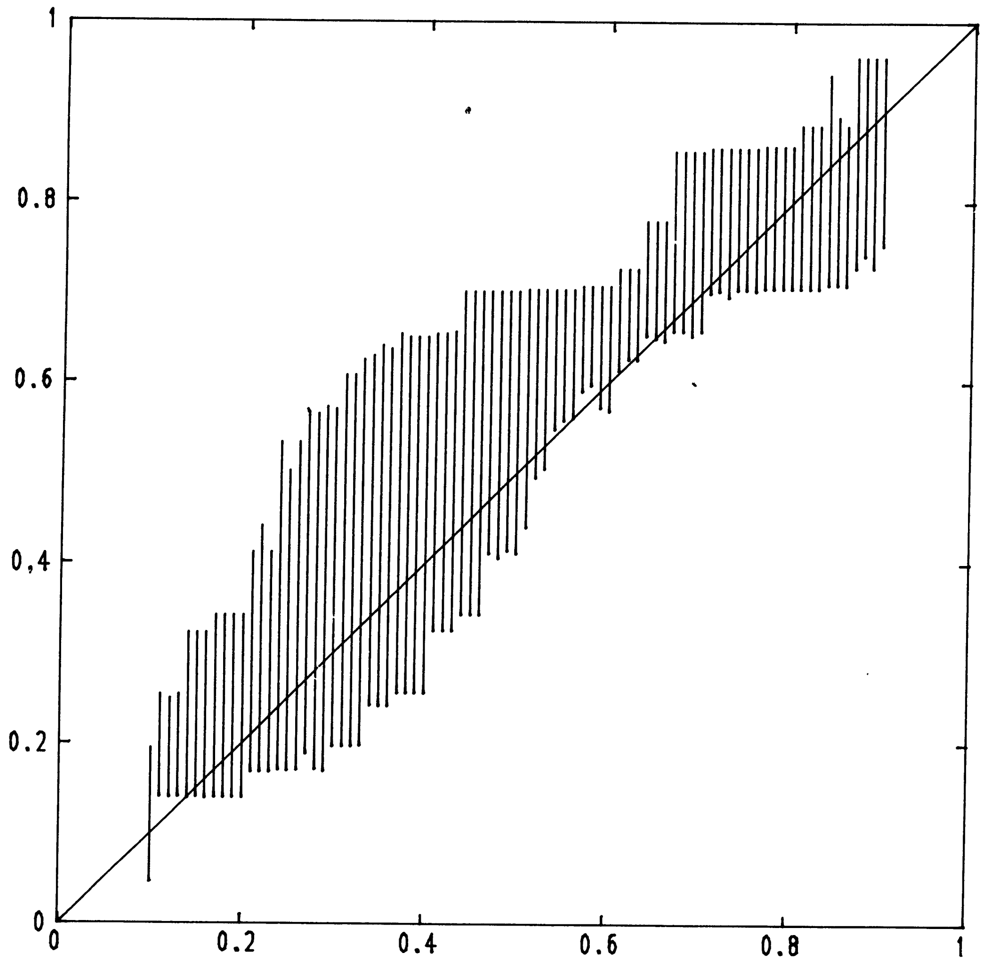

FIG. 2. *Confidence intervals derived by using backward critical points.*

lines display the c.i. of (approximate) level 0.9 for $F^{-1}(q)$ with q ranging from 0.1 to 0.9. Both figures are based on the same sample of 30 uniformly on $(0,1)$-distributed (pseudo-)random numbers, i.e., $F^{-1}(q) = q$. The c.i.'s with backward critical points (Figure 2) are typically more concentrated around the quantile function $F^{-1}(\cdot)$ than those derived by ordinary bootstrap (Figure 1), even for smaller sample sizes [Dohmann (1989)].

We remark that in the case of $T(F) = F^{-1}(q)$ a Monte Carlo simulation for the computation of bootstrap quantiles $G_n^{*-1}(\alpha)$, $\alpha \in (0,1)$, can be avoided. This has already been observed by Efron (1982). The representation $I_{nb} = [X_{r_1:n}, X_{r_2:n})$ implies

$$G_n^{*-1}(\alpha) = n^{1/2}\big(X_{r_1:n} - T(F_n)\big) = n^{1/2}\big(X_{r_1:n} - X_{\langle nq\rangle:n}\big).$$

This exact representation of $G_n^{*-1}(\alpha)$ as a spacing of two particular order statistics from the original sample $X_1, \dots, X_n$ allows the fast and accurate computation of I_n and I_{nb}; the simulation study by Dohmann (1989) benefitted very much from this fact.

We finally remark that under a suitable von Mises type condition on the upper tail of F as defined in Falk (1988) the preceding results carry over to the case of large quantiles, where $q = q_n$ converges to 1 but $n(1 - q_n)$ tends to ∞ as the sample size n increases. Details will be published elsewhere.

PROOF OF THE THEOREM. We prove the assertion only for $x \in (0, 1/2)$, the case $x \geq 1/2$ can be dealt with in a completely analogous way. Define $r := r_n(q) := \langle nq \rangle$. Denote as before by $\bar{F}_n$ the empirical d.f. of n independent and uniformly on (0, 1) distributed r.v.'s $U_1, \dots, U_n$ independent of $X_1, \dots, X_n$ and $H_n(t) = P\{\bar{F}_n^{-1}(q) \leq t\}$. Then, as in the proof of Theorem 1.3 in Falk and Reiss (1989),

$$P\left\{G_n^*\left(n^{1/2}(F_n^{-1}(q) - F^{-1}(q))\right) < x\right\}$$

$$= \int P\left\{ \frac{n^{1/2}}{\sigma_q}\left(\bar{F}_n \circ F(2F^{-1}(u_n) - F^{-1}(q)) - q\right) < \Lambda_n^{-1}(x) \middle| \bar{F}_n^{-1}(q) = u_n \right\} \Lambda_n(du),$$

where $\sigma_q = (q(1-q))^{1/2}$, $\Lambda_n(t) := H_n(t(\sigma_q/n^{1/2}) + q)$, $t \in \mathbb{R}$, and $u_n = q + u\sigma_q n^{-1/2}$.

An Edgeworth expansion of length 3 for the distribution of the sample q-quantile as given in Theorem 4.2.1 in Reiss (1989) yields

$$\Lambda_n(t) = \Phi(t) + \phi(t)\frac{-b_n - 2a_1 - a_1 t^2}{n^{1/2}} + O(n^{-1}).$$

By a Cornish–Fisher expansion this expansion can be inverted pointwise yielding

$$\Lambda_n^{-1}(x) = \Phi^{-1}(x) + \frac{b_n + 2a_1 + a_1(\Phi^{-1}(x))^2}{n^{1/2}} + O(n^{-1}).$$

Note that $\Lambda_n(-\log n) = O(n^{-1})$.

Lemma 1.4 in Falk and Reiss (1989) implies

$$P * \left(\bar{F}_n\left(F(2F^{-1}(u_n) - F^{-1}(q))\right) \middle| \bar{F}_n^{-1}(q) = u_n\right)$$

$$= \begin{cases} P * \left(\dfrac{r-1}{n}\bar{F}_{r-1}(t_n(u))\right), & -\log n \leq u < 0, \\ P * \left(\dfrac{r}{n} + \dfrac{n-r}{n}\bar{F}_{n-r}(t_n(u))\right), & 0 < u \leq \log n, \end{cases}$$

where

$$t_n(u) := \begin{cases} \dfrac{F\big(2F^{-1}(u_n) - F^{-1}(q)\big)}{u_n}, & -\log n \le u < 0, \\ \dfrac{F\big(2F^{-1}(u_n) - F^{-1}(q)\big) - u_n}{1 - u_n}, & 0 < u \le \log n. \end{cases}$$

Hence, with

$$A_n(u) := \begin{cases} P\left\{ \dfrac{n^{1/2}}{\sigma_q} \left(\dfrac{r-1}{n} \bar{F}_{r-1}(t_n(u)) - q \right) < \Lambda_n^{-1}(x) \right\}, \\ \qquad\qquad -\log n \le u < 0, \\ P\left\{ \dfrac{n^{1/2}}{\sigma_q} \left(\dfrac{n-r}{n} \bar{F}_{n-r}(t_n(u)) - q + \dfrac{r}{n} \right) < \Lambda_n^{-1}(x) \right\}, \\ \qquad\qquad 0 < u \le \log n, \end{cases}$$

we can write

$$P\big\{n^{1/2}\big(F_n^{-1}(q) - F^{-1}(q)\big) < G_n^{*-1}(x)\big\} = \int_{-\log n}^{\log n} A_n(u)\Lambda_n(du) + o(n^{-1/2}).$$

Fix now $x \in (0, 1/2)$. It follows from $\Lambda_n^{-1}(x) \to \Phi^{-1}(x) < 0$ and $r/n - q = O(n^{-1})$ for $n \to \infty$ that $A_n(u) = 0$ if $u > 0$ and n large. Hence, it suffices to show

$$\int_{-\log n}^{0} A_n(u)\Lambda_n(du) = x + C_n(F, q, x)n^{-1/2} + o(n^{-1/2}).$$

An iterated application of Taylor's formula yields uniformly for $u \in [-\log n, 0]$:

$$t_n(u) = 1 + \frac{u(1-q)^{1/2}}{(nq)^{1/2}} + \frac{g_n(u)}{nq},$$

where $g_n(u) = g(u) + O(c_n u^2)$ with $c_n \to 0$, $n \to \infty$. Furthermore, g_n is differentiable in $|u| \le \log n$ with $g_n'(u) = O(u)$ uniformly in n and $|u| \le \log n$.

Define $s := s_n := \langle \sigma_q n^{1/2} \Lambda_n^{-1}(x) + nq \rangle$. Then

$$\frac{s}{r} = 1 + \frac{\Lambda_n^{-1}(x)(1-q)^{1/2}}{(nq)^{1/2}} + \frac{R_n'(x)}{nq},$$

where $R_n'(x) = O(1)$. Put $\alpha_n := (s(r-s))^{1/2}/r^{3/2} = d_n(1-q)^{1/2}/(n^{3/4}q^{1/2})$, where $d_n = (-\Phi^{-1}(x)/\sigma_q)^{1/2} + O(n^{-1/4})$.

Moreover, write $\alpha_n^{-1}(t_n(u) - s/r) = v_n(u) + \Theta_n(u)$, where

$$v_n(u) := \frac{n^{1/4}}{d_n}(u - \Lambda_n^{-1}(x)) \quad \text{and} \quad \Theta_n(u) := \frac{n^{1/4}}{d_n}\left(\frac{g_n(u) - R_n'(x)}{n^{1/2}\sigma_q}\right).$$

Theorem 4.2.1 of Reiss (1989) together with Taylor's formula then yields uniformly for $|u| \le \log n$,

$$\begin{aligned} A_n(u) &= P\{\alpha_n^{-1}(U_{s:r-1} - s/r) > \alpha_n^{-1}(t_n(u) - s/r)\} \\ &= 1 - \Phi(v_n(u)) - \sum_{i=1}^{2} \Phi^{(i)}(v_n(u))\frac{\Theta_n^i(u)}{i!} \\ &\quad - \sum_{i=1}^{2}\sum_{j=0}^{2-i} I(L_{i,s,r-1})^{(j)}(v_n(u))\frac{\Theta_n^j(u)}{j!} + o(n^{-1/2}) \\ &=: 1 - \Phi(v_n(u)) + \gamma_n(u) + o(n^{-1/2}), \end{aligned}$$

where $L_{i,s,r-1}$, $i = 1, 2$, are polynomials with coefficients of order $n^{-i/4}$ and degree $\le 3i$, $I(L_{i,s,r-1})(t) := \int_{-\infty}^{t} L_{i,s,r-1}\, dN_{(0,1)}$ and

$$I(L_{1,s,r-1})(t) = \frac{r - 2s}{(s(r-s)r)^{1/2}}\phi(t)(1 - t^2)/3.$$

Note that $\Theta_n(u) = O((1 + u^2)n^{-1/4})$ and $\gamma_n(u) = O(n^{-3/4})$, if $u \in [-\log n, 2\Lambda_n^{-1}(x)]$. Using the representation $x = \int_{-\infty}^{\Lambda_n^{-1}} d\Lambda_n$, we have

$$\begin{aligned} &\int_{-\log n}^{0} A_n(u)\Lambda_n(du) - x \\ &\quad = \int_{\Lambda_n^{-1}(x)}^{0} (1 - \Phi(v_n(u)))\Lambda_n(du) + \int_{2\Lambda_n^{-1}(x)}^{\Lambda_n^{-1}(x)} - \Phi(v_n(u))\Lambda_n(du) \\ &\qquad + \int_{2\Lambda_n^{-1}(x)}^{0} \gamma_n(u)\Lambda_n(du) + o(n^{-1/2}) \\ &\quad =: I_n + II_n + III_n + o(n^{-1/2}). \end{aligned}$$

By Theorem 4.2.1 in Reiss (1989), the substitution $u \mapsto v_n^{-1}(u) = \Lambda_n^{-1}(x) + u d_n n^{-1/4}$ and Taylor's formula, we obtain that

$$\begin{aligned} I_n = \frac{d_n}{n^{1/4}} \int_0^{-n^{1/4}\Lambda_n^{-1}(x)/d_n} &\Phi(-u) \\ &\times \sum_{j=0}^{1} ((1 + Q_{1,n})\phi)^{(j)}(\Lambda_n^{-1}(x))\left(\frac{ud_n}{n^{1/4}}\right)^j \frac{1}{j!}du + o(n^{-1/2}), \end{aligned}$$

where $Q_{1,n}(x) = n^{-1/2}(a_1 x^3 + b_n x)$, $x \in \mathbb{R}$. Since the integral II_n is of the same structure as I_n, analogous computations yield

$$II_n = -\frac{d_n}{n^{1/4}} \int_0^{-n^{1/4}\Lambda_n^{-1}(x)/d_n} \Phi(-u) \times \sum_{j=0}^{1} ((1+Q_{1,n})\phi)^{(j)}(\Lambda_n^{-1}(x)) \left(\frac{-ud_n}{n^{1/4}}\right)^j \frac{1}{j!} du + o(n^{-1/2}).$$

By summing up I_n and II_n, the terms with $j = 0$ cancel each other and from the expansion $\int_0^t \Phi(-u)u\,du = 1/4 + O(t^{-3})$, $t > 0$, we obtain

$$\begin{aligned} I_n + II_n &= \frac{d_n^2}{n^{1/2}}((1+Q_{1,n})\phi)'(\Lambda_n^{-1}(x))/2 + o(n^{-1/2}) \\ &= \frac{|\Phi^{-1}(x)|}{2} \frac{\phi'(\Phi^{-1}(x))}{n^{1/2}\sigma_q} + o(n^{-1/2}). \end{aligned}$$

Next we turn to III_n. The preceding arguments yield

$$III_n = \sum_{l=0}^{1} \frac{((1+Q_{1,n})\phi)^{(e)}(\Lambda_n^{-1}(x))d_n^{l+1}}{l!n^{(l+1)/4}} \times \int_{n^{1/4}\Lambda_n^{-1}(x)/d_n}^{-n^{1/4}\Lambda_n^{-1}(x)/d_n} - u^l \left(\Phi^{(1)}(u) \frac{g_n(v_n^{-1}(u)) - R_n'(x)}{n^{1/4}\sigma_q d_n} + I(L_{1,s,r-1})(u) \right) du + o(n^{-1/2}).$$

For $l = 1$ the terms in the above sum are of order $o(n^{-1/2})$. With $z_n = -n^{1/4}\Lambda_n^{-1}(x)/d_n$, we have

$$\int_{-z_n}^{z_n} I(L_{1,s,r-1})(u)\,du = O(n^{-1/4}) \int_{-z_n}^{z_n} \phi(u) \frac{1-u^2}{3} du = o(n^{-1/2})$$

and

$$\int_{-z_n}^{z_n} \phi(u) \frac{g_n(v_n^{-1}(u)) - R_n'(x)}{n^{1/4}\sigma_q d_n} du = \frac{g_n(\Phi^{-1}(x)) - R_n'(x)}{n^{1/4}\sigma_q d_n} + o(n^{-1/4}).$$

Consequently,

$$III_n = -\frac{\phi(\Phi^{-1}(x))}{n^{1/2}\sigma_q} \left(g(\Phi^{-1}(x)) - R_n'(x)\right) + o(n^{-1/2}),$$

where

$$R_n'(x) = R_n(\Phi^{-1}(x)) + O(n^{-1/2}).$$

The assertion of Theorem 4 now follows by summing up the preceding representations of $I_n + II_n$ and III_n. This completes the proof of our theorem. □

PROOF OF THE LEMMA. Write for $x \in (0, 1)$,

$$P\{H_n(\bar{F}_n(q)) < x\} = P\{\bar{F}_n(q) < H_n^{-1}(x)\} = P\{U_{s:n} > q\},$$

where $s := \langle nH_n^{-1}(x)\rangle = \langle \sigma_q n^{1/2} \mathbb{A}_n^{-1}(x) + nq\rangle$. Theorem 4.2.1 in the book by Reiss (1989) together with elementary computations yields

$$P\{U_{s:n} > q\} = x + n^{-1/2} \frac{\phi(\Phi^{-1}(x))}{\sigma_q} \left((2q-1) \frac{2 + (\Phi^{-1}(x))^2}{6} + S_n(x) \right) + o(n^{-1/2}),$$

where

$$S_n(x) = R_n(\Phi^{-1}(x)) + \left(5 - 4q + (1-2q)(\Phi^{-1}(x))^2\right)/3 - 2\delta_n + O(n^{-1/2}).$$

This implies the assertion. □

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