The Annals of Statistics
1990, Vol. 18, No. 4, 1895–1899

IMPROVED BOUNDS FOR THE AVERAGE RUN LENGTH OF CONTROL CHARTS BASED ON FINITE WEIGHTED SUMS

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The average run length (ARL) is a key variable for assessing the properties of process control procedures. For continuous sampling procedures that are based on finite weighted sums (such as the moving sum technique) closed form expressions of the ARL are not available in the literature. For normally distributed random variables, Lai gives bounds for the ARL. In this paper we derive a lower bound of the ARL that (1) does not depend on normality and (2) in many situations is much sharper than the one obtained by Lai. Our results also imply that Lai's upper bound deviates from the true value less than the number of terms in the sum. Furthermore, we show that the applicability of Lai's bounds is not restricted to normally distributed control variables.

1. Introduction. Statistical methods of process control usually are assessed on the basis of their run length (RL), that is, the number of samples taken before an out-of-control signal occurs at a certain quality level. The run length should be large if the process is under control, and it should be small otherwise. Due to the complexity of the problem, only the average run length (ARL), that is, the expectation of the run length, is discussed in the literature for most quality control procedures. But even for the ARL, it is difficult to find closed form expressions. For weighted sum schemes like the moving sum technique, the mathematical treatment is made difficult by the fact that the increments in the test statistics are not independent. For normally distributed control variables, Lai (1974) gives upper and lower bounds for the ARL.

In this note we derive bounds which do not assume normality of the control variables. In particular, we derive a new lower bound which allows us to assess the quality of Lai's upper bound. These results are based on an inequality that is given in a lemma in Section 2.

2. An upper bound for ARL. Suppose the random variables $X_1, X_2, \dots$ are i.i.d. with density $f(x)$, $E\{X\} = \theta$ and $\operatorname{Var}\{X\} = \sigma^2 < \infty$. We construct the sequence $Y_n = \sum_{r=1}^{n} c_{n-r}(X_r - \theta)$, $n = k, k+1, \dots,$ where the weights satisfy $0 < c_i < \infty$ for $i = 0, \dots, k-1$ and $c_i = 0$ for $i \geq k$. The sequence $Y_k, Y_{k+1}, \dots$ is stationary with mean zero and $\operatorname{Cov}\{Y_n, Y_{n+s}\} = \sigma^2 \sum_{i=0}^{k-1-s} c_i c_{i+s} > 0$ if $0 \leq s < k$ and 0 if $s \geq k$. As $X_1, X_2, \dots$ are i.i.d. random variables and the Y's are nondecreasing functions of the X's, the Y's are associated random variables [see Esary, Proschan and Walkup (1967)].

Received February 1989; revised November 1989.
AMS 1980 *subject classifications*. Primary 62N10; secondary 60G40, 60G50.
Key words and phrases. Quality control, nonnormal control variable, weighted sums, average run length.

The quantity of interest is the run length (or stopping time) RL $= \inf\{n \geq k: Y_n \geq h\}$. Let us define $\lambda_n(x) = \text{Prob}\{Y_{k+1} < x, \dots, Y_{k+n} < x\}$ for $n \geq 0$, with $\lambda_0 = 1$. The average run length ARL, that is, the expectation of RL, can be written as ARL $= E\{\text{RL}\} = k + \sum_{n=1}^{\infty} \lambda_n(h)$. Define $\rho_n(x) = \lambda_{n-1}(x) - \lambda_n(x) = \text{Prob}\{Y_{k+1} < x, \dots, Y_{k+n-1} < x,\ Y_{k+n} \geq x\}$ for $n \geq 1$, that is, the probability that the sequence $Y_k, Y_{k+1}, \dots$ exceeds x the first time at $k+n$. It should be noted that the sequence $\rho_1(x), \rho_2(x), \dots$ is nonincreasing for any x. This can be seen from $\rho_n(x) \leq \text{Prob}\{Y_{k+2} < x, \dots, Y_{k+n-1} < x,\ Y_{k+n} \geq x\} = \rho_{n-1}(x)$. The last identity is due to the stationarity of the Y's.

LEMMA. *Let $Y_k, Y_{k+1}, \dots$ be the previously defined sequence of random variables. Then for all $n \geq k$,*

$$\rho_n(x) \geq \lambda_{n-k}(x)\rho_k(x). \tag{1}$$

PROOF. Let Z_i, $i = 1, 2, 3$, be the indicator functions of the events $\{Y_{k+1} < x, \dots, Y_n < x\}$, $\{Y_{n+1} < x, \dots, Y_{k+n-1} < x\}$, and $\{Y_{k+n} < x\}$, respectively. We can rewrite $\rho_n(x)$ as

$$\begin{aligned}
\rho_n(x) &= \text{Prob}\{Y_{k+1} < x, \dots, Y_{k+n-1} < x, Y_{k+n} \geq x\} \\
&= \text{Prob}\{Z_1 = 1,\ Z_2 = 1,\ Z_3 = 0\} \\
&= \text{Prob}\{Z_2 = 1,\ Z_3 = 0\} - \text{Prob}\{Z_1 = 0,\ Z_3 = 0\} \\
&\quad + \text{Prob}\{Z_1 = 0,\ Z_2 = 0,\ Z_3 = 0\}.
\end{aligned}$$

From independence of Z_1 and Z_3 follows $\text{Prob}\{Z_1 = 0,\ Z_3 = 0\} = \text{Prob}\{Z_1 = 0\}\text{Prob}\{Z_3 = 0\}$. From the fact that $Y_k, Y_{k+1}, \dots$ are associated random variables, it follows that $\text{Prob}\{Z_1 = 0,\ Z_2 = 0,\ Z_3 = 0\} \geq \text{Prob}\{Z_1 = 0\}\text{Prob}\{Z_2 = 0,\ Z_3 = 0\}$ [see Esary, Proschan and Walkup (1967)]. So, we get

$$\begin{aligned}
\rho_n(x) &\geq \text{Prob}\{Z_2 = 1,\ Z_3 = 0\} - \text{Prob}\{Z_1 = 0\}\text{Prob}\{Z_3 = 0\} \\
&\quad + \text{Prob}\{Z_1 = 0\}\text{Prob}\{Z_2 = 0,\ Z_3 = 0\} \\
&= \text{Prob}\{Z_2 = 1,\ Z_3 = 0\} - \text{Prob}\{Z_1 = 0\}\text{Prob}\{Z_2 = 1,\ Z_3 = 0\} \\
&= \text{Prob}\{Z_1 = 1\}\text{Prob}\{Z_2 = 1,\ Z_3 = 0\} \\
&= \lambda_{n-k}(x)\rho_k(x).
\end{aligned}$$

□

For normally distributed control variables X, Lai (1974) gives an upper bound for the average run length using the inequality (1). As a consequence of our lemma, Lai's upper bound also holds for any i.i.d. control variable with finite second moment: For any probability distribution of the control variable that has a finite second moment, an *upper bound of the ARL* is given by

$$E\{\text{RL}\} \leq k + \frac{\lambda_k(h)}{\rho_k(h)} = L_u. \tag{2}$$

The proof of this result uses the above lemma and follows the steps in Lai.

3. Lower bounds for ARL. Next, we give a lower bound for the ARL: For any probability distribution of the control variable with finite second moment, a *lower bound of the ARL* is given by

$$E\{\mathrm{RL}\} \geq 1 + \frac{\lambda_k(h)}{\rho_k(h)} = L_1. \tag{3}$$

This can be proved as follows: For $k \leq n < 2k$, we can make use of the inequality $\rho_n(h) \leq \rho_k(h)$ which is fulfilled for all $n \geq k$; applying the definition of ρ_n leads to

$$\lambda_n(h) \geq \lambda_{n-1}(h) - \rho_k(h) \quad \text{for } k \leq n < 2k. \tag{4}$$

For $n \geq 2k$, we obtain $\rho_n(h) \leq \lambda_{n-2k+1}(h)\rho_k(h)$:

$$\begin{aligned}
\rho_n(h) &= \mathrm{Prob}\{Y_{k+1} < h, \dots, Y_{k+n-1} < h, Y_{k+n} \geq h\} \\
&\leq \mathrm{Prob}\{Y_{k+1} < h, \dots, Y_{n-k+1} < h, Y_{n+1} < h, \dots, Y_{n+k-1} < h, Y_{n+k} \geq h\} \\
&= \mathrm{Prob}\{Y_{k+1} < h, \dots, Y_{n-k+1} < h\} \\
&\quad \times \mathrm{Prob}\{Y_{n+1} < h, \dots, Y_{n+k-1} < h, Y_{n+k} \geq h\} \\
&= \lambda_{n-2k+1}(h)\rho_k(h).
\end{aligned}$$

The transition between the last two lines makes use of the independence of Y_{k+1} and Y_{k+i} for $i \geq k$. This leads to

$$\lambda_n(h) \geq \lambda_{n-1}(h) - \lambda_{n-2k+1}(h)\rho_k(h) \quad \text{for } n \geq 2k. \tag{5}$$

Adding the sum over (4) for all $n = k, \dots, 2k-1$ and that over (5) for all $n \geq 2k$ gives

$$\sum_{n=k+1}^{\infty} \lambda_n(h) \geq \sum_{n=k}^{\infty} \lambda_n(h) - (k-1)\rho_k(h) - \rho_k(h) \sum_{n=1}^{\infty} \lambda_n(h)$$

or

$$k + \sum_{n=1}^{\infty} \lambda_n(h) \geq \frac{\lambda_k(h)}{\rho_k(h)} + 1.$$

This completes the proof, since $E\{\mathrm{RL}\} = k + \sum_{n=1}^{\infty}\lambda_n(h)$. As we made no use of further assumptions, this lower bound is valid for any distribution of the i.i.d. control variables that has a finite second moment.

A related lower bound L_2,

$$E\{\mathrm{RL}\} \geq k + [1 - \lambda_k(h)]^{-1} \sum_{n=1}^{k} \lambda_n(h) = L_2, \tag{6}$$

can be derived as follows. Taking the association of the Y's into account, one can show that $\lambda_n(x) \geq \lambda_{n-k}(x)\lambda_k(x)$ for all $n \geq k$ and for all x. This leads to

$$E\{\mathrm{RL}\} = k + \sum_{n=1}^{\infty} \lambda_n(h) \geq k + \sum_{n=1}^{k} \lambda_n(h)\left[1 + \lambda_k(h) + \lambda_k(h)^2 + \cdots\right]$$

$$= k + \left[1 - \lambda_k(h)\right]^{-1} \sum_{n=1}^{k} \lambda_n(h).$$

A similar lower bound $L_3 = k[1 - \lambda_k(h)]^{-1}$ is given by Lai (1974). It is obtained from L_1 by replacing $\sum_{n=1}^{k}\lambda_n(h)$ by $k\lambda_k(h)$ so that $L_3 \leq L_2$ for all $k \geq 1$. In most cases the improvement of L_2 on L_3 is rather small.

The relative merits of L_1 and L_2 or L_3 depend on the ratio $\lambda_k(h)/\lambda_{k-1}(h)$. If the process is out of control and this ratio approaches zero, L_2 and L_3 converge to k (they are "asymptotically sharp"), whereas L_1 tends to 1, which underestimates the true ARL. If the process is under control or only slightly disturbed and this ratio is near 1, L_2 and L_3 show large deviations from the true ARL. However, in all cases the difference $L_u - L_1$ is always $k - 1$. This implies that L_u cannot exceed the true ARL by more than $k - 1$ and L_1 cannot fall short by more than $k - 1$. Therefore, the value of the bound L_1 is twofold: First, it shows that the upper bound L_u is a rather sharp one as it deviates at most by $k - 1$ from the true ARL. Second, it is a considerable improvement on other lower bounds; this fact is particularly useful for in-control or nearly in-control cases.

It should be noted that the necessary effort for computing L_3 is less than that for L_2. Only $\lambda_k(h)$ must be known for L_3, whereas $\lambda_n(h)$, $n = 1, \dots, k$, are involved in computing L_2.

4. A numerical illustration.

Here, we consider the cases $k = 2$, 3, and 4 with weights $c_0 = \cdots = c_{k-1} = 1$ and $c_i = 0$, $i \geq k$: The sequences of the Y's consist of overlapping moving sums of two, three, and four terms each, respectively. Table 1 gives, for normally distributed control variables ($\sigma^2 = 1$), the upper and the three lower bounds of the average run length. Numerical integration by means of the NAG FORTRAN subroutine D01FCF, using an

TABLE 1

Bounds for the ARL of a sequence of weighted sums of normally distributed random variables for $k = 2$, 3, and 4, and weights $c_0 = \cdots = c_{k-1} = 1$ and $c_i = 0$ for $i \geq k$; $\sigma^2 = 1$

k = 2					**k = 3**					**k = 4**				
$h/\sqrt{2}$	L_u	L_1	L_2	L_3	$h/\sqrt{3}$	L_u	L_1	L_2	L_3	$h/\sqrt{4}$	L_u	L_1	L_2	L_3
3	788.6	787.6	764.5	764.0	3	872.4	870.4	822.1	821.8	3	966.8	963.8	894.2	893.3
2	53.3	52.3	48.7	48.3	2	64.0	62.0	55.5	55.2	2	75.0	72.0	63.7	62.9
1	9.8	8.8	8.2	7.9	1	12.9	10.9	10.0	9.8	1	16.1	13.1	12.4	11.8
0	4.0	3.0	3.3	3.0	0	5.7	3.7	4.2	4.1	0	7.4	4.4	5.6	5.2

adaptive subdivision algorithm, was used to compute the bounds. The accuracy in evaluating the integrals was chosen so that the values of the ARL bounds reported in the table are exact.

Table 1 makes clear that the knowledge of the lower bound L_1 increases the information on the ARL considerably. The lower bound L_1 is preferable to the bounds L_2 and L_3, even for small values of h. The superiority of L_1 over L_2 and L_3 is increased with increasing k. There are no great differences between the bounds L_2 and L_3.

Acknowledgment. The authors wish to thank Johannes Ledolter for many helpful comments.

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