The Annals of Statistics
1990, Vol. 18, No. 4, 1840–1850

PARTIAL ORDERS ON PERMUTATIONS AND DEPENDENCE ORDERINGS ON BIVARIATE EMPIRICAL DISTRIBUTIONS[1]

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Three well-known partial orderings and one new one, designated by $\geq_{b_t}$, $t = 1, 2, 3, 4$, are defined on permutations of $\{1, 2, \ldots, n\}$ through a unified approach. Various formulations of these partial orderings are also considered. With the aid of these formulations we show that the four orderings on permutations are equivalent to positive dependence orderings defined over empirical distributions based on rank data. In particular, we show that the orderings b_1, b_2, b_3 and b_4 are equivalent, respectively, to more concordant, more row regression dependent, more column regression dependent and more associated.

1. Introduction. Let $\mathbf{C}(F, G)$ denote the class of all bivariate c.d.f.'s $H(x, y)$ with x-marginal and y-marginal c.d.f.'s being, respectively, $F(x)$ and $G(y)$. There have been a variety of approaches for comparing $H_1, H_2 \in \mathbf{C}(F, G)$ to see if one c.d.f. is more positively dependent than another. Most of these approaches lead to a partial order $\to$ on $\mathbf{C}(F, G)$, where $H_1 \to H_2$ means that the random variables corresponding to H_1 are more positively related in some sense than the random variables corresponding to H_2. Examples of such partial orders are the more concordant or more positive quadrant dependent (PQD) ordering [Yanagimoto and Okamoto (1969) and Tchen (1980)], the more associated ordering [Schriever (1985)], the more row-regression dependent ordering [Schriever (1985)], the more column regression dependent ordering [Schriever (1985)] and the larger regression dependent ordering [Yanagimoto and Okamoto (1969)].

When $H_1 \in \mathbf{C}(F_1, G_1)$ and $H_2 \in \mathbf{C}(F_2, G_2)$, the comparison is more problematic. An approach taken by Schriever [(1985), Section 4.1] is the following. Let (X_1, Y_1) have c.d.f. H_1 and (X_2, Y_2) have c.d.f. H_2. We can then attempt to compare H_1 and H_2 if $F(X_1) \sim F_2(X_2)$ and $G_1(Y_1) \sim G_2(Y_2)$, where $\sim$ means distributed as. Two cases which allow this comparison are continuous marginals and when (X_1, Y_1) and (X_2, Y_2) are bivariate random variables corresponding to empirical distributions of samples of the same size with no ties in the marginals. In either case we write $U_H(u, v) = P\{F(X) \leq u,\ G(Y) \leq v\}$, where X has c.d.f. F and Y has c.d.f. G. In the first case (F and G continuous), U will have continuous uniform marginals on $(0, 1)$ and in the second case (F and

Received October 1987; revised January 1990.

[1]Research supported by the Air Force Office of Scientific Research under Contract AFOSR 84-0113. Reproduction in whole or part is permitted for any purpose of the United States Government.

AMS 1980 *subject classifications*. 62H05, 20B99.

Key words and phrases. Permutation, partial ordering, more concordant, more column or row regression dependent, more associated, L-superadditive, ranks, empirical distribution function, positive dependence, concordance, inversion.

G are empirical c.d.f.'s of the same number of data points with no ties), U will have discrete uniform marginals on $1/n, 2/n, \dots, 1$. Schriever (1985) [and also Kimeldorf and Sampson (1987)] then compare H_1 and H_2 by comparing via $\to$ the two c.d.f.'s U_{H_1} and U_{H_2}, which now have the same marginals.

We now discuss the empirical c.d.f. case in more detail. Suppose $(X_1, Y_1), \dots, (X_n, Y_n)$ is a random sample from a c.d.f. H with continuous marginals F and G. Rank order the X's as $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$, and let $j(k)$ denote the rank among $Y_1, \dots, Y_n$ of the Y_k corresponding to $X_{(k)}$. Similarly, rank order the Y's and let $i(k)$ denote the corresponding rank of the X_k. Let $\hat{H}, \hat{F}, \hat{G}$ be, respectively, the natural empirical estimators of H, F, G. We rescale the support of $U_{\hat{H}}(u, v)$ by multiplying by n and write the resulting distribution as

$$\hat{U}_H^R(u,v) = n^{-1} \sum_{k=1}^{n} \{I_{[k,n]}(u) \times I_{[j(k),n]}(v)\}, \qquad 0 \le u \le n, 0 \le v \le n, \tag{1.1}$$

where $I_A(u)$ is the indicator function of the set A. This bivariate distribution, called the empirical rank distribution, places mass n^{-1} at the points $(1, j(1)), \dots, (n, j(n))$ or equivalently at $(i(1), 1), \dots, (i(n), n)$. Thus, $\hat{U}_H^R$ can be completely described by either of the permutations $\mathbf{j} = (j(1), \dots, j(n))$ or $\mathbf{i} = (i(1), \dots, i(n))$ which are called, respectively, the column or row representations of $\hat{U}_H^R$. Note that $\mathbf{i} = \mathbf{j}^{-1}$.

To compare (according to a positive dependence ordering) two bivariate c.d.f.'s H_1 and H_2 based upon random samples from each, it is natural to compare $\hat{H}_1$ and $\hat{H}_2$. Because these two empirical distributions have nonidentical marginals, this then leads to comparing $U_{\hat{H}_1}$ and $U_{\hat{H}_2}$, or, equivalently, $\hat{U}_{H_1}^R$ and $\hat{U}_{H_2}^R$. Under the assumption of no ties in the marginals, $\hat{U}_{H_1}^R$ and $\hat{U}_{H_2}^R$ have the same marginals only when the sample sizes from H_1 and H_2 are the same. Thus, describing $\hat{U}_{H_1}^R$ and $\hat{U}_{H_2}^R$, by, respectively, the permutations $\mathbf{j}_1$ and $\mathbf{j}_2$, we are naturally led to examining orderings on the set of all permutations of $1, \dots, n$.

In Section 2 we consider three partial orderings on permutations that have been discussed in the nonparametrics literature. In addition, we introduce a fourth ordering and study the properties and relationships among these four orderings. The correspondence of these partial orderings on permutations to positive dependence orderings on bivariate empirical rank distributions is established in Section 3. Implications of these results are discussed in Section 4 with particular attention given to classes of functions whose expectations are ordered according to these partial orders.

2. Partial orderings of permutations. In this section we examine some important results for three well-known partial orderings and one new partial ordering over all permutations of $1, \dots, n$. We follow a unified approach in the spirit of Hájek (1970). Throughout this paper whenever we write the vectors $\mathbf{i}$ and $\mathbf{j}$ we mean $\mathbf{i} = (i(1), \dots, i(n))$ and $\mathbf{j} = (j(1), \dots, j(n))$ are permutations of $1, \dots, n$ for some fixed positive integer n. The collection S_n of all such permutations forms a noncommutative group with respect to the product operation, denoted by $\cdot$, and is defined by $\mathbf{i} \cdot \mathbf{j} = (i(j(1)), \dots, i(j(n)))$. The inverse of a permutation $\mathbf{i}$ will be denoted by $\mathbf{i}^{-1}$ and we write $\mathbf{i}^{-1} = (i^{-1}(1), \dots, i^{-1}(n))$. The permutation $1, \dots, n$ will be denoted by $\mathbf{e}$. In order to define several partial orderings over S_n through a unified approach, we need the following standard definition.

DEFINITION 2.1. An interchange of two components $i(k)$ and $i(l)$ of $\mathbf{i}$ is said to be: (i) a correction of an inversion of type 1 if $\Delta_{kl} < 0$; (ii) a correction of an inversion of type 2 if $\Delta_{kl} < 0$ and $|i(k) - i(l)| = 1$; and (iii) a correction of inversion of type 3 if $\Delta_{kl} < 0$ and $|k - l| = 1$, where $\Delta_{kl} = (k - l)(i(k) - i(l))$.

Definition 2.1 is used to define several different ordering relations on S_n.

DEFINITION 2.2. For $t = 1, 2, 3$, a permutation $\mathbf{i}$ is said to be better ordered in the sense of ordering b_t than $\mathbf{j}$, written as $\mathbf{i} \geq_{b_t} \mathbf{j}$, if $\mathbf{i} = \mathbf{j}$ or if $\mathbf{i}$ is obtainable from $\mathbf{j}$ in a number of steps each of which consists of correcting an inversion of type t.

REMARK 2.3. For fixed t, by definition, $\mathbf{i} \geq_{b_t} \mathbf{j}$ implies that either $\mathbf{i} = \mathbf{j}$ or that there exists a sequence $\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(p)}$ of *distinct* permutations such that $\mathbf{i} = \mathbf{i}^{(1)} \geq_{b_t} \mathbf{i}^{(2)} \geq_{b_t} \cdots \geq_{b_t} \mathbf{i}^{(p)} = \mathbf{j}$, where $\mathbf{i}^{(k)}$ is obtainable from $\mathbf{i}^{(k+1)}$ by correcting an inversion of type t. The intermediate permutations or path $\mathbf{i}^{(2)}, \dots, \mathbf{i}^{(p-1)}$, if they exist, may not be unique. However, for fixed $\mathbf{i}$ and $\mathbf{j}$, the number p above is unique for the b_2 and b_3 orderings.

The above orderings have been used in the past under different names and notations by several authors. See, for example, Savage (1957), for the b_1 ordering; Lehmann (1966) and Hájek (1970) for the b_2 ordering; and Yanagimoto and Okamoto (1969) for the b_3 ordering.

Definition 2.4 introduces a new order relation.

DEFINITION 2.4. A permutation $\mathbf{i}$ is said to be better ordered in the sense of ordering b_4 than $\mathbf{j}$, written as $\mathbf{i} \geq_{b_4} \mathbf{j}$ if $\mathbf{i} = \mathbf{j}$ or if $\mathbf{i}$ is obtainable from $\mathbf{j}$ in a number of steps, each of which consists of correcting an inversion of type 2 or of type 3.

It is direct to show that b_1, b_2, b_3 and b_4 are all partial orderings on S_n, i.e., each of these four order relations are reflexive, transitive and antisymmetric. We now establish duality results involving the four orderings. Observe that for a fixed permutation $\mathbf{i}$ and for fixed k and l, $k > l$, $i(k) > (<) i(l) \Leftrightarrow a > b$, $i^{-1}(a) > (<) i^{-1}(b)$, where a and b satisfy the conditions $i^{-1}(a) = k$ and $i^{-1}(b) = l$. From this it is clear that the correction of an inversion of type 2 (type 3) in $\mathbf{i}$ is equivalent to the correction of an inversion of type 3 (type 2) in $\mathbf{i}^{-1}$. This result immediately establishes the duality of two orderings b_2 and b_3 and self-duality of two orderings b_1 and b_4 as

$$\begin{aligned} &\mathbf{i} \geq_{b_2} \mathbf{j} \Leftrightarrow \mathbf{i}^{-1} \geq_{b_3} \mathbf{j}^{-1}, \\ &\mathbf{i} \geq_{b_1} \mathbf{j} \Leftrightarrow \mathbf{i}^{-1} \geq_{b_1} \mathbf{j}^{-1}, \\ &\mathbf{i} \geq_{b_4} \mathbf{j} \Leftrightarrow \mathbf{i}^{-1} \geq_{b_4} \mathbf{j}^{-1}. \end{aligned} \tag{2.1}$$

The implications among the orderings b_t, $t = 1, 2, 3, 4$, are given in Theorem 2.5.

THEOREM 2.5. *Let* $\mathbf{i}$ *and* $\mathbf{j}$ *be two permutations in* S_n. *Then,*

$$\begin{pmatrix} \mathbf{i} \geq_{b_2} \mathbf{j} \\ or \\ \mathbf{i} \geq_{b_3} \mathbf{j} \end{pmatrix} \Rightarrow \mathbf{i} \geq_{b_4} \mathbf{j} \Rightarrow \mathbf{i} \geq_{b_1} \mathbf{j}, \tag{2.2}$$

and no other implications are possible.

PROOF. The result follows from the definitions. The fact no other implications are possible can be readily checked for the case $n = 4$. □

We now consider various formulations of the four orderings. In this direction we have the following theorem stated without proof by Yanagimoto and Okamoto (1969), which characterizes the b_1 ordering. A simple intuitive proof is given in Metry and Sampson (1988).

THEOREM 2.6. *For any positive integers* $m \leq n$, *let* $i(1, m) < \cdots < i(m, m)$ *be the increasing rearrangement of the first* m *components of a permutation* $\mathbf{i}$ *and* $j(l, m) < \cdots < j(m, m)$ *be the increasing rearrangement of the first* m *components of a permutation* $\mathbf{j}$. *Then* $\mathbf{i} \geq_{b_1} \mathbf{j}$ *iff* $i(k, m) \leq j(k, m)$ *for any* k *and* m *such that* $1 \leq k \leq m \leq n$.

To characterize the b_2 and b_3 orderings, we introduce the following ordering. This ordering is motivated by the better ordering of Lehmann (1966) and leads to an important partial ordering on S_n. A partial ordering of Schriever (1987a) is similar, but stronger.

DEFINITION 2.7. A permutation $\mathbf{i}$ is said to be better ordered in the sense of ordering $\mathbf{a}$ than a fixed permutation $\mathbf{j}$, denoted by $\mathbf{i} \geq_a \mathbf{j}$, if there exists a permutation $\mathbf{r} = \mathbf{r}(\mathbf{i}, \mathbf{j})$ such that

$$k < l, j(k) < j(l) \Rightarrow r(k) < r(l), i(r(k)) < i(r(l)), \tag{2.3a}$$

$$r(k) < r(l), i(r(k)) > i(r(l)) \Rightarrow k < l, j(k) > j(l). \tag{2.3b}$$

When (2.3a) and (2.3b) hold for fixed $\mathbf{i}, \mathbf{j}$ and $\mathbf{r}$ we write $\mathbf{i} \to_{\mathbf{r}} \mathbf{j}$.

REMARK 2.8. (i) Some but not all of our orderings satisfy the condition (2.3a) and (2.3b) for each ordered pair $\mathbf{i}$ and $\mathbf{j}$. For example, $\mathbf{i} = (1432) \geq_{b_1} (3412) = \mathbf{j}$, but it can easily be shown that there does not exist an $\mathbf{r}$ satisfying (2.3a) and (2.3b) for this $\mathbf{i}$ and $\mathbf{j}$.

(ii) When $\mathbf{r} = \mathbf{e}$, the condition (2.3a) is equivalent to the condition (2.3b). This implication is not necessarily the case when $\mathbf{r} \neq \mathbf{e}$ as can be seen by taking $\mathbf{i} = (231)$, $\mathbf{j} = (312)$ and $\mathbf{r} = (312)$.

(iii) Note that

$$\mathbf{i} \to_{\mathbf{r}} \mathbf{j} \quad \text{and} \quad \mathbf{k} \to_{\mathbf{s}} \mathbf{i} \Rightarrow \mathbf{k} \to_{\mathbf{s}\cdot\mathbf{r}} \mathbf{j}. \tag{2.4}$$

To obtain the main result of this section, Theorem 2.11, we need Lemma 2.9 whose proof is straightforward. This lemma can also be used to characterize the b_2 and b_3 orderings and these results are given in Theorem 2.10.

LEMMA 2.9. *Let* $\mathbf{i}$ *and* $\mathbf{j}$ *be two permutations in* S_n. *Then, the following four statements are equivalent.*

$$\text{(a)} \qquad \mathbf{i} \to_{\mathbf{r}} \mathbf{j},$$

$$\text{(b)} \qquad \mathbf{i} \to_{\mathbf{r}} \mathbf{i}\cdot\mathbf{r} \to_{\mathbf{e}} \mathbf{j},$$

$$\text{(c)} \qquad \mathbf{i} \to_{\mathbf{e}} \mathbf{j}\cdot\mathbf{r}^{-1} \to_{\mathbf{r}} \mathbf{j},$$

$$\text{(d)} \qquad \mathbf{i}^{-1} \to_{\mathbf{i}\cdot\mathbf{r}\cdot\mathbf{j}^{-1}} \mathbf{j}^{-1}.$$

THEOREM 2.10. *Let* $\mathbf{i}$ *and* $\mathbf{j}$ *be two permutations. Then*

$$\text{(i)} \qquad \mathbf{i} \geq_{b_2} \mathbf{j} \Leftrightarrow \mathbf{i} \to_{\mathbf{e}} \mathbf{j},$$

$$\text{(ii)} \qquad \mathbf{i} \geq_{b_3} \mathbf{j} \Leftrightarrow \mathbf{i} \to_{\mathbf{i}^{-1}\cdot\mathbf{j}} \mathbf{j}.$$

PROOF. Part (i) is proved by Hájek (1970). To prove (ii), let $\mathbf{i}$ and $\mathbf{j}$ be two permutations. Then by (2.1), part (i) and Lemma 2.9(a), (d),

$$\mathbf{i} \geq_{b_3} \mathbf{j} \Leftrightarrow \mathbf{i}^{-1} \geq_{b_2} \mathbf{j}^{-1} \Leftrightarrow \mathbf{i}^{-1} \to_{\mathbf{e}} \mathbf{j}^{-1} \Leftrightarrow \mathbf{i} \to_{\mathbf{i}^{-1}\mathbf{j}} \mathbf{j}. \qquad \square$$

THEOREM 2.11. *The partial ordering* $\geq_{b_4}$ *is equivalent to the partial ordering* $\geq_{\mathbf{a}}$.

PROOF. Suppose that $\mathbf{i} \geq_{b_4} \mathbf{j}$ with $\mathbf{i} \neq \mathbf{j}$. Then by Definition 2.4 and Theorem 2.10, there exist two sequences $\mathbf{i}^{(1)}, \dots, \mathbf{i}^{(p)}$ and $\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(p-1)}$ of permutations in S_n such that $\mathbf{i} = \mathbf{i}^{(1)} \to_{\mathbf{r}^{(1)}} \mathbf{i}^{(2)} \to_{\mathbf{r}^{(2)}} \cdots \to_{\mathbf{r}^{(p-1)}} \mathbf{i}^{(p)} = \mathbf{j}$, where each $\mathbf{r}^{(k)}$ is either $\mathbf{e}$ or $(\mathbf{i}^{(k)})^{-1}\cdot\mathbf{i}^{(k+1)}$. By transitivity $\mathbf{i} \to_{\mathbf{r}} \mathbf{j}$, where $\mathbf{r} = \mathbf{r}^{(1)}\cdot\mathbf{r}^{(2)} \cdots \mathbf{r}^{(p-1)}$, and, hence, $\mathbf{i} \geq_a \mathbf{j}$. To prove the converse, note by Lemma 2.9 and Theorem 2.10 that $\mathbf{i} \geq_a \mathbf{j}$ implies $\mathbf{i} \geq_{b_3} \mathbf{i}\cdot\mathbf{r} \geq_{b_2} \mathbf{j}$ and hence $\mathbf{i} \geq_{b_4} \mathbf{j}$. $\square$

Corollary 2.12 is an important result obtained by extracting the "bridge" result from the previous proof.

COROLLARY 2.12. *For permutations* $\mathbf{i}$ *and* $\mathbf{j}$, $\mathbf{i} \geq_{b_4} \mathbf{j}$ *if and only if there exists a permutation* $\mathbf{k}$ *such that* $\mathbf{i} \geq_{b_2} \mathbf{k} \geq_{b_3} \mathbf{j}$ *or* $\mathbf{i} \geq_{b_3} \mathbf{k} \geq_{b_3} \mathbf{j}$.

3. Relationships to positive dependence orderings. In this section we consider some applications of the orderings discussed in Section 2. In particular, we establish the connection between these partial orderings on permutations and the partial orderings defined over bivariate empirical distributions based on ranks.

Four special orderings of positive dependence are of most interest in this paper, namely, more concordant, more associated, more row regression dependent and more column regression dependent. In giving the definitions of these orderings, we follow Schriever (1985).

Let the bivariate random variables $(X^{(1)}, Y^{(1)})$ and $(X^{(2)}, Y^{(2)})$ have joint c.d.f.'s $H_1(x, y)$ and $H_2(x, y)$, respectively. Denote the corresponding pairs of marginal c.d.f.'s by $F_1(x), G_1(y)$ and $F_2(x), G_2(y)$. We employ the following standard definitions (and related notation) which can be found in Schriever [(1985) and (1987b), page 1209]: *more concordant* ($\geq_c$), *more quadrant dependent* ($\geq_q$), *more associated* ($\geq_A$), *more row regression dependent* ($\geq_{rr}$) and *more column regression dependent* ($\geq_{cr}$). [Schriever (1987b) calls more row regression dependent just *more regression dependent*; more column regression dependent is defined by interchanging the roles of x and y in Schriever's definition of more regression dependent.] We note that formally the usual definition of more row regression dependent requires $F_1(x) = F_2(x)$. However, we can easily extend this to compare any H_1 and H_2 if $F_1(X_1) \sim F_2(X_2)$.

The notion of more concordant is discussed by Yanagimoto and Okamoto (1969) and Tchen (1980) among others. The notion of more quadrant dependent is discussed by Schriever (1985) and Kimeldorf and Sampson (1987). The notions of more associated and more column regression dependent are due to Schriever (1985) and more regression dependent is discussed by Schriever (1985) and Yanagimoto and Okamoto (1969). It is known, e.g., Schriever (1985), (1987b), that the relationships among these dependence orderings assuming $H_1, H_2 \in C(F, G))$ are

$$\begin{pmatrix} \geq_{rr} \\ \text{or} \\ \geq_{cr} \end{pmatrix} \Rightarrow \geq_A \Rightarrow \geq_c, \tag{3.1}$$

and that the implications are strict.

The result of (3.1) is strikingly similar to (2.2). To establish the connection, consider samples of size n drawn from each of H_1 and H_2 with no ties in the marginals. Let $\hat{H}_1$ and $\hat{H}_2$ again denote the bivariate empirical distributions. As in (1.1), $\mathbf{j}^{(1)}$ and $\mathbf{i}^{(1)}$ are the column and row representations of $\hat{U}^R_{H_1}$, and similarly $\mathbf{j}^{(2)}$ and $\mathbf{i}^{(2)}$ for $\hat{U}^R_{H_2}$. Again, $\hat{U}^R_{H_1}$ and $\hat{U}^R_{H_2}$ will be viewed as the joint distribution functions of discrete random variables $(R^{(1)}, S^{(1)})$ and $(R^{(2)}, S^{(2)})$, both of which take values on the $n \times n$ lattice $\{1, \dots, n\} \times \{1, \dots, n\}$. In this notation, for $m = 1, 2$,

$$P\big(R^{(m)} = k, S^{(m)} = j^{(m)}(k)\big) = n^{-1} \quad \text{for all } k = 1, 2, \dots, n, \tag{3.2}$$

and

$$\hat{U}^R_{H_m}(\alpha, \beta) = P\left(R^{(m)} \le \alpha, S^{(m)} \le \beta\right) = n^{-1}\#\left(A^{(m)}_{\alpha\beta}\right),$$

where $\#(A^{(m)}_{\alpha\beta})$ is the cardinality of the set $A^{(m)}_{\alpha\beta} = \{(k, j^{(m)}(k))\colon\ k \le \alpha, j^{(m)}(k) \le \beta\}$. The following theorems establish connections between positive dependence orderings on the empirical distribution functions and orderings on the permutations representing the bivariate empirical distribution functions. Proofs are given only for orderings between the rank distributions $\hat{U}^R_{H_1}$ and $\hat{U}^R_{H_2}$, since these are strictly increasing functions of the empirical distribution functions on their supports. The first result is well known and gives the connection between the ordering $\ge_q$ on the empirical distributions and the ordering $\ge_{b_1}$ on S_n. It establishes the pattern for the theorems which follow it.

THEOREM 3.1. *The following three statements are equivalent.*

(a) $$\hat{H}_2 \ge_q \hat{H}_1,$$

(b) $$\mathbf{j}^{(2)} \ge_{b_1} \mathbf{j}^{(1)},$$

(c) $$\mathbf{i}^{(2)} \ge_{b_1} \mathbf{i}^{(1)}.$$

The proof follows from results in Tchen (1980); see also Marshall and Olkin [(1979), page 382, proof of Proposition M3]. Rüschendorf (1986) has also considered a very similar class of results.

The next theorem establishes the connection between the $\ge_{rr}$, $\ge_{cr}$ and $\ge_A$ orderings and the b_2, b_3 and b_4 orderings.

THEOREM 3.2.

(a) $$\hat{H}_2 \ge_{rr} \hat{H}_1 \Leftrightarrow \mathbf{j}^{(2)} \ge_{b_2} \mathbf{j}^{(1)} \Leftrightarrow \mathbf{i}^{(2)} \ge_{b_3} \mathbf{i}^{(1)}.$$

(b) $$\hat{H}_2 \ge_{cr} \hat{H}_1 \Leftrightarrow \mathbf{j}^{(2)} \ge_{b_3} \mathbf{j}^{(1)} \Leftrightarrow \mathbf{i}^{(2)} \ge_{b_2} \mathbf{i}^{(1)}.$$

(c) $$\hat{H}_2 \ge_A \hat{H}_1 \Leftrightarrow \mathbf{j}^{(2)} \ge_{b_4} \mathbf{j}^{(1)} \Leftrightarrow \mathbf{i}^{(2)} \ge_{b_4} \mathbf{i}^{(1)}.$$

PROOF. Schriever [(1985), Proposition 4.1.5(i)] has shown that $\hat{H}_2 \ge_A \hat{H}_1$ is equivalent to a formulation in terms of the data (x_i, y_i) for $i = 1, \dots, n$. This can be shown to be equivalent to our formulation of $\ge_a$ in terms of permutations. The crucial step follows from Theorem 2.11 and also (2.1). Similar remarks apply to (a) and (b). □

To handle the more associated ordering or to check the conditions in $\mathbf{j}^{(2)} \ge_a \mathbf{j}^{(1)}$ is not so easy. The following theorems provide an apparently easier way to handle these orderings using conditional c.d.f.'s. For simplicity, we have the following definition which is a minor modification of Yanagimoto and Okamoto [(1969), equation (5.3)].

DEFINITION 3.3. The empirical distribution function $\hat{H}_2$ is said to have *more regression dependence on the first variable* than $\hat{H}_1$, denoted by $\hat{H}_2 \geq_{rd(1)} \hat{H}_1$ if

$$P(S^{(1)} \leq l | R^{(1)} = k) > P(S^{(2)} \leq l' | R^{(2)} = k) \tag{3.3}$$
$$\Rightarrow P(S^{(1)} \leq l | R^{(1)} = k') \geq P(S^{(2)} \leq l' | R^{(2)} = k'),$$

for all $k' > k$ and any l and l'.

REMARK 3.4. Interchanging the roles of S and R in (3.3), we get the dual version of the definition, i.e., the definition of $\hat{H}_2$ having *more regression dependence on the second variable* than $\hat{H}_1$, which is denoted by $\hat{H}_2 \geq_{rd(2)} \hat{H}_1$.

Note that by a simple argument we can show that

$$P(S \leq \beta | R = k) = I_{A_\beta}(j(k)) \quad \text{and} \quad P(R \leq \beta | S = k) = I_{A_\beta}(j^{-1}(k)), \tag{3.4}$$

where $S, R, j(\cdot)$, are one of $S^{(1)}, R^{(1)}, j^{(1)}(\cdot)$ or $S^{(2)}, R^{(2)}, j^{(2)}(\cdot)$ and $A_\beta = \{k: 1 \leq k \leq \beta\}$.

THEOREM 3.5. *For any bivariate empirical rank distribution functions* $\hat{H}_1$ *and* $\hat{H}_2$ *of the same sample size, we have*

(a) $$\hat{H}_2 \geq_{rr} \hat{H}_1 \Leftrightarrow \hat{H}_2 \geq_{rd(1)} \hat{H}_1,$$

(b) $$\hat{H}_2 \geq_{cr} \hat{H}_1 \Leftrightarrow \hat{H}_2 \geq_{rd(2)} \hat{H}_1.$$

PROOF. Based on Theorem 3.2 and (3.4), we start by proving the equivalence of the statement

$$I_{A_\beta}(j^{(1)}(k)) > I_{A_{\beta^*}}(j^{(2)}(k)) \Rightarrow I_{A_\beta}(j^{(1)}(k^*)) \geq I_{A_{\beta^*}}(j^{(2)}(k^*)), \tag{3.5}$$

for $k^* > k$ and any β, β^* and the statement

$$k < k^*, \qquad j^{(1)}(k) < j^{(1)}(k^*) \Rightarrow j^{(2)}(k) < j^{(2)}(k^*). \tag{3.6}$$

If (3.6) is false, i.e., there exist k_1, k_2 such that $k_1 < k_2$, $j^{(1)}(k_1) < j^{(1)}(k_2)$ but $j^{(2)}(k_1) > j^{(2)}(k_2)$. Choose $k = k_1$, $k^* = k_2$, $\beta^* = j^{(2)}(k_2)$, $\beta = j^{(1)}(k_1)$. Then (3.5) becomes the statement $1 > 0 \Rightarrow 0 \geq 1$. This contradictory statement proves (3.5) $\Rightarrow$ (3.6).

To show the other direction, note that $I_{A_\beta}(j^{(1)}(k)) > I_{A_{\beta^*}}(j^{(2)}(k))$ means $j^{(1)}(k) \leq \beta$ and $j^{(2)}(k) > \beta^*$ by definition of $I_{A_\beta}(\cdot)$. Assume that (3.5) is false. Then there is a $k < k^*$ such that $I_{A_\beta}(j^{(1)}(k)) > I_{A_{\beta^*}}(j^{(2)}(k))$ and $I_{A_\beta}(j^{(1)}(k^*)) < I_{A_{\beta^*}}(j^{(2)}(k^*))$. This implies $j^{(1)}(k) \leq \beta$, $\beta^* < j^{(2)}(k)$ and $\beta < j^{(1)}(k^*)$, $j^{(2)}(k^*) \leq \beta^*$, i.e., $j^{(1)}(k) \leq \beta < j^{(1)}(k^*)$ and $j^{(2)}(k^*) \leq \beta^* < j^{(2)}(k)$. This gives for $k < k^*$, $j^{(1)}(k) < j^{(1)}(k^*)$ and $j^{(2)}(k^*) < j^{(2)}(k)$ which is not (3.6). Part (a) now follows using Theorem 2.10(i).

By the duality (2.1) of the orderings $\geq_{b_2}$ and $\geq_{b_3}$ part (b) follows. □

THEOREM 3.6. *$\hat{H}_2 \geq_A \hat{H}_1$ if and only if there exist empirical rank distribution functions $\hat{H}^*$ or $\hat{H}_*$ such that $\hat{H}_2 \geq_{rd(1)} \hat{H}^* \geq_{rd(2)} \hat{H}_1$ or $\hat{H}_2 \geq_{rd(2)} \hat{H}_* \geq_{rd(1)} \hat{H}_1$.*

PROOF. The proof follows from Theorem 3.2, Corollary 2.12 and Theorem 3.5. □

4. Implications of results. We now consider the concept of functions which preserve the various positive dependence orderings. A bivariate function $g(x, y)$ is called *L-superadditive*, if for all $x_1 \leq x_2$ and $y_1 \leq y_2$ in its domain of definition, $g(x_1, y_1) + g(x_2, y_2) \geq g(x_1, y_2) + g(x_2, y_1)$. The concept of L-superadditive has been called superadditive by Tchen (1980). The L-superadditive terminology has been used in the mathematical literature and is used in Marshall and Olkin (1979). It is known that under slight regularity conditions, e.g., Tchen (1980) or Cambanis, Simons and Stout (1976), that one bivariate distribution is more concordant than another if and only if the ordering is preserved for the expectations of all L-superadditive functions. More specifically, let $H_1, H_2 \in C(F, G)$. Then under these regularity conditions,

$$H_2 \geq_c H_1 \Leftrightarrow E_{H_2}(g(X_2, Y_2)) \geq E_{H_1}(g(X_1, Y_1)), \tag{4.1}$$

for all L-superadditive functions g.

It seems natural to try to extend this result to the other orderings of Section 3. We now present some results for the $\geq_{rr}$ ordering (and by analogue the $\geq_{cr}$ ordering).

We first observe that if g is defined on a lattice L in R^2 and has a one-step L-superadditive property, i.e., for all $(x_1, y_1) \in \{1, 2, \ldots, n-1\} \times \{1, 2, \ldots, n-1\}$, $g(x_1, y_1) + g(x_1 + 1, y_1 + 1) \geq g(x_1, y_1 + 1) + g(x_1 + 1, y_1)$ (where we take the lattice to be $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ for simplicity of notation), then it is easily shown that g is L-superadditive in the lattice.

From (3.1), one distribution being more row regression dependent than another implies that the first is more concordant than the second. Consequently, the ordering of the distributions is preserved for all L-superadditive functions because of (4.1). Since there are distributions which are more concordant, but not more regression dependent, the class of L-superadditive functions cannot characterize the more regression dependent $\geq_{rr}$ ordering. It seems reasonable then, to try to find a larger class which characterizes this ordering. The following result, for empirical distributions, shows that this is impossible.

THEOREM 4.1. *Let g be a function of two variables on the lattice $L = \{1, \ldots, n\} \times \{1, \ldots, n\}$. Suppose that g satisfies $E_{\hat{H}_2}(g(X_2, Y_2)) \geq E_{\hat{H}_1}(g(X_1, Y_1))$, for every two bivariate empirical rank distributions $\hat{H}_1$ and $\hat{H}_2$ satisfying $\hat{H}_2 \geq_{rr} \hat{H}_1$. Then it follows that g must be L-superadditive.*

PROOF. Choose any $(x, y) \in \{1, \dots, n-1\} \times \{1, \dots, n-1\}$. Construct $\hat{H}_2$ to have mass n^{-1} at each of the points (x, y), $(x+1, y+1)$ and at $n-2$ other points so that $\hat{H}_2$ is an empirical rank distribution. Define $\hat{H}_1$ to put mass n^{-1} at the same $n-2$ points and also at the points $(x, y+1), (x+1, y)$ and note that $\hat{H}_2 \geq_{rr} \hat{H}_1$. Now by assumption, $E_{\hat{H}_2} g(X, Y) \geq E_{\hat{H}_1} g(X, Y)$, and since $n-2$ of the mass points of $\hat{H}_1$ and $\hat{H}_2$ are the same, this reduces to $g(x, y) + g(x+1, y+1) \geq g(x, y+1) + g(x+1, y)$. Consequently, g is L-superadditive by the remarks preceding the theorem. □

It should be mentioned that Yanagimoto and Okamoto (1969) and Schriever (1985) have considered measures which preserve the orderings $\geq_{rr}$ and $\geq_A$ on the sample spaces. It is clear that Theorem 4.1 of Schriever (1987b) can be modified for the $\geq_{rr}$ and $\geq_{cr}$ orderings. The class of functions in the resulting theorems for these two cases will then be the ones which preserve the b_2 and b_3 orderings.

In summary, we note that the more associated partial ordering introduced by Schriever (1987b) seems to be a valid and interesting positive dependence ordering, but is difficult to apply in concrete situations since it involves the existence of certain increasing functions. For the case of bivariate rank data, Schriever (1985) considered the discrete analogue of the above ordering applied to two data sets; the discrete analogue is slightly easier to deal with. In the present paper we have shown that this ordering can be reduced to a partial order (b_4) between single permutations and that this is quite useful for most statistical applications. Furthermore, we have demonstrated the relationships between this partial order and various other partial orders (b_2 and b_3) previously considered in the literature, and have also operationally characterized this b_4 partial order in terms of b_2 and b_3.

Acknowledgment. The authors would like to thank a referee for a very careful reading and also for helping to streamline the results.

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