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AN ASYMPTOTIC LOWER BOUND FOR THE LOCAL MINIMAX REGRET IN SEQUENTIAL POINT ESTIMATION

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Let Ω be an interval and let F_ω, $\omega \in \Omega$, denote a one-parameter exponential family of probability distributions on $\mathscr{R} = (-\infty, \infty)$, each of which has a finite mean θ, depending on some unknown parameter $\omega \in \Omega$. The main results of this paper determine an asymptotic lower bound for the local minimax regret, under a general smooth loss function and for a general class of estimators of θ. This bound is obtained by first determining the limit of the Bayes regret and then maximizing with respect to the prior distribution of ω.

1. Introduction. Let Ω be an interval and let F_ω, $\omega \in \Omega$, denote a one-parameter exponential family of probability distributions on $\mathscr{R}$; that is,

$$F_\omega\{dx\} = \exp\{\omega x - \psi(\omega)\}\lambda\{dx\}$$

for $-\infty < x < \infty$ and $\omega \in \Omega$, where λ is a nondegenerate, sigma-finite measure on $\mathscr{R}$, $\exp\{\psi(\omega)\} = \int e^{\omega x}\lambda\{dx\}$ and Ω is the set of all ω for which this integral is finite. Recall that the mean and the variance of F_ω are $\theta = \psi'(\omega)$ and $\sigma^2 = \psi''(\omega)$ for each $\omega \in \Omega^0$, the interior of Ω, where the prime denotes differentiation. Next, set $\Theta = \psi'(\Omega^0)$ and let K be a nonnegative function on $\Theta \times \overline{\Theta}$, satisfying the following conditions:

1. $K(\theta, \theta) = 0$ for all θ.
2. $K(\theta, a)$ is decreasing in $a < \theta$ and increasing in $a > \theta$, for fixed θ.
3. $K(\theta, a)$ is continuous in θ, four times continuously differentiable with respect to a and $K_{02}(\theta, a) > 0$ for all a in a neighborhood of θ, where

$$K_{ij} = \frac{\partial^{i+j}}{\partial\theta^i\,\partial a^j}K$$

denote the partial derivatives of K.
4. There are constants $C > 0$ and $q \geq 2$ for which

$$K(\theta, a) \leq C[1 + |\theta - a|^q]$$

for all $\theta \in \Theta$ and all $a \in \overline{\Theta}$. Finally, let $X_1, X_2, \ldots$ be independent and identically distributed random variables with common distribution F_ω, under a probability measure P_ω, for some unknown $\omega \in \Omega$.

Suppose that $X_1, X_2, \ldots$ may be observed sequentially, that observation must cease at some (possibly random) time n, that the population mean θ is estimated

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by $\hat{\theta}_n$ and that if observation is terminated at time n, then the loss incurred is of the form

$$L_A(n, \omega) = AK(\theta, \hat{\theta}_n) + n \tag{1.1}$$

for $\omega \in \Omega$, where $A > 0$ and $\hat{\theta}_n$ is an estimator of θ. The parameter A determines the importance of estimation error relative to the cost of a single observation.

Recall that a stopping time t is a positive, extended, integer-valued random variable for which the event $\{t = n\}$ is determined by $X_1, \dots, X_n$, for each $n \geq 1$. If t is a stopping time, then the risk function is defined to be

$$R_A(t, \omega) = E_\omega[L_A(t, \omega)],$$

where E_ω denotes expectation with respect to the probability measure P_ω. If t is a fixed sample size, say $t = n$, $K(\theta, a) = (\theta - a)^2$ and $\hat{\theta}_n = \overline{X}_n$ is the sample mean, then

$$R_A(n, \omega) = \frac{A}{n}\sigma^2 + n \geq 2\sqrt{A}\,\sigma.$$

The regret of a stopping time t is defined to be the additional risk incurred by using t instead of the best fixed sample size, that is,

$$r_A(t, \omega) = R_A(t, \omega) - 2\sqrt{A}\,\sigma \tag{1.2}$$

for $\omega \in \Omega$ and $A > 0$. Let

$$\tau = \inf\{n \geq m: n \geq \sqrt{A}\,\hat{\sigma}_n\},$$

where $m \geq 1$ is an initial sample size and $\hat{\sigma}_n^2$, $n \geq 2$, is the sequence of maximum likelihood estimators of σ^2. When F_ω is the normal distribution, Starr (1966) shows that $R_A(\tau, \omega)/2\sqrt{A}\,\sigma \to 1$ as $A \to \infty$ for all $\omega \in \Omega$ if $m \geq 3$, and Woodroofe (1977) shows that $r_A(\tau, \omega) \to \frac{1}{2}$ as $A \to \infty$ for all $\omega \in \Omega$ if $m \geq 6$.

Woodroofe (1985) introduced an optimality property called asymptotic local minimax regret. A family $t = t_A$, $A > 0$, of stopping times is said to have asymptotic local minimax regret if and only if

$$\lim_{\Omega_1 \downarrow \omega} \limsup_{A \to \infty} [M_A(t, \Omega_1) - M_A(\Omega_1)] = 0,$$

for all $\omega \in \Omega$ and compact subsets $\Omega_1 \subset \Omega^0$ where

$$M_A(t, \Omega_1) = \sup_{\omega \in \Omega_1} r_A(t, \omega)$$

and

$$M_A(\Omega_1) = \inf_s M_A(s, \Omega_1). \tag{1.3}$$

(Here the infimum is taken over stopping times s.) He derived an asymptotic lower bound for $M_A(\Omega_1)$ for multiparameter exponential families and the nonparametric case, under a weighted squared error loss.

In the sequential context, $\overline{X}_t$ may be a biased estimator of θ. For example, suppose that F_ω is the normal distribution with mean $\theta = \omega$ and unit variance and let

$$t = \inf\{n \geq 1: |S_n| \geq a\sqrt{n}\}$$

for $a > 0$, where $S_n = X_1 + \cdots + X_n$. Also, let m be a positive integer and set $T = \min\{m, t\}$. Then

$$E_\omega[\bar{X}_T] = \begin{cases} \theta(1 + 2/a^2) + o(1/a^2) & \text{if } |\theta| > \theta_1, \\ \theta + o(1/a^2) & \text{if } |\theta| < \theta_1, \end{cases}$$

as $a \to \infty$ and $m \to \infty$ in such a way that $a = \theta_1\sqrt{m}$, for some $\theta_1 > 0$. To reduce the order of magnitude of the bias, define

$$\hat{\theta} = \begin{cases} \bar{X}_t/(1 + 2/a^2) & \text{if } t \leq m, \\ \bar{X}_m & \text{if } t > m. \end{cases}$$

Then $E_\omega[\hat{\theta}] = \theta + o(1/a^2)$ for all $\theta \neq \theta_1$ [see Siegmund (1978)]. More generally, let b_n, $n \geq 1$, be a sequence of bounded, continuous functions on $\mathscr{R}$, which converges to a continuously differentiable function b, uniformly on compact subsets of $\mathscr{R}$, and such that

$$\sup_{x \in \mathscr{R}} |b_n(x)| = o(\sqrt{n})$$

as $n \to \infty$. Then consider estimators of the mean θ of the form

$$\hat{\theta}_n = \bar{X}_n - \frac{1}{n} b_n(\bar{X}_n) \tag{1.4}$$

for $n \geq 1$. These estimators have been considered by Ghosh, Sinha and Wieand (1980) for nonrandom sample sizes. These authors prove an asymptotic, second order complete class theorem for such estimators.

The formulation of the problem assumes that there are potential observations $X_1, X_2, \ldots$ which are independent and identically distributed with common distribution F_ω, for some unknown $\omega \in \Omega$, and that the mean θ is to be estimated by estimators $\hat{\theta}_n$ of the form (1.4) with the loss function given by (1.1). Thus, a Taylor series expansion suggests that the risk of a fixed sample size procedure, say $t = n$, is approximately

$$R_A(n, \omega) \approx \frac{A}{2n} K_{02}(\theta, \theta)\psi''(\omega) + n \geq 2\sqrt{A}\,\gamma(\omega)$$

for all sufficiently large n, by minimizing with respect to n, where

$$\gamma(\omega) = \sqrt{\tfrac{1}{2} K_{02}(\theta, \theta)\psi''(\omega)} \tag{1.5}$$

for $\omega \in \Omega$. The regret of a stopping time is as in (1.2) with σ replaced by γ.

The main results of the present paper determine an asymptotic lower bound, as $A \to \infty$, for the minimax regret, $\mathsf{M}_A(\Omega_1)$, defined by (1.3). In order to do so, it is convenient to consider some related Bayesian optimal stopping problems. If π

is a prior distribution for which $\int_\Omega \gamma \, d\pi < \infty$, let

$$\bar{r}_A(t,\pi) = \int_\Omega r_A(t,\omega)\, d\pi(\omega)$$

and

$$\bar{r}_A(\pi) = \inf_t \bar{r}_A(t,\pi)$$

for $A > 0$, where the infimum is taken over stopping times t. Then it follows from the minimax theorem that

$$M_A(\Omega_1) = \sup\{\bar{r}_A(\pi) \colon \pi(\Omega_1) = 1\}$$

for all $A > 0$ and all compact $\Omega_1 \subset \Omega$. Thus, an asymptotic lower bound for $M_A(\Omega_1)$, as $A \to \infty$, may be obtained by determining the limit of $\bar{r}_A(\pi)$ as $A \to \infty$ and then maximizing with respect to π.

Throughout this paper π denotes a prior distribution for which $\int_\Omega \gamma \, d\pi < \infty$ and E^π denotes expectation with respect to a probability measure P^π under which ω has distribution π and $X_1, X_2, \ldots$ are conditionally independent and identically distributed random variables with common distribution F_ω, given ω.

2. Preliminaries. In this section, asymptotic expansions are obtained for the Bayes regret of a stopping time. This result requires the following proposition which is adapted from Rehailia (1983).

PROPOSITION 2.1. *Suppose that π has a k-times continuously differentiable density ξ with compact support in Ω and let e be a nonnegative function for which $e\xi$ is a k times continuously differentiable function on Ω. Next, let*

$$M_{n,k}(e) = E^\pi\{e(\omega)(\theta - \bar{X}_n)^k \mid \bar{X}_n\}$$

for $k \geq 0$ and $n \geq 1$. Then for $k \geq 2$,

$$M_{n,k}(e) = \sum_{j=1}^{k-1} \frac{k-j}{n^j} M_{n,k-j-1}\left[(e\xi)^{(j-1)} \frac{\psi''}{\xi}\right] + \frac{1}{n^k} M_{n,0}\left[\frac{(e\xi)^{(k)}}{\xi}\right]$$

and

$$E^\pi\{e(\omega)(\theta - \hat{\theta}_n)^k \mid \bar{X}_n\} = \sum_{j=0}^{k} \binom{k}{j} \frac{1}{n^{k-j}} [b_n(\bar{X}_n)]^{k-j} M_{n,j}(e)$$

with

$$M_{n,1}(e) = \frac{1}{n} M_{n,0}\left[\frac{(e\xi)'}{\xi}\right]$$

for all $n \geq 1$, where $f^{(j)}$ denotes the jth derivative of the function f with respect to ω.

A Taylor series expansion for K about $a = \theta$ shows that the Bayes regret, $\bar{r}_A(t, \pi)$, of a stopping time t with respect to the prior distribution π may be written as

$$\begin{aligned}\bar{r}_A(t,\pi) &= E^{\pi}\left[AK(\theta,\hat{\theta}_t) + t - 2\sqrt{A}\,\gamma(\omega)\right]\\ &= E^{\pi}\left[\frac{1}{2}AK_{02}(\theta,\theta)(\hat{\theta}_t-\theta)^2 + t - 2\sqrt{A}\,\gamma(\omega)\right]\\ &\quad+\frac{1}{3!}AE^{\pi}\left[K_{03}(\theta,\theta)(\hat{\theta}_t-\theta)^3\right] + \frac{1}{4!}AE^{\pi}\left[K_{04}(\theta,\theta_t^*)(\hat{\theta}_t-\theta)^4\right],\end{aligned}$$

where θ_t^* is an intermediate point between $\hat{\theta}_t$ and θ. Next, let

$$\xi_i(\omega) = \frac{1}{\xi(\omega)}\frac{d^i}{d\omega^i}\left[\frac{1}{2}K_{02}(\theta,\theta)\xi(\omega)\right]I\{\xi(\omega) > 0\} \tag{2.1}$$

for $i = 1, 2$, $\theta = \psi'(\omega)$ and $\omega \in \Omega$, where $I\{\cdot\}$ denotes the indicator function of the set $\{\cdot\}$. Then conditioning on $\bar{X}_t$ and applying Proposition 2.1 with $e(\omega) = K_{02}(\theta, \theta)$ yields

$$E^{\pi}\left[K_{02}(\theta,\theta)(\theta-\bar{X}_t)\frac{b_t(\bar{X}_t)}{t}\right] = 2E^{\pi}\left[\frac{b_t(\bar{X}_t)}{t^2}E^{\pi}\{\xi_1(\omega)\mid\bar{X}_t\}\right]$$

and

$$E^{\pi}\left[\frac{1}{2}K_{02}(\theta,\theta)(\theta-\bar{X}_t)^2\right] = E^{\pi}\left[\frac{1}{t}E^{\pi}\{\gamma^2(\omega)\mid\bar{X}_t\}\right] + E^{\pi}\left[\frac{1}{t^2}E^{\pi}\{\xi_2(\omega)\mid\bar{X}_t\}\right]$$

w.p. 1 (P^{π}), where γ is defined by (1.5). Now let

$$\begin{aligned}U_n &= E^{\pi}\{\gamma^2(\omega)\mid\bar{X}_n\},\\ V_n &= E^{\pi}\{\gamma(\omega)\mid\bar{X}_n\},\\ M_n &= E^{\pi}\{\xi_2(\omega) + 2b_n(\bar{X}_n)\xi_1(\omega) + \tfrac{1}{2}K_{02}(\theta,\theta)b_n^2(\bar{X}_n)\mid\bar{X}_n\},\\ W_n &= E^{\pi}\{K_{03}(\theta,\theta)(\hat{\theta}_n-\theta)^3\mid\bar{X}_n\},\\ R_n &= E^{\pi}\{K_{04}(\theta,\theta_n^*)(\hat{\theta}_n-\theta)^4\mid\bar{X}_n\},\end{aligned} \tag{2.2}$$

for $n \geq 1$. Then since $E^{\pi}[V_t] = E^{\pi}[\gamma(\omega)]$, it follows easily that

$$\begin{aligned}\bar{r}_A(t,\pi) &= E^{\pi}\left[\frac{A}{t}(U_t - V_t^2) + \frac{A}{t^2}M_t + \frac{1}{t}(\sqrt{A}\,V_t - t)^2\right]\\ &\quad+\frac{1}{3!}AE^{\pi}[W_t] + \frac{1}{4!}AE^{\pi}[R_t]\end{aligned} \tag{2.3}$$

for any stopping time t. The following two lemmas are proved in Section 4.

LEMMA 2.1. *Let ε be any positive constant and let*

$$R_n^* = \int_{|\hat{\theta}_n - \theta| \le \varepsilon} n^2 K_{04}(\theta, \theta_n^*)(\hat{\theta}_n - \theta)^4 \, d\pi_n(\omega)$$

for $n \ge 1$, where π_n denotes the posterior distribution of ω, given $\overline{X}_n$. If π has a four times continuously differentiable density, then R_n^*, $n \ge 1$, are uniformly integrable with respect to P^π and $R_n^* \to 3K_{04}(\theta, \theta)[\psi''(\omega)]^2$ w.p. 1 (P^π) as $n \to \infty$.

LEMMA 2.2. *Suppose that b_n, $n \ge 1$, satisfy the conditions listed in Section 1. If π has compact support in Ω, then $E^\pi \sup_{n \ge 1} |b_n(\overline{X}_{\hat{n}})|^p < \infty$ for all $p \ge 1$.*

3. An asymptotic lower bound for the minimax regret. This section provides an asymptotic lower bound for the minimax regret defined by (1.3). In the remainder of this paper, Π will denote the class of prior distributions π having a twice continuously differentiable density ξ with compact support $\Omega_0 = [\omega_0, \omega_1]$, where $-\infty < \omega_0 < \omega_1 < \infty$ and such that $\inf_{\omega \in \Omega} \xi_2(\omega) > -\infty$. Also, $\Theta_0 = \psi'(\Omega_0) = [\theta_0, \theta_1]$ with $\theta_i = \psi'(\omega_i)$ for $i = 0, 1$.

THEOREM 3.1. *Suppose that K satisfies the conditions listed in Section 1 and that $K(\theta, a)$ is convex in a for fixed θ. Suppose also that $\pi \in \Pi$ has a density ξ of the form*

$$\xi(\omega) = (\omega - \omega_0)_+^p (\omega_1 - \omega)_+^p \xi_0(\omega) \tag{3.1}$$

for some $p \ge 2$ and all $\omega \in \Omega$, where $(f)_+ = \max\{f, 0\}$ and ξ_0 is a positive, twice continuously differentiable function on Ω. Then there are stopping times $s = s(A, \pi)$ which minimize $\bar{r}_A(t, \pi)$ with respect to stopping times t for $A > 0$ and satisfy the following:

(i) *There exists a constant $\delta = \delta(\pi) > 0$ for which $s \ge \delta\sqrt{A}$ w.p. 1 (P^π) for all $A > 0$.*

(ii) *If, in addition, ξ is four times continuously differentiable, then $s/\sqrt{A} \to \gamma$ in P^π-probability as $A \to \infty$.*

The proof of this theorem is presented in Section 5.

THEOREM 3.2. *Suppose that the hypotheses of Theorem 3.1 are satisfied and that ξ is four times continuously differentiable. Then*

$$\liminf_{A \to \infty} \bar{r}_A(\pi) \ge \int_\Omega G(\omega)\pi(d\omega), \tag{3.2}$$

where

$$G = \frac{1}{\gamma^2}\frac{[\gamma']^2}{\psi''} + \frac{1}{2}K_{02}(\theta,\theta)\frac{d^2}{d\omega^2}[\gamma^{-2}]$$
$$-K_{02}(\theta,\theta)\frac{d}{d\omega}[\gamma^{-2}b(\theta)] - \frac{1}{2}\frac{1}{\gamma^2}K_{03}(\theta,\theta)b(\theta)\psi'' + \frac{b^2}{\psi''}$$
$$+\frac{1}{3}K_{03}(\theta,\theta)\psi''\frac{d}{d\omega}[\gamma^{-2}] + \frac{1}{6}K_{03}(\theta,\theta)\frac{d}{d\omega}[\gamma^{-2}\psi'']$$
$$+\frac{1}{8}\frac{1}{\gamma^2}K_{04}(\theta,\theta)[\psi'']^2$$

for $\theta = \psi'(\omega)$ *and* $\omega \in \Omega$, *with* γ *being defined by* (1.5).

PROOF. Let s be as in Theorem 3.1. Then there exists a sequence A_n, $n \ge 1$, along which $A \to \infty$, the lim inf in (3.2) is attained and $s/\sqrt{A} \to \gamma$ w.p. 1 (P^π) as $A \to \infty$. Attention is restricted to such a sequence. It follows from (2.3) that

$$\bar{r}_A(\pi) \ge E^\pi\left[\frac{A}{s}(U_s - V_s^2)\right] + E^\pi\left[\frac{A}{s^2}M_s\right] \tag{3.3}$$
$$+\frac{1}{3!}AE^\pi[W_s] + \frac{1}{4!}AE^\pi[R_s]$$
$$= Q_{1A} + Q_{2A} + Q_{3A} + Q_{4A},$$

say, where U_n, V_n, M_n, W_n and R_n are defined by (2.2). Now

$$\liminf_{A\to\infty} Q_{1A} \ge \int_\Omega \frac{1}{\gamma^2(\omega)}\frac{[\gamma'(\omega)]^2}{\psi''(\omega)}\xi(\omega)\,d\omega. \tag{3.4}$$

See the proof of Theorem 1 of Woodroofe (1985). Next,

$$\frac{A}{s^2}M_s \to \frac{1}{\gamma^2(\omega)}\left[\xi_2(\omega) + 2b(\theta)\xi_1(\omega) + \gamma^2(\omega)\frac{b^2(\theta)}{\psi''(\omega)}\right],$$

w.p. 1 (P^π) as $A \to \infty$, by Theorem 3.1, the martingale convergence theorem and the conditions imposed on b_n, $n \ge 1$. Moreover, AM_s/s^2, $A > 0$, are uniformly integrable since A/s^2, $A > 0$, are bounded [see assertion (i) of Theorem 3.1] and M_n, $n \ge 1$, are uniformly integrable. The uniform integrability of M_n, $n \ge 1$, follows from the definition of M_n and Lemma 2.2. Thus,

$$Q_{2A} \to \int_\Omega \frac{1}{\gamma^2(\omega)}\left[\xi_2(\omega) + 2b(\theta)\xi_1(\omega) + \gamma^2(\omega)\frac{b^2(\omega)}{\psi''(\omega)}\right]\xi(\omega)\,d\omega$$
$$= \int_\Omega\left[\frac{1}{2}K_{02}(\theta,\theta)\frac{d^2}{d\omega^2}[\gamma^{-2}(\omega)] - K_{02}(\theta,\theta)\frac{d}{d\omega}[\gamma^{-2}(\omega)b(\theta)] + \frac{b^2(\theta)}{\psi''(\omega)}\right]\xi(\omega)\,d\omega \tag{3.5}$$

as $A \to \infty$, by an integration by parts. Also, Proposition 2.1 implies that

$$Q_{3A} \to \int_{\Omega} \frac{1}{\gamma^2(\omega)} \left[-\frac{1}{2} K_{03}(\theta, \theta) b(\theta) \psi''(\omega) + \frac{1}{3} K_{03}(\theta, \theta) \psi''(\omega) \frac{d}{d\omega} [\gamma^{-2}(\omega)] + \frac{1}{6} K_{03}(\theta, \theta) \frac{d}{d\omega} [\gamma^{-2}(\omega) \psi''(\omega)] \right] \xi(\omega)\, d\omega \tag{3.6}$$

as $A \to \infty$, by an integration by parts. Finally,

$$Q_{4A} \to \frac{1}{8} \int_{\Omega} \frac{1}{\gamma^2(\omega)} K_{04}(\theta, \theta) [\psi''(\omega)]^2 \xi(\omega)\, d\omega \tag{3.7}$$

as $A \to \infty$, by Lemma 2.1, Theorem 3.1 and the martingale convergence theorem. The theorem now follows by taking the lim inf in (3.3) and using (3.4)–(3.7). □

COROLLARY 3.1. *Suppose that the hypotheses of Theorem 3.2 are satisfied. If Ω_1 is an open set with compact closure in Ω, then*

$$\liminf_{A \to \infty} M_A(\Omega_1) \geq \sup_{\omega \in \Omega_1} G(\omega),$$

where G is as in Theorem 3.2.

PROOF. The proof is similar to that of Corollary 1 of Woodroofe (1985) and is therefore omitted. □

EXAMPLE. This example provides an asymptotic lower bound for the minimax regret under a weighted squared error loss. In fact, suppose that K is of the form

$$K(\theta, a) = \gamma_0^2(\omega)(a - \theta)^2$$

for $\theta = \psi'(\omega)$, $\omega \in \Omega$ and $a \in \overline{\Theta}$, where γ_0 is a positive, twice continuously differentiable function on Ω. Then, clearly K satisfies all the conditions listed in Section 1 and therefore, it follows from Theorem 3.2 that

$$G = \frac{1}{\gamma^2} \frac{[\gamma']^2}{\psi''} + \gamma_0^2 \frac{d^2}{d\omega^2} [\gamma^{-2}] - 2\gamma_0^2 \frac{d}{d\omega} [\gamma^{-2} b(\theta)] + \frac{b^2(\theta)}{\psi''}$$

where $\gamma^2 = \gamma_0^2 \psi''$.

4. Proofs of Lemmas 2.1 and 2.2. The proof of Lemma 2.1 makes use of the following result.

LEMMA 4.1. *If ε is any positive constant, then*

$$\int_{|\hat{\theta}_n - \theta| > \varepsilon} n^2 (\hat{\theta}_n - \theta)^4 \, d\pi_n(\omega) \to 0$$

w.p. 1 (P^{π}) *as $n \to \infty$, where π_n denotes the posterior distribution of ω, given $\overline{X}_n$, for $n \geq 1$.*

Proof. Since $\sup_x |b_n(x)| = o(\sqrt{n})$ as $n \to \infty$, it suffices to show that

$$\int_{|\hat{\theta}_n - \theta| > \varepsilon} n^2 (\overline{X}_n - \theta)^4 \, d\pi_n(\omega) \to 0 \tag{4.1}$$

w.p. 1 (P^π) as $n \to \infty$. So, let $H(\omega, \hat{\omega}_n) = \psi(\omega) - \psi(\hat{\omega}_n) - (\omega - \hat{\omega}_n)\psi'(\hat{\omega}_n)$, for $\omega \in \Omega$, where $\hat{\omega}_n$ satisfies the equation $\psi'(\hat{\omega}_n) = \overline{X}_n$, for all sufficiently large n. Then

$$\text{LHS (4.1)} = \frac{\int_{|\hat{\theta}_n - \theta| > \varepsilon} n^{5/2} (\overline{X}_n - \theta)^4 \exp\{-nH(\omega, \hat{\omega}_n)\} \xi(\omega) \, d\omega}{\sqrt{n} \int_\Omega \exp\{-nH(\omega, \hat{\omega}_n)\} \xi(\omega) \, d\omega} = \frac{N_n}{D_n},$$

say, where ξ denotes the density of π. Next, it can easily be shown that

$$D_n \to \sqrt{2\pi / \psi''(\omega^*)} \, \xi(\omega^*),$$

w.p. 1 (P_{ω^*}) for fixed ω^*, as $n \to \infty$. To estimate N_n, first observe that $H(\omega, \hat{\omega}_n)$ can be rewritten as $H(\omega, \hat{\omega}_n) = \frac{1}{2}\psi''(\omega_n^*)(\omega - \hat{\omega}_n)^2$, where ω_n^* is an intermediate point between $\hat{\omega}_n$ and ω, and note that $H(\omega, \hat{\omega}_n)$ is convex in ω, for fixed $\hat{\omega}_n$, since ψ'' is positive on Ω^0. Also, for any $\alpha > 0$, $\overline{X}_n \in [\theta_0 - \alpha, \theta_0 + \alpha]$, for all sufficiently large n, where $\theta_0 = \psi'(\omega_0)$. Furthermore, the convexity of H in ω and the continuity of ψ'' imply that for any $\eta > 0$, $H(\omega, \hat{\omega}_n) \geq \eta$ whenever $|\hat{\theta}_n - \theta| > \varepsilon$ and $\overline{X}_n \in [\theta_0 - \alpha, \theta_0 + \alpha]$, for all sufficiently large n. Thus, there exists a constant $c > 0$ for which

$$N_n \leq c n^{5/2} e^{-n\eta} \int_\Omega (\theta^4 + 1) \xi(\omega) \, d\omega$$

for all sufficiently large n and therefore $N_n / D_n \to 0$ w.p. 1 (P^π) as $n \to \infty$. □

Proof of Lemma 2.1. Since $\sup_x |b_n(x)| = o(\sqrt{n})$, it is enough to prove the uniform integrability of

$$\int_{|\hat{\theta}_n - \theta| \leq \varepsilon} n^2 K_{04}(\theta, \theta_n^*) (\overline{X}_n - \theta)^4 \, d\pi_n(\omega). \tag{4.2}$$

Let M denote an upper bound for K_{04} on $\{(\theta, a): \theta \in \Theta_0, |a - \theta| \leq \varepsilon\}$. Then

$$|\text{LHS (4.2)}| \leq M n^2 E^\pi \{ (\overline{X}_n - \theta)^4 \mid \overline{X}_n \}$$

w.p. 1 (P^π) for all $n \geq 1$. Next, applying Proposition 2.1 with $e \equiv 1$ and $k = 4$ yields

$$\begin{aligned} E^\pi\{(\theta - \overline{X}_n)^4 \mid \overline{X}_n\} &= \frac{3}{n^2} E^\pi \{ [\psi''(\omega)]^2 \mid \overline{X}_n \} \\ &\quad + \frac{1}{n^3} E^\pi \bigg\{ 3 \frac{(\xi(\omega)\psi''(\omega))''}{\xi(\omega)} + 2 \frac{(\xi'(\omega)\psi''(\omega))'}{\xi(\omega)} \\ &\qquad\qquad + \frac{\xi''(\omega)\psi''(\omega)}{\xi(\omega)} \bigg| \overline{X}_n \bigg\} \\ &\quad + \frac{1}{n^4} E^\pi \bigg\{ \frac{\xi^{(4)}(\omega)}{\xi(\omega)} \bigg| \overline{X}_n \bigg\} \end{aligned} \tag{4.3}$$

w.p. 1 (P^{π}) for all $n \geq 1$, after some algebra. Hence, it is easily seen from (4.3) that $n^2 E^{\pi}\{(\theta - \overline{X}_n)^4 \mid \overline{X}_n\}$, $n \geq 1$, are uniformly integrable. This completes the proof of the first assertion. To establish the second assertion, let η be an arbitrary positive constant and let $\alpha > 0$ be so small that $|K_{04}(\theta, \theta^*) - K_{04}(\theta, \theta)| \leq \eta$ whenever $|\theta^* - \theta| \leq \alpha$ for all $\theta \in \Theta_0$. Then the region of integration $|\theta - \hat{\theta}_n| \leq \varepsilon$ may be replaced by $|\theta - \hat{\theta}_n| \leq \alpha$, by Lemma 4.1. With this change

$$R_n^* \leq n^2 E^{\pi}\Big\{\big[K_{04}(\theta, \theta) + \eta\big]\big(\hat{\theta}_n - \theta\big)^4 \mid \overline{X}_n\Big\}$$
$$= n^2 E^{\pi}\Big\{K_{04}(\theta, \theta)\big(\hat{\theta}_n - \theta\big)^4 \mid \overline{X}_n\Big\} + n^2 \eta E^{\pi}\Big\{\big(\hat{\theta}_n - \theta\big)^4 \mid \overline{X}_n\Big\}.$$

Next, applying Proposition 2.1 with $e(\omega) = K_{04}(\theta, \theta)$ to the first term on the right side of the last equality and (4.3) to the second yields

$$\limsup_{n \to \infty} R_n^* \leq 3\big[K_{04}(\theta, \theta) + \eta\big]\big[\psi''(\omega)\big]^2.$$

A similar argument yields

$$\liminf_{n \to \infty} R_n^* \geq 3\big[K_{04}(\theta, \theta) - \eta\big]\big[\psi''(\omega)\big]^2.$$

The last assertion of the lemma now follows by letting η go to zero in the lim sup and lim inf above. □

Proof of Lemma 2.2. Let Ω_0 denote the support of π. Then there is a constant $\beta > 0$ for which

$$P^{\pi}\big\{|\overline{X}_n - \theta| > \varepsilon\big\} \leq e^{-\beta n \varepsilon^2} \tag{4.4}$$

for sufficiently small $\varepsilon > 0$ and sufficiently large n. Next, choose $\varepsilon > 0$ so small that $\Lambda_n = \{|\overline{X}_n - \theta| \leq \varepsilon\} \subset \Theta$. Also, let B_1 denote an upper bound for b_n on Λ_n and let B_2 be an upper bound for b_n/n on $\overline{\Theta}$, for $n \geq 1$. Then

$$\sup_{n \geq 1} |b_n(\overline{X}_n)|^p \leq B_1^p + B_2^p \sum_{n \geq 1} n^p I(\tilde{\Lambda}_n)$$

for all $p > 0$, where $\tilde{J}$ denotes the complement of the event J. Therefore, there exists a constant $C > 0$ for which

$$E^{\pi} \sup_{n \geq 1} |b_n(\overline{X}_n)|^p \leq C + C \sum_{n \geq 1} n^p P^{\pi}(\tilde{\Lambda}_n) < \infty$$

for all $p > 0$, by (4.4). □

5. Proof of Theorem 3.1. The proof of assertion (i) of the theorem is similar to that of the second assertion of Lemma 2 of Woodroofe (1985), although some of the details are slightly different [see Tahir (1987)]. However, the proof of assertion (ii) requires the following result.

Lemma 5.1. *If π has compact support in Ω, then $\bar{r}_A(\pi) = o(\sqrt{A})$ as $A \to \infty$.*

Proof. Let Ω_0 denote the support of π and let $m = m_A$ be an integer such that $m \sim A^{1/4}$ as $A \to \infty$. Also, let

$$t = t_A = \max\big\{m, \big[\sqrt{A}\,\gamma(\hat{\omega}_m)\big] + 1\big\},$$

where $\hat{\omega}_n$ denotes the maximum likelihood estimator of ω, restricted to Ω_0, and $[\cdot]$ denotes the greatest integer function. Then $t/\sqrt{A} \to \gamma(\omega)$ w.p. 1 as $A \to \infty$ and $t/\sqrt{A}$, $A > 0$, are bounded below. Next, let Θ_1 be a neighborhood of Θ_0 with compact closure in Θ. Then

$$\begin{aligned}\frac{1}{\sqrt{A}}\bar{r}_A(\pi) \le {} & E^\pi\left[\frac{1}{2}\sqrt{A}\,K_{02}(\theta,\theta_t^*)\big(\hat{\theta}_t-\theta\big)^2 I\{\hat{\theta}_t\in\Theta_1\}\right] \\ & + E^\pi\left[\frac{t}{\sqrt{A}}\right] - 2E^\pi[\gamma(\omega)] + \sqrt{A}\,E^\pi\big[K(\theta,\hat{\theta}_t)I\{\hat{\theta}_t\notin\Theta_1\}\big],\end{aligned} \tag{5.1}$$

where θ_t^* is an intermediate point between θ and $\hat{\theta}_t$. The second term on the right side of (5.1) is $o(1)$, by condition 4 on K (see Section 1), Schwarz's inequality and Lemma 1 of Woodroofe (1985). Furthermore, for fixed ω,

$$\frac{1}{2}\sqrt{A}\,K_{02}(\theta,\theta_t^*)\big(\hat{\theta}_t-\theta\big)^2 I\{\hat{\theta}_t\in\Theta_1\} \to \frac{1}{2}K_{02}(\theta,\theta)\frac{\psi''(\omega)}{\gamma(\omega)}Z^2 = \gamma(\omega)Z^2$$

in distribution with respect to P_ω, as $A \to \infty$, where Z is a random variable having the standard normal distribution. Moreover, by conditioning on $X_1,\dots,X_m$, it is easy to see that the higher moments are bounded. Hence

$$E^\pi\left[\tfrac{1}{2}\sqrt{A}\,K_{02}(\theta,\theta_t^*)\big(\hat{\theta}_t-\theta\big)^2 I\{\hat{\theta}_t\in\Theta_1\}\right] \to E^\pi[\gamma(\omega)]$$

as $A \to \infty$, since $\sqrt{A}/t$, $A > 0$, are bounded and $(S_t - t\theta)^2/t$, $A > 0$, are uniformly integrable. The lemma now follows by taking the limit as $A \to \infty$ in (5.1). □

To establish assertion (ii) of Theorem 3.1, let s be as in assertion (i). Then it follows from (2.3) and Lemma 5.1 that

$$\begin{aligned}o(\sqrt{A}) &\ge E^\pi\left[\frac{A}{s^2}M_s\right] + E^\pi\left[\frac{1}{s}\big(\sqrt{A}\,V_s - s\big)^2\right] \\ &\quad + \frac{1}{3!}AE^\pi[W_s] + \frac{1}{4!}AE^\pi[R_s] \\ &= T_1 + T_2 + T_3 + T_4,\end{aligned} \tag{5.2}$$

say. Next, it is easy to see that $T_1 \ge O(1)$ as $A \to \infty$ since A/s^2, $A > 0$, are bounded by assertion (i) and ξ_2 is bounded below. Also, T_3 is $O(1)$, by Schwarz's inequality, Proposition 2.1 and the boundedness of $(\sqrt{A}/s)^r$ for all $r > 0$. Finally, $T_4 = O(1)$ by assertion (i) and Lemma 2.1. It follows from these observations and (5.2) that $T_2/\sqrt{A} = o(1)$ as $A \to \infty$, which requires $s/\sqrt{A} \to \gamma$ in mean square as $A \to \infty$. □

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