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ASYMPTOTIC ANALYSIS OF MINIMAX STRATEGIES IN SURVEY SAMPLING

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Suppose that real numbers y_i are associated with the units $i = 1, 2, \dots, N$ of a population U and that the vector $y = (y_1, y_2, \dots, y_N)$ is known to be an element of the parameter space Θ. The statistician has to select a sample $s \subset U$ of n units and to employ y_i, $i \in s$, to estimate $\bar{y} = \sum y_i/N$. We propose to base this decision on an asymptotic version of the minimax principle. The asymptotically minimax principle is applied to three parameter spaces, including the parameter space considered by Scott and Smith and a space discussed by Cheng and Li. It turns out that stratified sampling is asymptotically minimax if the allocation is adapted to the parameter space. In addition we show that the commonly used ratio strategy [i.e., simple random sampling (srs) together with ratio estimation] and the RHC-strategy (see Rao, Hartley and Cochran) are asymptotically minimax with respect to parameter spaces chosen appropriately.

1. Introduction. We consider a population $U = \{1, 2, \dots, N\}$ and a parameter vector $y = (y_1, y_2, \dots, y_n)$. We define $\bar{y} = \sum y_i/N$, $\sigma_{yy} = \sum(y_i - \bar{y})^2/N$, where $\sum$ includes all $i \in U$.

A sample s is a nonempty subset of the population U. A sampling design p is defined to be a probability distribution on the set S of all samples. If p is a sampling design and if $p_s > 0$, $p_{s'} > 0$ implies $|s| = |s'|$ (i.e., the *sizes* $|s|$, $|s'|$ of s and s' are equal), then p is said to be of *fixed size*. p is a simple random sampling design (srs) of size n, $n \in \mathbb{N}$, if $p_s = (N-n)!\, n!/N!$ for $|s| = n$ and if $p_s = 0$ for $|s| \neq n$.

Following Godambe (1955) we consider (linear) estimators

$$t = t(s, y) = \sum t_{si} y_i$$

with $t_{si} = 0$ for $i \notin s$. The sample mean $\bar{y}_s = \sum_{i \in s} y_i/|s|$ is an estimator of special importance. The mean square error (MSE) of a strategy (p, t), p a design and t an estimator, is

$$\mathrm{MSE}(y; p, t) = \sum_{s \in S} p_s (t(s, y) - \bar{y})^2.$$

It may be meaningful to associate with $s \in S$ and $i \in s$ a random variable T_{si} and to use the randomized (linear) estimator $t = \sum_{i \in s} T_{si} y_i$ for which

$$\mathrm{MSE}(y; p, t) = E \sum_s p_s \Big(\sum_{i \in s} T_{si} y_i - \bar{y} \Big)^2.$$

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Let a subset $\Theta \subset \mathbb{R}^N$, called *parameter space*, be determined and consider a class Δ of strategies for which the minimax value

$$r(\Theta, \Delta) = \inf_{(p,t)\in\Delta} \sup_{y\in\Theta} \mathrm{MSE}(y; p, t)$$

is finite. Then $(p, t) \in \Delta$ is defined to be minimax if

$$\sup_{y\in\Theta} \mathrm{MSE}(y; p, t) = r(\Theta, \Delta).$$

In a number of papers properties of symmetry of Θ have been the basis for finding minimax strategies [see Blackwell and Girshick (1954), Joshi (1979), Royall (1970b) and Stenger (1979)]. Especially, the parameter space

$$\left\{ y \in \mathbb{R}^N : \frac{1}{N}\sum (y_i - \bar{y})^2 \le c^2 \right\}, \tag{1.1}$$

$c \neq 0$, has been discussed by Aggarwal (1959) and by Bickel and Lehmann (1981). For

$$\left\{ y \in \mathbb{R}^N : 0 \le y_i \le c \text{ for all } i \in U \right\}, \tag{1.2}$$

$c > 0$, see Hodges and Lehmann (1982). Our question is, what can be done if Θ is asymmetrical? Let us consider, for $x_i > 0$, $i \in U$, known, the spaces

$$\Theta_1 = \left\{ y \in \mathbb{R}^N : 0 \le y_i \le x_i \text{ for all } i \in U \right\}, \tag{1.3}$$

$$\Theta_2 = \left\{ y \in \mathbb{R}^N : \frac{1}{N}\sum \left(y_i - \frac{\bar{y}}{\bar{x}} x_i \right)^2 \le c^2 \right\}, \tag{1.4}$$

$$\Theta_3 = \left\{ y \in \mathbb{R}^N : \frac{1}{N}\sum \frac{\bar{x}}{x_i} \left(y_i - \frac{\bar{y}}{\bar{x}} x_i \right)^2 \le c^2 \right\}. \tag{1.5}$$

(1.3) contains (1.2) as a special case, while (1.4) and (1.5) are generalizations of (1.1). For (1.3) Scott and Smith (1975) have determined a minimax strategy in the class $\Delta = \{(p, t): p$ of fixed size 1, $t_{si} = \bar{x}/x_i$ for $s = \{i\}\}$. Unfortunately, no conclusive results seem to be within reach for sample sizes greater than 1. The difficulties arising from the spaces (1.4) and (1.5) seem to be unsurmountable.

So Cheng and Li (1983), interested especially in the space (1.5), propose to search for approximately minimax strategies. In fact they prove the approximate minimaxity of the RHC-strategy [see Rao, Hartley and Cochran (1962)] with respect to (1.5). [See 1.1 and 1.2 with $g(x_i) = \sqrt{x_i}$ and $L = L_2$ in Cheng and Li (1983).] They also derive approximately minimax solutions for representative strategies with sample sizes $n \ge 1$, given the parameter space (1.3).

In the present paper we propose an asymptotically minimax approach. We show how to expand asymptotically the minimax value and introduce the concept of an asymptotically minimax strategy. In this approach the RHC-strategy is not asymptotically minimax (with respect to Θ_3), in general. It is, however, asymptotically minimax if the sampling fraction n/N is small, i.e., converges to 0. This may be surprising in view of the inadmissibility of the strategy (which is a consequence of its randomization nature).

2. Results. We consider sequences (i) of populations $U^{(1)}, U^{(2)}, \dots,$ with sizes $N^{(1)} < N^{(2)} < \cdots$, (ii) of parameter spaces $\Theta^{(1)}, \Theta^{(2)}, \dots,$ with $\Theta^{(\nu)}$ equal to

$$\Theta_1^{(\nu)} = \left\{ y^{(\nu)} \in \mathbb{R}^{N^{(\nu)}} : 0 \le y_i^{(\nu)} \le x_i^{(\nu)} \text{ for all } i \in U^{(\nu)} \right\}$$

or

$$\Theta_2^{(\nu)} = \left\{ y^{(\nu)} \in \mathbb{R}^{N^{(\nu)}} : \frac{1}{N^{(\nu)}} \sum \left(y_i^{(\nu)} - \frac{\bar{y}^{(\nu)}}{\bar{x}^{(\nu)}} x_i^{(\nu)} \right)^2 \le c^2 \right\},$$

or $\Theta_3^{(\nu)}$ defined analogously, and (iii) of sample sizes $n^{(1)}, n^{(2)}, \dots,$ with existing limit

$$f = \lim \frac{n^{(\nu)}}{N^{(\nu)}} \tag{2.1}$$

and define for $\nu = 1, 2, \dots,$

$$r^{(\nu)} = r(\Theta^{(\nu)}, \Delta^{(\nu)}) = \inf_{\Delta^{(\nu)}} \sup_{\Theta^{(\nu)}} \operatorname{MSE}(y^{(\nu)}; p^{(\nu)}, t^{(\nu)}),$$

where $\Delta^{(\nu)}$ is the class of strategies with sample size $n^{(\nu)}$.

Associated with the vector $x^{(\nu)}$ defining $\Theta^{(\nu)}$ is a distribution function

$$G^{(\nu)}(\xi) = \left| \left\{ i \in U^{(\nu)} : x_i^{(\nu)} \le \xi \right\} \right| / N^{(\nu)},$$

i.e., $G^{(\nu)}(\xi)$ is the fraction of units of $U^{(\nu)}$ with x-values not greater than ξ. We assume that the limit

$$\Gamma(\xi) = \lim_{\nu} G^{(\nu)}(\xi) \tag{2.2}$$

exists for all $\xi \in \mathbb{R}$.

It will be shown that, under appropriate conditions, the limit $\lim_\nu n^{(\nu)} r^{(\nu)}$ exists and is a function of Γ and f. If this function is denoted by $\rho(\Gamma, f)$ it seems natural to use

$$\frac{1}{n^{(\nu)}} \rho\left(G^{(\nu)}, \frac{n^{(\nu)}}{N^{(\nu)}} \right)$$

as an approximation for $r(\Theta^{(\nu)}, \Delta^{(\nu)})$.

Let $(p^{(\nu)}, t^{(\nu)})$, $\nu = 1, 2, \dots,$ be a sequence of strategies. $(p^{(\nu)}, t^{(\nu)})$ is defined to be asymptotically minimax if

$$\lim_{\nu} n^{(\nu)} \sup_{\Theta^{(\nu)}} \operatorname{MSE}(y^{(\nu)}; p^{(\nu)}, t^{(\nu)}) = \rho(\Gamma, f).$$

It would be cumbersome to indicate permanently the sequence of populations, parameter spaces, etc. under consideration. In most circumstances it is not misleading to suppress the superscript ν. Subsequently we state our main results in this simplified notation.

We derive under weak assumptions

$$\rho(\Gamma, f) = \frac{1}{4} \left(\left[\int \xi \, d\Gamma \right]^2 - f \int \xi^2 \, d\Gamma \right) \tag{2.3}$$

for $\Theta = \Theta_1$ (and f small; see Theorems 4.1 and 5.1). Consequently, $r(\Theta_1, \Delta)$ may be approximated by

$$\frac{1}{4n}\left(\left[\int \xi \, dG\right]^2 - \frac{n}{N}\int \xi^2 \, dG\right) = \frac{1}{4n}\left(\bar{x}^2 - \frac{n}{N}\frac{1}{N}\sum x_i^2\right).$$

Further results are derived for stratified populations (see Section 4). We prove that $r(\Theta_2, \Delta)$ and $r(\Theta_3, \Delta)$ may be approximated by

$$\frac{c^2}{n}\left(1 - \frac{n}{N}\right) \quad \text{and} \quad \frac{c^2}{n}\frac{z}{\bar{x}} \le \frac{c^2}{n}\left(1 - \frac{n}{N}\right),$$

respectively, where z is defined in (3.10); x-proportionate and proportionate allocations define strategies which are asymptotically minimax for Θ_1 and Θ_2, respectively. A more involved allocation is shown to lead to a strategy with the asymptotic minimax property for Θ_3.

In addition we prove that the ratio strategy is asymptotically minimax with respect to Θ_2 (see Theorem 4.2).

REMARK 2.1. In an earlier paper [see Stenger (1983)] $(p^{(\nu)}, t^{(\nu)}) \in \Delta^{(\nu)}$ has been defined to be asymptotically minimax if

$$\sup \lim_{\nu} n^{(\nu)} \mathrm{MSE}(y^{(\nu)}; p^{(\nu)}, t^{(\nu)}) \le \sup \lim_{\nu} n^{(\nu)} \mathrm{MSE}(y^{(\nu)}; p^{(\nu)}, t^{(\nu)})$$

for all $(p^{(\nu)}, t^{(\nu)}) \in \Delta^{(\nu)}$, $\nu = 1, 2, \ldots$. An important drawback of such an approach is that the usual minimaxity of a strategy $(p^{(\nu)}, t^{(\nu)})$ does not imply its asymptotic minimaxity.

It should be noted that the use of the operator $\lim_\nu \sup$ (instead of $\sup \lim_\nu$ used in the earlier approach) is closely related to the asymptotic expansion of the minimax value $r^{(\nu)}$.

REMARK 2.2. It is known that the ratio estimator can be justified by other methods too, by Bayesian methods [see Ericson (1969)] and by superpopulation methods [see Royall (1970a)]. But in both cases it is impossible to justify srs at the same time.

3. Stratification of the population U. This section deals with one population U, one parameter space Θ and one sample size n (not with sequences). The smallest value occurring in $x_1, x_2, \ldots, x_N > 0$ is denoted by $c(1)$, the next greater value by $c(2), \ldots$ and the greatest by $c(H)$. Then

$$0 < c(1) < c(2) < \cdots < c(H).$$

It will be convenient to form the strata

$$U(h) = \{i \in U: x_i = c(h)\}, \qquad h = 1, 2, \ldots, H,$$

and to define $N(h) = |U(h)|$ and

$$\bar{y}(h) = \frac{1}{N(h)} \sum_{i \in U(h)} y_i, \qquad \sigma_{yy}(h) = \frac{1}{N(h)} \sum_{i \in U(h)} (y_i - \bar{y}(h))^2$$

for $h = 1, 2, \ldots, H$ and $y \in \Theta$. Then $N = \Sigma N(h)$, $\bar{x} = \Sigma N(h)c(h)/N$ and

$$\bar{y} = \sum \frac{N(h)}{N} \bar{y}(h), \qquad \sigma_{yy} = \sum \frac{N(h)}{N} \left(\sigma_{yy}(h) + [\bar{y}(h) - \bar{y}]^2 \right).$$

Furthermore [see (1.3), (1.4) and (1.5)],

$$(3.1) \quad \Theta_1 = \left\{ y \in \mathbb{R}^N : 0 \le y_i \le c(h) \text{ for } i \in U(h);\ h = 1, 2, \ldots, H \right\},$$

$$(3.2) \quad \Theta_2 = \left\{ y \in \mathbb{R}^N : \sum \frac{N(h)}{N} \left[\sigma_{yy}(h) + \left[\bar{y}(h) - \frac{\bar{y}}{\bar{x}} c(h) \right]^2 \right] \le c^2 \right\},$$

$$(3.3) \quad \Theta_3 = \left\{ y \in \mathbb{R}^N : \bar{x} \sum \frac{N(h)}{N} \frac{1}{c(h)} \left[\sigma_{yy}(h) + \left[\bar{y}(h) - \frac{\bar{y}}{\bar{x}} c(h) \right]^2 \right] \le c^2 \right\}.$$

We observe a random H-vector $\mathbf{n}$, called *allocation*, with integer valued components $n(1), n(2), \ldots, n(H)$, summing up to n, and draw samples of sizes $n(1), n(2), \ldots, n(H)$ from the strata $U(1), U(2), \ldots, U(H)$ by srs. We use as an estimator

$$t(s, y) = \sum_h \tau(\mathbf{n}, h) \bar{y}_s(h),$$

where $\tau(\mathbf{n}, h) = 0$ if $n(h) = 0$ and where $\bar{y}_s(h)$ is the sample mean for $U(h)$, provided $n(h) > 0$, $h = 1, 2, \ldots, H$. For the stratified strategy (p, t) so defined we have

$$(3.4) \quad \begin{aligned} \mathrm{MSE}(y; p, t) &= E \sum{}' \tau^2(\mathbf{n}, h) \frac{\sigma_{yy}(h)}{n(h)} \frac{N(h) - n(h)}{N(h) - 1} \\ &\quad + E \left[\sum \left(\tau(\mathbf{n}, h) - \frac{N(h)}{N} \right) \bar{y}(h) \right]^2, \end{aligned}$$

where Σ' includes all h with $N(h) > 1$ and $n(h) > 0$. With

$$(3.5) \qquad \tilde{t}(s, y) = \sum \frac{N(h)}{N} \bar{y}_s(h)$$

and nonstochastic $n(1), n(2), \ldots, n(H) > 0$ (which implies $n \ge H$), (3.4) simplifies to

$$(3.6) \qquad \mathrm{MSE}(y; p, \tilde{t}) = \sum \left[\frac{N(h)}{N} \right]^2 \frac{\sigma_{yy}(h)}{n(h)} \frac{N(h) - n(h)}{N(h) - 1}.$$

Three nonrandomized allocations are of special importance for our discussion.

Define for $h = 1, 2, \ldots, H$,

$$a_1(h) = \frac{N(h)c(h)}{\sum N(k)c(k)}, \tag{3.7}$$

$$a_2(h) = \frac{N(h)}{N}, \tag{3.8}$$

$$a_3(h) = \frac{N(h)c(h)}{Nz + nc(h)}, \tag{3.9}$$

where z is the unique solution of

$$\sum \frac{N(h)c(h)}{Nz + nc(h)} = 1. \tag{3.10}$$

By the convexity of $(a + bx)^{-1}$ and by the concavity of $x(a + bx)^{-1}$, where $a, b > 0$, we obtain

$$1 - \frac{n}{N} \frac{\sum[N(h)/N]c^2(h)}{\bar{x}^2} \leq \frac{z}{\bar{x}} \leq 1 - \frac{n}{N}. \tag{3.11}$$

Define for $j = 1, 2, 3$,

$$n_j'(h) = \langle na_j(h) \rangle,$$

where $\langle \kappa \rangle$ is the greatest integer not greater than κ. We choose $\tilde{n}_j(h)$ such that

$$n_j'(h) \leq \tilde{n}_j(h) \leq n_j'(h) + 1,$$

$$\sum_h \tilde{n}_j(h) = n.$$

It is not difficult to see that the conditions

$$\frac{c(H)}{c(1)} \frac{1}{N(h)} < \frac{n}{N} \leq \frac{c(1)}{c(H)} \frac{N(h) - 1}{N(h)}, \tag{3.12}$$

$$\frac{1}{N(h)} < \frac{n}{N} \leq \frac{N(h) - 1}{N(h)}, \tag{3.13}$$

$$\frac{c(H)}{c(1)} \frac{1}{N(h)} < \frac{n}{N} \leq \frac{c^2(1)}{2c^2(H)} \frac{N(h) - 1}{N(h)}, \tag{3.14}$$

for $j = 1, 2, 3$, respectively, imply $1 < na_j(h) \leq N(h) - 1$ and, consequently,

$$1 \leq \tilde{n}_j(h) \leq N(h), \qquad h = 1, 2, \ldots, H. \tag{3.15}$$

[Note that (3.11) is used to derive the sufficiency of (3.14). (3.14) is more restrictive than (3.12), (3.12) more restrictive than (3.13).] Therefore, $\tilde{n}_j(1), \tilde{n}_j(2), \ldots, \tilde{n}_j(H)$ is a (nonrandomized) allocation, for $j = 2$ the usual proportionate allocation and for $j = 1$ the so-called x-proportionate allocation [see Raj (1968), page 67].

LEMMA 3.1. *For the designs* $\tilde{p}_1$, $\tilde{p}_2$ *and* $\tilde{p}_3$ *associated with the allocations defined by* (3.7), (3.8) *and* (3.9) *and for the estimator* $\tilde{t}$ *defined by* (3.5) *we have*

$$\sup_{\Theta_j} \mathrm{MSE}(y; \tilde{p}_j, \tilde{t})$$

$$\le \frac{1}{4}\sum\left[\frac{N(h)}{N}\right]^2 \frac{N(h)}{N(h)-1} c^2(h)\left[\frac{\bar{x}}{nN(h)c(h)/N-\bar{x}} - \frac{1}{N(h)}\right] \quad (\text{for } j=1) \tag{3.16}$$

$$\le c^2 \max_h \frac{N(h)}{N}\frac{N(h)}{N(h)-1}\left[\frac{1}{nN(h)/N-1} - \frac{1}{N(h)}\right] \quad (\text{for } j=2) \tag{3.17}$$

$$\begin{aligned}&\le \frac{c^2}{\bar{x}} \max_h \frac{N(h)}{N}\frac{N(h)}{N(h)-1} c(h)\\ &\quad\times\left[\frac{1}{n[N(h)c(h)]/[Nz+nc(h)]-1} - \frac{1}{N(h)}\right] \quad (\text{for } j=3)\end{aligned} \tag{3.18}$$

[z *defined in* (3.10)].

PROOF. We derive from (3.6) for $j = 1,2,3$,

$$\begin{aligned}\mathrm{MSE}(y; \tilde{p}_j, \tilde{t}) &= \sum\left[\frac{N(h)}{N}\right]^2 \frac{N(h)}{N(h)-1}\sigma_{yy}(h)\left[\frac{1}{\tilde{n}_j(h)} - \frac{1}{N(h)}\right]\\ &\le \sum\left[\frac{N(h)}{N}\right]^2 \frac{N(h)}{N(h)-1}\sigma_{yy}(h)\left[\frac{1}{na_j(h)-1} - \frac{1}{N(h)}\right]\end{aligned} \tag{3.19}$$

[see the left-hand part of (3.15)]. (3.19) achieves its maximum at $\sigma_{yy}(h) = c^2(h)/4$ if $j = 1$. For $j = 2$ and $j = 3$ we have the restrictions

$$\sum \frac{N(h)}{N}\sigma_{yy}(h) \le c^2 \quad [\text{see } (3.2)],$$

$$\bar{x}\sum \frac{N(h)}{N}\frac{\sigma_{yy}(h)}{c(h)} \le c^2 \quad [\text{see } (3.3)],$$

respectively, and the maximum of (3.19) is achieved in a corner. The lemma follows. □

4. Sequences of stratified populations. In this section we consider sequences of populations, parameter spaces and sample sizes as stated in Section 2. In addition we assume that the auxiliary variable only has finitely many possible values as the population size goes to infinity. These values are denoted by $c(1), c(2), \dots, c(H)$ and $0 < c(1) < c(2) < \cdots < c(H)$ is assumed as earlier. We define for $h = 1,2,\dots,H$ and $\nu = 1,2,\dots,$

$$U^{(\nu)}(h) = \{i \in U^{(\nu)};\ x_i^{(\nu)} = c(h)\}$$

and use $N^{(\nu)}(h)$, $\bar{y}^{(\nu)}(h)$ and $\sigma_{yy}^{(\nu)}(h)$ in a straightforward sense. Hence we have to replace Θ_1 by

$$\Theta_1^{(\nu)} = \{y^{(\nu)}: 0 \le y_i^{(\nu)} \le c(h) \text{ for } i \in U^{(\nu)}(h);\ h = 1,2,\dots,H\}$$

and analogously Θ_2 by $\Theta_2^{(\nu)}$ and Θ_3 by $\Theta_3^{(\nu)}$.

Under the conditions of this section the distribution functions $G^{(\nu)}$, $\nu = 1,2,\dots$ (defined by $x^{(\nu)}$, $\nu = 1,2,\dots,$ in Section 2), are step functions with steps at $c(1), c(2),\dots, c(H)$. We assume

$$\beta(h) = \lim_{\nu} \frac{N^{(\nu)}(h)}{N^{(\nu)}} > 0 \tag{4.1}$$

for $h = 1,2,\dots,H$ [see (2.2)].

Subsequently (in Section 4) the superscript ν will be suppressed.

LEMMA 4.1. *Consider a sequence of stratified strategies (p, t) with existing limits*

$$a(h) = \lim_{\nu} E\frac{n(h)}{n}, \qquad h = 1,2,\dots,H. \tag{4.2}$$

Then $a(1), a(2),\dots, a(H) > 0$ implies

$$\liminf_{\nu} n \sup_{\Theta_j} \mathrm{MSE}(y; p, t) \ge \frac{1}{4}\sum \beta(h)c^2(h)\left[\frac{\beta(h)}{a(h)} - f\right] \qquad (\textit{for } j = 1)$$

$$\ge c^2 \max_h \left[\frac{\beta(h)}{a(h)} - f\right] \qquad (\textit{for } j = 2)$$

$$\ge \frac{c^2}{\sum \beta(k)c(k)} \max_h c(h)\left[\frac{\beta(h)}{a(h)} - f\right] \qquad (\textit{for } j = 3).$$

If h exists with $a(h) = 0$, then for $j = 1,2,3$,

$$\liminf_{\nu} n \sup_{\Theta_j} \mathrm{MSE}(y; p, t) = \infty.$$

PROOF. Define for $i = 1,2,\dots,N$; $h = 1,2,\dots,H$; $\varepsilon > 0$ (and $\nu = 1,2,\dots$),

$$y_i(h, \varepsilon) = \begin{cases} \varepsilon, & \text{for } i \in U(h), \\ 0, & \text{otherwise.} \end{cases}$$

For the vector $y(h, \varepsilon) \in \mathbb{R}^N$ with these components we compute $\sigma_{yy}(1) = \sigma_{yy}(2) = \cdots = \sigma_{yy}(H) = 0$ and

$$\bar{y}(k) = \begin{cases} \varepsilon, & \text{if } k = h, \\ 0, & \text{otherwise.} \end{cases}$$

Hence for ε sufficiently small, $y(h, \varepsilon) \in \Theta_j$ for $j = 1, 2, 3$, $h = 1, 2, \ldots, H$, $\nu = 1, 2, \ldots,$ and

$$\sup_{\Theta_j} \mathrm{MSE}(y; p, t) \geq \mathrm{MSE}(y(h, \varepsilon); p, t) = \varepsilon^2 E\left[\tau(\mathbf{n}, h) - \frac{N(h)}{N}\right]^2. \tag{4.3}$$

We assume $\liminf n \sup \mathrm{MSE}(y; p, t) < \infty$. By (4.3) we derive for $h = 1, 2, \ldots, H$ and K sufficiently large,

$$K > nE\left[\tau(\mathbf{n}, h) - \frac{N(h)}{N}\right]^2 = n \operatorname{var} \tau(\mathbf{n}, h) + n\left[E\tau(\mathbf{n}, h) - \frac{N(h)}{N}\right]^2$$

and consequently

$$\lim E\tau(\mathbf{n}, h) = \beta(h), \tag{4.4}$$

$$\lim E\tau^2(\mathbf{n}, h) = \beta^2(h). \tag{4.5}$$

From (3.4) we get

$$\begin{aligned} n\,\mathrm{MSE}(y; p, t) &\geq E{\sum}' \sigma_{yy}(h) \tau^2(\mathbf{n}, h)\left(\frac{N}{n(h)} - \frac{n}{N}\frac{N}{N(h)}\right) \\ &= \sum \sigma_{yy}(h)\left[E\tau^2(\mathbf{n}, h)\frac{n}{n(h)} - \frac{n}{N}\frac{N}{N(h)} E\tau^2(\mathbf{n}, h)\right], \end{aligned} \tag{4.6}$$

where $\tau^2(\mathbf{n}, h)/n(h)$ is defined to be 0 for $n(h) = 0$. With the last definition,

$$\sqrt{\frac{n(h)}{n}}\sqrt{\tau^2(\mathbf{n}, h)\frac{n}{n(h)}} = \tau(\mathbf{n}, h)$$

and therefore,

$$E\frac{n(h)}{n} E\tau^2(\mathbf{n}, h)\frac{n}{n(h)} \geq [E\tau(\mathbf{n}, h)]^2. \tag{4.7}$$

By (4.1) and (4.4), $\lim En(h)/n > 0$ and

$$E\tau^2(\mathbf{n}, h)\frac{n}{n(h)} \geq \frac{[E\tau(\mathbf{n}, h)]^2}{E[n(h)]/n}$$

for $h = 1, 2, \ldots, H$ and for ν sufficiently large. Consequently, (4.6) and (4.7) give

$$n\,\mathrm{MSE}(y; p, t) \geq \sum \sigma_{yy}(h)\left(\frac{[E\tau(\mathbf{n}, h)]^2}{E[n(h)]/n} - \frac{n}{N}\frac{N}{N(h)} E\tau^2(\mathbf{n}, h)\right).$$

With

$$A(h) = \frac{[E\tau(\mathbf{n}, h)]^2}{E[n(h)]/n} - \frac{n}{N}\frac{N}{N(h)} E\tau^2(\mathbf{n}, h)$$

we have, therefore,

$$\sup_{\Theta_j} n\,\mathrm{MSE}(y; p, t) \geq \sum \frac{c^2(h)}{4} A(h) \qquad (\text{for } j = 1)$$

$$(4.8) \qquad \geq \max_h \frac{c^2}{N(h)/N} A(h) \qquad (\text{for } j = 2)$$

$$\geq \max_h \frac{c^2 c(h)}{\bar{x} N(h)/N} A(h) \qquad (\text{for } j = 3).$$

By (4.4) and (4.5),

$$\lim_\nu A(h) = \beta(h)\left[\frac{\beta(h)}{a(h)} - f\right] \quad \text{for } h = 1, 2, \dots, H$$

and the lemma follows from (4.8). □

Lemma 4.2. *For an arbitrary strategy sequence* (p, t) *we have*

$$\liminf_\nu n \sup_{\Theta_j} \mathrm{MSE}(y; p, t) \geq \frac{1}{4}\left[\left[\sum \beta(h) c(h)\right]^2 - f \sum \beta(h) c^2(h)\right] \quad (\text{for } j = 1)$$

$$\geq c^2(1 - f) \quad (\text{for } j = 2)$$

$$\geq \frac{c^2 \zeta}{\sum \beta(h) c(h)} \quad (\text{for } j = 3),$$

where ζ *is the unique solution of* [*see* (3.10)]

$$(4.9) \qquad \sum \frac{\beta(h) c(h)}{\zeta + f c(h)} = 1.$$

Proof. The set

$$A = \left\{(a(1), a(2), \dots, a(H)) : a(1), \dots, a(H) \geq 0; \sum a(h) = 1\right\}$$

is compact. So there exists a subsequence of strategies with existing limits (4.2) and it is sufficient to prove the lemma for this subsequence.

It is not difficult to see that, starting with an arbitrary strategy (p, t), we find a stratified strategy $(\bar{p}, \bar{t})$ with the same sample size as (p, t) and with the property

$$\sup_{\Theta_j} \mathrm{MSE}(y; \bar{p}, \bar{t}) \leq \sup_{\Theta_j} \mathrm{MSE}(y; p, t) \quad \text{for } j = 1, 2, 3.$$

[This is a consequence of the symmetry properties of the sets Θ_j, $j = 1, 2, 3$. See Blackwell and Girshick (1954), page 226.] So it is sufficient to prove the lemma for a sequence of stratified strategies (p, t) with (4.2), as considered in Lemma 4.1.

It is easily seen that the minima of the functions

$$\sum \beta(h)c^2(h)\left[\frac{\beta(h)}{a(h)} - f\right], \qquad \max_h \left[\frac{\beta(h)}{a(h)} - f\right], \qquad \max_h c(h)\left[\frac{\beta(h)}{a(h)} - f\right]$$

defined on the set A are

$$\left[\sum \beta(h)c(h)\right]^2 - f\sum \beta(h)c^2(h), \qquad 1 - f, \qquad \zeta,$$

respectively. The assertion follows with Lemma 4.1. □

We evaluate the limits of (3.16), (3.17) and (3.18) and obtain by Lemma 4.2:

THEOREM 4.1. *Let* $c(1), c(2), \dots, c(H)$ *be the possible values of the auxiliary variable and assume that* H *is fixed as the population size goes to infinity. Assume, furthermore,* (2.1), (2.2) *and* (4.1). *Then*

$$\begin{aligned}
\lim_\nu nr(\Theta_j, \Delta) &= \lim_\nu \inf_\Delta \sup_{\Theta_j} n\,\mathrm{MSE}(y; p, t)\\
&= \frac{1}{4}\left[\left[\sum \beta(h)c(h)\right]^2 - f\sum \beta(h)c^2(h)\right] && \left(\text{for } j = 1 \text{ provided } f \le \frac{c(1)}{c(h)}\right)\\
&= c^2(1 - f) && (\text{for } j = 2 \text{ provided } f < 1)\\
&= \frac{c^2\zeta}{\sum \beta(h)c(h)} && \left(\text{for } j = 3 \text{ provided } f \le \frac{c^2(1)}{2c^2(H)}\right)
\end{aligned}$$

[*see* (4.9)].

From Lemma 3.1 and Theorem 4.1 it is evident that stratified sampling with x-proportionate allocation is asymptotically minimax with respect to Θ_1. Stratified sampling with proportionate allocation is asymptotically minimax with respect to Θ_2, and the allocation defined in (3.9) leads to a strategy which is asymptotically minimax with respect to Θ_3. We are interested in finding other strategies with the asymptotic minimax property.

THEOREM 4.2. *Assume the conditions stated in Theorem* 4.1. *Then the ratio strategy* (p, t), *where* p *denotes srs of size* n *and* $t = \bar{x}\bar{y}_s/\bar{x}_s$, *is asymptotically minimax with respect to* Θ_2.

PROOF. Define $z(h) = \bar{y}(h) - \bar{y}c(h)/\bar{x}$ and $\tau(\mathbf{n}, h) = \bar{x}n(h)/\sum n(k)c(k)$. Then, by (3.4), with $u(h)$ and $\nu(h, k)$ appropriately defined,

$$\mathrm{MSE}(y; p, t) = \sum_h u(h)\sigma_{yy}(h) + \sum_{h,k} \nu(h, k)z(h)z(k).$$

As is well known, $n\,\mathrm{MSE}(y;\,p,t)$ converges to

$$(1-f)\sum\beta(h)\big[\sigma_{yy}(h)+z^2(h)\big].$$

Since $\mathrm{MSE}(y;\,p,t)$ is a polynomial in $z(1),\dots,\sigma_{yy}(1),\dots,$ the convergence is uniform on every compact set, especially on

$$C(\varepsilon)=\Big\{\big(z(1),\dots,z(H),\sigma_{yy}(1),\dots,\sigma_{yy}(H)\big):\ \sum\beta(h)\big[\sigma_{yy}(h)+z^2(h)\big]\le c^2+\varepsilon\Big\},\qquad \varepsilon>0.$$

Now, for ν sufficiently large,

$$C^{(\nu)}=\left\{\big(z(1),\dots,z(H),\sigma_{yy}(1),\dots,\sigma_{yy}(H)\big):\sum\frac{N(h)}{N}\big[\sigma_{yy}(h)+z^2(h)\big]\le c^2\right\}$$

is a subset of $C(\varepsilon)$ and, taking into account the uniformity of the convergence, we derive

$$\begin{aligned}\lim_{\nu}\sup_{C^{(\nu)}} n\,\mathrm{MSE}(y;\,p,t)&\le\lim_{\nu}\sup_{C(\varepsilon)} n\,\mathrm{MSE}(y;\,p,t)\\&=\sup_{C(\varepsilon)}\lim_{\nu} n\,\mathrm{MSE}(y;\,p,t)=(c^2+\varepsilon)(1-f).\end{aligned}$$

The minimaxity of (p,t) is, therefore, a consequence of Theorem 4.1. □

THEOREM 4.3. *Assume the conditions stated in Theorem* 4.1. *Then the RHC-strategy is asymptotically minimax with respect to* Θ_3 *if* (*and only if*) *the sampling fraction* n/N *converges to* 0.

PROOF. For the RHC-strategy (p,t) we have

$$\mathrm{MSE}(y;\,p,t)=\frac{1}{n}\left(1-\frac{n}{N}\right)\frac{N}{N-1}\frac{1}{N}\sum\frac{\bar{x}}{x_i}\left(y_i-\frac{\bar{y}}{\bar{x}}x_i\right)^2$$

and, therefore,

$$\sup_{\Theta_3}\mathrm{MSE}(y;\,p,t)=\frac{c^2}{n}\left(1-\frac{n}{N}\right)\frac{N}{N-1}.$$

Consequently

$$\lim_{\nu} n\sup_{\Theta_3}\mathrm{MSE}(y;\,p,t)=c^2(1-f).$$

Now, it may be shown that for ζ defined in (4.9),

$$c^2(1-f)\le\frac{c^2\zeta}{\Sigma\beta(h)c(h)}$$

with equality if (and only if) $f=0$. Hence the RHC-strategy is (only) asymptotically minimax for $\Theta=\Theta_3$ if the sampling function n/N is small, i.e., converges to $f=0$. □

5. Sequences of nonstratified populations. It is desirable to drop the assumption that the populations $U^{(\nu)}$, $\nu = 1, 2, \dots$, are divided into a fixed number of strata (and that the parameter spaces are symmetric with respect to permutations within the strata). This may be done without difficulty for the Scott–Smith parameter space Θ_1.

THEOREM 5.1. *Assume* (2.1), (2.2) *and the existence of* x_0, x_∞ *with* $0 < x_0 < x_i^{(\nu)} \le x_\infty$ *for all* $i \in U^{(\nu)}$ *and* $\nu = 1, 2, \dots$. *Assume, furthermore,*

$$0 < f < x_0/x_\infty, \tag{5.1}$$

$$\Gamma \text{ is strictly increasing on } (x_0, x_\infty]. \tag{5.2}$$

Then, for Θ_1, (2.3) *is true.*

PROOF. (Superscripts suppressed.) The interval $(x_0, x_\infty]$ is divided into H parts of equal length $(x_\infty - x_0)/H$. We define

$$c(h) = x_0 + h\frac{x_\infty - x_0}{H}, \qquad h = 1, 2, \dots, H.$$

For $i \in U$ we find $h = 1, 2, \dots, H$ with

$$c(h) - \frac{x_\infty - x_0}{H} < x_i \le c(h)$$

and define $x_i^+ = c(h)$. x^+ is used to define the approximations

$$\Theta_1^+ = \{y : 0 \le y_i \le x_i^+ \text{ for all } i \in U\},$$

$$\Theta_1^- = \left\{y : 0 \le y_i \le x_i^+ - \frac{x_\infty - x_0}{H} \text{ for all } i \in U\right\}$$

of Θ_1. Evidently $\Theta_1^- \subset \Theta_1 \subset \Theta_1^+$.

Denote the minimax values for the spaces $\Theta_1^-, \Theta_1, \Theta_1^+$ by r^-, r, r^+, respectively. Then

$$r^- \le r \le r^+. \tag{5.3}$$

Now (5.1) implies (3.12) for ν sufficiently large and according to Theorem 4.1 we have

$$4\lim_\nu nr^+ = \left[\sum \beta(h)c(h)\right]^2 - f\sum \beta(h)c^2(h),$$

where [see (5.2)]

$$\beta(h) = \Gamma(c(h)) - \Gamma\left(c(h) - \frac{x_\infty - x_0}{H}\right) > 0.$$

Hence

$$4\lim_H \lim_\nu nr^+ = \left[\int \xi \, d\Gamma\right]^2 - f\int \xi^2 \, d\Gamma$$

and by analogy

$$4 \lim_H \lim_\nu nr^- = \left[\int \xi \, d\Gamma \right]^2 - f \int \xi^2 \, d\Gamma$$

so that the theorem follows by (5.3). □

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